In differential geometry one wants to extend the notions of calculus to surfaces. In Section 7.9 we have extended the notion of smoothness of a map, but what is the proper generalization of the *derivative* of such a map?

Recall that for a smooth map $F: \mathbb{R}^n \to \mathbb{R}^m$, the notion of the derivative at a point q is expressed in the Jacobian matrix $JF(q)$. For such a map F, the linear map $\mathbb{R}^n \to \mathbb{R}^m$ having $JF(q)$ as its matrix is called the *differential* of F at q, and it is denoted

$$dF_q: \mathbb{R}^n \to \mathbb{R}^m.$$

Notice that with this notion, the *chain rule* asserts that

$$d(G \circ F)_q = dG_{F(q)} \circ dF_q$$

for all smooth maps $F: \mathbb{R}^n \to \mathbb{R}^m$ and $G: \mathbb{R}^m \to \mathbb{R}^l$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be smooth surfaces, and let $f: \mathcal{S}_1 \to \mathcal{S}_2$ be smooth (see Definition 7.9). Let $p \in \mathcal{S}_1$ be given. We denote by $T_p\mathcal{S}_1$ and $T_{f(p)}\mathcal{S}_2$ the tangent spaces of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, at p and $f(p)$ (see Definition 7.6). Recall that these are 2-dimensional linear subspaces of $\mathbb{R}^3$.

Lemma. *Let $p \in \mathcal{S}_1$ and let $v \in T_p\mathcal{S}_1$. If F is any smooth extension of f at p, as in Theorem 7.9(c), then $dF_p(v) \in T_{f(p)}\mathcal{S}_2$.*

Furthermore, if γ is any smooth curve γ on $\mathcal{S}_1$ with tangent vector $\gamma'(0) = v$ at $p = \gamma(0)$ (such a curve exists according to Thm. 7.6) *then*

$$dF_p(v) = (f \circ \gamma)'(0). \tag{1}$$

Proof. Let γ be as above. It follows that

$$dF_p(v) = dF_p(\gamma'(0)) = (F \circ \gamma)'(0) = (f \circ \gamma)'(0),$$

the second equality by the chain rule. This proves (1). Since $f \circ \gamma$ is a smooth curve on $\mathcal{S}_2$ with $f \circ \gamma(0) = f(p)$, it follows from Thm. 7.6 that $(f \circ \gamma)'(0)$ belongs to $T_{f(p)}\mathcal{S}_2$. □

Definition. The *differential* (or *tangent map*) of f at $p \in \mathcal{S}_1$ is the linear map

$$df_p: T_p\mathcal{S}_1 \to T_{f(p)}\mathcal{S}_2$$

given by $v \mapsto dF_p(v)$, where F is a smooth extension of f at p.

It follows from (1) that

$$df_p(v) = (f \circ \gamma)'(0),$$

where γ is any smooth curve on $\mathcal{S}_1$ with $\gamma(0) = p$ and $\gamma'(0) = v$. Hence $df_p(v)$ does not depend on the choice of the extension F.

Example Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(q) = Aq + b$ for $q \in \mathbb{R}^3$ where A is a linear map and b a fixed vector. The differential dF_p of F at p is the linear map A, for all $p \in \mathbb{R}^3$. If $\mathcal{S}_1, \mathcal{S}_2$ are such that $F(\mathcal{S}_1) \subseteq \mathcal{S}_2$, then $f = F|_{\mathcal{S}_1}$ is a smooth map $\mathcal{S}_1 \to \mathcal{S}_2$, and it follows that the differential $df_p: T_P\mathcal{S}_1 \to T_{f(p)}\mathcal{S}_2$ is the restriction of $dF_p = A$ to $T_p\mathcal{S}_1$, for each $p \in \mathcal{S}_1$.

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