Suppose some tangent vectors $\mathbf{v} \in T_P\mathcal{S}$ and $\mathbf{w} \in T_Q\mathcal{S}$ at different points P, Q of a surface are given. It is a natural question to ask when $\mathbf{v}$ and $\mathbf{w}$ should be considered parallel. If $\mathcal{S}$ is a plane, the answer is obvious, the tangent vectors are parallel if and only if they are parallel when considered as vectors in $\mathbb{R}^3$. However, for other surfaces the tangent spaces $T_P\mathcal{S}$ and $T_Q\mathcal{S}$ are not equal, and a more sophisticated notion has to be introduced. In fact, a reasonable definition only exists if a curve is given that connects P and Q. The notion of parallelism between vectors in $T_P\mathcal{S}$ and $T_Q\mathcal{S}$ will in general depend on the curve.

Let γ be a regular parametrized curve on $\mathcal{S}$. A (tangent) *vector field* $\mathbf{V}$ *along* γ is a smooth map which to each t assigns a vector $\mathbf{V}(t) \in \mathbb{R}^3$ such that $\mathbf{V}(t) \in T_{\gamma(t)}\mathcal{S}$. We will define the notion of parallelism of such a vector field. For simplicity we consider only vector fields of constant length $\|\mathbf{V}(t)\|$.

Definition 1. A vector field $\mathbf{V}(t)$ along γ of constant length is called *parallel* if the derivative $\mathbf{V}'(t)$ is normal to the tangent plane $T_P\mathcal{S}$ at each point $P = \gamma(t)$ of the curve.

The idea is that since the tangent space varies along γ, the vector field is forced to vary too, but we require that this variation is invisible from the surface. Thus a parallel vector field will be conceived as constant by a resident of the surface (a surface bug).

Notice that the condition that $\mathbf{V}'(t)$ is normal to the tangent plane for each t implies that $\frac{d}{dt}(\mathbf{V} \cdot \mathbf{V}) = 2\mathbf{V}' \cdot \mathbf{V} = 0$. The condition of constant length is thus actually a consequence of the other condition.

In our further analysis we will assume throughout that $\mathcal{S}$ is oriented, so that a unit normal vector $\mathbf{N}$ is given at each point.

Example. The unit tangent vectors to a great circle on a sphere form a parallel vector field. This is easily seen by simple geometric considerations, but it also follows from the next lemma.

Lemma 1. *A unit speed curve γ on a surface $\mathcal{S}$ is a geodesic if and only if its tangent vectors $\gamma'(t)$ form a parallel field.*

Proof. Immediate from Pressley's Definition 8.1. □

The definition of a parallel vector field makes use of the extrinsic concept of the normal to the surface. However, the discussion after Definition 1 suggests that it is *intrinsic*. Indeed, this follows from the next proposition, since it expresses the concept in terms of the first fundamental form alone.

Proposition 1. *Let γ be a regular curve on an orthogonal parametrized surface (σ, U), and let $\mathbf{V}(t)$ be a vector field along γ with constant non-zero length. Let $\varphi(t)$ denote the angle from σ_u to $\mathbf{V}(t)$ (determined up to multiples of 2π), and assume that it depends differentiably on t.*

Then $\mathbf{V}$ is parallel if and only if

$$\varphi' = \frac{1}{2\sqrt{EG}} \{u'E'_v - v'G'_u\} \tag{1}$$

at each point of the curve.

Proof. As in Note 3, let

$$\mathbf{X} = \frac{\sigma'_u}{\sqrt{E}} \quad \text{and} \quad \mathbf{Y} = \frac{\sigma'_v}{\sqrt{G}},$$

then at each point P of $\sigma(U)$, the triple $(\mathbf{X}, \mathbf{Y}, \mathbf{N})$ constitutes a positively oriented orthonormal basis for $\mathbb{R}^3$. It was shown in Note 3, that $\mathbf{X} \cdot \mathbf{Y}'$ is equal to the expression on the right side of (1).

