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Discrete Crossed product C*-algebras

Adam Sierakowski

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Advisor Mikael Rørdam
University of Copenhagen, Denmark

Assessment Committee Søren Eilers (chair)
University of Copenhagen, Denmark

Wojciech Szymanski
University of Southern Denmark, Denmark

Nadia S. Larsen
University of Oslo, Norway

Adam Sierakowski
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen
adam@math.ku.dk

Abstract

Classification of C*-algebras has been an active area of research in mathematics for at least half a century.

In this thesis, we consider classification results related to the class of crossed product C*-algebras. We investigate the ideal structure of the reduced crossed product C*-algebra and in particular we determine sufficient − and in some cases also necessary − conditions for A to separate the ideals in $A \rtimes_r G$. When A separates the ideals in $A \rtimes_r G$, then there is a one-to-one correspondence between the ideals in $A \rtimes_r G$ and the invariant ideals in A. We extend the concept of topological freeness and present a generalization of the Rokhlin property. Exactness properties of the underlying dynamical systems turns out to be crucial in these investigations.

When imposing properties that ensure A separates the ideals in $A \rtimes_r G$ we are able to show that the crossed product $A \rtimes_r G$ is purely infinite if and only if the non-zero positive elements in A are properly infinite viewed as elements in $A \rtimes_r G$ − in the case A is separable and has the ideal properly. As an application of this result we show that for a particular class of crossed products, where a discrete group G acts in a particular way on the Cantor set X, the C*-algebra $C(X) \rtimes_r G$ is purely infinite if and only if it is traceless.

Resumé

Klassifikation af C*-algebraer have været et aktivt område indenfor forskning indenfor matematik i mindst et halvt århundrede.

I denne afhandling fokuserer vi på klassifikations resultater relateret til krydsprodukt C*-algebraer. Vi undersøger idealstrukturen af den reducerede krydsprodukt C*-algebra og finder tilstrækkelige – og i nogle tilfælde også nødvendige – betingelser for at A adskiller idealer i $A \rtimes_r G$. Når A adskiller idealer i $A \rtimes_r G$, så der er en en-til-en korrespondance mellem idealer i $A \rtimes_r G$ og invariante idealer i A. Vi udvider begrebet topologisk frihed og præsenter en generalisering af Rokhlin egenskaben. Eksakthed egenskaber ved de underliggende dynamiske systemer viser sig at være afgørende i disse undersøgelser.

Ved indførelse af egenskaber der sikrer at A adskiller idealer i $A \rtimes_r G$ er vi i stand til at vise, at et krydsprodukt $A \rtimes_r G$ er rent uendelig, hvis og kun hvis de positive elementer forskellig fra nul i A er egentlig uendelige set som elementer i $A \rtimes_r G$ – forudsat at A er separabel og har ideal egenskaben. Som en anvendelse af dette resultat viser vi, at for en bestemt klasse af krydsprodukter, hvor en diskret gruppe G virker på en passende måde på Cantor mængden X, er C*-algebraen $C(X) \rtimes_r G$ rent uendelig, hvis og kun hvis den er sporløs.

Contents

Preface

This dissertation concludes my work as a Ph.D. student in mathematics at the University of Copenhagen. The thesis is the result of research done from March 2005 to February 2009 under the supervision of Prof. Mikael Rørdam.

The work was funded by a grant from University of Southern Denmark with the support of 1/3 from the Ph.D. school OP-ALG-TOP-GEO. As a student of the Ph.D. school OP-ALG-TOP-GEO I was partially supported by the Danish Research Training Council. My advisor changed his work environment in the beginning of 2009. As a result part of my work was funded by a grant from University of Copenhagen where I have worked since September 2008.

My time as a Ph.D. student has primarily been spent focusing on crossed product C*-algebras associated with transformation groups. From July to December 2007 I was invited to The Fields Institute for Research in Mathematical Sciences, Toronto. During my visit at The Fields Institute I was inspired to focus my research on the ideal structure of crossed products C*-algebras. The outcome of this work has been accepted for publication.

I would like to thank, first and foremost, my advisor, Mikael Rørdam, for his help and support and for numerous valuably discussions we have had. My four years of Ph.D. have been a life changing experience for me in a very positive way. Also I am pleased to thank Eberhard Kirchberg for letting me visit him on several occasions (his wife is a great cook). His kindness and patience has resulted in many interesting discussions.

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Outline

The dissertation intends to present the most interesting results by the author in a combined and coherent way. We have decided to incorporate the three manuscripts [54, 55, 56] as a integrated part of the thesis. Many details have been added in order to make the dissertation more easily readable. Other parts are still left as an exercise. We encourage everyone to take their time to work out the omitted details.

Chapter 1 gives a short introduction to the notation and some of the basic definitions. Chapter 2 outlines the most prominent work concerning crossed products, including the author's own work carried out under supervision of Prof. Mikael Rørdam. The last parts of the chapter sketch out some of the naturally occurring examples of separable, simple, nuclear, purely infinite C*-algebras associated with transformation groups. Chapter 3 recaps the work made with the help of Prof. Eberhard Kirchberg on ideal structures of reduced crossed products. This work is essentially contained in [55], to be published in Münster Journal of Mathematics. Chapter 4-5 contains results related to pure infiniteness of crossed product C*-algebras. In the last chapter we try to relate pure infiniteness to lack of traces.

Chapter 1

Introduction

In this chapter we will introduce some basic framework for the study of reduced crossed product.

1.1 Notation

We generally follow the notation conventions of [61] and [9], except for a few exceptions. For example, we usually do not emphasize the action of a group G on a C*-algebra A and denote a C*-dynamical system by (A, G). We will make this more precise shortly.

Let e denote the unit of a group. Let $\mathcal{M}(A)$ denote the multiplier algebra of a C*-algebra A. Let A^+ and A^{sa} be the the positive cone of A and the set of selfadjoint elements in A. All homomorphisms between C*-algebras are assumed to be *-preserving, and ideals in C*-algebras are always closed and two-sided. Unless otherwise stated, a representation of a C*-algebra on a Hilbert space is presumed to be nondegenerate, and our Hilbert spaces are all complex. Let $\mathcal{I}(A)$ denote the set of ideals in a C*-algebra A and let $\mathrm{Ideal}_A[S]$, or simply $\mathrm{Ideal}[S]$, be the smallest ideal in A generated by $S \subseteq A$.

Fix now a C*-dynamical system (A, G) with G discrete. Let $A \rtimes_r G$ (resp. $A \rtimes G$) denote the reduced (resp. the full) crossed product. Recall that $A \subseteq C_c(G, A) \subseteq A \rtimes_r G$ in a canonical way. Elements in $C_c(G, A)$ will be written as sums $\sum_{t \in G} a_t u_t$, where only finite many a_t's are non-zero, and where u_t is, in a canonical way, a unitary element in $\mathcal{M}(A \rtimes_r G)$. Let $\mathcal{I}(A)^G$ denote the set of all *invariant* (or more precisely all G-*invariant*) ideals in A and let $\mathrm{Ideal}_A[S]^G$, or simply $\mathrm{Ideal}[S]^G$, be the smallest invariant ideal in A generated by $S \subseteq A$.

For a topological space X denote the collection of open sets by τ_X, the Borel sets of X by $\mathbb{B}(X)$, and the power set of X by $\mathcal{P}(X)$.

1.2 Full and reduced crossed product

Let G be a discrete group and A a C*-algebra. An *action* of G on A is a group homomorphism from G into $\mathrm{Aut}(A)$ — the group of automorphisms on A. The action is denoted by a dot in the following way

$$t \;\rightarrow\; (a \mapsto t.a), \quad t \in G,\; a \in A.$$

A *C*-dynamical system* (A, G) consist of a discrete group G, a C*-algebra A and an action of G on A. For a C*-dynamical system (A, G) with G discrete[1] let $C_c(G, A)$ be the linear span of finitely supported functions on G with values in A. A typical element a in $C_c(G, A)$ is written as a sum $a = \sum_{t\in G} a_t u_t$, where only finitely many a_t's are non-zero.

One equips $C_c(G, A)$ with a *-operation and a twisted product, in such a way that the action becomes "inner". By this we mean that $t.a = u_t a u_t^*$ for every $t \in G, a \in A$. More precisely we have the following product and *-operation

$$ab = \sum_{s,t\in G} a_t(t.b_s)u_{ts},\;\; a^* = \sum_{t\in G}(t^{-1}.a_t^*)u_{t^{-1}}, \quad a = \sum_{t\in G} a_t u_t, b = \sum_{s\in G} b_s u_s.$$

Fix a C*-dynamical system (A, G) with G discrete. A *covariant representation* (π, u, H) of (A, G) consist of a unitary representation $u\colon G \to B(H)$ of G and a representation $\pi\colon A \to B(H)$ of A such that $u(t)\pi(a)u(t)^* = \pi(t.a)$ for every $t \in G$ and $a \in A$. For a covariant representation (π, u, H) let $\pi \times u$ denote the associated representation of $C_c(G, A)$ on H.

The *full C*-algebra norm*[2] on $C_c(G, A)$ is given by

$$\|\cdot\| = \sup \|(\pi \times u)(\cdot)\|,$$

where the supremum is taken over all[3] covariant representations (π, u, H) of (A, G). The *full crossed product*, denoted $A \rtimes G$, is the completion of $C_c(G, A)$ with respect to the full C*-algebra norm.

The *reduced C*-algebra norm* on $C_c(G, A)$ is given by

$$\|\cdot\|_\lambda = \|\tilde{\pi} \times \lambda(\cdot)\|_{B(\mathscr{H})},$$

[1] We decided to emphasize the discreteness of G throughout the manuscript in order to remove any possible doubt regarding this fact.

[2] We will see in a moment why this is a C*-algebra norm and not just a semi-norm.

[3] The definition of the full C*-algebra norm is taken from [61, Lemma 2.27]. Note that D. Williams allows π to be degenerate. We only consider nondegenerate representations of C*-algebras. Hence our definition only includes nondegenerate representations of A. This however does not change the norm, cf. [61, Lemma 2.31].

where the *regular representation* $\tilde{\pi} \times \lambda : C_c(G, A) \to B(\mathscr{H})$ is the representation associated to the covariant representation $(\tilde{\pi}, \lambda, \mathscr{H})$ given by

$$\tilde{\pi}(a)\delta_{t,\xi} = \delta_{t,\pi(t^{-1}.a)\xi}, \quad \lambda(s)\delta_{t,\xi} = \delta_{st,\xi}, \quad a \in A,\ s, t \in G,\ \xi \in H$$

where $\pi \colon A \to B(H)$ is any faithful representation, $\mathscr{H} = l^2(G, H)$ and $\delta_{t,\xi} \in \mathscr{H}$ is the map $s \mapsto \delta_{t,s}\xi \in H$ ($\delta_{t,s}$ is the Kronecker delta). In this way $\tilde{\pi} \times \lambda$ becomes faithful, cf. [61, Lemma 2.26]. The reduced C*-algebra norm does not depend on the choice of the faithful representation π, cf. [9, Proposition 4.1.5].

The *reduced crossed product*, denoted $A \rtimes_r G$, is the completion of $C_c(G, A)$ with respect to the reduced C*-algebra norm. The inclusion of G into the group of unitaries in $\mathcal{M}(A \rtimes_r G)$ allows us to consider $u_t, t \in G$ as an unitary element in $\mathcal{M}(A \rtimes_r G)$, cf. [61, Proposition 2.34].

Let (π, u, H) be a covariant representation of a C*-dynamical system (A, G) with G discrete. Then $\pi \times u$ extends to a representation $\pi \rtimes u$ of $A \rtimes G$ on H. The map sending (π, u, H) into $\pi \rtimes u$ is a one-to-one correspondence between covariant representations of (A, G) and representations of $A \rtimes G$, cf. [61, Proposition 2.40].

Let us proof a few results to give the reader a better understanding of some of the well known results about crossed product C*-algebras.

Lemma 1.2.1. *Let (A, G) be a C*-dynamical system with G discrete. Then*

(i) The map $a \mapsto au_e$ defines a canonical inclusion of A in $A \rtimes_r G$ (cf. [9, Proposition 4.1.7])

(ii) The map $E : C_c(G, A) \to A : \sum_{t \in G} a_t u_t \mapsto a_e$ extends by continuity to a faithful positive conditional expectation $E : A \rtimes_r G \to A$ (cf. [9, Proposition 4.1.9]). Sometimes one write E_A instead of E

(iii) For every $I \in \mathcal{I}(A)^G$ the natural maps ι and ρ in the short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\rho} A/I \longrightarrow 0 \tag{1.2.1}$$

extends, in a canonical way (see how in the proof), to maps $\iota \rtimes_r id$ and $\rho \rtimes_r id$ at the level of reduced crossed products. We obtain the following commutative diagram (cf. [61, Remark 7.14])

$$\begin{tikzcd}
0 \arrow[r] & I \rtimes_r G \arrow[r, "\iota \rtimes_r id"] \arrow[d, "E_I"] & A \rtimes_r G \arrow[r, "\rho \rtimes_r id"] \arrow[d, "E_A"] & A/I \rtimes_r G \arrow[r] \arrow[d, "E_{A/I}"] & 0 \\
0 \arrow[r] & I \arrow[r, "\iota"] & A \arrow[r, "\rho"] & A/I \arrow[r] & 0
\end{tikzcd} \tag{1.2.2}$$

Proof. (*i*) By the definition of the regular representation we obtain that

$$\begin{aligned}\|au_e\|_\lambda^2 &\geq \sup_{\|\xi\|=1} \|\tilde{\pi} \times \lambda(au_e)\delta_{e,\xi}\|_{\mathscr{H}}^2 = \sup_{\|\xi\|=1} \|\delta_{e,\pi(a)\xi}\|_{\mathscr{H}}^2 \\ &= \sup_{\|\xi\|=1} \|\pi(a)\xi\|_H^2 = \|a\|^2\end{aligned}$$

$$\begin{aligned}\|au_e\|_\lambda^2 &= \sup_{\|\eta\|=1} \|\tilde{\pi} \times \lambda(au_e)\eta\|_{\mathscr{H}}^2 = \sup_{\|\eta\|=1} \|\tilde{\pi} \times \lambda(au_e) \sum_{t \in G} \delta_{t,\eta(t)}\|_{\mathscr{H}}^2 \\ &= \sup_{\|\eta\|=1} \|\sum_{t \in G} \delta_{t,\pi(t^{-1}.a)\eta(t)}\|_{\mathscr{H}}^2 = \sup_{\|\eta\|=1} \sum_{t \in G} \|\pi(t^{-1}.a)\eta(t)\|_H^2 \\ &\leq \|a\|^2 \sup_{\|\eta\|=1} \sum_{t \in G} \|\eta(t)\|_H^2 = \|a\|^2\end{aligned}$$

for every $a \in A$. We obtain an isometric homomorphism $A \to A \rtimes_r G : a \mapsto au_e$.

(*ii*) Fix $a = \sum_{t \in G} a_t u_t \in C_c(G, A)$, $\xi \in H$, $s \in G$ and the contractions $V_s \colon H \to \mathscr{H} \colon \xi \mapsto \delta_{s,\xi}$ and $V_s^* \colon \mathscr{H} \to H \colon \eta \mapsto \eta(s)$. From $V_e^*(\tilde{\pi} \times \lambda)(a_t u_t)V_e\xi = V_e^*\tilde{\pi}(a_t)\delta_{t,\xi} = \delta_t(e)\pi(a_t)\xi = \pi(E(a_t u_t))\xi$ we obtain that

$$\pi(E(b)) = V_e^*(\tilde{\pi} \times \lambda)(b)V_e, \quad \|E(b)\| \leq \|b\|_\lambda \quad b \in C_c(G, A),$$

Since $E(aa^*) = \sum_{t \in G} a_t a_t^*$ the map E is a positive conditional expectation on $C_c(G, A)$ and extends by continuity to a map with the same properties (since A^+ is closed in A). From the equality $s.E(a) = E(u_s a u_s^*)$ and continuity of E it follows that the conditional expectation is *equivariant* (sometimes also called *G-equivariant*), i.e.,

$$t.E(b) = E(u_t b u_t^*), \quad t \in G, b \in A \rtimes_r G.$$

Fix now $b \in A \rtimes_r G$ with $E(b^*b) = 0$. By the equivariance of E we have that $E(u_s^* b^* b u_s) = 0$. Since $\lambda(u_s)V_e = V_s$ we obtain the equality $T := V_s^*(\tilde{\pi} \times \lambda)(b^*b)V_s = 0$. From $\|(\tilde{\pi} \times \lambda)(b)\delta_{s,\xi}\|^2 = \langle(\tilde{\pi} \times \lambda)(b^*b)V_s\xi, V_s\xi\rangle = \langle T\xi, \xi\rangle = 0$ we conclude that $\|b\|_\lambda = 0$.

(*iii*) Given $I \in \mathcal{I}(A)^G$. Recall $\tilde{\pi} \times \lambda$ from the proof of (*i*). As the map $\pi_{|_I} \colon I \to B(H)$ is faithful it defines the reduced C^*-norm on $C_c(G, I)$, implying that the isometric map defined by

$$(C_c(G, I), \|\tilde{\pi_{|_I}} \times \lambda(\cdot)\|) \to (A \rtimes_r G, \|\tilde{\pi} \times \lambda(\cdot)\|) \colon au_s \mapsto au_s$$

extends to the inclusion map $\iota \rtimes_r id : I \rtimes_r G \to A \rtimes_r G$.

For a faithful representation $\varphi\colon A/I \to B(H')$ and the quotient map $\rho\colon A \to A/I$ we obtain that the map defined by

$$(C_c(G,A), \|\tilde{\pi}\times\lambda(\cdot)\|) \to (A/I \rtimes_r G, \|\tilde{\varphi}\times\lambda(\cdot)\|)\colon au_s \mapsto (a+I)u_s$$

is norm decreasing from the following calculation

$$\begin{aligned}
\|\sum_{t\in G}(a_t+I)u_t\|_\lambda &= \|(\tilde{\varphi}\times\lambda)(\sum_{t\in G}(a_t+I)u_t)\| = \|\widetilde{(\varphi\circ\rho)}\times\lambda(\sum_{t\in G}a_tu_t)\| \\
&\le \|(\widetilde{\varphi\circ\rho}\times\lambda)\oplus(\tilde{\pi}\times\lambda)(\sum_{t\in G}a_tu_t)\| \\
&= \|(\varphi\circ\rho\oplus\pi)\tilde{}\times\lambda(\sum_{t\in F}a_tu_t)\| = \|\sum_{t\in G}a_tu_t\|_\lambda.
\end{aligned}$$

We used here that the regular representations $(\varphi\circ\rho\oplus\pi)\tilde{}\times\lambda$ and $(\widetilde{\varphi\circ\rho}\times\lambda)\oplus(\tilde{\pi}\times\lambda)$ are canonically unitary equivalent and that $(\varphi\circ\rho)\oplus\pi$ is faithful. Hence one can extend the map by continuity to $\rho\rtimes_r id\colon A\rtimes_r G \to A/I\rtimes_r G$. The diagram commutes on a dense subset and all the maps are continuous giving a commutative diagram. □

With $I \in \mathcal{I}(A)^G$ one has the identities

$$I\rtimes_r G = \mathrm{Ideal}_{A\rtimes_r G}[I], \quad (I\rtimes_r G)\cap A = I = E_A(I\rtimes_r G),$$

and for $J\in\mathcal{I}(A\rtimes_r G)$ it follows that $J\cap A\in\mathcal{I}(A)^G$ is a subset of $E_A(J)$.

Recall that a discrete group G is *exact* provided the associated reduced group C*-algebra $C^*_r(G)$ is exact. The sequence in diagram (1.2.2) at the level of reduced crossed products is short exact provided that $\ker(\rho\rtimes_r id)\subseteq I\rtimes_r G$. This inclusion does not hold in general, but does hold when G is exact, cf. [33]. More about this later on. For a more thorough introduction to crossed products we refer to [9, 61].

1.3 Transformation group

A discrete group G *acts* on the left of a set X if there is a map

$$(s,x) \mapsto s.x \tag{1.3.1}$$

from $G\times X$ to X such that for all $s,t\in G$ and $x\in X$

$$e.x = x, \quad s.(t.x) = (st).x$$

In the case X is a locally compact Hausdorff space and the map in (1.3.1) is continuous from $G \times X$ to X then the pair (X, G) is called a *transformation group*.

