Note 8. Geodesics as shortest curves. MAT 3GE, F2003

This note contains a simplification of the proof of Theorem 8.2. We follow the proof on p. 192-194, but replace the expressions involving E, F, G by expressions involving the goedesic curvature κ_g. Equation (14) reads,

$$\frac{d}{d\tau}\mathcal{L}(\tau) = \frac{1}{2}\int_a^b g(\tau,t)^{-1/2}\frac{\partial g}{\partial \tau}\,dt,$$

where $g(\tau,t) = \|\dot\gamma^\tau\|^2 = \dot\gamma^\tau\cdot\dot\gamma^\tau$. Since γ has unit speed, $g(\tau,t)=1$ for $\tau=0$. Hence

$$\frac{d}{d\tau}\mathcal{L} = \frac{1}{2}\int_a^b \frac{\partial g}{\partial \tau}\,dt, \qquad \text{when } \tau = 0.$$

Now $\frac{\partial g}{\partial\tau} = 2\frac{\partial^2\gamma^\tau}{\partial\tau\partial t}\cdot\frac{\partial\gamma^\tau}{\partial t}$, hence using integration by parts

$$\frac{d}{d\tau}\mathcal{L} = \left[\frac{\partial\gamma^\tau}{\partial\tau}\cdot\frac{\partial\gamma^\tau}{\partial t}\right]_a^b - \int_a^b \frac{\partial\gamma^\tau}{\partial\tau}\cdot\frac{\partial^2\gamma^\tau}{\partial t^2}\,dt.$$

Since $\gamma^\tau(a)$ and $\gamma^\tau(b)$ are independent of τ (being equal to $\mathbf{p}$ and $\mathbf{q}$, respectively), we have

$$\frac{\partial\gamma^\tau}{\partial\tau} = 0 \quad \text{when } t=a \text{ or } t=b.$$

Hence the first term on the right-hand side is zero. Since $\gamma^\tau(t)\in\mathcal{S}$ for all τ, we have $\frac{\partial\gamma^\tau}{\partial\tau}\in T_P\mathcal{S}$ where $P=\gamma^\tau(t)$. Hence in the dot product in the second term we can replace $\ddot\gamma^\tau = \frac{\partial^2\gamma^\tau}{\partial t^2}$ by its orthogonal projection on $T_P\mathcal{S}$. For $\tau=0$ this equals $\kappa_g\mathbf{N}\times\dot\gamma$ (see p. 127, eq. (5)). In conclusion,

$$\frac{d}{d\tau}\mathcal{L} = -\int_a^b \frac{\partial\gamma^\tau}{\partial\tau}\cdot\kappa_g(\mathbf{N}\times\dot\gamma)\,dt, \qquad \text{when } \tau=0.$$

In particular, if γ is a geodesic, then $\kappa_g=0$ and we conclude that $\frac{d}{d\tau}\mathcal{L}=0$.

For the converse, we now have to show that if

$$\int_a^b \frac{\partial\gamma^\tau}{\partial\tau}\cdot\kappa_g(\mathbf{N}\times\dot\gamma)\,dt = 0$$

at $\tau=0$, for *all* families of curves γ^τ, then $\kappa_g=0$. Let $\gamma(t)=\sigma(u(t),v(t))$ and $\mathbf{N}\times\dot\gamma(t) = f(t)\sigma_u + g(t)\sigma_v$. Let $\phi(t)$ be an arbitrary smooth function defined for $a\le t\le b$ such that $\phi(a)=\phi(b)=0$, and define for τ sufficiently close to 0

$$\gamma^\tau(t) = \sigma(u(t)+\tau\phi(t)f(t), v(t)+\tau\phi(t)g(t)).$$

Then γ^τ is a family of the required type, and

$$\frac{\partial\gamma^\tau}{\partial\tau} = \phi(t)f(t)\sigma_u + \phi(t)g(t)\sigma_v = \phi(t)(\mathbf{N}\times\dot\gamma(t))$$

for $\tau=0$. Hence

$$0 = \int_a^b \phi(\mathbf{N}\times\dot\gamma)\cdot\kappa_g(\mathbf{N}\times\dot\gamma)\,dt = \int_a^b \phi\kappa_g\,dt,$$

because $\|\mathbf{N}\times\dot\gamma(t)\|=1$. Since ϕ was arbitrary, it follows easily that $\kappa_g=0$ (choose, for example, $\phi(t)=(t-a)(b-t)\kappa_g(t)$, so that $\phi\kappa_g\ge 0$).

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