Let

$$\hat{\mathbf{V}}(t) = \mathbf{N}(t) \times \mathbf{V}(t),$$

then $(\mathbf{V}, \hat{\mathbf{V}}, \mathbf{N})$ is also a positively oriented orthonormal basis for $\mathbb{R}^3$, at each point of γ. Since $\mathbf{V}$ has constant length, $\mathbf{V}' \perp \mathbf{V}$. Hence $\mathbf{V}'$ is a linear combination of $\hat{\mathbf{V}}$ and $\mathbf{N}$, and it is perpendicular to the tangent space if and only if the component along $\hat{\mathbf{V}}$ vanishes. Therefore, $\mathbf{V}$ is parallel if and only if $\mathbf{V}' \cdot \hat{\mathbf{V}} = 0$.

We claim that

$$\mathbf{V}' \cdot \hat{\mathbf{V}} = -\mathbf{X} \cdot \mathbf{Y}' + \varphi',$$

clearly this implies the statement of the proposition, in view of the preceding remarks.

By definition of $\varphi(t)$ we have

$$\mathbf{V} = \cos\varphi \mathbf{X} + \sin\varphi \mathbf{Y}. \tag{2}$$

Hence

$$\hat{\mathbf{V}} = -\sin\varphi \mathbf{X} + \cos\varphi \mathbf{Y}. \tag{3}$$

By differentiating (2) with respect to t and inserting (3) we obtain

$$\mathbf{V}' = -\varphi' \sin\varphi \mathbf{X} + \cos\varphi \mathbf{X}' + \varphi' \cos\varphi \mathbf{Y} + \sin\varphi \mathbf{Y}' = \varphi' \hat{\mathbf{V}} + \cos\varphi \mathbf{X}' + \sin\varphi \mathbf{Y}'$$

Using that $\mathbf{X}' \perp \mathbf{X}$ and $\mathbf{Y}' \perp \mathbf{Y}$ we conclude

$$\begin{aligned}\mathbf{V}' \cdot \hat{\mathbf{V}} &= \varphi' + (\cos\varphi \mathbf{X}' + \sin\varphi \mathbf{Y}') \cdot (-\sin\varphi \mathbf{X} + \cos\varphi \mathbf{Y}) \\ &= \varphi' + \cos^2\varphi \mathbf{X}' \cdot \mathbf{Y} - \sin^2\varphi \mathbf{Y}' \cdot \mathbf{X}.\end{aligned}$$

Notice that from $\mathbf{X} \cdot \mathbf{Y} = 0$ we obtain $\mathbf{X}' \cdot \mathbf{Y} + \mathbf{X} \cdot \mathbf{Y}' = 0$. Hence

$$\mathbf{V}' \cdot \hat{\mathbf{V}} = \varphi' - \cos^2\varphi \mathbf{X} \cdot \mathbf{Y}' - \sin^2\varphi \mathbf{Y}' \cdot \mathbf{X} = \varphi' - \mathbf{X} \cdot \mathbf{Y}'. \quad \square$$

Corollary 1. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be vector fields along* γ *of constant non-zero length, and suppose that* $\mathbf{V}$ *is parallel. Then* $\mathbf{W}$ *is parallel if and only if its angle with* $\mathbf{V}$ *is constant.*

Proof. If the vector fields make the angles $\varphi(t)$ and $\psi(t)$ with σ_u', then φ satisfies equation (1). Hence ψ satisfies the same equation if and only if $\psi' = \varphi'$, that is, if and only if the difference $\psi - \varphi$ is constant. The angle between $\mathbf{V}$ and $\mathbf{W}$ is exactly measured by that difference. $\square$

Corollary 2. *If* γ *is a geodesic, then a vector field* $\mathbf{W}$ *is parallel along* γ *if and only if its angle with* γ' *is constant.*

Proof. Combine Lemma 1 and Corollary 1 with $\mathbf{V} = \gamma'$. $\square$

The last corollary tells us exactly which vector fields are parallel along a geodesic. In the following proposition we generalize to other curves.

Proposition 2. *Let* γ *be a unit speed curve on* σ*, and let* $\mathbf{V}(t)$ *be a vector field along* γ *of constant non-zero length. Let* $\delta(t)$ *denote the angle from* $\gamma'(t)$ *to* $\mathbf{V}(t)$*, and assume that it depends differentiably on* t*. Then* $\mathbf{V}$ *is parallel if and only if*

$$\delta' = -\kappa_g$$

at each point of the curve.