Let (X, G) be a transformation group with G discrete. Then for each $t \in G$ the map $x \mapsto t.x$ is a homeomorphism of X. We obtain a homomorphism $G \to \mathrm{Aut}(C_0(X))$ defined by

$$(t.f)(x) := f(t^{-1}.x), \quad t \in G, x \in X.$$

In this way every transformation group (X, G) give raise to a C*-dynamical system $(C_0(X), G)$. Conversely if $(C_0(X), G)$ is a C*-dynamical system with G discrete and X locally compact Hausdorff space then G has an action on X which makes (X, G) into a transformation group fulfilling that $(t.f)(x) = f(t^{-1}.x)$, cf. [61, Proposition 2.7].

1.4 The spectrum and the primitive ideal space

Suppose that A is a C*-algebra. A subset S of A is described as *primitive ideal* if S is the kernel of some irreducible representation $\pi\colon A \mapsto B(H)$, i.e.

$$S = \ker \pi, \quad \pi(A)' = \mathbb{C}1_{B(H)}.$$

This implies that S is an ideal in A, and $S \neq A$ (since $A = \ker \pi \Rightarrow \pi = 0$). Let $\mathrm{Prim}(A)$ be the set of all primitive ideals in A. Given any subset S of A, the *hull* $h(S)$ of S is defined by

$$h(S) := \{P \in \mathrm{Prim}(A)\colon S \subseteq P\}.$$

When considering the family $\{h(S)\colon S \subseteq A\}$ as all the closed subsets of $\mathrm{Prim}(A)$ we obtain the *Jacobson topology* on $\mathrm{Prim}(A)$. For $S \subseteq \mathrm{Prim}(A)$ we have that $\overline{S} = h(\cap S)$, where $\cap S$ is the intersection of all elements in S. With this topology $\mathrm{Prim}(A)$ is called the *primitive ideal space*. The primitive ideal space is a T_0-space.

Lemma 1.4.1. *Given a C*-algebra A. There is a one-to-one correspondence between ideals in A and open subsets of* $\mathrm{Prim}(A)$ *given by*

$$I \mapsto h(I)^c = \{P \in \mathrm{Prim}(A)\colon I \nsubseteq P\}, \quad I \in \mathcal{I}(A).$$

Proof. Surjectivity: It is easy to see that $h(\{0\})^c = \emptyset$ and $h(A)^c = \mathrm{Prim}(A)$. Let U be an open subset of $\mathrm{Prim}(A)$, $U \neq \mathrm{Prim}(A)$. Then $S := U^c$ is non-empty closed subset of $\mathrm{Prim}(A)$ and $I := \cap S$ is an ideal in A. We obtain that

$$h(I) = h(\cap S) = \overline{S} = S, \quad h(I)^c = U.$$

Injectivity: With proper ideals $I_1, I_2 \in \mathcal{I}(A)$ we have, cf. [29, p. 791], that

$$\begin{aligned} I_1 \neq I_2 &\Rightarrow \cap h(I_1) = I_1 \neq I_2 = \cap h(I_2) \Rightarrow h(I_1) \neq h(I_2) \\ I_1 \neq A &\Rightarrow 0 \in I_1 = \cap h(I_1) \Rightarrow h(I_1) \neq \emptyset = h(A). \end{aligned}$$

Note that $h(S_1)^c \subseteq h(S_2)^c$ for $S_1 \subseteq S_2 \subseteq A$. □

Given a C*-algebra A. The *spectrum* of A, denoted by $\widehat{A}$, is the set of unitary equivalence classes of irreducible representations equipped with the (not necessarily separated) topology induced be the natural surjection onto $\mathrm{Prim}(A)$.

Lemma 1.4.2. *Given a C*-algebra A. A subset of $\widehat{A}$ is open if and only if it has the form*

$$\{[\pi] \in \widehat{A} \colon I \not\subseteq \ker \pi\},$$

for some $I \in \mathcal{I}(A)$.

Proof. Let $\varphi \colon \widehat{A} \to \mathrm{Prim}(A)$ be given by $\varphi([\pi]) = \ker \pi$. By an application of Lemma 1.4.1 together with the identity

$$\varphi^{-1}(\{P \in \mathrm{Prim}(A) \colon I \not\subseteq P\}) = \{[\pi] \in \widehat{A} \colon I \not\subseteq \ker \pi\},$$

valid for any $I \in \mathcal{I}(A)$ we are done. □

Let (A, G) be a C*-dynamical system with G discrete. Recall that the *action* of G on $\widehat{A}$ is defined by $t.[\pi] := [\pi \circ (t^{-1}.)]$ for every $t \in G, [\pi] \in \widehat{A}$. By [61, Proposition 2.8] the map $[\pi] \mapsto t.[\pi]$ is an homeomorphism.

Lemma 1.4.3. *Given an invariant ideal I in a C*-algebra A. The canonical map $\psi \colon \widehat{A/I} \to F := \{[\pi] \in \widehat{A} \colon I \subseteq \ker \pi\}$ is a equivariant homeomorphism.*

Proof. The map ψ is given by $[\pi] \mapsto [\pi \circ \rho]$, where $\rho \colon A \to A/I$ is the quotient map. Define $\varphi \colon F \to \widehat{A/I}$ by $\varphi([\pi]) = [a + I \mapsto \pi(a)]$. From the calculation

$$\psi(\varphi([\pi]))(a) = (\varphi([\pi]) \circ \rho)(a) = [a \mapsto \pi(a)](a) = [\pi](a) \quad [\pi] \in F, a \in A$$

$$\varphi(\psi([\pi]))(a+I) = \varphi([\pi \circ \rho])(a+I) = [\pi \circ \rho](a) = [\pi](a+I) \quad [\pi] \in \widehat{A/I}, a \in A$$

we obtain that ψ is bijective. Since I is invariant the identity

$$\begin{aligned} t.(\psi([\pi]))(a) &= t.[\pi \circ \rho](a) = [\pi \circ \rho \circ (t^{-1}.)](a) = [\pi](t^{-1}.a + I) \\ \psi(t.[\pi])(a) &= \psi([\pi \circ (t^{-1}.)])(a) = [\pi \circ (t^{-1}.) \circ \rho](a) = [\pi](t^{-1}.a + t^{-1}.I) \end{aligned}$$

gives that ψ is equivariant. Further let U be an open subset of $\widehat{A/I}$. Find an ideal $J \in \mathcal{I}(A/I)$ such that $U = \{\pi \in \widehat{A/I} \colon J \nsubseteq \ker \pi\}$ and define the ideal $J_0 := \rho^{-1}(J) \in \mathcal{I}(A)$ fulfilling that $\rho(J_0) = J$. From

$$\begin{aligned}\psi(U) &= \{[\pi \circ \rho] \in \widehat{A} \colon J \nsubseteq \ker \pi\} = \{[\pi \circ \rho] \in \widehat{A} \colon J_0 \nsubseteq \ker(\pi \circ \rho)\} \\ &= \{[\tilde{\pi}] \in \widehat{A} \colon J_0 \nsubseteq \ker \tilde{\pi})\} \cap \psi(\widehat{A/I}) = \{[\tilde{\pi}] \in \widehat{A} \colon J_0 \nsubseteq \ker \tilde{\pi})\} \cap F\end{aligned}$$

(where we use that $J \subseteq \ker \pi \Leftrightarrow \pi(J) = (\pi \circ \rho)(J_0) = 0 \Leftrightarrow J_0 \subseteq \ker(\pi \circ \rho))$ we obtain that ψ is open and continues. $\square$

1.5 Properly and purely infinite C*-algebras

Let A be a C*-algebra and let a, b be positive elements in A. Write $a \precsim b$ if there exists a sequence (r_n) in A such that $r_n^* b r_n \to a$, and write $a \sim b$ if there exist $r \in A$ such that $r^*r = a$ and $rr^* = b$. More generally for $a \in M_n(A)^+$ and $b \in M_m(A)^+$ write $a \precsim b$ if there exist a sequence (r_n) in $M_{m,n}(A)$ with $r_n^* b r_n \to a$ and write $a \sim b$ if there exist $r \in M_{m,n}(A)$ such that $r^*r = a$ and $rr^* = b$. For $a \in M_n(A)$ and $b \in M_m(A)$ let $a \oplus b$ denote the element $\mathrm{diag}(a, b) \in M_{n+m}(A)$.

For a positive element a in a C*-algebra A and $\epsilon > 0$, let $(a - \epsilon)_+$ be the positive part of the self-adjoint element $a - \epsilon \cdot 1$ where 1 is the unit of the *unitization* of A. Equivalently, $(a - \epsilon)_+ = f_\epsilon(a)$, where $f_\epsilon \colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $f_\epsilon(t) = \max\{t - \epsilon, 0\}$. For the spectrum of $a \in A$ write $\sigma(a)$.

A positive element a i a C*-algebra A is called *infinite* if there exist a non-zero positive element b in A such that $a \oplus b \precsim a$. If a is non-zero and if $a \oplus a \precsim a$, then a is said to be *properly infinite*. If we only consider projections in A the notion introduced here is equivalent to the classical definition of infinite and properly infinite projections, cf. [31, p. 642-643]. A unital C*-algebra is said to be *infinite*, respectively *properly infinite*, if the unit of A has this property.

A C*-algebra is said to be *purely infinite* if there are no characters on A and if for every pair of positive elements a, b in A such that $b \in \mathrm{Ideal}_A[a]$ one has $b \precsim a$. It was shown in [31, Theorem 4.16] by Kirchberg and Rørdam that a C*-algebra A is purely infinite *if and only if* every non-zero positive element a in A is *properly infinite*.

Chapter 2

Properties of crossed products

This chapter focuses on some of our first work related to reduced crossed products. Most of the results are contained in [54]. We have improved the work on non-amenable groups in [56] and included the result here.

2.1 Simplicity

A C*-algebra is *simple* if it does not contain any non-trivial ideals. One necessary condition for $A \rtimes_r G$ to be simple, is the *minimality* of the action of G on A, meaning that A does not contain any non-trivial invariant ideals. Indeed if I is a non-trivial invariant ideal in a C*-algebra A, then $I \rtimes_r G$ is a non-trivial ideal in $A \rtimes_r G$ (containing I and not equal to $A \rtimes_r G$ since $I \rtimes_r G \cap A = I \neq A = A \rtimes_r G \cap A$).

Recall that a discrete group G is called *amenable* if the associated reduced group C*-algebra $C_r^*(G)$ is nuclear, cf. [9, Theorem 2.6.8]. Given a transformation group (G, X) with G discrete. The *isotropy group* of $x \in X$ (also called the *stabilizer subgroup*) is the set of all elements in G that fix x. The action of G on X is said to be *free* if every point in X has trivial isotropy and *minimal* if every G-orbit $\mathcal{O}_x := \{t.x \colon t \in G\}$ is dense in X. For an abelian C*-algebra A the action of a discrete group G on A is minimal if and only if the action of G on $\widehat{A}$ is minimal.

We will in the following present several sufficient conditions for simplicity. Some of the first results go back to work of E. Effros and F. Hahn in the 1960's, cf. [17, Theorem 5.16]. They showed that a countable amenable group G acting minimally and freely on a compact Hausdorff space X gives a simple crossed product $C(X) \rtimes_r G$. We will return to this result later on. Zeller-Meier made some generalizations in the non-commutative case. We refer to his work [62] from 1968.

A group G is called a *Powers group* if for any finite $F \subseteq G \setminus \{e\}$ and $n \in \mathbb{N}$ there exist subsets $D, E \subseteq G$ and $t_1, \ldots, t_n \in G$ such that

$$(i)\ tD \cap D = \emptyset,\ t \in F, \quad (ii)\ t_j E \cap t_k E = \emptyset,\ j \neq k, \quad (iii)\ E \cup D = G, E \cap D = \emptyset.$$

An example of a Powers group is the free non-abelian group on n generators $\mathbb{F}_n$. Powers showed that $C_r^*(\mathbb{F}_n)$ is simple. Harpe and Skandalis extended this in [16, Theorem 1]. They proved that for any C*-dynamical system (A, G) with a unital C*-algebra A and a Powers group G, the reduced crossed product $A \rtimes_r G$ is simple provided that the action on A is minimal. They ask the following

Question 2.1.1. Given a simple unital C*-algebra A and a group G such that $C_r^*(G)$ is simple, is it true that the reduced crossed product $A \rtimes_r G$ is simple?

When the action is trivial, one has that $A \rtimes_r G \cong A \otimes C_r^*(G)$, thus giving a partial answer to the question of Harpe and Skandalis, cf. [59]. For other examples we refer to [16].

In [39, Theorem 6.5] it was shown that for an abelian group G the C*-algebra $A \rtimes_r G$ is simple if and only if the action is minimal and *the Connes spectrum* of the action equals the whole dual group of G. We omit the definition of Connes spectrum.

Recall that an automorphism α of A is called *properly outer* if for any non-zero *α-invariant* ideal I in A (meaning that $I \in \mathcal{I}(A)$ and $\alpha(I) = I$) and any inner automorphism β of I, $\|\alpha|_I - \beta\| = 2$, cf. [18]. Let (A, G) be a C*-dynamical system with G discrete. The action, say α, of G on A is called *properly outer* if α_t is properly outer for every $t \neq e$, cf. [40]. It was shown by Elliott in [18, Theorem 3.2] that $A \rtimes_r G$ is simple, provided that A is an *AF-algebra* (i.e. approximately finite dimensional, cf. [48]) and the action is minimal and properly outer.

The properly outerness condition does not imply a one-to-one correspondence between the invariant ideals in A and the ideals in $A \rtimes_r G$. Elliot showed this by example in [18, p. 309]. He considered the group $\mathbb{Z}^2$ acting on the C*-algebra generated by the characteristic functions of half-open intervals $[a, b[$ in $\mathbb{Z}^2$ with lexicographical order.

Recall that for any automorphism β of a C*-algebra A one can define *the dual action* $\hat{\beta}\colon \mathbb{T} \to \mathrm{Aut}(A \rtimes_r \mathbb{Z})$ is such a way that $\hat{\beta}_\lambda(a) = \sum_{t \in \mathbb{Z}} \lambda^t a_t u_t$ for $a = \sum_{t \in \mathbb{Z}} a_t u_t \in C_c(\mathbb{Z}, A)$. We refer to [43, Proposition 7.8.3] for details. The *the strong Connes spectrum* $\hat{\mathbb{T}}(\beta)$ is then defined to be

$$\hat{\mathbb{T}}(\beta) := \{\lambda \in \mathbb{T} : \hat{\beta}_\lambda(I) = I \text{ for any primitive ideal } I \text{ in } A \rtimes_r \mathbb{Z}\}.$$

Kishimoto showed in [34, Theorem 3.1] the simplicity of $A \rtimes_r G$ provided that the action, say α, of G on A is minimal and that the strong Connes spectrum $\hat{\mathbb{T}}(\alpha_t)$ does not equal 1 for any $t \neq e$.

Archbold and Spielberg generalized in [3] the result of Elliot by introducing *topological freeness*. For a C*-dynamical system (A, G) the action on A is called topologically free if $\cap_{t \in F}\{x \in \widehat{A} : t.x \neq x\}$ is dense in $\widehat{A}$ for any finite subset $F \subseteq G \setminus \{e\}$. They showed that if the action on A is minimal and topologically free, then $A \rtimes_r G$ is simple. For a C*-dynamical system (A, G) with A abelian and G discrete, the notion of topological freeness and proper outerness coincide. In the case when A is separable, Kishimito's condition implies topological freeness, which further implies proper outerness, cf. [3, p. 122].

It is known that the full crossed product $A \rtimes G$ is simple if and only if the action on A is minimal, topologically free and *regular*, i.e. the canonical surjection $\pi^A \colon A \rtimes G \to A \rtimes_r G$ is injective. Here we present a simple proof of this fact when considering the full crossed product associated with a transformation group (X, G). The main idea is a reformulation of topological freeness. The first part of the proof is partially motivated by [30, Theorem 4.1]. We have, however, shortened the proof by using properties of *definite states* (see [28] for the definition).

Lemma 2.1.2. *Consider a transformation group (X, G) with G discrete. Let I_π be the kernel of the canonical surjection $\pi \colon C_0(X) \rtimes G \to C_0(X) \rtimes_r G$. The action of G on $C_0(X)$ is topologically free if and only if*

$$\forall I \in \mathcal{I}(C_0(X) \rtimes G) : C_0(X) \cap I = 0 \Rightarrow I \subseteq I_\pi$$

Proof. '$\Rightarrow$' Fix $a \in I^+$, $\epsilon > 0$ and $x \in X$. Define $f := E(\pi(a))$, where $E := E_{C_0(X)} \circ \pi$ is the canonical conditional expectation on $B := C_0(X) \rtimes G$. Find $b = \sum_{t \in F} b_t u_t \in C_c(G, C_0(X))^+$ such that $\|a - b\| < \epsilon$ and define $g := E(\pi(b))$. Using the continuity of f, find an open non-empty set $U \subseteq X$ such that

$$|f(x) - f(z)| < \epsilon, \quad z \in U.$$

As α is topologically free, $\cup_{t \in F \setminus \{e\}}\{x \in X : t.x = x\}$ has empty interior and hence does not contain U. Therefore we can find $y \in U$ such that

$$t.y \neq y, \quad t \in F \setminus \{e\}.$$

As $C_0(X) \cap I = 0$, each element in $C_0(X) + I$ has a unique decomposition of the form $h + i$ with $h \in C_0(X)$ and $i \in I$. The map $\varphi(h + i) = h(y)$ $(h \in C_0(X), i \in I)$ is a state on $C_0(X) + I$ and extends to a state $\varphi : B \to \mathbb{C}$.

Using that φ is a definite state with respect to elements in $C_0(X)$, cf. [28, Exercise 4.6.16], it follows that

$$\varphi(i) = 0,\ \varphi(h) = h(y),\ \varphi(hu_t) = 0, \quad i \in I, h \in C_0(X), t \in F \setminus \{e\}.$$

The last equality follows from the following consideration: Fix $t \in F \setminus \{e\}$. By Urysohn's Lemma there exist $k \in C_0(X)^+$ such that $k(t^{-1}.y) \neq k(y)$. By [28, Exercise 4.6.16] (since $u_t k = (t.k)u_t$) we obtain that $\varphi(u_t) = 0$ from

$$k(t^{-1}.y)\varphi(u_t) = \varphi((t.k)u_t) = \varphi(u_t k) = \varphi(u_t)k(y).$$

Consequently,

$$\varphi(a) = 0, \quad |\varphi(a) - \varphi(b)| < \epsilon, \quad \varphi(b) = \varphi(b_e) = g(y), \quad |g(y) - f(y)| < \epsilon,$$

implying that $|f(x)| < 3\epsilon$. One now has that $f = E(\pi(a)) = 0$. Since E is faithful, it follows that $a \in I_\pi$.

'$\Leftarrow$' [3, Theorem 2] For a given $x \in X$ let $(\delta_{t.x})_{t \in G}$ be an orthonormal basis for $H_x = l^2(G.x)$. Define a covariant representation (U_x, π_x, H_x) by

$$\pi_x(f)(\delta_{t.x}) = f(t.x)\delta_{t.x}, \quad U_x(r)(\delta_{t.x}) = \delta_{rt.x}, \quad f \in C_0(X), \quad t, r \in G.$$

The universal property of B gives a representation $\pi_x \rtimes U_x \colon B \to B(H_x)$ extending $\pi_x \times U_x$. With $I = \bigcap_{x \in X} \ker \pi_x \rtimes U_x$ it follows that $C_0(X) \cap I = 0$ (as $\pi_x \rtimes U_x(f)\delta_{e.x} = f(x)\delta_{e.x}$) and hence $I \subseteq I_\pi$.

