This note contains alternative proofs of Prop. 6.3 and Cor. 6.1. The proofs are based on formulas (9) and (10) on p. 132, which express the significance of the two fundamental forms $\mathcal{F}_I$ and $\mathcal{F}_{II}$:

1. Let $\mathbf{t}_1, \mathbf{t}_2 \in T_P\mathcal{S}$ then

$$\mathbf{t}_1 \cdot \mathbf{t}_2 = T_1^t \mathcal{F}_I T_2 \tag{9}$$

where T_i is the column formed by the coordinates of $\mathbf{t}_i$ with respect to the basis vectors σ_u, σ_v for $T_P\mathcal{S}$.

2. Let $\mathbf{t} \in T_P\mathcal{S}$ be a unit vector, then the normal curvature κ_n of $\mathcal{S}$ at P in direction $\mathbf{t}$ is given by

$$\kappa_n = T^t \mathcal{F}_{II} T, \tag{10}$$

where T is the column of coordinates of $\mathbf{t}$.

Let $f\colon T_P\mathcal{S} \to T_P\mathcal{S}$ be the linear transformation, whose matrix with respect to the basis σ_u, σ_v is

$$\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II},$$

the so-called Weingarten matrix (note that $\mathcal{F}_I$ is invertible according to the remark below Prop. 5.2 (p. 113)). Thus, by definition, the coordinates of $f(\mathbf{t})$ are determined by the column $\mathcal{W}T$.

Lemma 1. *The principal curvatures are the eigenvalues of f and the principal vectors are the corresponding eigenvectors.*

Proof. (This is also discussed on p. 133.) Recall that by definition a number $\kappa \in \mathbb{C}$ is a principal curvature if and only if $\det(\mathcal{F}_{II} - \kappa\mathcal{F}_I) = 0$, and a (non-zero) tangent vector $\mathbf{t} \in T_P\mathcal{S}$ is a principal vector if and only if $(\mathcal{F}_{II} - \kappa\mathcal{F}_I)T = 0$.

On the other hand, it follows from linear algebra that κ is an eigenvalue of f if and only if $\det(\mathcal{W} - \kappa I) = 0$, and that $\mathbf{t}$ is an eigenvector if and only if $(\mathcal{W} - \kappa I)T = 0$ (where I denotes the identity matrix).

Observe that $\mathcal{F}_{II} - \kappa\mathcal{F}_I = \mathcal{F}_I(\mathcal{W} - \kappa I)$. Since $\mathcal{F}_I$ is invertible, the lemma follows immediately. $\square$

Lemma 2. *The linear operator f is symmetric, that is, for all $\mathbf{t}, \mathbf{t}' \in T_P\mathcal{S}$*

$$f(\mathbf{t}) \cdot \mathbf{t}' = \mathbf{t} \cdot f(\mathbf{t}').$$

Proof. Let T, T' be the columns of coordinates of $\mathbf{t}$ and $\mathbf{t}'$. Then by (9),

$$f(\mathbf{t}) \cdot \mathbf{t}' = (\mathcal{W}T)^t \mathcal{F}_I T' = T^t \mathcal{W}^t \mathcal{F}_I T'$$

and since $\mathcal{F}_I, \mathcal{F}_{II}$ are symmetric matrices, $\mathcal{W}^t = \mathcal{F}_{II}\mathcal{F}_I^{-1}$, so

$$f(\mathbf{t}) \cdot \mathbf{t}' = T^t \mathcal{F}_{II} T'. \tag{$*$}$$

Similarly,

$$\mathbf{t} \cdot f(\mathbf{t}') = T^t \mathcal{F}_I(\mathcal{W}T') = T^t \mathcal{F}_{II} T'. \quad \square$$

Notice that even though the operator f is symmetric, the matrix $\mathcal{W}$ that represents it need not be symmetric, if the basis vectors σ_u, σ_v are not orthogonal to each other.

In order to prove Prop. 6.3 and Cor. 6.1 we will apply to f the spectral theorem for symmetric linear operators on a finite dimensional space (see Messer p. 325-327). According to this theorem the eigenvalues of f are real, and eigenvectors for different eigenvalues are mutually orthogonal. The statements (i) and (iii) in Prop. 6.3 are direct consequences, in view of Lemma 1.

According to the spectral theorem there exists an orthonormal basis $\mathbf{t}_1, \mathbf{t}_2$ for $T_P\mathcal{S}$ consisting of eigenvectors for f. By Lemma 1 the corresponding eigenvalues are the principal curvatures κ_1 and κ_2, and $\mathbf{t}_1, \mathbf{t}_2$ are principal directions.

The matrix of f with respect to the basis $\mathbf{t}_1, \mathbf{t}_2$ is diagonal with κ_1, κ_2 in the diagonal. If κ_1 and κ_2 have a common value, say κ, then this diagonal matrix is κI and f is κ times the identity operator. Hence in that case the matrix of f with respect to *any* basis is κI, so $\mathcal{W} = \kappa I$. Number (ii) of Prop. 6.3 follows.

Let $\mathbf{t}$ be a unit tangent vector, then

$$\mathbf{t} = \cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2 \qquad (**)$$

where θ is the angle from $\mathbf{t}_1$ to $\mathbf{t}$. The statement in Cor. 6.1 is, that the normal curvature at P in direction $\mathbf{t}$ is given by

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

We shall now prove this. By (10), $(*)$ and $(**)$

$$\kappa_n = f(\mathbf{t}) \cdot \mathbf{t} = f(\cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2) \cdot (\cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2),$$

and since f is linear and $\mathbf{t}_1, \mathbf{t}_2$ are eigenvectors

$$f(\cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2) = \cos\theta f(\mathbf{t}_1) + \sin\theta f(\mathbf{t}_2) = \kappa_1 \cos\theta \mathbf{t}_1 + \kappa_2 \sin\theta \mathbf{t}_2.$$

Hence

$$\kappa_n = (\kappa_1 \cos\theta \mathbf{t}_1 + \kappa_2 \sin\theta \mathbf{t}_2) \cdot (\cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

as claimed.

Remark. Note that the entries of the Weingarten matrix $\mathcal{W}$ are given a geometric interpretation in Prop. 6.4. They express in a certain sense how the normal vector $\mathbf{N}$ varies, which can be seen as a measure of the shape of $\mathcal{S}$. For this reason the operator f with matrix $\mathcal{W}$ is sometimes called the *shape operator*. This terminology is also supported by Lemma 1 above.

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