Note 6. Areas and integrals MAT 3GE, F2004

This note gives a short introduction to the theory of plane integrals. This topic would most properly be treated by means of Lebesgue measure, but we do not assume any knowledge of measure theory here. Instead we will base it on calculus (Mat 1GB), but with omission of almost all proofs. The main purpose is to define the area of a set $D \subset \mathbb{R}^2$ (suitably nice) and the integral over D of a continuous function $D \to \mathbb{R}$. These notions are used in Pressley, Section 5.4.

1. Rectangles and block sets. Consider a plane set $D \subseteq \mathbb{R}^2$. If D is an (axes-parallel) *rectangle*, $D = [a,b] \times [c,d]$, where $a \le b$, $c \le d$, then we agree that it has the area $A(D) = (b-a)(d-c)$. Moreover, if $f : D \to \mathbb{R}$ is continuous we define the integral of f over D by

$$\int_D f \, dA = \int_a^b \int_c^d f(u,v) \, dv \, du.$$

One can prove that then also

$$\int_D f \, dA = \int_c^d \int_a^b f(u,v) \, du \, dv$$

(this is sometimes called the mini-Fubini theorem).

If D is not a rectangle, it is more complicated to define its area, and to define integrals over it. By a *block-set* we will understand a set K which is a finite union of (axes-parallel) closed rectangles. It is easily seen that such a set K can be cut up as a finite union of rectangles with mutually disjoint interiors. The area $A(K)$ is then defined as the sum of the areas of the subrectangles, and the integral $\int_K f$ of a continuous function f over K is defined as the sum of the integrals over the subrectangles. It is intuitively obvious, but somewhat messy to prove, that these notions are independent of the way the set is cut up.

2. Null sets. We will now consider more general compact sets $D \subseteq \mathbb{R}^2$. We say that a compact set D is a *null set* if for each $\epsilon > 0$ there exists a block-set K of area $< \epsilon$ such that $D \subset K$.

As an example, consider a curve $\gamma : [a,b] \to \mathbb{R}^2$, assumed to be C^1. This means that it is continuous on $[a,b]$ and C^1 on (a,b) with a derivative $\dot\gamma$ which has a continuous extension to $[a,b]$. Using the continuous arclength function $s(t)$, we can divide γ in N pieces of equal length L/N, where L is the total length. Each piece is contained in a square of side length L/N (take the square which has the same mid-point as the piece of curve). The union of these squares has area $\le N(L/N)^2 = L^2/N$, which is $\le \epsilon$ for N sufficiently large. Hence the image in $\mathbb{R}^2$ of the curve is a null set.

The same conclusion holds for a curve $\gamma : [a,b] \to \mathbb{R}^2$ which is *piecewise* C^1 (this means that $[a,b]$ can be written as a finite union of closed intervals on each of which γ is C^1), since a finite union of null sets is obviously a null set.

3. Compact sets with area. Let $D \subset \mathbb{R}^2$ be compact. By definition, a block-set is called *inner* (with respect to D) if is contained in D and *outer* (with respect to D) if it contains D. If k is an inner block-set and K an outer block-set then $k \subseteq K$ and hence $A(k) \leq A(K)$. Thus the numbers defined by

$$\underline{A}(D) = \sup_{k \text{ inner}} A(k), \quad \text{and} \quad \overline{A}(D) = \inf_{K \text{ outer}} A(K),$$

make sense and satisfy $\underline{A}(D) \leq \overline{A}(D)$. These values are called, respectively, the inner and outer *content* of D. We say that D *possesses an area* (or is contended) if its inner and outer contents are equal, and in that case we call the common value $A(D)$ the *area* of D. In particular, we see that if the compact set D is a null set, then it has area 0. It is also clear that a block-set has the area with which it was already equipped.

Notice that the boundary ∂D of D is squeezed between K and the interior k° of k. The set difference $K \setminus k^\circ$ is also a block-set, and it has area $A(K) - A(k)$. It follows that if D possesses an area, then ∂D is a null set. The converse is also true, *if ∂D is a null set, then D possesses an area.* We omit the proof.