Assuming α is not topologically free set $V_t := \{y \in X : t.y \neq y\}$. If every $V_t, t \neq e$ is dense in X the action is topologically free. Hence one can find $s \neq e$ such that $\overline{V_s} \neq \emptyset$ and let f be a non-zero function with support in the complement of $\overline{V_s}$. For $T = \pi_x \rtimes U_x(f - fu_s) \in B(H_x)$ any $t \in G$ we obtain

$$\begin{aligned} t.x \in \operatorname{supp} f \quad &\Rightarrow \quad st.x = t.x \Rightarrow T\delta_{t.x} = f(t.x)\delta_{t.x} - f(st.x)f\delta_{st.x} = 0 \\ t.x \notin \operatorname{supp} f \quad &\Rightarrow \quad st.x \notin \operatorname{supp} f \Rightarrow T\delta_{t.x} = f(t.x)\delta_{t.x} - f(st.x)f\delta_{st.x} = 0 \end{aligned}$$

Hence $T = 0$. As $x \in X$ was arbitrary $f - fu_s \in I \subseteq I_\pi$ giving that $f = E(f - fu_s) = E_{C_0(X)}(\pi(f - fu_s)) = E_{C_0(X)}(0) = 0$. This contradicts $f \neq 0$. □

Corollary 2.1.3 (Archbold and Spielberg)**.** *Consider a transformation group (X, G) with G discrete. The full crossed product $C_0(X) \rtimes G$ is simple if and only if the action is minimal, topologically free and regular.*

Proof. '$\Leftarrow$' Regularity implies $I_\pi = 0$. Assuming $I \in \mathcal{I}(B), I \neq 0$ the topological freeness gives that $C_0(X) \cap I$ is an non-zero invariant ideal in $C_0(X)$. By minimality we conclude that $C_0(X) \cap I = C_0(X)$. Hence $I = B$.

'$\Rightarrow$' Assuming that B is simple one gets minimality as before. Regularity follows from considering $I_\pi \in \mathcal{I}(B)$ and it is now obvious that for every $I \in \mathcal{I}(B), C_0(X) \cap I = 0$ one has $I = 0$, giving the topological freeness. □

Remark 2.1.4. For (X, G) with G countable amenable and X compact it was shown in [17, Theorem 5.16] that the primitive ideal space and the set X/G of (the closure of) G-orbits in X are in a natural way homeomorphic, provided the action is free.

Using this result of Effros-Hahn one can, by combining [17, Theorem 5.2] and [29, Theorem 3.4.1, Exercise 10.5.82], show the existence of a one-to-one correspondence between the invariant ideals in A and the ideals in $A \rtimes_r G$. We here give a different and rather simple proof of this fact in the case of a transformation group using exactness instead of amenability.

Theorem 2.1.5. *Consider a transformation group (X, G) with G discrete. If G is exact and the action is free, then the map $I \mapsto I \cap C_0(X)$ is a bijection between the ideals in $C_0(X) \rtimes_r G$ and the invariant ideals in $C_0(X)$.*

Proof. Set $B := C_0(X) \rtimes_r G$ and fix $x \in B^+$. We have that

$$(i) \text{ freeness implies } E(x) \in I[x], \quad (ii) \text{ exactness implies } x \in I[E(x)].$$

First we prove (i): First we use the one-to-one correspondence between ideals in $C_0(X)$ and closed sets in X. As $I[x] \cap C_0(X)$ is an invariant ideal in $C_0(X)$, there exist a closed set $F \subseteq X$ such that

$$I_F := \{f \in C_0(X) \ : \ f|_F = 0\} = I[x] \cap C_0(X).$$

Set $f = E(x)$. Assuming $f \notin I_F$ take $y \in F$ such that $\epsilon = |f(y)| > 0$. Since $C_0(X) \cap I[x] = I_F$ we can define a map $\varphi \colon C_0(X) + I[x] \to \mathbb{C}$ by $\varphi(h + i) = h(y)$ $(h \in C_0(X), i \in I[x])$. As φ is a state on $C_0(X) + I[x]$ it extends to a state $\varphi : B \to \mathbb{C}$. Since φ is a definite state with respect to elements in $C_0(X)$, cf. [28, Exercise 4.6.16]) it follows (using freeness) that

$$\varphi(i) = 0, \quad \varphi(h) = h(y), \quad \varphi(hu_t) = 0, \quad i \in I[x], h \in C_0(X), t \in G \setminus \{e\}.$$

Find $b = \sum_{t \in F'} b_t u_t \in C_c(G, C_0(X))^+$ such that $\|b - x\| < \frac{\epsilon}{2}$. We now get a contradiction from

$$|f(y)| \leq |f(y) - \varphi(b)| + |\varphi(b) - 0| = |E(x - b)(y)| + |\varphi(b - x)| < \epsilon = |f(y)|.$$

Hence $f \in I_F$, impying that $E(x) \in I[x]$.

Now we prove (ii): Given $x \in B^+$. Set $I = I[E(x)] \cap C_0(X)$. As G is exact we obtain the following commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & I \rtimes_r G \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C_0(X)/I \rtimes_r G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & C_0(X) \arrow[r] & C_0(X)/I \arrow[r] & 0
\end{tikzcd}$$

where both sequences are short exact, cf. [33]. Using the diagram on $x \in B$, with the condition that $E(x) \in I$, it follows that $x \in I \rtimes_r G$. As $I[E(x)] \cap C_0(X)$ is the smallest G-invariant ideal in $C_0(X)$ containing $E(x)$, one has that $I[E(x)] = I \rtimes_r G$. Therefore $x \in I[E(x)]$.

As mentioned before the map $I \to I \cap C_0(X), I \in \mathcal{I}(B)$ is surjective. We now show injectivity. For $I, J \in \mathcal{I}(B)$ with $I \cap C_0(X) = J \cap C_0(X)$ it is by symmetry enough to show $J^+ \subseteq I$. With $x \in J^+$ we have, by (i), that

$$E(x) \in I[x] \subseteq J$$

implying that $E(x) \in J \cap C_0(X) = I \cap C_0(X)$. Using (ii) we conclude that $x \in I[E(x)] \subseteq I$. $\square$

Remark 2.1.6. We have been able to extend Theorem 2.1.5 to crossed products $A \rtimes_r G$ where A is an arbitrary C*-algebra, cf. Theorem 3.3.7. This is done i a way that extends the work of Archbold and Spielberg (who showed that a minimal and topologically free action of a discrete group G on a C*-algebra A gives a simple reduced crossed product $A \rtimes_r G$).

Remark 2.1.7. We now show by an example that the freeness condition in Theorem 2.1.5 can not be replaced by topological freeness if we want the theorem to remain valid. To see this, consider the short exact sequence

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\varphi} B \longrightarrow 0$$

where $A = C(\mathbb{T})$, $I = \{f \in A : f|_{\{-1\}} = 0\}$, $B = \mathbb{C}$, ι is the inclusion map and φ the evaluation map $f \mapsto f(-1)$. Let $\alpha : \mathbb{Z} \to \mathrm{Aut}(A)$ be an arbitrary topologically free action on A with a fix point in -1. From the universal property we get the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & I \rtimes \mathbb{Z} \arrow[r, "\iota \rtimes id"] \arrow[d, "E_I"] & A \rtimes \mathbb{Z} \arrow[r, "\varphi \rtimes id"] \arrow[d, "E_A"] & B \rtimes \mathbb{Z} \arrow[r] \arrow[d, "E_B"] & 0 \\
0 \arrow[r] & I \arrow[r, "\iota"] & A \arrow[r, "\varphi"] & B \arrow[r] & 0
\end{tikzcd}$$

Identifying $B \rtimes \mathbb{Z} \cong C^*(\mathbb{Z}) \cong C(\hat{\mathbb{Z}}) = C(\mathbb{T})$ we can find $J_1, J_2 \in \mathcal{I}(C(\mathbb{T}))$ such that

$$J_1 \neq C(\mathbb{T}), \quad J_2 \neq C(\mathbb{T}), \quad J_1 \neq J_2.$$

With $K_i = (\varphi \rtimes id)^{-1}(J_i) \in \mathcal{I}(A \rtimes \mathbb{Z})$ $(i = 1, 2)$ one can show, using the diagram, that

$$K_1 \cap A = I, \quad K_2 \cap A = I, \quad K_1 \neq K_2.$$

This implies that the map $I \to I \cap C(\mathbb{T})$ associated with the transformation group $(\mathbb{T}, \mathbb{Z})$, where the action is topologically free action but has a fix point in -1, is not injective.

2.2 Nuclearity

A C*-algebra A is *nuclear* if for each C*-algebra B there is a unique C*-norm on the algebraic tensor product $A \odot B$. Among others, all abelian C*-algebras and the AF-algebras are nuclear. Nuclearity plays an significant role in the classification program. We refer to [47] for a review.

Let (A, G) be a C*-dynamical system with A unital. The action (of G on A) is called *amenable* if there exist a net of functions $(T_i)_{i \in I}$ in $C_c(G, A)$, such that

(i) $T_i(t)$ is positive and commutes with A for all $i \in I, t \in G$,

(ii) $\langle T_i, T_i \rangle = 1_A, \quad i \in I$,

(iii) $\|u_s T_i - T_i\|_2 \to 0, \quad s \in G$,

where $\langle \cdot, \cdot \rangle$, $\| \cdot \|_2$ is defined, cf. [9, p.107], by

$$\langle S, T \rangle = \sum S(t)^* T(t) \in A, \quad \|S\|_2 = \|\langle S, S \rangle\|^{1/2}, \quad S, T \in C_c(G, A).$$

In the case when A is nuclear it follows that $A \rtimes_r G$ (and $A \rtimes G$) becomes nuclear provided the action is amenable, cf. [9, Th.4.3.4]. For a transformation group (X, G) with X compact it was shown in [2] that one also has the converse implication.

Theorem 2.2.1 (Anantharaman-Delaroche)**.** *Let (X, G) be a transformation group with G discrete and X compact. Then the following properties are equivalent:*

(i) The C-algebra $C(X) \rtimes_r G$ is nuclear.*

(ii) The action of G on $C(X)$ is amenable.

(iii) There exists a net of continuous maps $m_i : X \to \mathrm{Prob}(G)$[1] such that

$$\lim_i (\sup_{x \in X} \|s.m_i^x - m_i^{s.x}\|_1) = 0, \quad s \in G,$$

where $s.m_i^x(t) = m_i^x(s^{-1}t)$ $(t \in G)$.

[1]By definition, $\mathrm{Prob}(G)$ is the set of probability measures on G which we identify with the set of positive, norm one elements in $l^1(G)$. Continuity means with respect to the restriction of the weak-* topology on $l^1(G)$. In other words, $m \colon X \to \mathrm{Prob}(G)$ is continuous if and only if for each convergent net $x_i \to x \in X$ we have $m^{x_i}(t) \to m^x(t)$ for all $t \in G$

(iv) For any C-algebra A with a equivariant embedding of C(X) into the center of A, we have $A \rtimes G \cong A \rtimes_r G$.*

Proof. We refer to [9, Th. 4.4.3] for the proof. $\square$

Theorem 2.2.1 $(i) \Leftrightarrow (ii)$ relates nuclearity of the crossed product with amenability. The most know definition of an action being amenable is stated in condition (iii). This formulation gives an easy way to relate the amenability of a group and amenability of an action. Let us remind the reader about this relation.

Lemma 2.2.2. *Given a transformation group (X, G) with G countable and X compact. If G is amenable then the action of G on X is amenable.*

Proof. Write $G = \{t_1, t_2, \dots\}$. Amenability of G implies that G admits an *approximate invariant mean*, cf. [9, Definition 2.6.3]. Hence for any finite $F \subseteq G$ and $\epsilon > 0$ there exists $\mu \in \mathrm{Prob}(G)$ such that

$$\max_{s \in F} \|s.\mu - \mu\|_1 < \epsilon.$$

For $i \in \mathbb{N}$ find $\mu_i \in \mathrm{Prob}(G)$ such that $\max_{s \in \{t_1, \dots, t_i\}} \|s.\mu_i - \mu_i\|_1 < 1/i$. With $m_i^x = \mu_i, x \in X$ the sequence (m_i) fulfills the property (ii) in Theorem 2.2.1. Hence the action af G on $C(X)$ is amenable. $\square$

Finally we se, from the implication $(ii) \Rightarrow (iv)$ in Theorem 2.2.1, that amenability of an action (of G on A) implies $A \rtimes G \cong A \rtimes_r G$. This observation can be slightly strengthen.

Definition 2.2.3. Let (A, G) be a C*-dynamical system. We say that the action (of G on A) is *exact* if every invariant ideal I in A induces a short exact sequence

$$0 \longrightarrow I \rtimes_r G \xrightarrow{\iota \rtimes_r \mathrm{id}} A \rtimes_r G \xrightarrow{\rho \rtimes_r \mathrm{id}} A/I \rtimes_r G \longrightarrow 0$$

at the level of reduced crossed products.

Proposition 2.2.4. *Let (A, G) be a C*-dynamical system with G discrete. The action is regular and exact if and only if the action on any quotient $A/I, I \in \mathcal{I}(A)^G$ is regular.*

In particular if A is unital and abelian then amenability of the action implies that the action is regular and exact.

Proof. Fix $I \in \mathcal{I}(A)^G$. We have the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & I \rtimes G \arrow[r, "\iota\rtimes\mathrm{id}"] \arrow[d, "\pi^I"] & A \rtimes G \arrow[r, "\rho\rtimes\mathrm{id}"] \arrow[d, "\pi^A"] & A/I \rtimes G \arrow[r] \arrow[d, "\pi^{A/I}"] & 0\ , \\ 0 \arrow[r] & I \rtimes_r G \arrow[r, "\iota\rtimes_r\mathrm{id}"] & A \rtimes_r G \arrow[r, "\rho\rtimes_r\mathrm{id}"] & A/I \rtimes_r G \arrow[r] & 0 \end{tikzcd}$$

where we will suppress the inclusion maps $\iota \rtimes_r \mathrm{id}$ and $\iota \rtimes \mathrm{id}$.

'$\Leftarrow$'. By regularity of the action on A/I the map $\pi^{A/I}$ is injective. We claim that $\ker(\rho \rtimes_r \mathrm{id}) \subseteq I \rtimes_r G$. To see this fix $a \in A \rtimes_r G$ such that $\rho \rtimes_r \mathrm{id}(a) = 0$. Find $b \in A \rtimes G$ such that $\pi^A(b) = a$. From

$$\pi^{A/I}(\rho \rtimes \mathrm{id}(b)) = \rho \rtimes_r \mathrm{id}(\pi^A(b)) = \rho \rtimes_r \mathrm{id}(a) = 0$$

we have that $b \in \ker(\rho \rtimes \mathrm{id}) = I \rtimes G$. Hence $a = \pi^A(b) = \pi^I(b) \in I \rtimes_r G$.

'$\Rightarrow$'. We show that $\pi^{A/I}$ is injective. Fix $a \in A/I \rtimes G$ such that $\pi^{A/I}(a) = 0$. Find $b \in A \rtimes G$ such that $(\rho \rtimes \mathrm{id})(b) = a$. From

$$\rho \rtimes_r \mathrm{id}(\pi^A(b)) = \pi^{A/I}(\rho \rtimes \mathrm{id}(b)) = \pi^{A/I}(a) = 0$$

we have that $\pi^A(b) \in \ker(\rho \rtimes_r \mathrm{id}) = I \rtimes_r G = \pi^I(I \rtimes G)$. Find $i \in I \rtimes G$ such that $\pi^A(b) = \pi^I(i)$. As $\pi^I(i) = \pi^A(i)$ and π^A is injective we conclude that $i = b$ and hence that $a = \rho \rtimes_r \mathrm{id}(b) = \rho \rtimes_r \mathrm{id}(i) = 0$.

Suppose that A is unital and abelian and the action of G on A is amenable. By [9, Exercise 4.4.3] the action of G on A/I, $I \in \mathcal{I}(A)^G$ is amenable. In particular the action on of G on A/I, $I \in \mathcal{I}(A)^G$ is regular. To see this fix $I \in \mathcal{I}(A)^G$, set $B := A/I$ and consider an element $x \in B \rtimes G$ such that $\pi^B(x) = 0$ and $\epsilon > 0$. Find $y \in C_c(G, B)$ such that $\|x - y\| < \epsilon/3$. Using the proof of [9, Theorem 4.2.4] there exist a contractive completely positive linear map $\varphi : A \rtimes_r G \to A \rtimes G$ such that $\|y - \varphi(\pi^B(y))\| < \epsilon/3$. By

$$\|x\| = \|x - y\| + \|y - \varphi(\pi^B(y))\| + \|\varphi(\pi^B(y)) - \varphi(\pi^B(x))\| < \epsilon,$$

we conclude that $x = 0$. This imply that the action of G on A/I is regular. By the observation above we have that the action of G on A is regular and exact. $\square$

Question 2.2.5. Does regularity of an action imply exactness?

2.3 Proper infiniteness

A projection p in a C*-algebra A is *infinite* if it is equivalent to a proper sub-projection of itself. If p has mutually orthogonal sub-projections q_1 and q_2

such that $p \sim q_1 \sim q_2$, then p is said to be *properly infinite*. This definitions is equivalent to the one mentioned in the introduction, cf. [31, p. 642-643]. Recall that a unital C*-algebra A is said to be infinite, respectively, properly infinite, if the unit of A has this property.

It was shown in [51] that there exists a unital, simple, nuclear and separable C^*-algebra D containing a finite and infinite projection p, q. Using [48, Exercise 4.7-4.9] it follows that $A = qDq$ is properly infinite, but not purely infinite. One may consider the following

Question 2.3.1. Given a C^*-dynamical system (A, G) with G discrete, when is the crossed product $A \rtimes_r G$ (properly) infinite?

Let (X, G) be a transformation group with G discrete. Recall the action of G on X is called a *local boundary action* is for every non-empty open set $Y \subseteq X$ there exist an open set $\Delta \subseteq Y$ and $t \in G$, such that $t.\bar{\Delta} \subsetneq \Delta$, cf. [36, Definition 6].

For A abelian Laca and Spielberg obtained a way to find infinite projections in hereditary sub-C*-algebras of $A \rtimes_r G$. They showed in [36] that for a transformation group (X, G) with G discrete every non-zero hereditary sub-C*-algebra in $C_0(X) \rtimes_r G$ contains an infinite projection if the action of G on $C_0(X)$ is topologically free and the action of G on X is a local boundary action. In [1] a similar result was obtained independently by Ananthraman-Delaroche.

The notion of a local boundary action suggest a way to ensure infiniteness of crossed products. It turns out that for a transformation group (X, G) with G discrete and X compact one has the implication

$$\exists \Delta \subseteq X, t \in G : \Delta \text{ is open and } t.\bar{\Delta} \subsetneq \Delta \quad \Rightarrow \quad C(X) \rtimes_r G \text{ is infinite}$$

We found an easy proof of this implication by showing an even more general result. First we present three equivalent formulations of the condition implying that $C(X) \rtimes_r G$ is infinite.

Lemma 2.3.2. *Given a transformation group (X, G) with G discrete. The following conditions are equivalent*

(1) There exists an open subset $\Delta \subseteq X$ and $t \in G$ such that

$$t.\bar{\Delta} \subsetneq \Delta.$$

(2) There exist open subsets $U_1, U_2 \subseteq X$ and elements $t_1, t_2 \in G$ such that

$$U_1 \cup U_2 = X, \quad t_1.U_1 \cap t_2.U_2 = \emptyset, \quad t_1.\bar{U}_1 \cup t_2.\bar{U}_2 \neq X.$$

(3) There exist open subsets $U_1, U_2 \subseteq X$ and elements $t_1, t_2 \in G$ such that

$$U_1 \cup U_2 = X, \quad t_1.U_1 \cap t_2.U_2 = \emptyset, \quad t_1.U_1 \cup t_2.U_2 \neq X.$$

Proof. For any $S \subseteq X$ and $t \in G$ we have that $t.S^c = (t.S)^c$ (since $X = t.S \cup t.S^c$), where S^c denotes the complement of S.

(1) $\Rightarrow$ (2). Set

$$U_1 = \Delta,\ U_2 = t.(\bar{\Delta}^c),\ t_1 = t^2,\ t_2 = t^{-1}.$$

Using $t.\bar{\Delta} \subsetneq \Delta$ we obtain

$$U_1^c = \Delta^c \subseteq (t.\bar{\Delta})^c = t.(\bar{\Delta}^c) = U_2 \Rightarrow U_1 \cup U_2 = X$$

$$t_2.U_2 = \bar{\Delta}^c \subseteq \Delta^c \Rightarrow t_2.\bar{U}_2 \subseteq \Delta^c, \quad t_1.U_1 = t^2.\Delta \subseteq t.\bar{\Delta} \subsetneq \Delta \Rightarrow t_1.\bar{U}_1 \subsetneq \Delta.$$

It is now evident that property (2) holds.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Set $\Delta := U_1$ and $t := t_2^{-1}t_1$. We have that $t_1.\bar{U}_1 \subseteq (t_2.U_1)^c$ and $U_2^c \subseteq U_1$ giving that

$$t.\bar{\Delta} = t_2^{-1}t_1.\bar{U}_1 \subseteq t_2^{-1}.(t_2.U_2)^c = U_2^c \subseteq U_1 = \Delta$$

If $t.\bar{\Delta} = \Delta$ then $t_1.\bar{U}_1 = t_2.U_1$ and $\bar{U}_1 = U_1$. This implies that $t_2.U_2 \cup t_1.U_1 = t_2.(U_2 \cup U_1) = X$ — contradiction. $\square$

The last formulation (3) in Lemma 2.3.2 can be generalized in the following way.

