Smooth maps

The concept of a smooth map between two smooth surfaces is defined on p. 69. The definition can be phrased as follows:

Definition 1. Let $\mathcal{S}_1, \mathcal{S}_2$ be smooth surfaces. A continuous map $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ is called *smooth* if the map

$$\sigma_2^{-1} \circ f \circ \sigma_1 : \mathbb{R}^2 \to \mathbb{R}^2$$

is smooth for all patches (U_1, σ_1) and (U_2, σ_2) in some smooth atlases on $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. To be more precise, the map is defined on $\sigma_1^{-1}(f^{-1}(\sigma_2(U_2))) \subseteq \mathbb{R}^2$; the assumption of continuity of f implies that this is an open set.

The formula at the bottom of p. 69 shows that if f is smooth then $\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth for all regular patches on $\mathcal{S}_1$ and $\mathcal{S}_2$, also those which do not belong to the chosen atlases. Thus, the notion is independent of that choice.

Similar definitions can be given for maps $\mathbb{R}^n \to \mathcal{S}$, and for maps $\mathcal{S} \to \mathbb{R}^n$:

Definition 2. A continuous map $f\colon W \to \mathcal{S}$ or $\mathcal{S} \to \mathbb{R}^n$, where $W \subseteq \mathbb{R}^n$ is open, is called *smooth* if

$$\sigma^{-1} \circ f : \mathbb{R}^n \to \mathbb{R}^2, \quad \text{respectively} \quad f \circ \sigma : \mathbb{R}^2 \to \mathbb{R}^n$$

is smooth (on its proper domain of definition) for all patches (U, σ) in some smooth atlas on $\mathcal{S}$.

In particular, it makes sense to speak of smooth curves and smooth functions on a surface:

A *smooth curve on* $\mathcal{S}$ is a smooth map $\gamma\colon (\alpha, \beta) \to \mathcal{S}$. According to Definition 2 this is a continuous curve $\gamma\colon (\alpha, \beta) \to \mathcal{S}$ whose intersection with any regular patch (U, σ) can written as

$$\gamma(t) = \sigma(\beta(t)),$$

where β is a smooth plane curve in U. This expression is called the *local expression* for γ. Likewise, a *smooth function on* $\mathcal{S}$ is a smooth map $\mathcal{S} \to \mathbb{R}$.

It is easily seen that composition of the various types of smooth maps in Definitions 1 and 2 yield maps which are again smooth.

The following three lemmas offer different characterizations of the smoothness of a map from, into or between smooth surfaces.

Lemma 1. *Let $f\colon \mathcal{S} \to \mathbb{R}^n$ be a continuous map. The following conditions are equivalent.*

(a) f is smooth (Definition 2).

(b) Around each point $P \in \mathcal{S}$ there is an open neighborhood $V \subseteq \mathbb{R}^3$ of P and a smooth map $F\colon V \to \mathbb{R}^n$ such that $f|_{V \cap \mathcal{S}} = F|_{V \cap \mathcal{S}}$.

Thus, f is smooth if and only if it may be locally extended to a smooth map on open sets in $\mathbb{R}^3$. In particular, the coordinate functions x, y and z are smooth $\mathcal{S} \to \mathbb{R}$, and so is the inclusion map $\iota\colon \mathcal{S} \to \mathbb{R}^3$.

Proof. (b)⇒(a). This is clear, since in the neighborhood of P, $f \circ \sigma$ is equal to the smooth map $F \circ \sigma\colon U \to \mathbb{R}^n$.

(a)⇒(b). Let (U, σ) be a graph-patch (see Note 3, Thm. 1) covering a neighborhood of P in $\mathcal{S}$, and let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection on the plane over which σ exhibits $\mathcal{S}$ as a graph. The image $\sigma(U)$ is open in $\mathcal{S}$, hence equal to $W \cap \mathcal{S}$ for some open subset W in $\mathbb{R}^3$. Let $V = \pi^{-1}(U) \cap W$, and let $F = f \circ \sigma \circ \pi$. Then F is smooth and equals f on $V \cap \mathcal{S}$. □

Lemma 2. *Let $f\colon W \subseteq \mathbb{R}^n \to \mathcal{S}$ be a continuous map. The following conditions are equivalent.*

(a) f is smooth (Definition 2)

(b) f is smooth, considered as a map $W \to \mathbb{R}^3$.

Proof. Choose a smooth atlas of $\mathcal{S}$. Then $f(W) \subseteq \mathcal{S} = \cup\sigma(U)$, union over all charts σ from the atlas, and hence W is the union of the open sets $f^{-1}(\sigma(U))$.

(a)⇒(b). If f is smooth into $\mathcal{S}$, then by definition $\sigma^{-1} \circ f$ is smooth into $\mathbb{R}^2$ for all σ. Hence $\sigma \circ (\sigma^{-1} \circ f) = f\colon f^{-1}(\sigma(U)) \to \mathbb{R}^3$ is smooth for all σ, and hence $f\colon W \to \mathbb{R}^3$ is smooth.

(b)⇒(a). We may assume that the chosen atlas consists of graph-patches. Let (U, σ) be an arbitrary patch in this atlas. We must prove that $\sigma^{-1} \circ f$ is smooth. Let π denote the projection on the plane over which σ exhibits $\mathcal{S}$ as a graph. Then σ^{-1} is the restriction of π to $\sigma(U) \subseteq \mathcal{S}$. Since π is smooth $\mathbb{R}^3 \to \mathbb{R}^2$, we conclude that $\sigma^{-1} \circ f = \pi \circ f$ is smooth $f^{-1}(\sigma(U)) \to \mathbb{R}^2$. □

Remark. The lemma above shows, in particular, that a smooth curve on $\mathcal{S}$ is the same as a smooth curve in $\mathbb{R}^3$ whose image is contained in $\mathcal{S}$.

