Exam, Differential Equations (Diff)

Friday, August 24, 2012.

Skriftlig eksamen, 3 timer. Alle sædvanlige hjælpemidler (d.v.s. skriftligt materiale, støjsvage lommeregnere, støjsvage computere og ikke-elektroniske skriveredskaber) er tilladt. Det er tilladt at skrive med blyant og benytte viskelæder, så længe skriften er læselig og udviskninger foretages grundigt. Overstregning trækker ikke ned og anbefales ved større ændringer.

Written exam, 3 hours. All "usual aids" (i.e. written material, quiet calculators, quiet computers and non-electronic writing instruments) are allowed during the exam. Answers written in pencil are accepted if the writing is legible and erasings are careful. Deletion is recommended for larger changes, instead of erasing.

NB: There are 2 pages with a total of 4 exercises.

1. Consider the differential equation

$$\frac{dy}{dt} = 4t\sqrt{y} \tag{1}$$

on the region $D_0 = \mathbf{R} \times (0, \infty)$.

(a) Determine all maximal solutions to (1) in D_0.

(b) Consider instead the equation (1) on the region $D = \mathbf{R} \times [0, \infty)$. Determine all maximal solutions to the initial value problem $y(1) = 1$.

2. Consider the differential equation

$$s^2 \frac{d^2x}{ds^2} + s\frac{dx}{ds} - x = 0 \tag{2}$$

on $\{(s, x) \in \mathbf{R}^2 : s > 0\}$, with the initial value condition

$$x(2) = 0, \quad x'(2) = 1. \tag{3}$$

(a) Apply the change of variables $t = \ln(s)$ and $y(t) = x(s)$ to (2) and prove that

$$\frac{d^2y}{dt^2} - y = 0 \tag{4}$$

on $\mathbf{R} \times \mathbf{R}$. What are the initial conditions for y?

(b) Solve (4) with the initial conditions you obtained in part (a).

(c) Derive the solution to the initial problem (2) and (3).

3. Consider the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

Compute e^{tA} for all $t \in \mathbf{R}$.

4. Consider the differential equation

$$\frac{d^2x}{dt^2} + t\frac{dx}{dt} - 2x = t \tag{5}$$

and the corresponding homogenous equation

$$\frac{d^2x}{dt^2} + t\frac{dx}{dt} - 2x = 0. \tag{6}$$

(a) Find all polynomials of order ≤ 2 which solve (5).

(b) Assume that $x(t) = \sum_{k=0}^{\infty} a_k t^k$, with $a_k \in \mathbf{R}$, is an absolutely convergent power series on $\mathbf{R}$. By substituting the power series into (6), write down a recurrence relation that the coefficients a_k have to satisfy in order for $x(t)$ to solve the equation. Deduce that then $a_2 = a_0$ and $a_k = 0$ for all even integers $k > 2$. Prove also that then for odd integers $k = 2j + 1$, with $j \geq 1$,

$$a_{2j+1} = \frac{(-1)^j \prod_{i=1}^{j}(2i-3)}{(2j+1)!} a_1. \tag{7}$$

(c) Now let $a_k = 0$ for all even $k \geq 0$, and define a_k for k odd by $a_1 = 1$ and by (7) for $k > 1$. Prove that the series

$$\sum_{j=0}^{\infty} a_{2j+1} t^{2j+1}$$

is absolutely convergent for all $t \in \mathbf{R}$, and that its sum $g(t)$ solves the homogeneous equation (6).

(d) Write down the complete solution to (5) in terms of the polynomials found in (a) and the function $g(t)$ defined in (c).