This note contains some improvements to Theorems 1.1 and 1.2.

Theorem 1.1a. *Let $f: U \to \mathbb{R}$ be a smooth function, where $U \subseteq \mathbb{R}^2$ is open. Let*

$$\mathcal{C} = \{(x, y) \in U \mid f(x, y) = 0\}$$

Let $P \in \mathcal{C}$ and assume that $\partial f/\partial x$ and $\partial f/\partial y$ are not both zero at P.

Then there exists an open neighborhood $W \subseteq U$ of P, an open interval $I \subseteq \mathbb{R}$ around 0, and a regular parametrized curve $\gamma: I \to \mathbb{R}^2$ which maps I bijectively onto $\mathcal{C} \cap W$. Moreover the inverse map $\gamma^{-1}: \mathcal{C} \cap W \to I$ is continuous.

That is, in a neighborhood of the point P, the level curve $\mathcal{C}$ can be parametrized as a regular curve without selfintersection. The shortcoming of the statement in Theorem 1.1 is that it only asserts that γ maps *into* $\mathcal{C}$.

The theorem is an easy consequence of the following result, after a possible interchange of x and y.

Theorem 1.1b. (The implicit function theorem.) *Let $f: U \to \mathbb{R}$ be a smooth function, where $U \subseteq \mathbb{R}^2$ is open. Let $\mathcal{C} = \{(x, y) \in U \mid f(x, y) = 0\}$ and let $P = (x_0, y_0) \in \mathcal{C}$. Assume that $\partial f/\partial y \neq 0$ at P.*

Then there exist open intervals I and J around x_0 and y_0, respectively, such that $W = I \times J \subseteq U$, and a smooth map $h: I \to J$ such that

$$\mathcal{C} \cap W = \{(x, h(x)) \mid x \in I\}, \tag{1}$$

that is, in a neighborhood of P the set $\mathcal{C}$ is the graph of h.

Proof. Assume for simplicity that $P = (0, 0)$ and that the value of $\frac{\partial f}{\partial y}$ at P is positive (the latter can be arranged by replacing f with $-f$, if necessary). Choose $\delta > 0$ such that the closed square $[-\delta, \delta] \times [-\delta, \delta]$ lies inside U and such that $\frac{\partial f}{\partial y} \geq c$ on this square, for some constant $c > 0$ (using the continuity of $\frac{\partial f}{\partial y}$). Then $y \mapsto f(x, y)$ is strictly increasing on $J = (-\delta, \delta)$ for each fixed $x \in J$.

In particular, since $P = (0, 0) \in \mathcal{C}$ we have $f(0, 0) = 0$, and hence $f(0, -\delta) < 0$ and $f(0, \delta) > 0$. By continuity of f there exists a positive number $\eta \leq \delta$ such that $f(x, -\delta) < 0$ and $f(x, \delta) > 0$ for all $x \in I = (-\eta, \eta)$.

Let $x \in I$. Since $y \mapsto f(x, y)$ is strictly increasing and continuous, there exists a unique y between $-\delta$ and δ where $f(x, y) = 0$. This value of y is denoted $h(x)$. Then $h: I \to J$ clearly satisfies (1), it remains to be seen that it is smooth.

We first prove that h is continuous. Fix $x \in I$ and let $y = h(x)$, then $f(x, y) = 0$. Let Δx be sufficiently small so that $x + \Delta x \in I$. Associated to Δx we define Δy such that $y + \Delta y = h(x + \Delta x)$, then also $f(x + \Delta x, y + \Delta y) = 0$.

