In this note two different notions of connectedness for sets are discussed. Connectedness appears, for example, in Pressley p. 72.

Let M be a non-empty metric space.

Definition. *1) M is called* connected *if it cannot be separated in two disjoint non-empty open subsets, that is, if $M = A_1 \cup A_2$ with A_1, A_2 open and disjoint, then A_1 or A_2 is empty (and A_2 or A_1 equals M).*

2) M is called pathwise (or arcwise) connected *if for each pair of points $a, b \in S$ there exists real numbers $\alpha \leq \beta$ and a continuous map $\gamma\colon [\alpha, \beta] \to M$ such that $\gamma(\alpha) = a$ and $\gamma(\beta) = b$ (in which case we say that a and b can be joined by a continuous path in M).*

3) A non-empty subset $E \subseteq M$ is called connected or pathwise connected if it has this property as a metric space with the restriction of the metric of M.

The above definition of "connected" is standard in the theory of metric spaces (and more generally in topology). However, the notion of "pathwise connected" is sometimes (for example by Pressley) also referred to as "connected". The precise relation between the two notions will be explained in this note.

For example, any convex subset $E \subseteq \mathbb{R}^n$ is pathwise connected, since by definition any two points from E can be joined by a straight line, hence a continuous curve, inside E. It follows from Theorem 2 below that such a subset is also connected.

It is easy to prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval, and likewise it is pathwise connected if and only if it is an interval. Thus for subsets of $\mathbb{R}$ the two definitions agree. As we shall see, this is not so in general.

The most fundamental property of connected sets is expressed in the following theorem, which generalizes the well-known fact that a continuous real function carries intervals to intervals (the intermediate value property).

Theorem 1. *Let $f\colon M \to N$ be a continuous map between metric spaces. If M is connected, then so is the image $f(M) \subseteq N$. Likewise, if M is pathwise connected then so is $f(M)$.*

Proof. 1) Assume $f(M) = B_1 \cup B_2$ with B_1, B_2 open and disjoint, and let $A_i = f^{-1}(B_i)$. Then A_1, A_2 are open, disjoint and with union M. Hence if M is connected then A_1 or A_2 is empty, and hence B_1 or B_2 is empty.

2) If $a, b \in M$ can be joined by a continuous path γ, then $f(a)$ and $f(b)$ are joined by the continuous path $\gamma \circ f$. $\square$

Theorem 2. *A pathwise connected metric space is also connected.*

Proof. Suppose M were pathwise connected but not connected. Then $M = A \cup B$ with A, B open, disjoint and nonempty. Let $a \in A$, $b \in B$, then there exists a continuous map $\gamma\colon [\alpha, \beta] \to M$ joining a to b. The image $C = \gamma([\alpha, \beta])$ is the disjoint union of $C \cap A$ and $C \cap B$. These sets are open relative to C, and they are nonempty since they contain a and b, respectively. Hence C is not connected. On the other hand, it follows from Thm. 1 that $C = \gamma([\alpha, \beta])$ is connected, so that we have reached a contradiction. $\square$

The converse statement is false. There exists subsets of, for example $\mathbb{R}^n$ ($n \geq 2$), which are connected but not pathwise connected (an example is given below). However, for open subsets of $\mathbb{R}^n$ the two notions of connectedness agree:

Theorem 3. *Each open connected subset E of $\mathbb{R}^n$ is also pathwise connected.*

Proof. The crucial property of $\mathbb{R}^n$ in this respect is that for each point $a \in \mathbb{R}^n$, all the open balls centered at a are pathwise connected.

A metric space is said to be *locally pathwise connected* if it has the following property. For each point $a \in \mathbb{R}^n$ and each $\epsilon > 0$ there exists an open pathwise connected set U such that $a \in U \subseteq K(a, \epsilon)$. It follows from the observation above that $\mathbb{R}^n$ is locally pathwise connected.

We will prove that in a locally pathwise connected metric space, all open connected sets E are pathwise connected.

For $a, b \in E$ we write $a \sim b$ if a and b can be joined by a continuous path in E. It is easily seen that this is an equivalence relation. Since E is open there exists for each $a \in E$ an open ball $K(a, \epsilon) \subseteq E$, hence an open pathwise connected set U with $a \in U \subseteq E$. For all points x in U we thus have $a \sim x$. It follows that the equivalence classes for $\sim$ are open. Let A be an arbitrary of these equivalence classes, and let B denote the union of all other equivalence classes. Then A and B are open, disjoint and have union E. Since E is connected, A or B is empty. Since $a \in A$, we conclude that $B = \emptyset$ and $A = E$. Hence all points of E are equivalent with each other, which means that E is pathwise connected. ☐

Theorem 4. *Let $S \subseteq \mathbb{R}^3$ be a surface. Each open connected subset E of S is also pathwise connected.*

Proof. According to the previous proof it suffices to prove that S is locally pathwise connected. This follows from the definition of a surface, since it shows that for each $a \in S$ there exists an open neighborhood of a in S which is homeomorphic to an open set in $\mathbb{R}^2$. ☐

Example. The graph of the function

$$f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is connected but not pathwise connected. The proof is left to the reader.

HS