The purpose of this note is to clarify the definition of a smooth surface. It is meant to be read alongside with Sections 4.1 and 4.2.

We start by recalling this definition, which Pressley gives in two steps. First (p. 60) he defines the concept of a *surface*, and later (p. 67) that of a *smooth* surface. However, in the rest of the book *all surfaces considered are assumed smooth, even if this is not explicitly mentioned* (see p. 72).

Definition Let $\mathcal{S} \subset \mathbb{R}^3$ be a set.

1. A pair (U, σ), where $U \subset \mathbb{R}^2$ is an open set and $\sigma\colon U \to \mathbb{R}^3$ a continuous map, is called a *patch* (or a *chart*) of $\mathcal{S}$, if

(a) σ is injective,

(b) the image $\sigma(U)$ open in $\mathcal{S}$, and

(c) $\sigma^{-1}\colon \sigma(U) \to U$ is continuous.

The pair (U, σ) is called a *regular patch* of $\mathcal{S}$ if in addition

(d) $\sigma\colon U \to \mathbb{R}^3$ is smooth, and

(e) the vectors σ_u and σ_v are linearly independent at all $(u, v) \in U$.

2. An *atlas* of S is a set of patches which cover S (that is, S is the union of the image sets $\sigma(U)$).

3. S is called a *surface* if there exists an atlas of it, and it is called a *smooth surface* if there exists an atlas of it consisting of regular patches.

All the conditions in 1 are important. Recall that (b) means that $\sigma(U)$ is an open set in $\mathcal{S}$, equipped with the restriction of the metric of $\mathbb{R}^3$, or equivalently, that there exists an open set $W \subset \mathbb{R}^3$ such that $\sigma(U) = S \cap W$. Especially (c) can be quite cumbersome to verify in practice. Often one has to write down a formula for the inverse, from which it can be seen that it is continuous (see however Thm. 2 below). Pressley is rather vague about this in some of the examples.

Example 1 (Pressley Ex. 4.1 and 4.4). The plane $\mathcal{S}$ through $\mathbf{a}$ spanned by the vectors $\mathbf{p}$ and $\mathbf{q}$ (assumed to be orthonormal). Here $\mathcal{S}$ is equipped with the atlas consisting of a single patch, given by

$$\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}, \quad (u, v) \in U = \mathbb{R}^2.$$

The conditions (a)-(e) are easily verified in this example. For (b) we can take $W = \mathbb{R}^3$ (this can always be done if $\mathcal{S} = \sigma(U)$), and (c) follows from the expression given on p. 61, which shows that σ^{-1} is the restriction to $\mathcal{S}$ of the continuous map $\mathbf{r} \mapsto ((\mathbf{r} - \mathbf{a}) \cdot \mathbf{p}, (\mathbf{r} - \mathbf{a}) \cdot \mathbf{q})$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Example 2 (Pressley Ex. 4.2 and 4.5). The sphere S^2 is equipped with the atlas consisting of the two patches σ and $\tilde{\sigma}$ (see p. 61-62). Let us consider the former map. It maps the open square $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$ onto the intersection of S^2 with W (see p. 62). The following formula can be shown to hold for $(x, y, z) \in S^2 \cap W$:

$$\sigma^{-1}(x, y, z) = (\sin^{-1}(z), 2\cot^{-1}(\frac{y}{\sqrt{x^2 + y^2} - x})), \tag{1}$$

where $\sin^{-1}$ and $\cot^{-1}$ are inverse to $\sin\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$ and $\cot\colon (0, \pi) \to \mathbb{R}$. It follows from (1) that σ^{-1} is continuous. The proof for $\tilde{\sigma}$ is similar.

Now we look at some examples that reveal the importance of conditions (b) and (c), if we want reasonable surfaces.

Example 3. Let $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \vee z = 0\}$, the union of the xy-plane and the yz-plane. The maps

$$\sigma_1(u, v) = (u, v, 0), \quad \sigma_2(u, v) = (0, u, v)$$

from $\mathbb{R}^2$ to $\mathcal{S}$ are easily seen to satisfy (a) and (c)-(e), and together they cover $\mathcal{S}$. Nevertheless, these maps are not patches on $\mathcal{S}$, since condition (b) fails - the planes are not open subsets of $\mathcal{S}$. It can be seen (for example from Prop. 4.4), that $\mathcal{S}$ is not a smooth surface.

