Note 3. An intrinsic formula for geodesic curvature. MAT 3GE, F2005

Let (σ, U) be an orthogonal surface parametrization (orthogonal means that $F = 0$), and let $\gamma(s) = \sigma(u(s), v(s))$ be a unit speed curve on σ. We will establish a formula for the geodesic curvature $\kappa_g(s)$ of γ, which involves only the first fundamental form of σ.

Let $\theta(t)$ be the angle from σ'_u to $\gamma'(s)$ in the tangent space $T_{(u,v)}\sigma$ (with respect to the orientation determined by σ). Then $\theta(s)$ is uniquely determined up to integral multiples of 2π. We assume that $s \mapsto \theta(s)$ is chosen to be differentiable (this can always be attained in a neighborhood of each s, since σ'_u and $\gamma'(s)$ depend differentiably on s).

Theorem. *The geodesic curvature is intrinsic and is given by*

$$\kappa_g = \theta' - \frac{1}{2\sqrt{EG}} \left\{ u' E'_v - v' G'_u \right\}.$$

Proof. We will first establish the following lemma. Let

$$\mathbf{X} = \frac{\sigma'_u}{\sqrt{E}} \quad \text{and} \quad \mathbf{Y} = \frac{\sigma'_v}{\sqrt{G}}$$

so that $(\mathbf{X}, \mathbf{Y}, \mathbf{N})$ is an orthonormal basis for $\mathbb{R}^3$ (depending on $(u, v) \in U$).

Lemma.

$$\begin{aligned}
\mathbf{X}'_u &= -\frac{E'_v}{2\sqrt{EG}}\mathbf{Y} + \frac{L}{\sqrt{E}}\mathbf{N}, & \mathbf{X}'_v &= \frac{G'_u}{2\sqrt{EG}}\mathbf{Y} + \frac{M}{\sqrt{E}}\mathbf{N}, \\
\mathbf{Y}'_u &= \frac{E'_v}{2\sqrt{EG}}\mathbf{X} + \frac{M}{\sqrt{G}}\mathbf{N}, & \mathbf{Y}'_v &= -\frac{G'_u}{2\sqrt{EG}}\mathbf{X} + \frac{N}{\sqrt{G}}\mathbf{N}.
\end{aligned}$$

Proof. Since $\mathbf{X}$ is a unit vector, the derivatives $\mathbf{X}'_u$ and $\mathbf{X}'_v$ are perpendicular to $\mathbf{X}$. Likewise, $\mathbf{Y}'_u$ and $\mathbf{Y}'_v$ are perpendicular to $\mathbf{Y}$. Since $(\mathbf{X}, \mathbf{Y}, \mathbf{N})$ is an orthonormal basis, it follows that

$$\begin{aligned}
\mathbf{X}'_u &= (\mathbf{X}'_u \cdot \mathbf{Y})\mathbf{Y} + (\mathbf{X}'_u \cdot \mathbf{N})\mathbf{N}, & \mathbf{X}'_v &= (\mathbf{X}'_v \cdot \mathbf{Y})\mathbf{Y} + (\mathbf{X}'_v \cdot \mathbf{N})\mathbf{N}, \\
\mathbf{Y}'_u &= (\mathbf{Y}'_u \cdot \mathbf{X})\mathbf{X} + (\mathbf{Y}'_u \cdot \mathbf{N})\mathbf{N}, & \mathbf{Y}'_v &= (\mathbf{Y}'_v \cdot \mathbf{X})\mathbf{X} + (\mathbf{Y}'_v \cdot \mathbf{N})\mathbf{N}.
\end{aligned}$$

It follows from the definition $\mathbf{X} = E^{-1/2}\sigma'_u$ that

$$\mathbf{X}'_u = -\frac{1}{2}E^{-3/2}E'_u\sigma'_u + E^{-1/2}\sigma''_{uu}$$

and hence, since $\mathbf{Y}$ and $\mathbf{N}$ are both perpendicular to σ'_u,

$$\begin{aligned}
\mathbf{X}'_u \cdot \mathbf{Y} &= E^{-1/2}\sigma''_{uu} \cdot \mathbf{Y} = (EG)^{-1/2}\sigma''_{uu} \cdot \sigma'_v, \\
\mathbf{X}'_u \cdot \mathbf{N} &= E^{-1/2}\sigma''_{uu} \cdot \mathbf{N} = E^{-1/2}L.
\end{aligned}$$