Definition 2.3.3. Given a transformation group (X, G) with G discrete, The action is called *n-paradoxical* if there exist open subsets $U_1, U_2 \ldots, U_n \subseteq X$ and elements $t_1, t_2, \ldots, t_n \in G$ such that[2]

$$\bigcup_{i=1}^{n} U_i = X, \quad \bigsqcup_{i=1}^{n} t_i.U_i \subsetneq X, \quad t_k.U_k \cap t_l.U_l = \emptyset \ (k \neq l)$$

We now prove that it is sufficient to assume that the action is n-paradoxical to get a infinite crossed product.

Theorem 2.3.4. *Given a transformation group (X, G) with G discrete and X compact. If the action is n-paradoxical for some $n \in \mathbb{N}$, then $C(X) \rtimes_r G$ is infinite.*

[2] We use $\bigsqcup$ to emphasize the union is disjoint.

Proof. As X is compact and Hausdorff let $h_1, \ldots, h_n$ denote a partition of unity on X subordinate to the cover $U_1, \ldots, U_n$, cf. [52, Th.2.12]. With

$$v = \sum_{i=1}^{n} u_{t_i} h_i^{1/2},$$

we have that

$$v^* v = \sum_{i,j=1}^{n} h_i^{1/2} u_{t_i}^* u_{t_j} h_j^{1/2} = \sum_{i=1}^{n} h_i^{1/2} u_{t_i}^* u_{t_i} h_i^{1/2} = 1,$$

using that for $k \neq l$ in $\{1,\ldots,n\}$ (as $t.\operatorname{supp}(f) = \operatorname{supp}(t.f), f \in C(X), t \in G$) we obtain

$$t_k.U_k \cap t_l.U_l = \emptyset \Rightarrow t_k.\operatorname{supp}(h_k^{1/2}) \cap t_l.\operatorname{supp}(h_l^{1/2}) = \emptyset \Rightarrow h_k^{1/2} u_{t_k}^* u_{t_l} h_l^{1/2} = 0.$$

We now show that $vv^* \neq 1$. Find $x \in X$ such that $x \notin \cup_{i=1}^{n} t_i.U_i$ and define $\Phi_x \colon C(X) \rtimes_r G \to \mathbb{C}$ as

$$\Phi_x(a) = (E(a))(x), \quad a \in C(X) \rtimes_r G,$$

where E denotes the conditional expectation $E : C(X) \rtimes_r G \to C(X)$. From

$$vv^* = \sum_{i,j=1}^{n} u_{t_j} h_j^{1/2} h_i^{1/2} u_{t_i}^* = \sum_{i,j=1}^{n} t_j.(h_j^{1/2} h_i^{1/2}) u_{t_j} u_{t_i}^*$$

it follows that

$$E(vv^*) = \sum_{i=1}^{n} E(t_i.(h_i^{1/2} h_i^{1/2}) u_{t_i} u_{t_i}^*) = \sum_{i=1}^{n} t_i.h_i, \quad \Phi_x(vv^*) = \sum_{i=1}^{n} h_i(t_i^{-1}.x).$$

For $i = 1, \ldots, n$ we know that $x \notin t_i.U_i$. This implies that $t_i^{-1}.x \notin U_i$ and $h_i(t_i^{-1}.x) = 0$. Hence $vv^* \neq 1$ from

$$\Phi_x(1) = 1, \quad \Phi_x(vv^*) = \sum_{i=1}^{n} h_i(t_i^{-1} x) = 0.$$

Thus $C(X) \rtimes_r G$ is infinite. $\square$

There are several examples of transformation groups with a n-paradoxical action. Recall that for a discrete group G there is a natural transformation group $(\beta G, G)$ coming from the isomorphism $C(\beta G) \cong l^\infty(G)$, where βG is the *Stone-Čech compactification* of G. The left multiplication of G extends uniquely to an action of G on βG.

The crossed product $C(\beta G) \rtimes_r G$ is properly infinite if and only if G is non-amenable (cf. Theorem 2.5.1). Hence one would expect that the action of a non-amenable group G on βG is n-paradoxical. This is indeed the case (cf. Theorem 2.5.1 or [60, Theorem 10.11]).

Note that G does not have to be non-amenable in order to induce a n-paradoxical action. To see this just apply $G = \mathbb{Z}$ on the following

Theorem 2.3.5. *Given a free product $G = G_1 * \cdots * G_n$ of non-trivial cyclic groups, with $n \geq 1$ and $|G| = \infty$, then the action of G on βG is 2-paradoxical.*

Proof. Using the same idea as Wagon, who considered the free group of two generators, cf. [60, Th.1.2], find $V_1, V_2, U_1, U_2 \subseteq G$ and $t_1, t_2 \in G$ such that

$$V_j = t_j.U_j \ (j = 1, 2), \quad V_1 \cap V_2 = \emptyset, \quad U_1 \cup U_2 = G, \quad V_1 \cup V_2 \neq G.$$

We leave this as an exercise. From [38, Th. 3.1] the projections 1_{V_1}, 1_{V_2}, 1_{U_1} and 1_{U_2} in $l^\infty(G)$ can be uniquely extended to continuous functions on βG. The uniqueness asserts that these maps are projections and hence they take the value one on some clopen sets $\tilde{V}_1, \tilde{V}_2, \tilde{U}_1, \tilde{U}_2 \subseteq \beta G$ and the value zero elsewhere. It is easy to show that

$$\tilde{V}_j = t_j.\tilde{U}_j \ (j = 1, 2), \quad \tilde{V}_1 \cap \tilde{V}_2 = \emptyset, \quad \tilde{U}_1 \cup \tilde{U}_2 = \beta G, \quad \tilde{V}_1 \cup \tilde{V}_2 \neq \beta G,$$

implying the action of G on βG is 2-paradoxical. □

As an example we demonstrate that $(\beta G, G)$, for $G = \mathbb{Z}_2 * \mathbb{Z}_3$, has a 2-paradoxical action. We will use a drawing as a proof. As mentioned it is sufficient to find subsets V_1, V_2, U_1, U_2 in G and elements t_1, t_2 in G such that

$$V_j = t_j.U_j \ (j = 1, 2), \quad V_1 \cap V_2 = \emptyset, \quad U_1 \cup U_2 = G, \quad V_1 \cup V_2 \neq G.$$

Each triangle in Figure 2.1 corresponds to a unique element in $G = \mathbb{Z}_2 * \mathbb{Z}_3$. These triangles are disjoint and make up the whole unit disk (forgetting the triangle boundary). In this way G *tessellate* the unit disk. The set V_1 and V_2 corresponds to the red (lower) and blue (upper) part of the tessellation (left disk). After translation (right disk) they cover the whole disk, with the green area (third quadrant) being the intersection $U_1 \cap U_2$. Thus the action of G on βG is 2-paradoxical.

Let us now consider how to construct a properly infinite reduced crossed product.

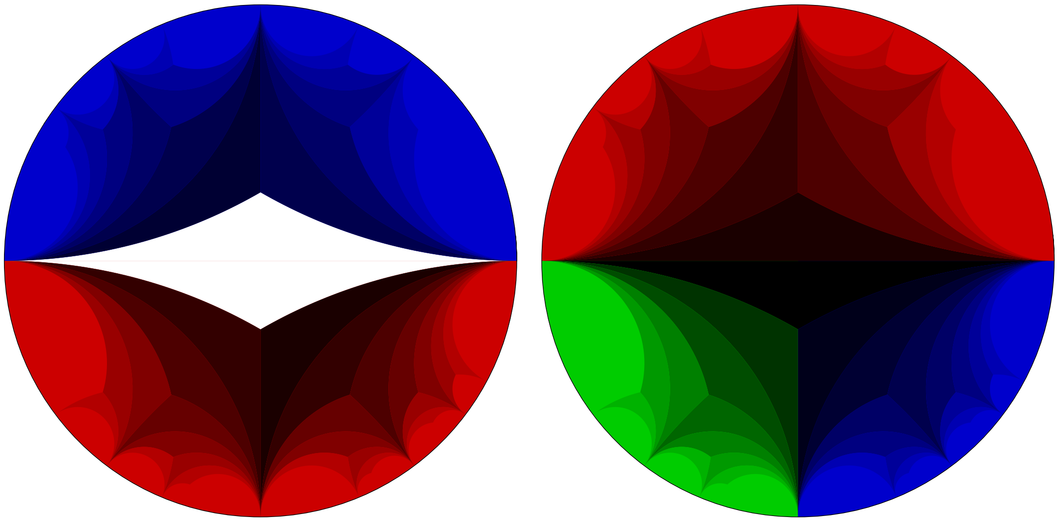

Figure 2.1: The action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on its Stone-Čech compactification is 2-paradoxical.

One way to approach this problem is by using a result of Kirchberg and Rørdam. They showed in [31, Th.3.14] that 1_A is properly infinite if and only if $1_A + I$ is infinite in A/I for every $I \in \mathcal{I}(A), I \neq A$. This reduces the problem to understanding the ideal structure of the crossed product. One could for example consider the case when the action is free and the group is exact, cf. Theorem 2.1.5.

There is another condition implying that the reduced crossed product associated with a transformation group is properly infinite. For $f, g \in C(X)^+$, with X compact Hausdorff, we have that

$$\operatorname{supp} f \subseteq \operatorname{supp} g \Leftrightarrow f \precsim g.$$

In order to ensure that the projections in $C(X)$ are properly infinite (relative to $C(X) \rtimes_r G$) it seems natural to consider a kind of duplication of subsets of X by the action of G on X:

Definition 2.3.6. Given a discrete transformation group (X, G) with G discrete and a family $\mathbb{E}$ of subsets of X. A non-empty set $V \subseteq X$ is called $\mathbb{E}$-*paradoxical* if there exist non-empty subsets $V_1, V_2, \ldots, V_{n+m} \in \mathbb{E}$ of V and elements $t_1, t_2, \ldots, t_{n+m}$ in G such that

$$\bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i = V, \quad \bigsqcup_{i=1}^{n+m} t_i.V_i \subseteq V, \quad t_k.V_k \cap t_l.V_l = \emptyset \ (k \neq l)$$

Recall that for a topological space X the family of open sets of X is denoted by τ_X. The following Lemma shows that there is — in some sense — a rather close connection between paradoxical sets and properly infinite projections.

Lemma 2.3.7. *Let (A,G) be a C^*-dynamical system with $A=C_0(X)$ abelian and G discrete. Suppose that U is a compact open subset of X. Consider the properties*

(i) U *is* τ_X*-paradoxical,*

(ii) $\exists x,y \in C_c(G,A^+) : x^*x = y^*y = 1_U,\ xx^* \perp yy^* \leq 1_U,$

(iii) $\exists x,y \in A \rtimes_r G : x^*x = y^*y = 1_U,\ xx^* \perp yy^* \leq 1_U,$

(iv) 1_U *is properly infinite in* $A \rtimes_r G$.

Then

$$(i) \Leftrightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv).$$

Proof. $(i) \Rightarrow (ii)$. Let $(V_i,t_i)_{i=1}^{n+m}$ be the subsets of U and elements in G that make U τ_X-paradoxical. One can assume that the n elements $t_1,\dots,t_n$ are different. (Otherwise make appropriate unions among of the sets $V_1,\dots,V_n$ if they are transformed by the same element in G). Define $V_{t_i} := V_i$ for every $i \in \{1,\dots,n\}$ and $F := \{t_1,\dots,t_n\}$. Find a partition of unity $(h_t)_{t\in F}$ for U relative to the open cover $(V_t)_{t\in F}$. With

$$x = \sum_{t\in F} u_t h_t^{1/2},$$

we have that

$$x^*x = \sum_{t,s\in F}^{n} h_t^{1/2} u_t^* u_s h_s^{1/2} = \sum_{t\in F} h_t^{1/2} u_t^* u_t h_t^{1/2} = \sum_{t\in F} h_t = 1_U,$$

using that for $s \neq t \in F$ (this notation includes that $s \in F$) we have that $s.V_s \cap t.V_t = \emptyset$ and hence $h_t^{1/2} u_t^* u_s h_s^{1/2} = 0$ (as in the proof of Theorem 2.3.4). We also have that

$$1_U x x^* 1_U = \sum_{t,s\in F} 1_U u_t h_t^{1/2} h_s^{1/2} u_s^* 1_U = \sum_{t,s\in F} u_t h_t^{1/2} h_s^{1/2} u_s^* = xx^*,$$

since the identity $\operatorname{supp} t.h_t^{1/2} = t.\operatorname{supp} h_t^{1/2} \subseteq t.V_t \subseteq U$ implies that $1_U u_t h_t^{1/2} = 1_U t.h_t^{1/2} u_t = t.h_t^{1/2} u_t = u_t h_t^{1/2}$. We conclude that $xx^* \leq 1_U$.

Using that $(V_i)_{i=n+1}^{n+m}$ is an open cover of U we can construct $y = \sum_{s\in F'} u_s g_s^{1/2}$ using the same method. We get that $xx^* \perp yy^*$ from the equality

$$x^*y = \sum_{t\in F, s\in F'} h_t^{1/2} u_t^* u_s g_s^{1/2} = \sum_{t\in F, s\in F'} u_t^* t.h_t^{1/2} s.g_s^{1/2} u_s = 0.$$

$(ii) \Rightarrow (i)$ From (ii) we have that $x = \sum_{t\in F} u_t h_t^{1/2}$ and $y = \sum_{t\in F'} u_s g_s^{1/2}$ for some finite $F, F' \subseteq G$ and some non-zero functions $h_t, g_s \in A^+, t \in F, s \in F'$. We now show that the sets

$$V_t := \{x \in X \colon h_t(x) \neq 0\}, \quad W_s := \{x \in X \colon g_s(x) \neq 0\}, \quad t \in F, s \in F'$$

make the set U τ_X-paradoxical. First note that

$$1_U = x^*x = E(x^*x) = \sum_{t,s\in F} E(h_t^{1/2} u_t^* u_s h_s^{1/2}) = \sum_{t\in F} E(h_t^{1/2} u_t^* u_t h_t^{1/2}) = \sum_{t\in F} h_t.$$

This imply that $z \in U \Leftrightarrow 1_U(z) = \sum_{t\in F} h_t(z) \neq 0 \Leftrightarrow z \in \cup_{t\in F} V_t$. Using the same argument on y we obtain that $\bigcup_{t\in F} V_t = \bigcup_{s\in F'} W_s = U$. Since the functions h_t, g_s are non-zero the sets V_t, W_s are non-empty.

For $r \neq e$ in G we have that

$$0 = E(1_U u_r) = E(x^*x u_r) = \sum_{t,s\in F} E(h_t^{1/2} u_t^* u_s h_s^{1/2} u_r).$$

Since the above sum is a sum of positive elements in A^+ we obtain that

$$E(h_t^{1/2} u_t^* u_s h_s^{1/2} u_r^*) = 0, \quad t, s \in F, e \neq r \in G.$$

It particular we have that

$$h_t^{1/2} u_t^* u_s h_s^{1/2} u_{t^{-1}s}^* = E(h_t^{1/2} u_t^* u_s h_s^{1/2} u_{t^{-1}s}^*) = 0, \quad t, s \in F, t \neq s.$$

This shows that $t.h_t \perp s.h_s$ for $t \neq s \in F$. In a similar way we obtain that $t.g_t \perp s.g_s$ for $t \neq s \in F'$ and $t.g_t \perp s.h_s$ for $t \in F, s \in s \in F'$ (the last property is obtained from the fact that $0 = E(x^*yu_r)$ for every $r \in G$). Suppose $x \in t.V_t \cap s.V_s$. Find $x = t.y_t = s.y_s$ such that $h_t(y_t) \neq 0$ and $h_s(y_s) \neq 0$. From $(t.h_t)(s.h_s)(x) = h_t(y_t)h_s(y_s) \neq 0$ we conclude $t = s$. By a similar argument we obtain that the sets $t.V_t, s.W_s \colon t \in F, s \in F'$ are all pairwise disjoint.

Finally we show that $t.V_t \subseteq U$ for every $t \in F$. Fix $x \in t.V_t$, $t \in F$. Find $y \in V_t$ such that $x = t.y$. We obtain that $t.h_t(x) = h_t(y) \neq 0$. Form

$$\sum_{r\in F} r.h_r^{1/2} u_r = \sum_{r\in F} u_r h_r^{1/2} = x = 1_U(xx^*)x = 1_U x = \sum_{r\in F} 1_U r.h_r^{1/2} u_r,$$

we obtain (when applying $E(\,\cdot\, u_s^*)^2, s \in F$ on the sum) that $s.h_s = 1_U s.h_s$ for every $s \in F$. In particular $0 \neq t.h_t(x) = 1_U t.h_t(x)$ implying that $x \in U$.

$(ii) \Rightarrow (iii)$. Trivial as $C_c(G, A^+) \subseteq A \rtimes_r G$.

$(iii) \Leftrightarrow (iv)$. Well know, cf. [31, Proposition 3.3]. □

Theorem 2.3.8. *Let (X, G) be a transformation group with X compact and G discrete. Suppose that X is τ_X-paradoxical. Then $C(X) \rtimes_r G$ is properly infinite.*

Proof. Use Lemma 2.3.7. □

Remark 2.3.9. Given a transformation group (X, G) with G discrete and X compact. If the action is 2-paradoxical and *transitive*, in the sense that $G.U = X$ for every open non-empty set $U \subseteq X$, then the action is τ_X-paradoxical. We leave this as an exercise.

2.4 Existence of a trace

A bounded *trace* on a C*-algebra A is a linear bounded map $\tau\colon A \to \mathbb{C}$ with the property that $\tau(ab) = \tau(ba)$ for any $a, b \in A$. A trace τ is *positive* if $\tau(a) \geq 0$ for every positive element a in A. If A is unital and τ is a positive trace with $\tau(1_A) = 1$, then τ is called a *tracial state*, cf. [48, p. 46].

Let (A, G) be a C*-dynamical system with G discrete and let τ be a state on A. We say that τ is *invariant* (or more precisely *G-invariant*) if $t.\tau = \tau$ for every $t \in G$, where $t.\tau$ is the state on A defined by $(t.\tau)(a) := \tau(t^{-1}.a)$ for $a \in A$.

Given a transformation group (X, G) with G discrete. A Borel measure $\mu\colon \mathbb{B}(X) \to [0, \infty]$ on X is said to be *invariant* (or more precisely *G-invariant*) provided that $t.\mu = \mu$ for every $t \in G$, where $t.\mu$ is the measure on X defined by

$$t.\mu(E) := \mu(t^{-1}.E), \quad E \in \mathbb{B}(X).$$

Let us recall a well known connection between traces on crossed products and measures on Borel spaces (our generalization to the unbounded case is presented in Lemma 5.2.2).

Lemma 2.4.1. *Consider a transformation group (X, G) with X compact and G discrete. The following conditions are equivalent*

(i) $C(X) \rtimes_r G$ admits a non-zero trace.

(ii) X admits a complete regular invariant Borel probability measure μ.

Proof. $(i) \Rightarrow (ii)$ Given a trace $\tau \neq 0$ on $A := C(X) \rtimes_r G$. We may assume that τ is a tracial state on A. To see this make τ hermitian and decompose it to a difference of two positive functionals $\tau_1 \neq 0, \tau_2$. Define $\tau_i'(a^*a) = \tau_i(aa^*), a \in A$ and extend canonically to A. Since τ_i and τ_i' are positive and agree on A^+ they have the same norm. Using [28, Th.4.3.6] $\tau_i = \tau_i'$. We conclude $\tau_1/\|\tau_1\|$ is a tracial state.

The restriction $\tau|_{C(X)}$ is an invariant tracial state and gives by means of Riesz Representation Theorem and Lusin's Theorem, cf. [52, Theorem 2.24]), the appropriate invariant measure. Lusin's Theorem is used to obtain invariance.