The asserted continuity amounts to the statement that $\Delta y \to 0$ when $\Delta x \to 0$. The function

$$t \mapsto \varphi(t) = f(x + t\Delta x, y + t\Delta y)$$

is zero both for $t = 0$ and $t = 1$. By the mean value theorem (Rolle's theorem) there exists a number $\theta \in (0, 1)$ (depending on Δx) such that

$$\varphi'(\theta) = 0.$$

Differentiating φ by means of the chain rule we thus obtain

$$\frac{\partial f}{\partial x}(x+\theta\Delta x, y+\theta\Delta y)\Delta x + \frac{\partial f}{\partial y}(x+\theta\Delta x, y+\theta\Delta y)\Delta y = 0.$$

Since $|\frac{\partial f}{\partial x}|$ is bounded, and since $\frac{\partial f}{\partial y} \geq c > 0$, it follows that $\Delta y \to 0$ when $\Delta x \to 0$, as claimed.

Next we prove that h is differentiable, which with the notation from above amounts to the convergence of $\Delta y/\Delta x$ as $\Delta x \to 0$. In fact, this follows immediately from the equation above, moreover the limit is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{\frac{\partial f}{\partial x}(x,y)}{\frac{\partial f}{\partial y}(x,y)}.$$

Hence h is differentiable and satisfies

$$h'(x) = -\frac{\frac{\partial f}{\partial x}(x,h(x))}{\frac{\partial f}{\partial y}(x,h(x))}. \tag{1}$$

Finally, we prove by induction that h is smooth. Assuming that $h \in C^r(I)$ for some natural number r, we see from equation (1) that also h' belongs to $C^r(I)$. Hence $h \in C^{r+1}(I)$. $\square$

Example 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = xy$. Then $\partial f/\partial x(x_0,y_0) = y_0$ and $\partial f/\partial y(x_0,y_0) = x_0$. Hence both derivatives are zero only at the origin $P = (0,0)$. Thus the theorem tells that the level curve $f(x,y) = 0$ can be parametrized as a regular curve near all its points, except possibly $P = (0,0)$. Obviously, this level curve is the union of the two axes, and the conclusion of the above theorem does indeed fail at the origin.

Example 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = y^4 - y^2 + x^2$, and consider the level curve $\mathcal{C} = \{(x,y) \mid f(x,y) = 0\}$. The derivatives $\partial f/\partial x = 2x$ and $\partial f/\partial y = 4y^3 - 2y$ are zero if and only if $x = 0$ and $y = 0$ or $y = \pm 1/\sqrt{2}$. Of the three points $(0,0)$, $(0, 1/\sqrt{2})$, $(0, -1/\sqrt{2})$, only the first one belongs to $\mathcal{C}$. Hence $\mathcal{C}$ can be parametrized as a regular curve near all its points, except possibly the origin. In fact, it can be shown that $\mathcal{C}$ is a curve that has the shape of the figure 8, with the intersection placed at the origin.

Remark Theorem 1.1b and the proof above is easily generalized to functions of more variables, say $f(x,y)$ where $(x,y) = (x_1,\dots,x_{n-1},y) \in U \subseteq \mathbb{R}^n$ with $\partial f/\partial y \neq 0$ at P. The conclusion is, that there exists a neighborhood $I \times J \subseteq U$ of P, where $I \subseteq \mathbb{R}^{n-1}$ and $J \in \mathbb{R}$ are open sets, such that the set of solutions $(x,y) \in I \times J$ to the equation $f(x,y) = 0$ is the graph of a smooth function $y = h(x)$, defined on I.

The conclusion of Theorem 1.2 can be similarly improved, so that γ actually maps an interval containing t_0 *onto* a neighborhood of (x_0,y_0) in the level curve. This follows immediately form the proof of Theorem 1.2 given in the book.

Theorem 1.2a. *Let $\gamma\colon I \to \mathbb{R}^2$ be a parametrized plane curve, regular at $t_0 \in I$. Then, there exist a neighborhood U of $P = \gamma(t_0)$, a smooth function $f\colon U \to \mathbb{R}$ with $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \neq (0,0)$ at P, and an open interval $J \subset I$ around t_0 such that $\gamma\colon J \to \{(x,y) \in U \mid f(x,y) = 0\}$ is bijective.*