As mentioned in Pressley, p. 229, a geometric property of a surface which 'can be measured by a bug living on the surface' is called *intrinsic*. This means that it should be possible to express the property using only arc lengths on the surface. The idea is that the bug is capable of measuring such lengths, but that it cannot detect the three-dimensional space in which the surface lies.

The surface bug is able to use local coordinates on the surface in much the same fashion as us. It associates a point $\sigma(u, v)$ on the surface to each pair of coordinates $(u, v)$, but it does not know the $(x, y, z)$-coordinates of $\sigma(u, v)$. A curve on the surface is then described by means of its coordinates $(u(t), v(t))$. Having chosen a coordinate system $\sigma(u, v)$, the bug can determine the quantities $E, F, G$ of the first fundamental form, as functions of $u$ and $v$, as follows. It measures the arclength $s(t)$ along the parameter curve $(u(t), v(t)) = (u_0 + t, v_0)$ where $u_0, v_0$ are fixed, and cleverly differentiates $s$ with respect to $t$. Then $E(u_0, v_0) = \dot{s}(0)^2$. Indeed, if $\gamma(t) = \sigma(u_0 + t, v_0)$ then $\dot{\gamma} = \dot{u}\sigma_u + \dot{v}\sigma_v = \sigma_u$ and hence

$$\dot{s}(t) = \|\dot{\gamma}(t)\| = \|\sigma_u\| = \sqrt{E}.$$

The determination of $G$ is similar, and after that $F$ is determined from the arc length along, for example, $(u(t), v(t)) = (u_0 + t, v_0 + t)$.

On the other hand, knowing the first fundamental form, we can determine all arc lengths by means of the equation on p. 98. Therefore, *an intrinsic property is a property which for any given patch can be determined from the first fundamental form alone.*

Examples of intrinsic properties, besides lengths of curves, are angles (see p. 106-107) and areas (see p. 113). The property of a curve on the surface, that it is a geodesic, is also intrinsic, since the geodesic equations are expressed with $E, F, G$, hence can be verified by the bug for a given curve $(u(t), v(t))$.

Examples of *extrinsic* properties (properties which are not intrinsic) are the $x, y, z$-coordinates of a point, the principal curvatures $\kappa_1$ and $\kappa_2$, and the corresponding principal directions. Thus, the surface bug is unable to determine these.

It is from this point view, that it is *remarkable* that the Gauss curvature

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2}$$

is intrinsic, as stated in Theorem 10.1, Theorema egregium. The mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ on the other hand is extrinsic.

Here follows a short proof of the Gauss theorem. In contrast to the proof in the book, we will employ the fact that an orthogonal surface patch can be chosen in a neighborhood of each point of the surface, see Proposition 7.2. In fact, it can also be arranged that $M = 0$, but this does not lead to any major simplification. Our aim is to prove the formula in Corollary 10.2 (i), from which the theorem follows, since it exhibits $K$ as a function of $E$ and $G$ alone.

The proof is based on the following lemma, in which we assume that $(\sigma, U)$ is an orthogonal patch on $\mathcal{S}$, that is, it has $F = 0$.

**Lemma.** *Let*

$$\mathbf{X} = \frac{\sigma_u}{\sqrt{E}} \quad \textit{and} \quad \mathbf{Y} = \frac{\sigma_v}{\sqrt{G}}$$

*so that* $(\mathbf{X}, \mathbf{Y}, \mathbf{N})$ *is an orthonormal basis for* $\mathbb{R}^3$. *Then*

$$\begin{aligned} \mathbf{X}_u &= -\frac{E_v}{2\sqrt{EG}}\mathbf{Y} + \frac{L}{\sqrt{E}}\mathbf{N}, \qquad & \mathbf{X}_v &= \frac{G_u}{2\sqrt{EG}}\mathbf{Y} + \frac{M}{\sqrt{E}}\mathbf{N}, \\ \mathbf{Y}_u &= \frac{E_v}{2\sqrt{EG}}\mathbf{X} + \frac{M}{\sqrt{G}}\mathbf{N}, \qquad & \mathbf{Y}_v &= -\frac{G_u}{2\sqrt{EG}}\mathbf{X} + \frac{N}{\sqrt{G}}\mathbf{N}. \end{aligned}$$

*Proof.* Since $\mathbf{X}$ is a unit vector, the derivatives $\mathbf{X}_u$ and $\mathbf{X}_v$ are perpendicular to $\mathbf{X}$. Likewise, $\mathbf{Y}_u$ and $\mathbf{Y}_v$ are perpendicular to $\mathbf{Y}$. Since $(\mathbf{X}, \mathbf{Y}, \mathbf{N})$ is an orthonormal basis, it follows that

$$\begin{aligned} \mathbf{X}_u &= (\mathbf{X}_u \cdot \mathbf{Y})\mathbf{Y} + (\mathbf{X}_u \cdot \mathbf{N})\mathbf{N}, \qquad & \mathbf{X}_v &= (\mathbf{X}_v \cdot \mathbf{Y})\mathbf{Y} + (\mathbf{X}_v \cdot \mathbf{N})\mathbf{N}, \\ \mathbf{Y}_u &= (\mathbf{Y}_u \cdot \mathbf{X})\mathbf{X} + (\mathbf{Y}_u \cdot \mathbf{N})\mathbf{N}, \qquad & \mathbf{Y}_v &= (\mathbf{Y}_v \cdot \mathbf{X})\mathbf{X} + (\mathbf{Y}_v \cdot \mathbf{N})\mathbf{N}. \end{aligned}$$

