Exam, Differential Equations (Diff)

Friday, June 22, 2012.

Skriftlig eksamen, 3 timer. Alle sædvanlige hjælpemidler (d.v.s. skriftligt materiale, støjsvage lommeregnere, støjsvage computere og ikke-elektroniske skriveredskaber) er tilladt. Det er tilladt at skrive med blyant og benytte viskelæder, så længe skriften er læselig og udviskninger foretages grundigt. Overstregning trækker ikke ned og anbefales ved større ændringer.

Written exam, 3 hours. All "usual aids" (i.e. written material, quiet calculators, quiet computers and non-electronic writing instruments) are allowed during the exam. Answers written in pencil are accepted if the writing is legible and erasings are careful. Deletion is recommended for larger changes, instead of erasing.

1. Consider the differential equation

$$x\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 + x^2\ln(x) = t^2x^2 \tag{1}$$

on $\{(t,x) \in \mathbf{R}^2 : x > 0\}$.

(a) Apply the change of variables $u = \ln(x)$ to (1) and show that x satisfies (1) if and only if u satisfies the differential equation

$$\frac{d^2u}{dt^2} + u = t^2 \tag{2}$$

on $\mathbf{R} \times \mathbf{R}$.

(b) Determine a particular maximal solution of (2). (Hint: try a polynomial.)

(c) Determine all maximal solutions of (2).

(d) Determine all maximal solutions of (1).

2. Consider the differential equation

$$t^2\frac{d^2x}{dt^2} + \sin(t)x = f(x) \tag{3}$$

on $\mathbf{R} \times \mathbf{R}$, where

$$f(x) = \begin{cases} \frac{\sin(x^2)}{|x|^{\frac{3}{2}}} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

(Note that f is continuous.) We impose the initial value conditions

$$x(t_0) = \eta_0, \quad \frac{dx}{dt}(t_0) = 0 \tag{4}$$

for some $t_0, \eta_0 \in \mathbf{R}$.

(a) Prove that for every $t_0 \neq 0$ and every η_0 there exists an open interval I containing 0, such that there exists a solution (I, x) of (3) and (4).

(b) Assume $t_0 \neq 0$ and $\eta_0 \neq 0$. Prove that there exists a subinterval J of I (from (a)), such that $(J, x\big|_J)$ is the unique solution of (3) and (4) on that subinterval.

3. Consider the partial differential equation on $\mathbf{R} \times (-\pi, \pi)$

$$\frac{\partial^2}{\partial t^2}\phi(t,x) - \frac{\partial^2}{\partial x^2}\phi(t,x) = f(x) \qquad \big((t,x) \in \mathbf{R} \times (-\pi,\pi)\big) \tag{5}$$

with initial value conditions

$$\phi(0,x) = \phi_0(x) \quad \text{and} \quad \frac{\partial}{\partial t}\phi(t,x)\Big|_{t=0} = \phi_1(x) \qquad \Big(x \in (-\pi,\pi)\Big), \tag{6}$$

where f, ϕ_0 and ϕ_1 are functions on (π, π). Assume that f, ϕ_0 and ϕ_1 can be extended to C^2, 2π-periodic functions on $\mathbf{R}$. We denote the Fourier coefficients of these functions by α_k, β_k and γ_k respectively. Hence

$$f(x) = \sum_{k\in\mathbf{Z}} \alpha_k e^{ikx}, \quad \phi_0(x) = \sum_{k\in\mathbf{Z}} \beta_k e^{ikx}, \quad \phi_1(x) = \sum_{k\in\mathbf{Z}} \gamma_k e^{ikx}, \qquad \Big(x \in (-\pi,\pi)\Big).$$

(a) Assuming that ϕ extends to a C^2-function on $\mathbf{R}\times\mathbf{R}$ that is 2π-periodic in the second variable, show that for every $k \in \mathbf{Z}$ there exists a C^2-function c_k such that $\phi(t,x) = \sum_{k\in\mathbf{Z}} c_k(t)e^{ikx}$. Show that the c_k satisfy the initial condition problem

$$\frac{d^2 c_k(t)}{dt^2} + k^2 c_k(t) = \alpha_k \qquad (t \in \mathbf{R}) \tag{7}$$

$$c_k(0) = \beta_k \quad \text{and} \quad \frac{dc_k}{dt}(0) = \gamma_k.$$

(b) Write down the maximal solution to (7) for each $k \in \mathbf{Z}$.

Now take

$$f(x) = \sin(x), \quad \phi_0(x) = \sin^2(x), \quad \phi_1(x) = 0 \qquad \Big(x \in (-\pi,\pi)\Big) \tag{8}$$

(c) Determine the Fourier series $\sum_{k\in\mathbf{Z}} \alpha_k e^{ikx}$ and $\sum_{k\in\mathbf{Z}} \beta_k e^{ikx}$ of f and ϕ_0 respectively.

(d) Determine a solution to (5), (6) and (8).