Proof. We consider the curve in a neighborhood of a given point, and we may assume that this neighborhood is contained in an orthogonal patch. As in Prop. 1, let φ be the angle from σ_u to $\mathbf{V}(t)$, then $\theta = \varphi - \delta$ measures the angle from σ_u' to $\gamma'(t)$. The proposition is obtained by combining Proposition 1 with the theorem in Note 3. $\square$

We will now show that parallel vector fields exist. In fact, we can uniquely determine a parallel vector field along γ from any given tangent vector at some initial point:

Proposition 3. *Let $\gamma : I \to \mathcal{S}$ be a unit speed curve on $\mathcal{S}$, and let $t_0 \in I$, $\mathbf{v} \in T_{\gamma(t_0)}\mathcal{S}$ be given. Then there exists a unique parallel vector field $\mathbf{V}(t)$ along γ of constant length and with $\mathbf{V}(t_0) = \mathbf{v}$.*

Proof. We may assume $\mathbf{v} \neq 0$ (otherwise take $\mathbf{V}(t) = 0$). Let δ_0 be the angle from $\gamma'(t_0)$ to $\mathbf{v}$ and define

$$\delta(t) = \delta_0 - \int_{t_0}^{t} \kappa_g(s)\, ds.$$

Let $\mathbf{V}(t)$ be the unit vector field along $\gamma(t)$ which makes the angle $\delta(t)$ with $\gamma'(t)$. Then it follows from Prop. 2 that $\mathbf{V}$ is parallel. The uniqueness is seen similarly. □

Let $P = \gamma(t_0)$ and $Q = \gamma(t_1)$ be points on γ. Using Prop. 3 we define the map

$$\mathrm{T} : T_P\mathcal{S} \to T_Q\mathcal{S},$$

which assigns to $\mathbf{v} \in T_P\mathcal{S}$ the vector $\mathrm{T}(\mathbf{v}) = \mathbf{V}(t_1) \in T_Q\mathcal{S}$, where $\mathbf{V}$ is the parallel vector field with $\mathbf{V}(t_0) = \mathbf{v}$. The map is called *parallel transport* along γ.

Definition 2. Two vectors $\mathbf{v} \in T_P\mathcal{S}$ and $\mathbf{w} \in T_Q\mathcal{S}$ are said to be *parallel* with respect to γ if $\mathbf{w}$ is the parallel transport along γ of $\mathbf{v}$.

Lemma 2. *Parallel transport is linear $T_P\mathcal{S} \to T_Q\mathcal{S}$.*

Proof. It suffices to prove the following. If $\mathbf{V}$ and $\mathbf{W}$ are parallel vector fields along the same curve γ, then the vector field $t \mapsto \lambda\mathbf{V}(t) + \mu\mathbf{W}(t)$ is also parallel, for all $\lambda, \mu \in \mathbb{R}$. This follows easily from Definition 1. □

Example Let $\mathcal{S} = S^2$, and let γ be the parallel circle of constant latitude $u = u_0$ (see p. 61), where $\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$. Let $a = \cos u_0, b = \sin u_0$ then

$$\gamma(t) = \sigma(u_0, t) = (a \cos t, a \sin t, b)$$

is a (non unit speed) parametrization of γ. Let $P = \gamma(0) = \sigma(u_0, 0) = (a, 0, b)$. We will determine the parallel transport along γ of the vector $\mathbf{v} = \sigma_u = (-b, 0, a) \in T_P\mathcal{S}$. If φ is the angle from $\sigma_u'(\gamma(t))$ to the vector $\mathbf{V}(t)$ of the parallel field, then by Prop. 1

$$\varphi' = b = \sin u_0$$

(because $u(t) = u_0$, $v(t) = t$, $E = 1$, $G = \cos^2 u$). Since $\varphi(0) = 0$ we obtain

$$\varphi(t) = bt.$$

In particular, when at $t = 2\pi$ the curve returns to the initial point P, the vector $\mathbf{v}$ has been displaced by the angle

$$\varphi(2\pi) - \varphi(0) = 2\pi b = 2\pi \sin u_0.$$