$(ii) \Rightarrow (i)$ The canonical state on $C(X)$ associated with μ is an invariant tracial state and extends in a canonical way (by composition with the conditional expectation) to a trace on $C(X) \rtimes_r G$. □

In [22] it was shown that a unital C*-algebra A does not admits a non-zero trace if and only if there exist $n \geq 2$ and $a_1, \ldots, a_n \in A$ such that $\sum_j a_j a_j^* = 1$ and $\| \sum_j a_j^* a_j \| < 1$. Another condition using finite sums of commutators is mentioned in [44].

Furstenberg defines the concept of a *strongly proximal action* implying that $C(X) \rtimes_r G$ admits no non-zero trace. An action is called strongly proximal if every invariant Borel probability measure *mu* the weak*-closure of the set $G.\mu$ contains a Dirac measure. If $C(X) \rtimes_r G$ admits a trace it follows from Lemma 2.4.1, $(i) \Rightarrow (ii)$ that there exists an invariant Borel probability measure μ. Consequently $\overline{G.\mu} = \{\mu\}$ and hence the action of G on X is not a strongly proximal action (for $|X| > 1, |G| > 1$).

A properly infinite C*-algebra obviously admits no non-zero traces, but the converse is not generally true. In [51] Rørdam presents a unital, simple, nuclear and separable C*-algebra D with finite and infinite projections p, q. It follows that $A = pDp$ is unital, simple, nuclear, finite and without non-zero traces. To see the last property note that $A \otimes \mathcal{K} \cong D \otimes \mathcal{K}$ contains a infinite projection. Hence $M_n(A)$ is properly infinite for some $n \geq 2$, cf. [50]. One conclude that A has no non-zero traces. This example however does not answer the following

Question 2.4.2. Let (X, G) be a transformation group with X compact and G discrete. Are the conditions mentioned below equivalent?

(i) $C(X)$ has no G-invariant state.

(ii) $C(X) \rtimes_r G$ has no tracial state.

(iii) $C(X) \rtimes_r G$ is properly infinite.

Using the proof of Lemma 2.4.1 one easily gets $(iii) \Rightarrow (ii) \Leftrightarrow (i)$. The last implication remains unsolved, except for some concrete examples. For example for the transformation group $(\beta G, G)$ with G discrete the conditions $(i)-(iii)$ are equivalent, cf. Theorem 2.5.1. It might be interesting to consider the action of a *word hyperbolic group* on the *Gromov boundary*. We will return to this example later on.

Hjelmborg showed in [23, Theorem 4.1, Theorem 4.10] that certain C*-algebras associated with Markov shifts and graphs are purely infinite if and only if they have no trace in sense of [31]. One may consider the following

Question 2.4.3. Let (X, G) be a transformation with X compact and G discrete. Is there an equivalence between the following conditions

(i) $C(X) \rtimes_r G$ has no trace in sense of [31].

(ii) $C(X) \rtimes_r G$ is purely infinite.

For the implication $(ii) \Rightarrow (i)$ see [31, Proposition 5.1]. As we will see later on, cf. Theorem 5.2.3, there is more to say regarding Question 2.4.3 when we consider a more specific class of transformation groups.

2.5 Non-amenable groups and $C(\beta G) \rtimes_r G$

Recall that a discrete group G is called *amenable* if the associated reduced group C*-algebra $C_r^*(G)$ is nuclear, cf. [9, Theorem 2.6.8]. When considering the action of a discrete group G on the *Stone-Čech compactification* βG of G we obtain the following

Theorem 2.5.1. *Consider the transformation group $(\beta G, G)$, where G acts canonically on the Stone-Čech compactification βG of G. Then the following conditions are equivalent*

(i) G is non-amenable

(ii) βG is $\tau_{\beta G}$-paradoxical

(iii) $C(\beta G) \rtimes_r G$ is properly infinite

(iv) $C(\beta G) \rtimes_r G$ has no tracial state

Proof. $(i) \Rightarrow (ii)$. Using the result of Wagon [60, Theorem 10.11] there exists subsets $U_1, U_2, \ldots, U_{n+m}$ of G and elements $t_1, t_2, \ldots, t_{n+m}$ in G such that

$$t_k.U_k \cap t_l.U_l = \emptyset \ (k \neq l), \quad \cup_{i=1}^n U_i = \cup_{i=n+1}^{n+m} U_i = G.$$

Adapting the proof of Theorem 2.3.5 we find clopen subsets $\tilde{U}_1, \tilde{U}_2, \dots, \tilde{U}_{n+m}$ of βG such that

$$t_k.\tilde{U}_k \cap t_l.\tilde{U}_l = \emptyset \ (k \neq l), \quad \cup_{i=1}^n \tilde{U}_i = \cup_{i=n+1}^{n+m} \tilde{U}_i = \beta G.$$

Hence the set βG is $\tau_{\beta G}$-paradoxical.

$(ii) \Rightarrow (iii)$. Theorem 2.3.8.

$(iii) \Rightarrow (iv)$. Easy.

$(vi) \Rightarrow (i)$. Use the fact that any reduced crossed product of a unital C*-algebra with a tracial state by a discrete amenable group admits a tracial state cf. [47, p. 83]. □

Using that $l_\infty(G)$ has real rank zero, non-amenability of infinite discrete groups can be related to actions on the Cantor set. Before making this more precise in Proposition 2.5.6 we recall a few partial results.

Lemma 2.5.2. *Any separable sub-C*-algebra A_0 of a (possibly non-separable) C*-algebra B of real rank zero is contained in a separable sub-C*-algebra A of real rank zero.*

Proof. Find a sequence (a_i) in A_0^{sa} such that $\{a_i\}^- = A_0^{sa}$. For every $n \in \mathbb{N}$ find $a_{1n}, \dots, a_{nn}$ in B^{sa} with finite spectrum such that $\|a_i - a_{in}\| < 1/n$. Set $A_1 = \lim_n C^*(A_0, \{a_{ij}\}_{1 \leq i \leq j \leq n})$. We obtain that

$$A_0^{sa} \subseteq \{b \in A_1^{sa} : b \text{ has finite spectrum }\}^-.$$

Successively we obtain an increasing sequence (A_j) fulfilling the property that $A_j^{sa} \subseteq \{c \in A_{j+1}^{sa} : c \text{ has finite spectrum }\}^-$. With $A := \overline{\bigcup_j A_j}$ we have that A has real rank zero. □

Remark 2.5.3. I order to find an invariant sub-C*-algebra $C(X)$ in $l_\infty(G)$ such that X is a Cantor set we need some characterization results of abelian C*-algebras. Let A be an abelian C*-algebra. By GNS A is isomorphic to $C_0(Y)$ for some locally compact Hausdorff space Y. If A is unital then Y is compact. If A is separable and unital then Y (has a countable basis, is regular and hence) is metrizable. If A has real rank zero then Y is *totally disconnected* (i.e. the only connected subspaces are one-point sets), cf. [41].

It was shown by Brouwer, cf. [7], that a compact Hausdorff metric totally disconnected *perfect* (i.e. no isolated points) space is homeomorphic to the Cantor set.

Lemma 2.5.4. *Let G be a infinite countable group. Then $l_\infty(G)$ contains an invariant sub-C*-algebra $C(X)$ such that X is a Cantor set.*

Proof. Let A_1 be the smallest invariant sub-C*-algebra in $l_\infty(G)$ containing $A_0 := C_0(G)$ and $1_{l_\infty(G)}$. Note that A_1 is separable since G is countable. Let A_2 be the separable real rank zero envelope of A_1 in $l_\infty(G)$, cf. Lemma 2.5.2. Let A_3 be the smallest invariant (separable) sub-C*-algebra in $l_\infty(G)$ containing A_2 and let A_4 be the separable real rank zero envelope of A_3 in $l_\infty(G)$. Continuing in this way we obtain a sequence of inclusions with an inductive limit $A := \overline{\cup_i A_i}$

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A \subseteq l_\infty(G)$$

The C*-algebra A is invariant: Fix $a \in A$, $t \in G$ and $\epsilon > 0$. Find $m \in \mathbb{N}$ odd and $b \in A_m$ such that $\|a - b\| < \epsilon$. As A_m is invariant we have that $\operatorname{dist}(t.a, A) < \epsilon$. We obtain that $t.a \in A$.

The C*-algebra A has real rank zero: Fix a self-adjoint element a in A and $\epsilon > 0$. Find $m \in \mathbb{N}$ even and $b \in A_m$ such that $\|a - b\| < \epsilon/2$. Since $\|a - \frac{b+b^*}{2}\| < \epsilon/2$ we can assume that b is self-adjoint. As A_m has real rank zero there exist a self-adjoint element $c \in A_m$ with the properties that $\|b - c\| < \epsilon/2$ and c has finite spectrum $\sigma_{A_m}(c)$. Using that $\|a - c\| < \epsilon$ and $\sigma_A(c)$ is finite we conclude that a belongs to the norm limit of self-adjoint elements in A of finite spectrum.

The C*-algebra A is separable and unital: Separability follows from the fact that A_i is separable for every $i \in \mathbb{N}$. Furthermore A is unital with the unit $1_A = 1_{l_\infty(G)}$.

By Remark 2.5.3 A is isomorphic to $C(Y)$ for some (invariant) compact Hausdorff metric totally disconnected space Y. We do however not know if the space Y is perfect.

Let X be the subset of Y where we have removed all the isolated points from X. We claim that $C(X)$ can be used as the desired algebra. It follows easily that X is invariant, compact, Hausdorff, metric, totally disconnected and perfect. We only need to verify that X is non-empty. As G is infinite A_0 has infinite spectrum. Hence also Y is infinite. If $X = \emptyset$ the singletons in Y produce an infinite open cover of Y. This contradicts the compactness of Y. Hence X is non-empty. Using the result by Brouwer we conclude that X is an invariant Cantor set. $\square$

Lemma 2.5.5. *Let (X, G) be a transformation group with G discrete. Suppose that X is τ_X-paradoxical. Then Y is τ_Y-paradoxical for every non-empty invariant subset Y of X.*

Proof. Find non-empty open subsets $V_1, V_2, \ldots, V_{n+m}$ of X and elements $t_1, t_2, \ldots, t_{n+m}$ in G such that

$$\bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i = X, \quad \bigsqcup_{i=1}^{n+m} t_i.V_i \subseteq X, \quad t_k.V_k \cap t_l.V_l = \emptyset \ (k \neq l).$$

With $U_i := V_i \cap Y \in \tau_Y$ we have that

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=n+1}^{n+m} U_i = Y, \quad \bigsqcup_{i=1}^{n+m} t_i.U_i \subseteq Y, \quad t_k.U_k \cap t_l.U_l = \emptyset \ (k \neq l).$$

We conclude that Y is τ_Y-paradoxical. □

Proposition 2.5.6. *Given an infinite countable group G. Then G is non-amenable if and only if $C(X) \rtimes_r G$ is properly infinite for some action of G on the Cantor set X.*

Proof. '$\Leftarrow$': Suppose G is amenable. By [47, p. 83] $C(X) \rtimes_r G$ admits a tracial state. Hence $C(X) \rtimes_r G$ in not properly infinite.

'$\Rightarrow$': Suppose G is non-amenable. By Theorem 2.5.1 βG is $\tau_{\beta G}$-paradoxical. Using Lemma 2.5.4 find a invariant sub-C*-algebra $C(X)$ of $l_\infty(G) = C(\beta G)$ such that X is an invariant Cantor set. Since X can be viewed as an invariant subset of βG, Lemma 2.5.5 ensures that X is τ_X-paradoxical. We conclude that $C(X) \rtimes_r G$ is properly infinite by Theorem 2.3.8. □

2.6 Pure infiniteness

Recall from [31] a C*-algebra A is *purely infinite* if and only if every non-zero positive element in A is properly infinite. This definition agrees with the definition of purely infiniteness by J. Cuntz (that every non-zero hereditary sub-C*-algebra of A contains an infinite projection, cf. [14]), provided A is simple.

There are several well known results about pure infiniteness of reduced crossed products. Let (A, G) be a C*-dynamical system with A purely infinite, simple and G discrete. It was shown by Kishimoto and Kumjian in [35] that with the additional assumption that the action α of G on A is *outer* (i.e. for each $t \neq e$ the automorphism α_t is outer) we obtain that $A \rtimes_r G$ is purely infinite and simple. If A is unital Jeong and Osaka showed that every non-zero hereditary sub-C*-algebra in $A \rtimes_r G$ contains an infinite projection provided the normal subgroup $N = \{t \in G \colon t. \text{ is inner on } A\}$ is finite, cf. [26].

Consider a transformation group (X, G) with X compact and G discrete. The action of G on X is called a *strong boundary action*, cf. [36, Definition 1], if X has at least three points and for every pair U, V of non-empty open subsets of X there exists $t \in G$ such that $t.U^c \subseteq V$. Laca and Spielberg showed in [36] that in the case the action of G on $C(X)$ is topologically free and the action of G on X is a strong boundary action then $A \rtimes_r G$ is purely infinite and simple.

Jolissaint and Robertson made a generalization valid in the non-abelian setting, cf. [27]. Let (A, G) be a C*-dynamical system with A separable, unital and G discrete. Recall, cf. [27, Definition 0.1] that the action of G on A is *n-filling* if for any positive norm one elements $a_1, \ldots, a_n$ in A and any $\epsilon > 0$ there exist $t_1, \ldots, t_n$ in G such that $\sum_{i=1}^{n} t_i.a_i \geq 1 - \epsilon$. They showed that for a discrete group G acting on a separable, unital C*-algebra A the crossed product $A \rtimes_r G$ is purely infinite and simple provided the action is properly outer and *n-filling* and every hereditary sub-C*-algebra in A generated by a non-zero projection in A is infinite dimensional.

If the $A = C(X)$ (X infinite) the n-filling property reduces to the following condition: For any non-empty open subsets $U_1, \ldots, U_n$ of X, there exist $t_1, ..., t_n \in G$ such that $t_1.U_1 \cup \cdots \cup t_n.U_n = X$. In this setting the strong boundary property is equivalent to the 2-filling property, cf. [27]. Further the condition that the hereditary C*-algebra generated by e is infinite dimensional for every nonzero projection $e \in A$ is equivalent the property that X has no isolated points. In this setting topological freeness the proper outerness coincide, cf. [3].

Before considering generalizations of the work of Jolissaint and Robertson to non-simple C*-algebras more work is needed. However we can already now relate the n-filling property to τ_X-paradoxical sets.

Lemma 2.6.1. *Given a transformation group (G, X) with X compact and G discrete. If the action is n-filling then every non-empty open subset of X is τ_X-paradoxical and the action is minimal.*

Proof. Let U be an non-empty open subset of X. Find $2n$ disjoint open subsets $U_1 \ldots, U_{2n}$ of U. Using the action is n-filling there exist $s_1, \ldots, s_{2n}$ such that

$$\bigcup_{i=1}^{n} s_i.U_i = \bigcup_{i=n+1}^{2n} s_i.U_i = X, \quad U_k \cap U_l = \emptyset \ (k \neq l)$$

With $V_i := s_i.U_i \cap U$ and $t_i = s_i^{-1}$ for $i = 1, \ldots, 2n$ we obtain that

$$\bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i = U, \quad \bigsqcup_{i=1}^{n+m} t_i.V_i \subseteq U, \quad t_k.V_k \cap t_l.V_l = \emptyset \ (k \neq l).$$

Suppose the action fails to be minimal. Hence one can find $x \in X$ and an open set U in the complement of the orbit $\mathcal{O}_x$. From the n-filling property there exist $t \in G$ such that $t.U \cap \mathcal{O}_x \neq \emptyset$. But this implies that $U \cap \mathcal{O}_x = t^{-1}.(t.U \cap \mathcal{O}_x) \neq \emptyset$. Contradiction. $\square$

Remark 2.6.2. It was shown in [31] that A is purely infinite if every non-zero hereditary sub-C*-algebra in any quotient of A contains an infinite projection. As we already mentioned Laca and Spielberg obtained a way to find infinite projections in hereditary sub-C*-algebras of $A \rtimes_r G$ in the case A is abelian. A similar result was considered by Clare Anantharaman-Delaroche in [1]. We will elaborate on this subject when discussing groupoids. Combining these observations together with Theorem 2.1.5 we can formulate the first result on purely infinite non-simple crossed products. (With more work this result can be improved, cf. Theorem 3.3.7 and Theorem 4.2.4.)

Proposition 2.6.3. *Let (G, X) be a transformation group with G exact, discrete. If the action on X is free and in addition is a local boundary action on every non-empty closed invariant subset Y of X, then $C_0(X) \rtimes_r G$ is purely infinite.*

Proof. Any quotient of $C_0(X) \rtimes_r G$ by an ideal has the form $C_0(Y)_r \rtimes G$ for some closed invariant subset Y of X. We conclude that every hereditary sub-C*-algebra in the quotient contains an infinite projection. □

Recently [12] Cuntz constructed a purely infinite, simple C*-algebra associated with the $ax + b$-semigroup over $\mathbb{N}$. It is a crossed product of the Bunce-Deddens algebra [10] associated to $\mathbb{Q}$ by the action of the multiplicative semigroup $\mathbb{N}$. Its stabilization is (isomorphic to) the crossed product $C_0(\mathrm{A}_f) \rtimes P_{\mathbb{Q}}^{+}$, the product of $C_0(\mathrm{A}_f)$ by the natural action of the $ax + b$-group $P_{\mathbb{Q}}^{+}$. Here A_f denotes the locally compact space of *finite adeles*.

Question 2.6.4. How does the new examples of Cuntz fit in with the definition of a local boundary action?

In the following sections we will consider a few classes of examples giving purely infinite crossed product C*-algebras.

2.7 Example 1: Free product group

Let $G = G_1 * \cdots * G_n, n \geq 2$ denote a free product of non-trivial cyclic groups. As in [58, Section 2] we do not consider the case $G = \mathbb{Z}_2 * \mathbb{Z}_2$ (this gives a finite boundary). There is a canonical *boundary* ∂G associated with G - the set of all "infinite reduced words" from the alphabet

$$\Omega := \bigcup_{i \in I_0} \{a_i, \ldots, a_i^{|G_i|-1}\} \cup \bigcup_{i \in I \setminus I_0} \{a_i, a_i^{-1}\},$$

with $I = \{1, \ldots, n\}, I_0 = \{i \in I : |G_i| < \infty\}$ and a_i being the generator for G_i. An *infinite reduced word* is a sequence $x_1x_2x_3\ldots \in \Omega^{\mathbb{N}}$ fulfilling

$$\forall m \in \mathbb{N} \forall i \in I \quad : \quad x_m, x_{m+1} \in G_i \Rightarrow x_m x_{m+1} \neq e \text{ and } i \notin I_0. \quad (2.7.1)$$

Note that G can be identified with the set of *finite reduced words*, i.e. set of finite parts of the infinite reduced words.

We equip Ω with the discrete topology, $\Omega^{\mathbb{N}}$ with product topology and $\partial G \subseteq \Omega^{\mathbb{N}}$ with the subset topology. The action α of G on ∂G is just the left multiplication followed by a canonical reduction

$$\alpha(a_i^k)(x_1x_2x_3\ldots) = \mathrm{Red}(a_i^k x_1x_2x_3\ldots), \quad a_i^k \in \Omega, \ x_1x_2x_3\ldots \in \partial G$$

where $\mathrm{Red}(\cdot)$ changes the sequence $a_i^k x_1x_2x_3\ldots$ only if $x_1 = a_i^l$ for some $l \in \mathbb{Z}$; (i) removing $a_i^k x_1$ if $a_i^k x_1 = e$ or (ii) replacing $a_i^k x_1$ with $a_i^{k+l \bmod |G_i|}$ if $a_i^k x_1 \neq e$ and $|G_i| < \infty$ (for $a_i^k x_1 \neq e$ and $|G_i| = \infty$ no change is needed).

Lemma 2.7.1. *Consider a free product $G = G_1 * \cdots * G_n, n \geq 2$ of non-trivial cyclic groups such that $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$. Equip G with the canonical action on the boundary ∂G of all infinite reduced words. Then $(\partial G, G)$ is a transformation group, where ∂G is a Cantor set with a basis for the topology consisting of the compact cylinder sets*

$$C(x_1 \ldots x_n) = \{(y_n)_{n\in\mathbb{N}} \in \partial G : (y_1, \ldots, y_n) = (x_1, \ldots, x_n)\},$$

where we run through all finite reduced words.