It follows from the definition $\mathbf{X} = E^{-1/2}\sigma_u$ that

$$\mathbf{X}_u = -\frac{1}{2}E^{-3/2}E_u\sigma_u + E^{-1/2}\sigma_{uu}$$

and hence, since $\mathbf{Y}$ and $\mathbf{N}$ are both perpendicular to $\sigma_u$,

$$\begin{aligned} \mathbf{X}_u \cdot \mathbf{Y} &= E^{-1/2}\sigma_{uu} \cdot \mathbf{Y} = (EG)^{-1/2}\sigma_{uu} \cdot \sigma_v, \\ \mathbf{X}_u \cdot \mathbf{N} &= E^{-1/2}\sigma_{uu} \cdot \mathbf{N} = E^{-1/2}L. \end{aligned}$$

From $\sigma_u \cdot \sigma_v = 0$ and $\sigma_u \cdot \sigma_u = E$ we get by differentiation with respect to $u$ and $v$, respectively, that $\sigma_{uu} \cdot \sigma_v = -\sigma_u \cdot \sigma_{vu} = -\frac{1}{2}E_v$. The equation for $\mathbf{X}_u$ follows.

The proof of the other three equations is similar. □

In order to establish the formula for $K$ we use the equations in the lemma to derive two expressions for $\mathbf{X}_u \cdot \mathbf{Y}_v - \mathbf{X}_v \cdot \mathbf{Y}_u$.

On the one hand, if we insert directly from the lemma, and use that $(\mathbf{X}, \mathbf{Y}, \mathbf{N})$ is orthonormal

$$\mathbf{X}_u \cdot \mathbf{Y}_v - \mathbf{X}_v \cdot \mathbf{Y}_u = \frac{LN - M^2}{\sqrt{EG}} = \sqrt{EG}\,K. \tag{1}$$

On the other hand,

$$\mathbf{X}_u \cdot \mathbf{Y}_v - \mathbf{X}_v \cdot \mathbf{Y}_u = (\mathbf{X} \cdot \mathbf{Y}_v)_u - (\mathbf{X} \cdot \mathbf{Y}_u)_v \tag{2}$$

by cancellation of terms $\mathbf{X} \cdot \mathbf{Y}_{vu} = \mathbf{X} \cdot \mathbf{Y}_{uv}$. Furthermore, again by insertion from the lemma

$$(\mathbf{X} \cdot \mathbf{Y}_v)_u - (\mathbf{X} \cdot \mathbf{Y}_u)_v = -\left(\frac{G_u}{2\sqrt{EG}}\right)_u - \left(\frac{E_v}{2\sqrt{EG}}\right)_v. \tag{3}$$

The formula for $K$ in Corollary 10.2 follows at once from these three equations. □

We end this note with a formula from which it follows that the geodesic curvature $\kappa_g$ of a curve on a surface is intrinsic. We have already argued that the property $\kappa_g = 0$, which is equivalent with the curve being a geodesic, is intrinsic. What we are claiming now is stronger. The formula is a generalization of that in Proposition 2.2 for plane curves.

**Proposition.** *Let $\gamma$ be a unit speed curve on $\mathcal{S}$, and let $\gamma(t) = \sigma(u(t), v(t))$ be its expression in an orthogonal patch. Let $\theta(t)$ denote a differentiable determination of the angle from $\sigma_u$ to $\dot{\gamma}(t)$, in the orientation given by* $\mathbf{N}$. *Then*

$$\kappa_g = \dot{\theta} - \frac{1}{2\sqrt{EG}} \left\{ \dot{u}E_v - \dot{v}G_u \right\}.$$

*Proof.* Let $\mathbf{X}$ and $\mathbf{Y}$ be as above, then $\theta(t)$ is determined up to constant multiples of $2\pi$ by

$$\dot{\gamma}(t) = \cos\theta(t)\mathbf{X} + \sin\theta(t)\mathbf{Y}.$$

The multiples of $2\pi$ play no role, as the assertion only involves the derivative of $\theta$.

The asserted formula for $\kappa_g$ is essentially given in the proof of Theorem 11.1. In that proof we can take $e' = \mathbf{X}$ and $e'' = \mathbf{Y}$. Then it is shown on p. 250 that

$$\kappa_g = \dot{\theta} - e' \cdot \dot{e}'' = \dot{\theta} - \mathbf{X} \cdot \dot{\mathbf{Y}},$$

so all we have to do is to compute $\mathbf{X} \cdot \dot{\mathbf{Y}}$. By the chain rule $\dot{\mathbf{Y}} = \dot{u}\mathbf{Y}_u + \dot{v}\mathbf{Y}_v$, and hence the equations of the previous lemma imply that

$$\mathbf{X} \cdot \dot{\mathbf{Y}} = \dot{u}\frac{E_v}{2\sqrt{EG}} - \dot{v}\frac{G_u}{2\sqrt{EG}}. \qquad \square$$

For example, we can determine the geodesic curvature along the coordinate curve $u \mapsto \sigma(u, v)$ as follows. If $t \mapsto \sigma(u(t), v)$ is a unit speed reparametrization, then $\|\dot{u}\sigma_u\| = 1$. Thus $\dot{u} = 1/\sqrt{E}$. Moreover, $\theta = 0$ and $\dot{v} = 0$, so we obtain from the formula above

$$\kappa_g = -\frac{E_v}{2E\sqrt{G}}.$$

Similarly, the geodesic curvature along $v \mapsto \sigma(u, v)$ is found to be

$$\kappa_g = \frac{G_u}{2G\sqrt{E}}.$$