Example 4. Let $\gamma : (0, a) \to \mathbb{R}^2$ be a plane regular parametrized smooth curve with the shape of the number 6 (starting at the top for $t = 0$). For example, one could use a suitable section of the limaçon on p. 20. The main point is that as $t \to a$, the point $\gamma(t)$ converges to a point, say $\gamma(b)$, where $0 < b < a$ (this is the point in the number 6 where the lines meet). Let $\sigma(u, v) = (\gamma(u), v) \in \mathbb{R}^3$, where $(u, v) \in U = (0, a) \times (-1, 1)$ and put

$$\mathcal{S} = \sigma(U) = \{(x, y, z) \mid (x, y) \in \gamma\big((0, a)\big), -1 < z < 1\}.$$

Then (a)-(b) and (d)-(e) are satisfied but (c) fails: $\sigma(u, v)$ converges to $\sigma(b, v)$ as $u \to a$, but (u, v) does not converge to (b, v), so σ^{-1} is not continuous.

The following example gives a very general construction of smooth surfaces.

Example 5. Let $f\colon U \to \mathbb{R}$ be a continuous function defined on an open set $U \subseteq \mathbb{R}^2$, then the map $\sigma\colon U \to \mathbb{R}^3$ given by

$$\sigma(u, v) = (u, v, f(u, v))$$

is continuous and injective. The inverse map is the restriction to $\sigma(U)$ of the projection $\pi\colon (x, y, z) \mapsto (x, y)$, hence it is continuous. It follows that the set $\mathcal{S} = \sigma(U)$, which is called the *graph* of f over the xy-plane, is a surface. It is easily seen that the patch (U, σ) is regular if f is smooth, so that in this case the graph is a smooth surface.

Let $\mathcal{S}$ be a smooth surface, and let (U, σ) be a regular patch on the open subset $\sigma(U) \subseteq \mathcal{S}$. We call σ a *graph-patch* if there exists a smooth function $f\colon U \to \mathbb{R}$ for which σ has one of the following three forms

$$(u, v, f(u, v)), \quad (u, f(u, v), v), \quad \text{or} \quad (f(u, v), u, v),$$

so that it is the graph of f over one of the coordinate planes (see Example 5 above).

Theorem 1. *Every smooth surface $\mathcal{S}$ has an atlas consisting of graph-patches.*

We say that the surface is *locally a graph.* An example of such an atlas is given for the sphere on page 65.

Proof. Let $P \in \mathcal{S}$. We want to find a graph patch (V, τ) with $P \in \tau(V)$. Let (U, σ) be an arbitrary regular patch on an open set $\sigma(U) \subseteq \mathcal{S}$ such that $P = \sigma(q) \in \sigma(U)$. Write

$$\sigma(u, v) = (f(u, v), g(u, v), h(u, v)).$$

Since σ is regular the columns of the Jacobian

$$J\sigma = \begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{pmatrix}$$

are linearly independent. By rearranging the order of the coordinates on $\mathbb{R}^3$ we may arrange that the two first rows of $J\sigma(q)$ are independent. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection $(x, y, z) \mapsto (x, y)$ and put

$$F = \pi \circ \sigma : U \to \mathbb{R}^2.$$

It follows that $\det JF(q) \neq 0$. By the inverse function theorem (see Note 2) there exists an open neighborhood W of q in U such that F is a diffeomorphism of W onto the open set $V = F(W) = \pi(\sigma(W)) \subseteq \mathbb{R}^2$ (so far, the proof was identical to the first part of that of Proposition 4.1, see p. 94).

Let $\Phi = F^{-1} : V \to W$ and put

$$\tau = \sigma \circ \Phi : V \to \mathcal{S}.$$

We will now prove that (V, τ) is a graph patch on a neighborhood of P in $\mathcal{S}$. We notice first that since W is open in U and σ is a homeomorphism, the image

$$\tau(V) = \sigma(\Phi(V)) = \sigma(W)$$

is open in $\sigma(U)$, hence also in $\mathcal{S}$. Hence (V, τ) is a regular patch (see Proposition 4.2). Moreover, it contains $P = \sigma(q)$.