Proof. A Cantor set is characterized as a compact, perfect, totally disconnected metric space, cf. Remark 2.5.3. The compactness follows from the Tychonoff theorem as Ω is finite. The topology on ∂G is metrizable by the metric $d((x_n), (y_n)) = \sum_{n\in\mathbb{N}} d_\Omega(x_n, y_n)/2^n$, using the canonical discrete metric on Ω. Using this metric one can show that the cylinder sets form a clopen basis for the topology and the space is totally disconnected and perfect. The last condition uses $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$. The action $\alpha_t : \partial G \to \partial G$ $(t \in G)$ is bijective and open, sending a cylinder set into a finite union of cylinder sets. Hence α_t is an homeomorphism of ∂G. Last it can be shown that α is a group homomorphism. $\square$

The transformation group $(\partial G, G)$ from Lemma 2.7.1 induces the reduced crossed product $C(\partial G) \rtimes_r G$. Spielberg showed in [58] that this crossed product is (isomorphic to) a Cuntz-Krieger algebra. It was therefore natural to consider if we could use properties of $(\partial G, G)$ to show simplicity, nuclearity and purely infiniteness of the C^*-algebra $C(\partial G) \rtimes_r G$.

Theorem 2.7.2. *Consider a free product $G = G_1 * \cdots * G_n, n \geq 2$ of non-trivial cyclic groups such that $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$. Equip G with the canonical action on the boundary ∂G of all infinite reduced words. Then $(\partial G, G)$ is a transformation group with an action that is (i) minimal, (ii) topologically free, (iii) amenable and (iv) a strong boundary action.*

Proof. (iv) Strong boundary action: From $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$ the boundary ∂G has at least three points.

Let U, V be two open non-empty subsets in ∂G. Find finite reduced words y, x such that $C(y) \subseteq U$, $C(x) \subseteq V$. It is enough to find $t \in G$ such that $t.C(y)^c \subseteq C(x)$ ($\Rightarrow t.U^c \subseteq V$). If x and y end with letters from the same cyclic group G_i, then replace x with xa_j for some $j \neq i$. Still $C(x) \subseteq V$ holds. As x and y now end with letters (from the alphabet Ω) from different cyclic groups the sequence xy^{-1} is a finite reduced word.

Let M be the set of finite reduced words of length $m := |y|$. Then

$$C(y)^c = \bigcup_{z \in M, z \neq y} C(z).$$

Define $t := xy^{-1}$ and let $z \in M, z \neq y$ be given. We get $t.C(z) \subseteq C(x)$ from

$$\begin{aligned} t.C(z) &= xy_m^{-1} \dots y_1^{-1}.C(z_1 \dots z_m) = xy_m^{-1} \dots y_{n+1}^{-1}.C(z_{n+1} \dots z_m) \\ &= C(xy_m^{-1} \dots y_{n+1}^{-1} z_{n+1} \dots z_m) \subseteq C(x), \end{aligned}$$

where n, the number of reductions while reducing $y^{-1}z$, is strictly less than m as $z \neq y$. Note that $y_{n+1}^{-1} z_{n+1}$ possibly merge to one letter if the elements y_{n+1}^{-1}, z_{n+1} belong to the same finite cyclic group.

(ii) Topologically free: Note first that ∂G is a *Baire space*, cf. [38, Th.7.2], hence it is enough to show that the open set $\{x \in \partial G : t.x \neq x\}$ is dense in ∂G for each $t \neq e$. Let therefore $y \in \partial G$, $\epsilon > 0$ and $t \in G \setminus \{e\}$ be given. We find $x \in \partial G$ close to y, such that $t.x \neq x$, using an aperiodic infinite word:

As $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$ we can find two generators $a = a_i$, $b = a_j$ such that $i \neq j$ and $a^2 \neq e$. Set $w = ab$ and $u = a^2 b$ (if $|a| < \infty$) or $u = aab$ (if $|a| = \infty$), and let z be a reduced word with length n such that $(z_1, \dots, z_n) = (y_1, \dots, y_n)$ and $1/2^n < \epsilon$. If it is necessary, extend z so it ends with b. With

$$x = zu^{p_1} wu^{p_2} wu^{p_3} wu^{p_4} \dots\dots \in \partial G,$$

where (p_n) is an increasing sequence of primes, it follows that $t.x \neq x$ and $d(y, x) < \epsilon$.

(iii) Amenable action: Using the idea from [9, p.149] it follows that

$$\|s.\mu^x - \mu^{s.x}\|_1 = 2 \max(n + \Delta, m - n)/N, \quad N \geq 2 + 2m,$$

where $\mu^x = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{x_1 \dots x_k}$ (is a convex combination of Dirac measures), $s \in G$, $x \in \partial G$, $m = |s|$, n denotes the number of reductions[3] of $s.x$ and finally $\Delta = 1$ if $|s_{m-n}| < \infty$ and $\Delta = 0$ otherwise[4]. We leave this as an exercise. By Theorem 2.2.1 (iii) we obtain (using the constant net $m_i := (x \mapsto \mu^x)$) that the action is amenable.

(i) Minimality: Follows from (iv). With $\emptyset \neq M = \bar{M} \subsetneq \partial G$, there exists $t \in G$ such that $t.(M^c)^c \subseteq M^c$, implying M is not G-invariant. □

Lemma 2.7.3. *Given a n-tuple $(m_1, \dots, m_n)$ of increasing numbers in $\mathbb{N} \cup \{\infty\}$. Let k (resp. l) count the occurrence of finite (resp. infinite) numbers in the n-tuple. With $N = \sum_{i=1}^{k}(m_i - 1) + 2l$ let A be the $N \times N$ matrix*

$$\begin{pmatrix} 0_{m_1-1} & & & & & \\ & \ddots & & & 1 & \\ & & 0_{m_k-1} & & & \\ & & & 1_2 & & \\ & 1 & & & \ddots & \\ & & & & & 1_2 \end{pmatrix}$$

with n smaller matrices $0_{m_1-1}, \dots, 0_{m_k-1}, 1_2, \dots, 1_2$ along the diagonal and with all other entries being 1. Using elementary matrix operations[5] the matrix $1_N - A^T$ can be reduced to the following form

$$\begin{pmatrix} 0_l & & & & & \\ & 1_{N-(n-1)} & & & & \\ & & m_1 & & & -m_k \\ & & & \ddots & & \vdots \\ & & & & m_{k-2} & -m_k \\ & & m_{k-1} & \dots & m_{k-1} & S \end{pmatrix},$$

with $S = m_{k-1} + m_k - m_{k-1}m_k(n-1)$.

[3]By a reduction we mean removal of a sequence aa^{-1} or $a^{-1}a$ with $a \in \Omega$.

[4]The case where two letters merge to one is accounted for using Δ.

[5]The elementary matrix operations are: (i) add an integer multiple of one row/column to another, (ii) interchange two rows/columns and (iii) multiply a row/column by -1. We did not use the operation (iii).

Proof. The matrix $1_N - A^T$ is given by

$$\begin{pmatrix} 1_{m_1-1} & & & & & \\ & \ddots & & & -1 & \\ & & 1_{m_k-1} & & & \\ & & & 0_2 & & \\ & -1 & & & \ddots & \\ & & & & & 0_2 \end{pmatrix}$$

where along the diagonal one first have k identity matrices of size $m_1 - 1 \times m_1 - 1, \ldots, m_k - 1 \times m_k - 1$ followed by l 2×2 matrices with entries 0. All the other entries are -1.

For each (of the n) sub-matrix take the first row and subtract from the other rows in the sub-matrix. For each of the first k sub-matrices take the last $m_i - 2$ columns and add to the first column. For each of the last l sub-matrices take the first column and subtract from the second column. Each of the first k sub-matrices contains $m_i - 2$ rows with only one non-zero entry equal to 1. Use these rows to change all the entries below and under the entry 1 into zero. Take one entry in the diagonal at the time. If the entire row and column around the diagonal contains only zeros, then move this diagonal entry to the top — collecting first all the zero-entries (and then the one-entries). We obtain $0_l \oplus 1_{\sum_{i=1}^k (m_i-2)} \oplus B$, with B being

$$\begin{pmatrix} 1 & 1-m_2 & \cdots & 1-m_k & -1 & -1 & \cdots & -1 \\ 1-m_1 & 1 & \cdots & 1-m_k & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-m_1 & 1-m_2 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ 1-m_1 & 1-m_2 & \cdots & 1-m_k & 0 & -1 & \cdots & -1 \\ 1-m_1 & 1-m_2 & \cdots & 1-m_k & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-m_1 & 1-m_2 & \cdots & 1-m_k & -1 & -1 & \cdots & 0 \end{pmatrix}$$

Working with B take the k-th row and subtract from all the others rows. Add the lower l rows to the k-th row. Use the last l rows to change the last l entries in the k-th column to zero. Move the last l diagonal entries to the upper left. We obtain $0_l \oplus 1_{\sum_{i=1}^k (m_i-2)+l} \oplus C$, with C being

$$\begin{pmatrix} m_1 & & & & -m_k \\ & m_2 & & & -m_k \\ & & \ddots & & -m_k \\ & & & m_{k-1} & -m_k \\ 1-m_1 & 1-m_2 & \cdots & 1-m_{k-1} & 1-lm_k \end{pmatrix}$$

Working with C add the last row to the first row. Add the first row to the last row $m_1 - 1$ times. Use the first column to change 2nd to kth entry in the top row to zero. Multiply the 2nd to $(k-1)$th row (in total $k-2$ rows) by m_1 and add to the k-th row. We obtain $0_l \oplus 1_{\sum_{i=1}^{k}(m_i-2)+l+1} \oplus D$, with D being

$$\begin{pmatrix} m_2 & & & -m_k \\ & \ddots & & -m_k \\ & & m_{k-1} & -m_k \\ m_1 & \cdots & m_1 & S \end{pmatrix}$$

for $S := m_1 + m_k - m_1 m_k(l+1+k-2)$. By a permutation of the numbers in the n-tuple (if necessary) we are done. $\square$

Here we present a calculation of the K-theory (K_0, K_1) in some interesting cases. The rest in Corollary 2.7.4 are well known results. The proof is included as an application of the tools we have discussed in previous part of the thesis.

Corollary 2.7.4. *Consider a free product $G = G_1 * \cdots * G_n, n \geq 2$ of non-trivial cyclic groups such that $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$. Equip G with the canonical action on the boundary ∂G of all infinite reduced words. Then the crossed product $C(\partial G) \rtimes_r G = C(\partial G) \rtimes G$ is a separable, simple, nuclear and purely infinite C^*-algebra in the UCT-class. Further*

$$\begin{aligned} K_0(C(\partial G) \rtimes_r G) &= G_1 \oplus \cdots \oplus G_{k-2} \oplus \mathbb{Z}/(N) \oplus G_{k+1} \oplus \cdots \oplus G_n \\ K_1(C(\partial G) \rtimes_r G) &= G_{k+1} \oplus \cdots \oplus G_n \end{aligned}$$

assuming G_j has order $m_j < \infty$ for $j = 1, \ldots, k$ with $m_{j-1} | m_j$ and the other l groups are infinite. Here $N = m_{k-1} + m_k - m_k m_{k-1}(n - 1 - \sum_{j=1}^{k-2} 1/m_j)$.

Proof. It follows from Lemma 2.7.2 that $C(\partial G) \rtimes G$ is simple, nuclear and purely infinite and that $C(\partial G) \rtimes_r G = C(\partial G) \rtimes G$. For the UCT-property we refer to [47, p. 79]. It is shown in [58] that $C(\partial G) \rtimes G$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$ with A defined as in Lemma 2.7.3.

As $K_0(\mathcal{O}_A) \cong \text{Coker}(I - A^T)$ and $K_1(\mathcal{O}_A) \cong \text{Ker}(I - A^T)$, cf. [47, p. 79], it is relevant to consider the matrix $I - A^T$. For simplicity assume $k > 1$ and $l > 0$, the other cases follow in a similar way. By Lemma 2.7.3 the matrix

$I - A^T$ can be reduced to the following form

$$\begin{pmatrix} 0_l & & & & & \\ & 1_{M-(n-1)} & & & & \\ & & m_1 & & & -m_k \\ & & & \ddots & & \vdots \\ & & & & m_{k-2} & -m_k \\ & & m_{k-1} & \dots & m_{k-1} & S \end{pmatrix},$$

with only one row and column with non-zero entries outside the diagonal, with $M := \sum_{i=1}^{k}(m_i - 1) + 2l$ and $S := m_{k-1} + m_k - m_{k-1}m_k(n-1)$. If $k = 2$ the matrix is diagonal. Otherwise use $m_j | m_{k-1}$ and $m_j | m_k$ for $j = 2, \dots, k-2$ to simplify the last row and the last column. Finally move 0_l down in the diagonal obtaining

$$\begin{pmatrix} 1_{M-(n-1)} & & & & & \\ & m_1 & & & & \\ & & \ddots & & & \\ & & & m_{k-2} & & \\ & & & & N & \\ & & & & & 0_l \end{pmatrix}.$$

This is the Smith Normal Form of $I - A^T$ giving, cf. [37, Corollary 2.2],

$$\begin{aligned} K_0(\mathcal{O}_A) &\cong \operatorname{Coker}(I - A^T) = \mathbb{Z}^M/(I - A^T)\mathbb{Z}^M \\ &\cong \big(\mathbb{Z}/(1)\big)^{M-(n-1)} \oplus \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(N) \oplus \big(\mathbb{Z}/(0)\big)^l, \\ K_1(\mathcal{O}_A) &\cong \operatorname{Ker}(I - A^T) \cong \mathbb{Z}^l, \end{aligned}$$

concluding the proof. $\square$

Remark 2.7.5. As an application of Corollary 2.7.4 we consider the following

Question 2.7.6. Fix an integer $m \geq 2$. For which group of the form $G = G_1 * \cdots * G_n$, $n \geq 2$, $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$ do we get a crossed product $C(\partial G) \rtimes_r G$ isomorphic to the Cunts algebra $\mathcal{O}_m$?

We made an algorithm to find K-theory (K_0, K_1) for $C(\partial G) \rtimes_r G$ with the order of the cyclic groups $|G_1|, \dots, |G_n|$ as input. For $m = 2, \dots, 10$ the following groups came up

m	G inducing $C(\partial G) \rtimes_r G$ with the same K_0, K_1 as $\mathcal{O}_m$
2	$\mathbb{Z}_2 * \mathbb{Z}_3$
3	$\mathbb{Z}_2 * \mathbb{Z}_4$
4	$\mathbb{Z}_2 * \mathbb{Z}_5, \mathbb{Z}_3 * \mathbb{Z}_3$
5	$\mathbb{Z}_2 * \mathbb{Z}_6$
6	$\mathbb{Z}_2 * \mathbb{Z}_7, \mathbb{Z}_3 * \mathbb{Z}_4$
7	$\mathbb{Z}_2 * \mathbb{Z}_8$
8	$\mathbb{Z}_2 * \mathbb{Z}_9, \mathbb{Z}_3 * \mathbb{Z}_5$
9	$\mathbb{Z}_2 * \mathbb{Z}_{10}, \mathbb{Z}_4 * \mathbb{Z}_4, \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_3$
10	$\mathbb{Z}_2 * \mathbb{Z}_{11}, \mathbb{Z}_3 * \mathbb{Z}_6$

The above list is complete in the sense that any other free product group $G = G_1 * \cdots * G_n$, $n \geq 2$, $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$ gives an reduced crossed product $C(\partial G) \rtimes_r G$ with K-theory different from the K-theory of $\mathcal{O}_2, \ldots, \mathcal{O}_{10}$.

For $G = \mathbb{Z}_3 * \mathbb{Z}_3$ one has that $C(\partial G) \rtimes_r G \cong M_3(\mathcal{O}_4) \not\cong \mathcal{O}_4$. Therefore the table does not give a complete answer to the Question 2.7.6 above, not even in the case when $m = 2, \ldots, 10$.

We end this section with following

Proposition 2.7.7. *Consider a free product $G = G_1 * \cdots * G_n, n \geq 2$ of non-trivial cyclic groups such that $G \neq \mathbb{Z}_2 * \mathbb{Z}_2$. Equip G with the canonical action on the set ∂G of all infinite reduced words. Then the set ∂G is $\tau_{\partial G}$-paradoxical using the smallest number of sets, i.e. with $n = m = 2$ in Definition 2.3.6.*

Proof. We leave this as an exercise. □

2.8 Example 2: Fuchsian group of the first kind

A *Fuchsian group*[6] is a discrete group of conformal isometries on the hyperbolic plane $H^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, cf. [24, Chapter IV].

Occasionally we identify H^2 with the Poincare disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ using the *Cayley transformation* $K\colon H^2 \to \mathbb{D}, z \mapsto (z-1)/(z+i)$. Then the set $\mathcal{G}$ of conformal isometries on $\mathbb{D}$ is simply, cf. [4, Theorem 7.4.1 + p.56 + p.188],

$$\mathcal{G} = \Big\{ z \mapsto \frac{az + \bar{b}}{bz + \bar{a}} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1, z \in \mathbb{C} \cup \{\infty\} \Big\}.$$

[6]The Polish mathematician Lazarus Fuchs (1833-1902) lent his name to this definition.

The construction of the *boundary* ∂G for a Fuchsian group G uses a certain set $R_G \subseteq \mathbb{D}$ associated with G

(i) R_G is open, connected and convex,

(ii) the sets $t.R_G, t \in G$, *tessellate* $\mathbb{D}$ (they are disjoint and $\mathbb{D} \subseteq \cup_{t \in G} t.\bar{R}_G = \bar{\mathbb{D}}$),

(iii) R_G is *locally finite* (every compact subset of $\mathbb{D}$ is covered by finitely many G-translates of $\bar{R}_G$)

(iv) R_G is bounded by *hyperbolic sides* (i.e. subsets of circles and lines orthogonal to $\mathbb{T}$) $s_1, s_1', \ldots, s_{k-1}, s_{k-1}', t_1, u_1, t_1', u_1', \ldots, t_g, u_g, t_g', u_g'$ with vertices $v_1, w_1, v_{k-1}, w_{k-1}, v_k, v_{k+1}, \ldots, v_{k+4g-1}$ counterclockwise, where g is the genus of $\bar{R_G}/G$ and k is the number of conjugacy classes of parabolic or elliptic cyclic subgroups,

(v) R_G has the *side pairing* of ∂R_G given by $r \to r'$ (with $r'' = r$),

(vi) the *vertex cycles* for R_G are $\{v_1, \ldots, v_{k+4g-1}\}, \{w_1\}, \ldots, \{w_{k-1}\}$ (with z_1 a vertex of R_G, let r_1 be the side extending from z_1 counterclockwise, $g_1 = t_{r_1} \in G$ ($r_1 \to r_1'$), $z_2 = g_1(z_1)$ and r_2 the other side ending at z_2; continuing this process gives a vertex cycle).

A Fuchsian group G is of the *first kind* if the *limit set* (i.e. the closure of all limit points in $\mathbb{R} \cup \{\infty\} \subseteq \mathbb{C} \cup \{\infty\}$ of orbits $\mathcal{O}_x, x \in H^2$) equals the whole boundary $\partial H^2 = \mathbb{R} \cup \{\infty\}$. A non-trivial element in $\mathcal{G}$ is called *elliptic*, *parabolic* or *hyperbolic* if tr^2 is $< 4, = 4$ or > 4, or equivalent if the element has one fixed point in $\mathbb{D}$, $\mathbb{T}$ or two fixed points in $\mathbb{T}$.

It is shown in [57, Theorem 1.1] that in the case G is a finitely generated Fuchsian group of the first kind containing parabolic elements, then a set R_G exists.

Remark 2.8.1. Let us make a simple example in Euclidian geometry for the reader unfamiliar with the existence of R_G in the Hyperbolic geometry. Let $G := \mathbb{Z}^2$ act canonically on $\mathbb{R}^2$ by translation in the two orthogonal directions. Notice the G is a discrete group consisting of conformal (angle-preserving) isometries. The set $R := (-\frac{1}{2}, \frac{1}{2})^2$ is open, connected and convex and tessellates $\mathbb{R}^2$. Also here we have a paring of the four sides of R using the generators for G and a vertex cycle consisting of all four vertices.

To see that the number k in property (iv) make sense and because we will need to find k in a specific example recall the following

Theorem 2.8.2. *(cf. [4, Theorem 10.3.3]) Let G a finitely generated Fuchsian group of the first kind. There exist a finite number of cyclic subgroups $\langle g_1 \rangle, \dots, \langle g_k \rangle$ of G fulfilling that every elliptic or parabolic element in G is conjugate to precisely one element in precisely one of these groups.*

Proof. Recall that G has an open, connected and convex set $R \subseteq \mathbb{D}$ — with finite number of hyperbolic sides and a side paring $s \to s'$ — that tessellates $\mathbb{D}$, cf. [4, Theorem 10.1.2].