Lemma 3. *Let $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ be a continuous map. The following conditions are equivalent.*

(a) f is smooth (Definition 1).

(b) f is smooth, considered as a map from $\mathcal{S}_1$ to $\mathbb{R}^3$ (Definition 2).

(c) Around each point $P \in \mathcal{S}_1$ there is an open neighborhood $V \subseteq \mathbb{R}^3$ of P and a smooth map $F\colon V \to \mathbb{R}^3$ such that $f|_{V\cap\mathcal{S}_1} = F|_{V\cap\mathcal{S}_1}$.

A smooth map F as in (c) is called a *local smooth extension* of f at P.

Proof. (a) ⇔ (b): Apply Lemma 2 to $f \circ \sigma_1 : U_1 \subseteq \mathbb{R}^2 \to \mathcal{S}_2$ for each (σ_1, U_1).

(b) ⇔ (c): Immediate from Lemma 1. □

Examples. 1. The identity map of a smooth surface to itself is smooth. This is essentially the content of Proposition 4.1, but it also follows from (c)⇒(a) (choose F to be the identity map on $\mathbb{R}^3$).

2. Let $\mathcal{S}_1 = S^2$ be the unit sphere given by $x^2 + y^2 + z^2 = 1$, and let $\mathcal{S}_2$ be the ellipsoid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (see Exercise 4.9). The map $f\colon (x, y, z) \mapsto (ax, by, cz)$ is smooth $\mathcal{S}_1 \to \mathcal{S}_2$ since it is the restriction of the smooth map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the same expression.

3. The antipodal map $P \mapsto -P$ of S^2 to itself is a smooth map, since it is the restriction of the smooth map, multiplication by -1, on $\mathbb{R}^3$.

4. If $\mathcal{S}$ is a surface of revolution, the rotations about its axis are smooth $\mathcal{S} \to \mathcal{S}$ since they are restrictions of rotations of $\mathbb{R}^3$.

The differential of a map

In differential geometry one wants to extend the notions of calculus to surfaces. We have extended the notion of smoothness of a map, but what is the proper generalization of the *derivative* of such a map? Recall that for a smooth map $f: \mathbb{R}^n \to \mathbb{R}^m$, the notion of the derivative at a point q is expressed in the Jacobian matrix $Jf(q)$, or equivalently, in the differential df_q which is the linear map $\mathbb{R}^n \to \mathbb{R}^m$ having $Jf(q)$ as its matrix.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be smooth surfaces, and let $f: \mathcal{S}_1 \to \mathcal{S}_2$ be smooth. Let $P \in \mathcal{S}_1$ be given. We denote by $T_P\mathcal{S}_1$ and $T_{f(P)}\mathcal{S}_2$ the tangent spaces of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, at P and $f(P)$ (see p. 74). Recall that these are 2-dimensional linear subspaces of $\mathbb{R}^3$.

Lemma 4. *Let $P \in \mathcal{S}_1$ and let $v \in T_P\mathcal{S}_1$. If F is a local smooth extension of f at P, as in Lemma 3(c), then $dF_P(v) \in T_{f(P)}\mathcal{S}_2$.*

Furthermore, if γ is a smooth curve γ on $\mathcal{S}_1$ with tangent vector $\dot\gamma(0) = v$ at $P = \gamma(0)$, then

$$dF_P(v) = (f \circ \gamma)^{\cdot}(0).$$

Proof. Let $v \in T_P\mathcal{S}_1$, then by definition there exists a smooth curve γ on $\mathcal{S}_1$ with $\gamma(0) = P$ and $\dot\gamma(0) = v$. It follows that

$$dF_P(v) = dF_P(\dot\gamma(0)) = (F \circ \gamma)^{\cdot}(0) = (f \circ \gamma)^{\cdot}(0),$$

the second equality by the chain rule. Since $f \circ \gamma$ is a smooth curve on $\mathcal{S}_2$ it follows that $(f \circ \gamma)^{\cdot}(0)$ belongs to $T_{f(P)}\mathcal{S}_2$. $\square$

Definition 3. The *differential* (or *tangent map*) of f at $P \in \mathcal{S}_1$ is the linear map

$$df_P: T_P\mathcal{S}_1 \to T_{f(P)}\mathcal{S}_2$$

given by $v \mapsto dF_P(v)$, where F is a local smooth extension of f at P.

It follows from Lemma 4 that

$$df_P(v) = (f \circ \gamma)^{\cdot}(0),$$

where γ is any smooth curve on $\mathcal{S}_1$ with $\gamma(0) = P$ and $\dot\gamma(0) = v$. Hence $df_P(v)$ does not depend on the choice of the local extension F.

A similar definition of the differential of f can be given in each of the situations in Definition 2.

Example 5. Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be given by $F(q) = Aq + b$ for $q \in \mathbb{R}^3$ where A is a linear map and b a fixed vector. The differential dF_P of F at P is the linear map A, for all $P \in \mathbb{R}^3$. If $\mathcal{S}_1, \mathcal{S}_2$ are such that $F(\mathcal{S}_1) \subseteq \mathcal{S}_2$, then $f = F|_{\mathcal{S}_1}$ is a smooth map $\mathcal{S}_1 \to \mathcal{S}_2$, and it follows that the differential $df_P: T_P\mathcal{S}_1 \to T_{f(P)}\mathcal{S}_2$ is the restriction of $dF_P = A$ to $T_P\mathcal{S}_1$, for each $P \in \mathcal{S}_1$.