4. Integrals. Assume that the compact set D does possess an area. We define the *plane integral* over D of a continuous function $f : D \to [0, \infty)$ by

$$\int_D f\, dA = \sup_{k \text{ inner}} \int_k f\, dA.$$

It is easily seen that $\int_k f\, dA \leq A(D) \sup f$ for all inner block-sets k, hence the supremum over k exists and is also bounded by $A(D) \sup f$. For a continuous function $f : D \to \mathbb{R}$ we put $f_+(x) = \max\{0, f(x)\}$ and $f_-(x) = \max\{0, -f(x)\}$, so that $f = f_+ - f_-$, and we then define

$$\int_D f = \int_D f_+ - \int_D f_-.$$

5. Example. Let $\phi : [a, b] \to \mathbb{R}$ be a continuous function with $0 < \phi(x)$ for $x \in (a, b)$, then the set of points between the x-axis and the graph of ϕ,

$$D = \{(x, y) \mid a \leq x \leq b,\, 0 \leq y \leq \phi(x)\}$$

has the area

$$A(D) = \int_a^b \phi(x)\, dx.$$

This follows from the definition of the integral as a common limit of lower and upper Riemann sums, which correspond exactly to the areas of inner and outer block-sets. Furthermore, the plane integral of a continuous function f over D is

$$\int_D f\, dA = \int_a^b \int_0^{\phi(x)} f(x, y)\, dy\, dx$$

(we will not prove this). When it comes to computation of plane integrals in practice, it is this formula which is used (not the definition given above).

For example, let D be the triangle $\{(x,y) \mid 0 \leq x \leq 1,\, 0 \leq x+y \leq 1\}$, then $A(D) = \int_0^1 1 - x\, dx = \frac{1}{2}$. Furthermore, if $f(x,y) = y$, then

$$\int_D f\, dA = \int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \frac{1}{2}(1-x)^2\, dx = \frac{1}{6}.$$

6. Remark. Not all compact sets D possess an area. An example is given in the following number. For the benefit of those readers who know measure theory, we mention that in contrast all compact sets are Lebesgue measurable. In fact, the outer content defined above equals the Lebesgue measure of the set. Moreover, it can be seen from the monotone convergence theorem, that our plane integral of a continuous function is identical with the integral with respect to Lebesgue measure of the funcion.

7. A compact set that does not have an area. Let $\{q_1, q_2, \dots\}$ be a countable dense subset of the unit square $B = [0,1] \times [0,1]$ (e.g. the set of points with rational coordinates). For each n, we cut away from B the open square with midpoint q_n and side length $2^{-(n+1)/2}$. The remaining set D is compact, being an intersection of closed subsets of B. Moreover, it has empty interior since its complement contains all the points q_n. Therefore, the inner content of D is zero. However, since the areas of the sets that were cut away sum up to at most $\sum_{n=1}^{\infty} 2^{-(n+1)} = 1/2$, it can be shown that the outer content of D is at least $1/2$.

8. Compact regions. In this course we consider areas and integrals over compact sets $D \subset \mathbb{R}^2$ of the following type. First of all we assume that the boundary ∂D of D is the image of a closed simple piecewise C^1 curve γ (i.e. $[a,b] \to \mathbb{R}^2$ piecewise C^1 with $\gamma(a) = \gamma(b)$ and $\gamma|_{[a,b)}$ one-to-one). It then follows from the Jordan curve theorem (Pressley p. 48) that ∂D divides the plane in two connected sets $\mathrm{int}(\gamma)$ and $\mathrm{ext}(\gamma)$. It can be shown that $D = \partial D$ or $D = \partial D \cup \mathrm{int}(\gamma)$. In the latter case, we call D a *compact region*. It follows from observations in numbers 2 and 3 that ∂D is a null set and hence that D possesses area.

Thus, in conclusion, a compact region D has area, and the plane integral $\int_D f\, dA$ of a continuous function over it makes sense.