Let h be a parabolic or elliptic element in G and $v \in \bar{\mathbb{D}}$ be the unique fixed point for h. One can find $g \in G$ such that $g(v) \in \partial R$. To see this suppose that $t(v) \notin \partial R$ for every $t \in G$. Since $v \in \cup_{t \in G} t.\bar{R}$ there exist $g \in G$ such that $w := g(v) \in R$. As w is a fixed point for $f := ghg^{-1}$ we obtain that $w = f(w) \in R \cap f(R) = \emptyset$ giving a contradiction.

If $g(v)$ is not a vertex one can change the side paring such that it becomes a vertex. To see this let $s \subseteq \partial R$ be the hyperbolic side of R containing w. Recall that the side transformation g_s is the quince non-trivial element fulfilling that $s = \bar{R} \cap g_s(\bar{R})$. Furthermore the side paring sends s to $s' = g_s^{-1}(s) = \bar{R} \cap g_s^{-1}(\bar{R})$. Since w is a fixed point for f it follows that $w \in \bar{R} \cap f(\bar{R})$. Suppose now that w is not a vertex. This imply that w belongs to a unique side giving that $f = g_s$. We obtain now that

$$w \in \bar{R} \cap f(\bar{R}) = s, \quad w \in \bar{R} \cap f^{-1}(\bar{R}) = s'.$$

This is only possible if $s = s'$ and $f = f^{-1}$ implying that w is the unique fixed point for the side transformation g_s of order 2, sending s to itself. Making w into a vertex adds one vertex cycle $\{w\}$. As G is finitely generated every R has a finite number of sides, cf. [4, Theorem 10.1.2]. Hence this process of adding vertices is finite.

By possibly a small change of vertices of R we obtain that every parabolic or elliptic element h is conjugate to element in G having a vertex as a fixed point. Let $V := \{v_1, \dots, v_n\}$ be such a set of vertices.

Recall that $G_i := \{g \in G \colon g(v_i) = v_i\}$ is a cyclic group and let g_i be the generator. If necessary remove some of the subgroups of G until they are pairwise non-conjugate and reindex the generators.

We now show that different powers of a generator g_i are non-conjugate. Suppose $f g_i^n f^{-1} = g^m$ for some element $f \in G$, generator g_i and $n, m \in \mathbb{N}$. This implies that $f g_i^n(v_i) = f(v_i) = g_i^m(f(v_i))$. As $f(v_i)$ is the unique a fixed point for g_i^m we have that $f(v_i) = v_i$. Hence f is itself a power of g_i implying that $n = m$. □

We are now ready to define the boundary on which the Fuchsian group acts. For simplicity we assume here, that $g = 0$. The boundary ∂G with topology τ is then defined as

$$\partial G = \mathbb{T}, \quad \tau = \mathrm{Top}(\{t.I(i,j) \subseteq \mathbb{T} : t \in G, (i,j) \in J\}),$$

with (for an example se Figure 2.3)

$$J = \big\{(i,j) \in I \times \mathbb{Z} : j \in \{1,\dots,m_i - 1\} \text{ if } i \in I_0 \text{ and } j \in \{1,-1\} \text{ if } i \in I \backslash I_0\big\},$$

$$I(i,j) = [t_i^{-j+1}.v_i, t_i^{-j}.v_i[\quad \text{if } i \in I_0, \text{ counterclockwise}$$
$$I(i,1) = [v_i, w_i[, \quad I(i,-1) = [w_i, v_{i+1}[\quad \text{if } i \in I \setminus I_0,$$

where t_i is the element carrying s_i to s_i' of order m_i $(i = 1 \dots, k-1)$, $I = \{1,\dots,k-1\}$ and $I_0 = \{i \in I : m_i < \infty\}$. Noticing that $\mathbb{T}$ is $\mathcal{G}$-invariant we have a canonical action of G on ∂G. For non-zero genus g one can also define the the boundary ∂G. We refer to [57] for the details.

Lemma 2.8.3. *Let G be a finitely generated Fuchsian group G of the first kind containing parabolic elements. Then $(\partial G, G)$ is a transformation group with a minimal, topologically free, amenable and a strong boundary action.*

Proof. All except the last condition are mentioned in [57, Section 3]. Further, cf. [24, Theorem 7.6] or [57, Remark 1.2], G admits the presentation

$$G = \langle t_1, \dots, t_{2g+k-1} : t_i^{m_i} = e \text{ for } i \in I_0\rangle,$$

where $t_i = t_{s_i}^{-1}$ is side-parring carrying s_i to s_i' of order m_i $(i = 1,\dots,k-1)$. Let $\tilde{\partial G}$ be the canonical boundary associated with G as a free product of cyclic groups, cf. Lemma 2.7.1. In the proof of [57, Theorem 2.2] a G-equivariant homeomorphism $\varphi : \tilde{\partial G} \to \partial G$ is constructed implying that $(\tilde{\partial G}, G)$ is a strong boundary action[7]. $\square$

A Fuchsian group G with R_G satisfying the mentioned conditions $(i)-(vi)$ is equipped with a *signature* as follows. The conjugacy classes mentioned in (iv) are all cyclic groups. Let $m_1 \le \dots \le m_k$ denote the order of these groups. Together with the genus g also mentioned in (iv) we get the signature $(g, k, m_1, \dots, m_k)$. The signature defines the *Euler characteristic* for G:

$$\chi := 2g - 2 + k - \sum_{i=1}^{k} 1/m_i.$$

[7] One should consider (i) why the representation of G as a free product has at least two non-trivial cyclic groups and (ii) why $G \ncong \mathbb{Z}_2 * \mathbb{Z}_2$. We leave this as an exercise.

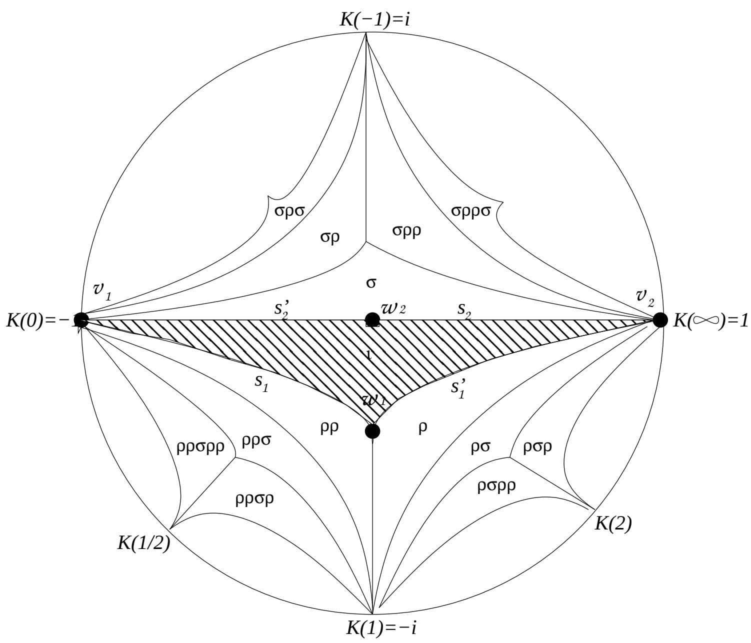

Figure 2.2: The set $R_{\mathbb{Z}_2 * \mathbb{Z}_3}$ is pointed out.

Let $(g, k, m_1, \ldots, m_k)$ be any signature with at least one $m_i = \infty$ and $\chi > 0$. There exist a finitely generated Fuchsian group of the first kind containing parabolic elements with the specified signature. Any finitely generated Fuchsian group of the first kind containing parabolic elements fulfills that $\chi > 0$, cf. [4].

Corollary 2.8.4. *Consider a finitely generated Fuchsian group G of the first kind containing parabolic elements. Let $(g, k, m_1, \ldots, m_k)$ be the signature of G and $A = C(\partial G)$. Then the crossed product $A \rtimes_r G = A \rtimes G$ is a separable, simple, nuclear and purely infinite C^*-algebra in the UCT-class. Further*

$$\begin{aligned} K_0(A \rtimes_r G) &= \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{l-2}} \oplus \mathbb{Z}_N \oplus \mathbb{Z}^{2g+k-l-1} \\ K_1(A \rtimes_r G) &= \mathbb{Z}^{2g+k-l-1} \end{aligned}$$

assuming that $m_j < \infty$ for $j = 1, \ldots, l$ with $m_{j-1} | m_j$ and $m_j = \infty$ for $j = l+1, \ldots, k$. Here $N = m_{l-1} + m_l - m_l m_{l-1}(2g - 2 + k - \sum_{j=1}^{l-2} 1/m_j)$.

Proof. By the proof of Lemma 2.8.3 the K-theory can be calculated simply by considering the action of

$$\begin{aligned} G &= \langle t_1, \ldots, t_{2g+k-1} : t_i^{m_i} = e \text{ for } i \in I_0 \rangle \\ &= \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_l} * \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2g+k-l-1} \end{aligned}$$

on the space of all infinite reduced words. By Corollary 2.7.4 we are done. $\square$

The best known example of a finitely generated Fuchsian group G of the first kind containing parabolic elements is the modular group

$$G = \mathrm{PSL}_2(\mathbb{Z}) \cong \langle \sigma, \rho \in \mathcal{G} : \sigma(z) = -z,\ \rho(z) = \frac{(1-2i)z+1}{z+(1+2i)} \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

For completeness we argue why G is a finitely generated Fuchsian group

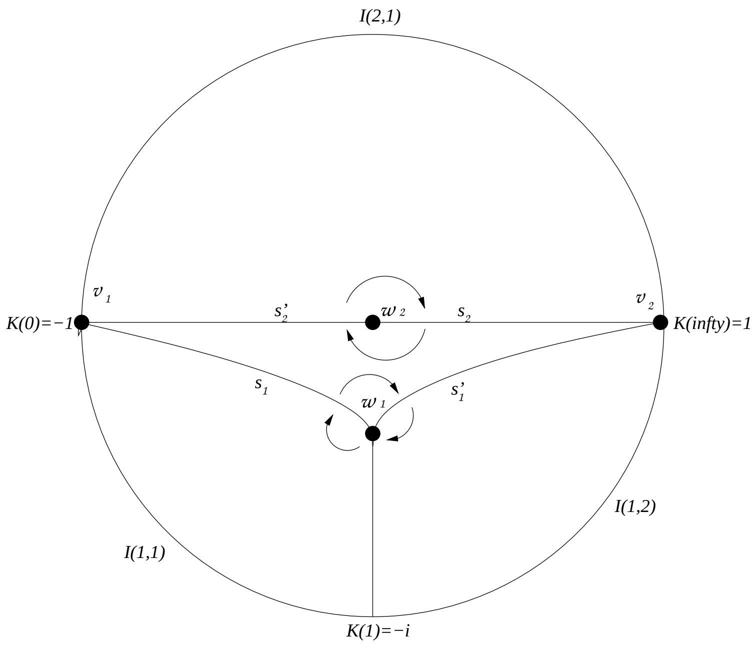

Figure 2.3: Here we see the side paring for R_G.

of the first kind containing parabolic elements. These properties imply the existence of R_G. As $\mathbb{Z}$ is a discrete subgroup of $\mathbb{R}$ it follows that $G = \mathrm{PSL}_2(\mathbb{Z})$ is a discrete subgroup of $\mathcal{G} \cong \mathrm{PSL}_2(\mathbb{R})$, cf. [60, p. 61]. From

$$|-1|^2 + |0|^2 = 1, \quad |(1+2i)/2|^2 - |1/2|^2 = 1,$$

σ, ρ are conformal isometries. The element $\sigma\rho$ is parabolic with the fixed point $1 \in \mathbb{T}$. As isometries of H^2 $\rho\sigma$, σ have the form

$$a : z \to z+1, \quad b : z \to -1/z.$$

For $n_0, n_1, \ldots \in \mathbb{Z}$, one gets

$$0 \rightarrow^{a^{n_0}} n_0 \rightarrow^{b} \frac{1}{-n_0} \rightarrow^{a^{-n_1}} -n_1 + \frac{1}{-n_0} \rightarrow^{b} \frac{1}{-(-n_1 + \frac{1}{-n_0})} \rightarrow^{a^{n_2}} n_2 + \frac{1}{n_1 + \frac{1}{n_0}}.$$

Using continued fraction representation and the fact that $\mathbb{Q} \cup \infty$ is dense in $\partial H^2 \subseteq \mathbb{C} \cup \infty$ it follows that G is of the first kind.

Figure 2.2 shows the set R_G associated with $G = \mathbb{Z}_2 * \mathbb{Z}_3$. Let us now calculate the signature of G. When identifying the sides of $\bar{R_G}$ using the side parring we get a sphere. Hence the genus g for $\bar{R_G}/G$ is zero. All the elliptic/parabolic elements are conjugate to $\sigma\rho, \sigma\rho^{-1}, \sigma$ or ρ, as they fix the four vertices of R_G. Further $\sigma\rho$ is conjugate to a power of $\sigma\rho^{-1}$, implying that $\sigma\rho$ and $\sigma\rho^{-1}$ belongs to the same conjugate class. Consequently $k = 3$ (and not lover as $\sigma\rho$, σ and ρ have different order). The signature for G is therefore equal to $(0, 3, 2, 3, \infty)$.

Question 2.8.5. Can the construction described above be extended to Fuchsian groups of the *second kind*, where the limit set is a Cantor subset of $\mathbb{T}$?

Remark 2.8.6. Let us emphasize one possible obstruction in the case we consider a Fuchsian group of second kind. When G is of the first kind we have the following

Lemma 2.8.7. *Let G be a finitely generated Fuchsian group G of the first kind containing parabolic elements. The action of G on ∂G is minimal.*

Proof. Recall that the boundary ∂G with topology τ is defined as

$$\partial G = \mathbb{T}, \quad \tau = \mathrm{Top}(\{t.I(i,j) \subseteq \mathbb{T} : t \in G, (i,j) \in J\}),$$

where $\{I(i,j) \colon (i,j) \in J\}$ is a partition of ∂G in disjoint intervals. Let F be the set of all fixed points of parabolic elements in G. Spielberg shows, cf. [57, p. 588], that the translation of the endpoints of the intervals $\{I(i,j) \colon (i,j) \in J\}$ by G equals the set F and that

$$\tau = \mathrm{Top}(\{[a,b[\subseteq \mathbb{T} : a \neq b \in F\}), \quad \overline{F}^{\tau_{\mathbb{T}}} = \mathbb{T}.$$

Note that the usual topology

$$\tau_{\mathbb{T}} = \mathrm{Top}(\{(a,b) \subseteq \mathbb{T} : a \neq b \in \mathbb{T}\})$$

is contained in τ since for every $y \in (a,b) \subseteq \mathbb{T}, a \neq b \in \mathbb{T}$ there exist $a_0, b_0 \in F$ such that $y \in [a_0, b_0[\subseteq (a,b)$. Here we use that F is dense in $\mathbb{T}$. Since G is of the first kind the action of G on $(\mathbb{T}, \tau_{\mathbb{T}})$ is minimal (since the limit set contains F and the limit set is the smallest non-empty closed invariant subset of $\mathbb{C} \cup \{\infty\}$, cf. [4, Definition 8.1.1.]). By the identity $\tau_{\mathbb{T}} \subseteq \tau$ we conclude that the action of G on $(\partial G, \tau)$ is minimal. $\square$

In the case we consider a finitely generated Fuchsian group G of the second kind containing parabolic elements the proof of minimality in the above Lemma 2.8.7 fails (since $\overline{F}^{\tau_{\mathbb{T}}} \neq \mathbb{T}$). The action of G on $(\mathbb{T}, \tau_{\mathbb{T}})$ and on $(\partial G, \tau)$ (with $\tau = \mathrm{Top}(\{[a, b[\subseteq \mathbb{T} : a \neq b \in F\})$ fails to be minimal. We leave this as an exercise. We refer to [1] for related results.

2.9 Example 3: Word hyperbolic group

Consider a group G with a symmetric finite set S of generators not containing the identity e. We equip G with the *word metric*

$$|s - t|_S = min\{n : s^{-1}t = a_1 a_2 \dots a_n,\ a_i \in S\}, \quad s, t \in G,$$

corresponding to the shortest path in the *Cayley graph* $\mathcal{G}(G, S)$ for G (the graph with vertices G and edges (t, s) whenever $s^{-1}t \in S$).

Identifying every edge with the interval $[0, 1]$ gives a metric on $\mathcal{G}(G, S)$. The space $\mathcal{G}(G, S)$ is *geodesic*, meaning that every pair of points $x, y \in \mathcal{G}(G, S)$ can be joined by a *geodesic segment* $[x, y]$ (i.e. a map $\sigma : I = [0, a] \to \mathcal{G}(G, S)$ such that $\sigma(0) = x$, $\sigma(a) = y$ and $|\sigma(s) - \sigma(t)| = |s - t|$ for $s, t \in I$). Futher $\mathcal{G}(G, S)$ is *proper*, meaning that all the bounded subsets in $\mathcal{G}(G, S)$ are relatively compact.

A geodesic metric space X is called *hyperbolic* provided that the *insize* of all geodesic triangles are bounded above by some finite constant. The insize of a geodesic triangle $\Delta = [x, y, z]$ is defines as follows. There exist unique non-negative real numbers a, b, c such that

$$|x - y| = a + b, \quad |y - z| = b + c, \quad |z - x| = c + a.$$

When diviving the sides of Δ into 6 geodesic segments of length a, b, b, c, c and a three vertices are added. The insize of Δ is the maximal distance between any two of these three vertices.

Any group for which the Cayley graph can be nicely embedded in the hyperbolic plane is a *word hyperbolic group*. For example, $\mathbb{Z}_2 * \mathbb{Z}_3$ is a word hyperbolic group. Figure 4 shows a drawing of the Cayley graph for $\mathbb{Z}_2 * \mathbb{Z}_3$. The figure seems symmetric, but the edges have two different lengths. If the basepoint x for the $\mathbb{Z}_2 * \mathbb{Z}_3$-orbit is

$$x = \frac{2\,i\,\left(-1 - e + \sqrt{1 + e + e^2}\right) + \sqrt{(1 + e)\,\left(-7 - 7\,e + 8\,\sqrt{1 + e + e^2}\right)}}{(1 + \sqrt{e})^2}$$

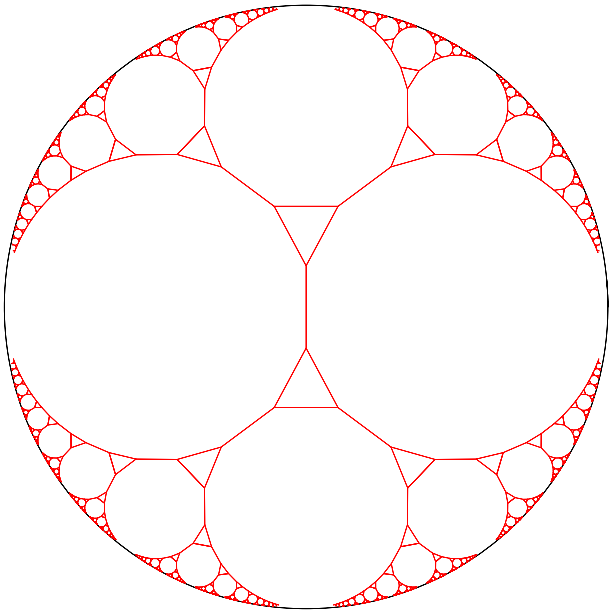

Figure 2.4: An embedding of the Cayley graph for $\mathbb{Z}_2 * \mathbb{Z}_3$ into the hyperbolic space.

then all the edges have unit length. We omit the calculation, but note that we simply used the constrain on the distance from x to $\sigma(x)$ and then on the distance from x to $\rho(x)$, with $\mathbb{Z}_2 * \mathbb{Z}_3 \cong \langle \sigma, \rho \rangle$ from the previews section. The result can be seen in Figure 2.9.