Next we remark that

$$\pi \circ \tau = \pi \circ \sigma \circ \Phi = I,$$

the identity map on V. Hence $\tau(x, y) = (x, y, f(x, y))$, where $f(x, y)$ is defined as the third coordinate of $\tau(x, y)$. Thus indeed (V, σ) is a graph-patch.

Hence each point $P \in \mathcal{S}$ lies in the image of a graph-patch. The collection of all these graph-patches is then an atlas of $\mathcal{S}$ $\square$

The following theorem, which is derived from Theorem 1, is useful when we consider examples of surfaces. It shows that in many cases we do not have to carry out the detailed investigation of σ^{-1}, as in Examples 4.1-4.2, in order to check whether σ is a patch. The point is, that if it is known already (for example from Theorem 4.1) that $\mathcal{S}$ is a smooth surface, then (b) and (c) are implied by (a), (d) and (e).

Theorem 2. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a smooth surface and let $\sigma: U \to \mathcal{S}$ be a smooth injective map of an open set $U \subseteq \mathbb{R}^2$ into $\mathcal{S}$ such that the vectors σ_u and σ_v are linearly independent at all points $(u, v) \in U$. Then $\sigma(U)$ is open in $\mathcal{S}$ and σ^{-1} is continuous $\sigma(U) \to U$ (so that σ is a surface patch on $\mathcal{S}$).*

Proof. Let $P = \sigma(q) \in \sigma(U)$. By Thm. 1 there exists a graph-patch on an open neighborhood of P in S, say, $\tau : V \to \tau(V) \subseteq \mathcal{S}$ where $\tau(u, v) = (u, v, f(u, v))$. Then if $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the projection $(x, y, z) \mapsto (x, y)$,

$$\tau \circ \pi|_{\tau(V)} = I, \tag{2}$$

the identity map $\tau(V) \to \tau(V)$.

The set $U_1 = \sigma^{-1}(\tau(V)) \subseteq U$ is open since σ is continuous, and the map $h = \pi \circ \sigma : U_1 \to V$ is smooth since σ and π are both smooth. Since $\sigma(U_1) \subseteq \tau(V)$ we see from (2) that

$$\sigma|_{U_1} = \tau \circ \pi \circ \sigma|_{U_1} = \tau \circ h.$$

By the chain rule, $d\sigma_q = d\tau_{h(q)} \circ dh_q$, and hence, since $d\sigma_q$ is injective, so is dh_q. It follows from the inverse function theorem that there exists a neighborhood $U_2 \subseteq U_1$ of q such that $h(U_2) \subseteq V$ is open and such that h is a diffeomorphism of U_2 onto its image $h(U_2)$. Now $\sigma(U_2) = \tau(h(U_2))$, and since τ is a homeomorphism, this is an open subset of $\mathcal{S}$. Hence $\sigma(U)$ contains an open neighborhood of P in $\mathcal{S}$.

Furthermore, since both τ and h are homeomorphisms, then so is $\sigma|_{U_2} = \tau \circ h|_{U_2}$. Hence σ^{-1} is continuous on a neighborhood of P. Since P was arbitrary in $\sigma(U)$, this completes the proof. □

Example 6. It follows from Theorem 4.1 that the sphere is a smooth surface (see Example 4.7). In order to verify that σ and $\tilde{\sigma}$ (see Example 4.2) are surface patches, we need only verify that they are smooth, injective and regular, whereas the conditions of open image and continuous inverse follow from Theorem 2. Thus, after all, it is not necessary to write down the expression (1) for σ^{-1} and the corresponding one for $\tilde{\sigma}^{-1}$.

In the final part of this note we discuss one of the examples from Section 4.4 (Example 4.13) more carefully.

