It follows from Theorem 8.2 that if a unit-speed curve is a shortest path on a smooth surface $\mathcal{S}$, then the curve is a geodesic. The converse is false, as is easily seen from the following example:

Let P and Q be two points on S^2 which are not antipodal. Then there is a unique great circle through P and Q, and by separating this circle at P and Q we obtain two geodesics which both join P to Q. Only one of these will be shortest.

However, using geodesic coordinates we can prove the following

Theorem. *Let μ be a unit-speed geodesic on $\mathcal{S}$, and let P be a point on it, say $P = \mu(0)$. Then there exists $\epsilon > 0$ such that for for all $Q = \mu(t)$ where $t \in (-\epsilon, \epsilon)$, μ is the unique shortest regular curve on $\mathcal{S}$ from P to Q (uniqueness being up to reparametrization).*

Proof. Let $\mathbf{t} \in T_P\mathcal{S}$ be a unit vector orthogonal to the tangent vector $\dot{\mu}(0)$ to μ at P ($\mathbf{t}$ is unique up to change of sign), and let γ be the unique geodesic through P with tangent vector $\mathbf{t}$. Let $\sigma(u, v)$, $(u, v) \in U$, be a system of geodesic coordinates around P, constructed from the curve γ as in Proposition 8.7. It follows from that construction that $\gamma(t) = \sigma(0, t)$. Moreover, the geodesic $t \mapsto \sigma(t, 0)$, is perpendicular to γ at P, hence identical with μ (up to change of direction).

By Prop. 8.7 the first fundamental form satisfies $E = 1$, $F = 0$ and $G(0, v) = 1$. In particular, $G = 1$ at P, so by shrinking U we may assume $G \geq \frac{1}{2}$ on $\sigma(U)$. Let $\epsilon > 0$ be such that the ball in $\mathbb{R}^2$ of radius 2ϵ around $(0, 0)$ is contained in U.

Let $t_0 \in (-\epsilon, \epsilon)$ and let $Q = \mu(t_0) = \sigma(t_0, 0)$. Let α be an arbitrary regular smooth curve on $\mathcal{S}$ from $P = \alpha(a)$ to $Q = \alpha(b)$. At first we assume that α is contained in $\sigma(U)$, say $\alpha(t) = \sigma(u(t), v(t))$ for $a \leq t \leq b$. Then $(u(a), v(a)) = (0, 0)$ and $(u(b), v(b)) = (t_0, 0)$. Now

$$\ell(\alpha) = \int_a^b \|\dot{\alpha}(t)\| \, dt$$

and since $\dot{\alpha} = \dot{u}\sigma_u + \dot{v}\sigma_v$ and $E = 1$, $F = 0$, we have

$$\|\dot{\alpha}\| = (\dot{u}^2 + G\dot{v}^2)^{1/2} \geq |\dot{u}|. \tag{1}$$

Hence

$$\ell(\alpha) \geq \int_a^b |\dot{u}(t)| \, dt \geq |\int_a^b \dot{u}(t) \, dt| = |u(b) - u(a)| = |t_0|.$$

Since $t \mapsto \gamma(t) = \sigma(t, 0)$ is unit-speed, $|t_0|$ is exactly the length of γ from P to Q. We have thus shown $\ell(\alpha) \geq \ell(\gamma)$.

The inequality (1) is strict unless $\dot{v} = 0$, hence $\ell(\alpha) > \ell(\mu)$ unless v =constant along α. Since $v(a) = v(b) = 0$ this would imply that $v = 0$ everywhere, so that α has the same trace as γ. This proves the asserted uniqueness.

It remains to be seen that a path α from P to Q, which is not contained in $\sigma(U)$, must be strictly longer. Such a path must necessarily cross the image by σ of the circle of radius 2ϵ. Let $c \in (a, b]$ be the smallest value of t for which $\alpha(t)$ belongs to this image, and put $Q_1 = \alpha(c)$. The length of α from P to Q is $\geq$ the length of

α from P to Q_1, and by a computation similar to the one above, the length of the latter part of α is

$$\int_a^c (E\dot{u}(t)^2 + G\dot{v}(t)^2)^{1/2}\,dt \geq (\tfrac{1}{2})^{1/2} \int_a^c (\dot{u}^2 + \dot{v}^2)^{1/2}\,dt.$$

Here we used that $E = 1 \geq \frac{1}{2}$ and $G \geq \frac{1}{2}$. The integral on the right is the length of the curve $(u(t), v(t))$ in $\mathbb{R}^2$ which joins $(0,0)$ with $(u(c), v(c))$ on the circle of radius 2ϵ. Hence that integral is $\geq 2\epsilon$, and we conclude that $\ell(\alpha) \geq (\frac{1}{2})^{1/2} 2\epsilon > \epsilon$. $\square$

For example on S^2 a geodesic of length $< \pi$ is the unique shortest path between its end points, hence $\epsilon = \pi$ can be used in the theorem above. Likewise, on a cylinder of radius 1 the value $\epsilon = \pi$ can be used.

HS