Assignment 2
Due Wednesday, February 20.

In this assignment we consider the differential equation

\[(1 + s^2) \frac{d^2 u}{ds^2} + s \frac{du}{ds} - u = \frac{1}{\sqrt{1 + s^2}}\]  

on \(\mathbb{R} \times \mathbb{R}\).

(a) Apply the temporal substitution of variables \(s = \sinh t\) to rewrite (1) as the linear inhomogeneous differential equation

\[\frac{d^2 x}{dt^2} = x + \frac{1}{\cosh t}\]  

on \(\mathbb{R} \times \mathbb{R}\), where \(x = u \circ \sinh\). Hint: use the identity

\[\cosh^2 t - \sinh^2 t = 1\]  

to determine \(\cosh t\) in terms of \(s\).

(b) Rewrite (2) as a first order differential equation of the form

\[\frac{dy}{dt} = Ay + b(t)\]  

on \(\mathbb{R} \times \mathbb{R}^2\), where \(A\) is a \(2 \times 2\)-matrix and \(b : \mathbb{R} \to \mathbb{R}^2\).

(c) Show that

\[e^{tA} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\].

Hint: Use the power series for \(\sinh\) and \(\cosh\):

\[\sinh t = \sum_{k \text{ odd} \geq 0} \frac{1}{k!} t^k; \quad \cosh t = \sum_{k \text{ even} \geq 0} \frac{1}{k!} t^k,\]

or use the result from Exercise 6.9.

(d) Find a particular maximal solution of (4). Hint: Use the formula for \(y_0\) in Theorem 6.28, but take the factor \(e^{tA}\) outside of the integral. Use that \(\int \tanh s \, ds = \ln(\cosh s)\).

(e) Determine all maximal solutions of (4).

(f) Determine all maximal solutions of (1).