Surfaces of revolution Consider a surface obtained by rotating about the z-axis a profile curve

$$\mathcal{C} \subset \{(x, z) \in \mathbb{R}^2 \mid x > 0\}.$$

The points that we obtain from rotating a given point (r, z) in the xz-plane have the form (x, y, z) with $\sqrt{x^2 + y^2} = r$. Hence the surface is given by

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2}, z) \in \mathcal{C}\}.$$

Following Pressley, we assume that the profile curve $\mathcal{C}$ has a smooth, injective and regular parametrization

$$(x, z) = \gamma(u) = (f(u), g(u))$$

for u in an open interval I, and we define

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)), \quad (u, v) \in I \times \mathbb{R}. \tag{3}$$

Then σ maps $I \times \mathbb{R}$ onto $\mathcal{S}$. The map σ is not injective, but its restriction to $U = I \times J$ is injective for each open interval $J \subset \mathbb{R}$ of length $\ell(J) \leq 2\pi$. Pressley's verification that $\mathcal{S}$ is a smooth surface, and that (U, σ) is a regular patch, leaves out the continuity of σ^{-1}. In fact, *this is false in general.* Thus, the statement made at the bottom of p. 82 is false. A counterexample can be obtained by rotating a profile curve of shape 6, in analogy with Example 4. In the following proposition the error is repaired by adding an extra assumption on γ.

Theorem 3. *Let $\gamma: I \to \mathcal{C} = \gamma(I)$ be an injective regular smooth curve in the halfplane $\{(x,z) \in \mathbb{R}^2 \mid x > 0\}$ for which the inverse $\gamma^{-1}: \mathcal{C} \to I$ is continuous. Then the surface of revolution $\mathcal{S}$ with profile curve $\mathcal{C}$ is a smooth surface, and each pair (U,σ) with σ given by (3) and $U = I \times J$, $\ell(J) \leq 2\pi$, is a regular patch.*

Proof. Let (U,σ) be as above with $U = I \times J$. We will show that $\sigma^{-1}: \sigma(U) \to U$ is continuous. It then follows from the arguments given on p. 82 that (U,σ) is a regular patch. Since $\mathcal{S}$ is covered by two such patches, for example with $J = (-\pi, pi)$ and $J = (0, 2\pi)$, it then follows that $\mathcal{S}$ is a smooth surface.

Let $\phi: J \to S^1$ be given by $\phi(v) = (\cos v, \sin v)$. Then ϕ is injective and $\phi^{-1}: \phi(J) \to J$ is continuous. Indeed, composing ϕ with a suitable rotation of S^1 we may assume $J \subseteq (-\pi, \pi)$, and using the formula $\tan(\frac{v}{2}) = \frac{\sin v}{1+\cos v}$ we then see that $\phi^{-1}(x,y) = 2\tan^{-1}(\frac{y}{1+x})$. This is clearly continuous in x and y.

Now if $(x,y,z) = \sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$ then $u = \gamma^{-1}(r,z)$ and $v = \phi^{-1}(x/r, y/r)$ where $r = \sqrt{x^2+y^2}$. Hence $(x,y,z) \mapsto (u,v)$ is continuous. □

Example 7. The *torus* (see p. 73) is the surface of revolution with profile curve

$$\mathcal{C} = \{(x,z) \mid (x-a)^2 + z^2 = b^2\}.$$

The profile curve is parametrized by $\gamma(u) = (a + b\cos u, b\sin u)$, where $u \in \mathbb{R}$. The theorem above cannot be used directly, because γ is not injective, but it can be applied to the restriction $\gamma|_I$ for each open interval I with $\ell(I) \leq 2\pi$. The continuity of the inverse of $\gamma|_I$ can be seen by an argument similar to the one given in the proof above, involving the function ϕ. It then follows from Theorem 3 that

$$\sigma(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u), \quad (u,v) \in I \times J,$$

is a regular patch. One thus obtains an atlas with four charts, each of the intervals I and J being either $(-\pi, \pi)$ or $(0, 2\pi)$, for example.

Alternatively, it follows from Theorem 4.1, applied to the equation

$$(\sqrt{x^2+y^2} - a)^2 + z^2 = b^2,$$

that the torus is a smooth surface, and hence from Theorem 2 above that $(I \times J, \sigma)$ is indeed a regular patch.

A similar error to the mentioned one in Example 4.13 occurs in Examples 4.10-4.12. Also here one must add the assumption on the parametrisation γ that it has a continuous inverse.

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