From $\sigma'_u \cdot \sigma'_v = 0$ and $\sigma'_u \cdot \sigma'_u = E$ we get by differentiation with respect to u and v, respectively, that $\sigma''_{uu} \cdot \sigma'_v = -\sigma'_u \cdot \sigma''_{vu} = -\frac{1}{2}E'_v$. The equation for $\mathbf{X}'_u$ follows, and the proof of the other three equations is similar. □

In order to establish the formula for κ_g we use the equations in the lemma to derive an expression for the dot product of $\mathbf{X}(u(s), v(s))$ with the derivative $\mathbf{Y}'$ of $s \mapsto \mathbf{Y}(u(s), v(s))$. By the chain rule $\mathbf{Y}' = u'\mathbf{Y}'_u + v'\mathbf{Y}'_v$, and hence the equations of the previous lemma imply that

$$\mathbf{X} \cdot \mathbf{Y}' = u' \frac{E'_v}{2\sqrt{EG}} - v' \frac{G'_u}{2\sqrt{EG}}.$$

We now come to the proof of the formula. By definition, the angle $\theta(s)$ is determined up to constant multiples of 2π by

$$\gamma'(s) = \cos\theta(s)\mathbf{X} + \sin\theta(s)\mathbf{Y}.$$

The multiples of 2π play no role, as the assertion only involves the derivative of θ.
Since $\mathbf{N} = \mathbf{X} \times \mathbf{Y}$ we obtain

$$\mathbf{N} \times \gamma' = -\sin\theta\mathbf{X} + \cos\theta\mathbf{Y}.$$

Furthermore, by differentiation with respect to s,

$$\gamma''(s) = -\theta' \sin\theta\mathbf{X} + \cos\theta\mathbf{X}' + \theta' \cos\theta\mathbf{Y} + \sin\theta\mathbf{Y}'$$

so that

$$\begin{aligned}
\kappa_g &= (\mathbf{N} \times \gamma') \cdot \gamma'' \\
&= (-\sin\theta\mathbf{X} + \cos\theta\mathbf{Y}) \cdot (-\theta' \sin\theta\mathbf{X} + \cos\theta\mathbf{X}' + \theta' \cos\theta\mathbf{Y} + \sin\theta\mathbf{Y}') \\
&= (\sin^2\theta + \cos^2\theta)\theta' - \sin^2\theta\mathbf{X} \cdot \mathbf{Y}' + \cos\theta^2\mathbf{X}' \cdot \mathbf{Y}
\end{aligned}$$

where in the last step ioot is used that $\mathbf{X} \cdot \mathbf{Y} = \mathbf{X} \cdot \mathbf{X}' = \mathbf{Y} \cdot \mathbf{Y}' = 0$. Since $\mathbf{X} \cdot \mathbf{Y} = 0$ we also have by differentiation that $\mathbf{X}' \cdot \mathbf{Y} = -\mathbf{X} \cdot \mathbf{Y}'$. Application of the identity $\cos^2 + \sin^2 = 1$ now yields

$$\kappa_g = \theta' - \mathbf{X} \cdot \mathbf{Y}'.$$

The theorem follows immediately. □

For example, we can determine the geodesic curvature along the coordinate curve $u \mapsto \sigma(u, v)$ as follows. If $t \mapsto \sigma(u(t), v)$ is a unit speed reparametrization, then $\|u'\sigma_u\| = 1$. Thus $u' = 1/\sqrt{E}$. Moreover, $\theta = 0$ and $v' = 0$, so we obtain from the formula above

$$\kappa_g = -\frac{E'_v}{2E\sqrt{G}}.$$

Similarly, the geodesic curvature along $v \mapsto \sigma(u, v)$ is found to be

$$\kappa_g = \frac{G'_u}{2G\sqrt{E}}.$$