9. Transformation of integrals. We mention the important theorem of transformation of plane integrals by a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $\Phi : \Omega \to \mathbb{R}^2$ be a C^1-map, which is a C^1-diffeomorphism onto its image $\Phi(\Omega)$. Then, if $D \subset \Omega$ is a compact region, so is its image $\Phi(D)$. The theorem asserts that

$$\int_{\Phi(D)} f\, dA = \int_D (f \circ \Phi)\, |\det(J\Phi)|\, dA$$

for $f : \Phi(D) \to \mathbb{R}$ continuous. In particular, thus the area of $\Phi(D)$, which is the plane integral over $\Phi(D)$ of $f = 1$, is obtained as the integral over D of the absolute of the Jacobian determinant of Φ.

10. Green's theorem (Pressley, p. 50). Green's theorem is the analog for plane integrals of the fundamental theorem of calculus, that $\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$. It expresses an integral of derivatives by means of values on the boundary of the domain of integration.

Let $D \subset \mathbb{R}^2$ be a compact region with boundary ∂D, and let $f(x, y), g(x, y)$ be C^1 functions on D. This means that they are continuous on D and C^1 on $\operatorname{int}(D)$ with derivatives having continuous extensions to D. Assume that ∂D is parametrized counterclockwise by $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$. Then

$$\int_D \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \, dA = \oint_\gamma f \, dx + g \, dy,$$

where the integral on the right side is defined as $\int_a^b f \frac{dx}{dt} + g \frac{dy}{dt} \; dt$. Another common form of the theorem is the divergence form

$$\int_D \operatorname{div} F \, dA = \oint_{\partial D} F \cdot n \, dt,$$

where $F = (g, -f)$ and $n = (\frac{dy}{dt}, -\frac{dx}{dt})$ is the outward normal field on ∂D.

11. Areas and integrals over surfaces. Based on the theorem of number 9, we extend the definitions of areas and integrals to smooth surfaces.

Let $\mathcal{S}$ be a smooth surface. We call a subset $D \subseteq \mathcal{S}$ for a *compact region* if there exists a regular patch (U, σ) and a compact region $R \subset U$ such that $D = \sigma(R)$. The area $\mathcal{A}(D)$ of such a region D is defined in Definition 5.3 as

$$\mathcal{A}(D) = \int_R (EG - F^2)^{1/2} \, dA$$

and it is shown in Proposition 5.3 that it is independent of σ. The proof invokes the theorem of number 9 above.

Furthermore, let $f \colon D \to \mathbb{R}$ be a continuous function. Then we define similarly

$$\int_D f \, d\mathcal{A} = \int_R (f \circ \sigma)(EG - F^2)^{1/2} \, dA.$$

The proof of Proposition 5.3 can be repeated with the extra factor of $f \circ \sigma$ inside the integral, to show that this definition is also independent of σ.

12. Example. The area of the torus. The torus is parametrized by $\sigma(\theta, \varphi) = ((a + b\cos\theta)\cos\varphi, (a + b\cos\theta)\sin\varphi, b\sin\theta)$ (see p. 73). If we restrict both variables θ and φ to the interval $(0, 2\pi)$, then σ is a regular patch covering all of the torus except one meridian and one parallel. An elementary computation shows that $E = b^2$, $F = 0$ and $G = (a + b\cos\theta)^2$, and $(EG - F^2)^{1/2} = b(a + \cos\theta)$. Hence the area of the torus is

$$\mathcal{A} = \int_0^{2\pi} \int_0^{2\pi} b(a + b\cos\theta) \, d\theta \, d\varphi = 2\pi b \int_0^{2\pi} a + b\cos\theta \, d\theta = 4\pi^2 ab.$$

(Strictly speaking, the method does not apply directly. We should compute the area of $D_\epsilon = \sigma(R_\epsilon)$ where $R_\epsilon = [\epsilon, 2\pi - \epsilon] \times [\epsilon, 2\pi - \epsilon]$, and then compute the area as the limit of $\mathcal{A}(D_\epsilon)$ for $\epsilon \to 0$. The result would be unchanged.)