We define the boundary ∂G for a word hyperbolic group G to be the "Gromov boundary" $\partial \mathcal{G}(G, S)$ for the hyperbolic space $\mathcal{G}(G, S)$. Given a hyperbolic space X the *Gromov boundary* ∂X is the set of equivalence classes of sequences which *converge at infinity* in the Gromov sense (here $(\cdot, \cdot)$ is the Gromov product with respect to some fixed basepoint)

$$\partial X = \{(x_n) \in X^{\mathbb{N}} : \lim_{(n,p)} (x_n, x_p) = \infty\} / \sim, \quad (x_n) \sim (y_n) \Leftrightarrow \lim_n (x_n, y_n) = \infty.$$

When X is geodesic and proper, Ascoli's theorem [38, Th.6.1] implies that for $x \in X$, $y \in \partial X$ there exists a *geodesic ray* "joining" these points (i.e. an isometry $\sigma : I = [0, \infty[\to X$ with $\sigma(0) = x$, $(\sigma(n))_{n \in \mathbb{N}} \sim y$). Further we can equip $X \cup \partial X$ with a topology[8] such that X is open and dense in $X \cup \partial X$,

[8]The topology is the quotient topology induced by the natural surjection

$$\pi : Y \to X \cup \partial X, \quad \pi(\sigma) = (\sigma(n))_{n \in \mathbb{N}},$$

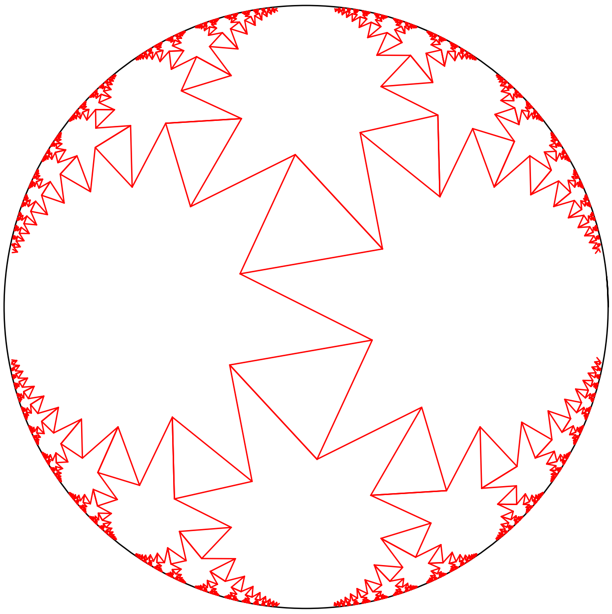

Figure 2.5: The Cayley graph for $\mathbb{Z}_2 * \mathbb{Z}_3$ with all edges of unit length.

$X \cup \partial X$ is compact and finally such that a sequence (x_n) in X converges to $y \in \partial X$ iff the sequence belongs to y.

Theorem 2.9.1 (Laca, Spielberg and Anantharaman-Delaroche)**.** *Let G be a finitely generated word hyperbolic group without any cyclic subgroup of finite index. Then $(\partial G, G)$ is a transformation group with an action that is (i) minimal, (ii) topologically free, (iii) amenable and a (iv) strong boundary action.*

2.10 Groupoids

A groupoid is similar to a group, except that its composition is not defined everywhere. More precisely, a *groupoid* is a set G together with a subset $G^2 \subseteq G \times G$, a product $(a, b) \mapsto ab$ from G^2 to G, an inverse $a \mapsto a^{-1}$ from

where Y is the set of geodesic segments/rays starting at the basepoint equipped with the topology of uniform convergence on compact sets, cf. [11, p. 11]. Segments $\sigma : I = [0, a] \to X$ are extended to rays by taking $\sigma(t) = \sigma(a)$ for every $t \geq a$.

G to G fulfilling

$$\begin{array}{ll} (i) & a \in G \Rightarrow (a, a^{-1}) \in G^2, \quad (a^{-1})^{-1} = a \\ (ii) & (a,b),(b,c) \in G^2 \Rightarrow (ab,c),(a,bc) \in G^2, \quad (ab)c = a(bc) \\ (iii) & (a,b) \in G^2 \Rightarrow a^{-1}(ab) = b, \quad (ab)b^{-1} = a \end{array}$$

An equivalent definition can be made when regarding G as a set of arrows together with G^0 (the set of endpoints) and the associated domain and range maps

$$d, r : G \to G^0 = r(G) = d(G), \quad r(a) = aa^{-1}, \quad d(a) = a^{-1}a, \quad a \in G.$$

The composition and the inverse can then be regarded as a composition of arrows and as an inverse arrow cf. [1, Definition 1.1].

A *topological* groupoid is a groupoid G equipped with a topology making the product and the inverse continuous. If each $t \in G$ has an open neighborhood V such that $r(V)$ is open and $r|_V : V \to r(V)$ is a homeomorphism, then G is called *r-discrete*. We refer to [42], [45] for details.

Let us consider some simple examples of groupoids:

(1) Given $n \geq 2$. Define $S = \{1, \ldots, n\}$ and let $Z = S^{\mathbb{N}}$ and $W = \cup_{n \in \mathbb{N}_0} S^n$ be the infinite and finite sequences (words) in S. Define G_n as all triples of the form

$$(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma),$$

where $\alpha, \beta \in W$, $\gamma \in Z$ and $l(\alpha)$ is the length of α. With

$$G_n^2 = \{((x,k,x'),(y,l,y')) \in G_n \times G_n \ : \ x' = y\},$$

$$(x,k,x')(x',l,x'') = (x, k+l, x''), \quad (x,k,x')^{-1} = (x', -k, x),$$

G_n is a groupoid. For S discrete and Z equipped with product topology it follows that the sets

$$U_{\alpha,\beta} = \{(\alpha\gamma, l(\alpha) - l(\beta), \beta\gamma) \ : \ \gamma \in Z\}, \quad \alpha, \beta \in W,$$

define a basis for a topology on G_n. In this way G_n becomes a locally compact Hausdorff r-discrete groupoid.

(2) Given a transformation group (X, G) with a right action[9]. Define $G(X,G) = X \times G$ equipped with product topology. With

$$G(X,G)^2 = \{((x,t),(y,s)) \ : \ x.t = y\},$$

[9] Following the notation of [1] we let the group act on the right. This is only a change in notation and does not add anything new.

$$(x,t)(x.t,s) = (x,ts), \quad (x,t)^{-1} = (x.t,t^{-1}),$$

$G(X,G)$ is a locally compact Hausdorff r-discrete groupoid.

Note that a groupoid G is not necessarily second countable or Hausdorff. A locally compact Hausdorff r-discrete groupoid gives raise to a reduced C^*-algebra $C^*_r(G)$ and a universal C^*-algebra $C^*(G)$ as the completion of the *-algebra $C_c(G)$ with respect to the reduced and the universal C^*-norm.

The universal C^*-norm is the supremum of all norms associated with representations of $C_c(G)$ and the reduced C^*-norm can be found in [1, p. 205]. It follows, cf. [42, p.155], [42, p.11] and [1, p.205], that

$$C^*(G_n) \cong \mathcal{O}_n, \quad C^*(X,G) \cong C(X) \rtimes G, \quad C^*_r(X,G) \cong C(X) \rtimes_r G.$$

An r-discrete groupoid G has a basis for its topology consisting of open *G-sets*, i.e, subsets S of G such that $r|_S$ and $d|_S$ are bijective. A G-set S induces a map $\alpha_S \colon r(S) \to d(S)$ given by $\alpha_S(r(x)) = d(xS), x \in S$. G is called *locally contracting* if there for every non-empty open $U \subseteq G^0$ exists an open set $V \subseteq U$ and a G-set S such that $\bar{V} \subseteq r(V)$ and $\alpha_S(\bar{V}) \subsetneq V$. Further G is called *essentially free* provided that $\{u \in G^0 : r^{-1}(u) \cap d^{-1}(u) = \{u\}\}$ is dense in G^0.

It was shown in [1, Th.2.4] that every non-zero hereditary C^*-subalgebra of $C^*_r(G)$ contains an infinite projection whenever G is r-discrete, locally contracting and essentially free.

Further Renault showed in [45, Th.4.6] that there is a one-to-one correspondence between ideals in $C^*_r(G)$ and *invariant* open subsets of G^0 provided G is *essentially principal* (the definition can be found in [45]).

Question 2.10.1. Can these results help to a better understanding of simplicity and purely infiniteness of the reduced/universal crossed product associated with a transformation group?

Chapter 3

Ideal structure

This chapter is the offspring of the work on the ideal structure of reduced crossed products. Parts of the work were initiated in [54]. However that biggest part regarding the generalization of the Rokhlin property however made during fall 2007 at The Field Institute in Toronto. These results can be found in [55].

3.1 Reformulation

Definition 3.1.1. Let (A, G) be a C*-dynamical system with G discrete. We say that A *separates the ideals* in $A \rtimes_r G$ if the map

$$\mathcal{I}(A \rtimes_r G) \to \mathcal{I}(A)^G : J \mapsto J \cap A$$

in injective.

We showed in Theorem 2.1.5 that for a transformation group (X, G) with G discrete, exact the algebra $C_0(X)$ separates the ideals in $C_0(X) \rtimes_r G$ provided the action on X is free. We will now consider the generalization to the non-abelian case.

Note that the map in Definition 3.1.1 is automatically surjective (using that $(I \rtimes_r G) \cap A = I$ for $I \in \mathcal{I}(A)^G$). Hence, if A separates the ideals in $A \rtimes_r G$, then there is a ono-to-one correspondence between ideals in the crossed product $A \rtimes_r G$ and the invariant ideals in A.

Proposition 3.1.2. *Let (A, G) be a C*-dynamical system with G discrete. The following properties are equivalent*

(i) For every $J \in \mathcal{I}(A \rtimes_r G)$ we have that $J = \mathrm{Ideal}_{A \rtimes_r G}[J \cap A]$.

(ii) The map $\mathcal{I}(A \rtimes_r G) \to \mathcal{I}(A)^G : J \mapsto J \cap A$ is injective.

(iii) The map $\mathcal{I}(A)^G \to \mathcal{I}(A \rtimes_r G) : I \mapsto I \rtimes_r G$ *is surjective.*

(iv) For every $J \in \mathcal{I}(A \rtimes_r G)$ *we have that* $J = \mathrm{Ideal}_{A\rtimes_r G}[E(J)]$.

Proof. $(i) \Rightarrow (ii)$ Fix ideals $J_1, J_2 \in \mathcal{I}(A \rtimes_r G)$ having the same intersection with A, one gets

$$J_1 = \mathrm{Ideal}[J_1 \cap A] = \mathrm{Ideal}[J_2 \cap A] = J_2.$$

$(ii) \Rightarrow (iii)$ Let $J \in \mathcal{I}(A \rtimes_r G)$. Using the general fact that

$$((J \cap A) \rtimes_r G) \cap A = J \cap A$$

together with (ii) it follows that $(J \cap A) \rtimes_r G = J$.

$(iii) \Rightarrow (iv)$ Let $J \in \mathcal{I}(A \rtimes_r G)$. From (iii) it follows that $J = I \rtimes_r G$ for some $I \in \mathcal{I}(A)^G$. We get that

$$J = I \rtimes_r G = \mathrm{Ideal}[I] = \mathrm{Ideal}[E_A(I \rtimes_r G)] = \mathrm{Ideal}[E_A(J)].$$

$(iv) \Rightarrow (i)$ Let $J \in \mathcal{I}(A \rtimes_r G)$. By (iv) we have that

$$J \cap A \subseteq E_A(J) \subseteq \mathrm{Ideal}[E_A(J)] \cap A = J \cap A.$$

Using (iv) once more we obtain that $J = \mathrm{Ideal}[E_A(J)] = \mathrm{Ideal}[J \cap A]$. □

The equality in part (i) of Proposition 3.1.2 has one trivial inclusion. This is not the case when we consider the equality in (iv). Here we have two non-trivial inclusion. However, as we will see, one of the inclusions corresponds to exactness of the action (cf. Definition 2.2.3), and hence it is automatic provided that G is exact.

3.2 Residual intersection property

Let us now consider the two inclusions corresponding to the last equality in Proposition 3.1.2 separately. Let (A, G) be a C*-dynamical system. Recall that the action (of G on A) is exact if every invariant ideal I in A induces a short exact sequence

$$0 \longrightarrow I \rtimes_r G \xrightarrow{\iota \rtimes_r \mathrm{id}} A \rtimes_r G \xrightarrow{\rho \rtimes_r \mathrm{id}} A/I \rtimes_r G \longrightarrow 0$$

at the level of reduced crossed products.

Proposition 3.2.1. *Let* (A, G) *be a C*-dynamical system with* G *discrete. Then the following properties are equivalent* *(i) For every $J \in \mathcal{I}(A \rtimes_r G)$ we have that $J \subseteq \mathrm{Ideal}_{A\rtimes_r G}[E(J)]$.*

(ii) For every $x \in (A \rtimes_r G)^+$ we have that $x \in \mathrm{Ideal}_{A\rtimes_r G}[E(x)]$.

(iii) The action of G on A is exact.

Proof. $(ii) \Rightarrow (i)$ From

$$J^+ = \bigcup_{x\in J^+} \{x\} \subseteq \bigcup_{x\in J^+} \mathrm{Ideal}[E(x)] \ \subseteq \ \mathrm{Ideal}[E(J)]$$

it follows that $J = \mathrm{Ideal}[J^+] \subseteq \mathrm{Ideal}[E(J)]$.

$(i) \Rightarrow (iii)$ Let $I \in \mathcal{I}(A)^G$. It is sufficient to verify the inclusion $\ker(\rho \rtimes_r \mathrm{id}) \subseteq I \rtimes_r G$ for the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \rtimes_r G \arrow[r, "\iota\rtimes_r\mathrm{id}"] \arrow[d, "E_I"] & A \rtimes_r G \arrow[r, "\rho\rtimes_r\mathrm{id}"] \arrow[d, "E_A"] & A/I \rtimes_r G \arrow[r] \arrow[d, "E_{A/I}"] & 0 \\
0 \arrow[r] & I \arrow[r, "\iota"] & A \arrow[r, "\rho"] & A/I \arrow[r] & 0.
\end{tikzcd}$$

With $J := \ker(\rho \rtimes_r \mathrm{id})$ we get that $\rho(E_A(J)) = E_{A/I}(\rho \rtimes_r \mathrm{id}(J)) = 0$. This implies that $E_A(J) \subseteq \ker \rho = I$ and hence that

$$J \subseteq \mathrm{Ideal}[E_A(J)] \subseteq \mathrm{Ideal}[I] = I \rtimes_r G.$$

$(iii) \Rightarrow (ii)$ Let $x \in (A \rtimes_r G)^+$. Set $I := \mathrm{Ideal}_A[E_A(x)]^G$ and consider the two, using (iii), short exact sequences in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \rtimes_r G \arrow[r, "\iota\rtimes_r\mathrm{id}"] \arrow[d, "E_I"] & A \rtimes_r G \arrow[r, "\rho\rtimes_r\mathrm{id}"] \arrow[d, "E_A"] & A/I \rtimes_r G \arrow[r] \arrow[d, "E_{A/I}"] & 0 \\
0 \arrow[r] & I \arrow[r, "\iota"] & A \arrow[r, "\rho"] & A/I \arrow[r] & 0.
\end{tikzcd}$$

As $E_A(x) \in I$, it follows that $E_{A/I}(\rho \rtimes_r \mathrm{id}(x)) = \rho(E_A(x)) = 0$. As $E_{A/I}$ is faithful on positive elements, it follows that $\rho \rtimes_r \mathrm{id}(x) = 0$ and hence

$$x \in \ker(\rho \rtimes_r \mathrm{id}) = I \rtimes_r G = \mathrm{Ideal}[I] = \mathrm{Ideal}[\mathrm{Ideal}_A[E_A(x)]^G] = \mathrm{Ideal}[E_A(x)].$$

□

Let us present the first application of Proposition 3.2.1. In general it is unknown whenever a family of invariant ideals (I_i) fulfills that $\bigcap_i (I_i \rtimes_r G) = (\bigcap_i I_i) \rtimes_r G$. However when the action is exact the equality is easily shown.

Proposition 3.2.2. *Let (A, G) be a C*-dynamical system with G discrete. If the group G (or the action) is exact, then every family of ideals (I_i) in $\mathcal{I}(A)^G$ fulfills the identity*

$$\bigcap_i (I_i \rtimes_r G) = (\bigcap_i I_i) \rtimes_r G.$$

Proof. Note that $E(\bigcap_i (I_i \rtimes_r G)) \subseteq \bigcap_i I_i$. By Proposition 3.2.1 we get the inclusion $\bigcap_i (I_i \rtimes_r G) \subseteq \mathrm{Ideal}[E(\bigcap_i (I_i \rtimes_r G))]$ and hence

$$\begin{aligned} (\bigcap_i I_i) \rtimes_r G \;&\subseteq\; \bigcap_i (I_i \rtimes_r G) \;\subseteq\; \mathrm{Ideal}[E(\bigcap_i (I_i \rtimes_r G))] \\ &\subseteq\; \mathrm{Ideal}[\bigcap_i I_i] = (\bigcap_i I_i) \rtimes_r G. \end{aligned}$$

□

Remark 3.2.3. A classical question of non-commutative harmonic analysis is how the irreducible representations of $A \rtimes G$ or at least of $A \rtimes_r G$ look like. Since any proper, closed, two-sided ideal of a C*-algebra is the intersection of the primitive ideals containing it, we can conclude from Proposition 3.2.1 and the semi-continuity property in Proposition 3.2.2 that at least the following holds: *If the action of G on A is exact, then every irreducible representation D of $A \rtimes_r G$ has kernel $J_D = (A \cap J_D) \rtimes_r G$ if and only if A separates the ideals of $A \rtimes_r G$.*

The second inclusion in part (*iv*) of Proposition 3.1.2 is closely related to the way ideals in the reduced crossed product intersect the original algebra.

Definition 3.2.4. Let (A, G) be a C*-dynamical system with G discrete. We say that the action (of G on A) has the *intersection property* if every non-zero ideal in $A \rtimes_r G$ has a non-zero intersection with A. If the induced action of G on A/I has the intersection property for every invariant ideal I in A, we say that action (of G on A) has the *residual intersection property.*

Lemma 3.2.5. *Let (A, G) be a C*-dynamical system with G discrete and let π be the surjection $A \rtimes G \to A \rtimes_r G$. The following properties are equivalent*

(i) The action has the intersection property.

(ii) Every representation of $A \rtimes_r G$ with a faithful restriction on A is itself faithful.

(iii) $\forall J \in \mathcal{I}(A \rtimes G)$ *with* $\pi^{-1}(\pi(J)) = J : J \cap A = 0 \Rightarrow \pi(J) = 0.$

Proof. $(i) \Rightarrow (ii)$. Given a representation $\pi\colon A \rtimes_r G \to B(H)$ with a faithful restriction on A. Assume $J = \ker \pi$ is non-zero. By (i) there exist $a \in J \cap A, a \neq 0$. But this contradicts $\pi_{|A}$ beeing faithful. Hence $J = \ker \pi = 0$.

$(ii) \Rightarrow (i)$. Given $J \in \mathcal{I}(A \rtimes_r G)$, assume $J \neq 0$. Find (cf. GSN) a representation $\pi\colon A \rtimes_r G \to B(H)$ with kernel J. From (ii) the kernel of $\pi_{|A}$ is an non-zero implying that $\ker \pi_{|A} \subseteq J \cap A \neq 0$.

$(i) \Rightarrow (iii)$. Assume the action has the intersection property and take $J \in \mathcal{I}(A \rtimes G)$ with $\pi^{-1}(\pi(J)) = J$ and $J \cap A = 0$. With $J_1 = \pi(J)$ one have $\pi^{-1}(J_1) \cap A = 0$ and hence that $J_1 \cap A = 0$. By assumption $J_1 = 0$. Hence $\pi(J) = 0$.

$(iii) \Rightarrow (i)$. Assume that for any $J \in \mathcal{I}(A \rtimes G)$ with $\pi^{-1}(\pi(J)) = J$ one has $J \cap A = 0 \Rightarrow \pi(J) = 0$. For $J_1 \in \mathcal{I}(A \rtimes_r G)$ with $J_1 \cap A = 0$ define $J = \pi^{-1}(J_1)$. As $\pi^{-1}(\pi(J)) = J$ and $J \cap A \subseteq \pi(J \cap A) \subseteq \pi(J) \cap \pi(A) = J_1 \cap A = 0$ we get $\pi(J) = 0$ by assumption. Hence $J_1 = 0$. □

Remark 3.2.6. Let (A, G) be a C*-dynamical system with G discrete. By Lemma 3.2.5 we see that a way to ensure that the action has the intersection property is to show that every representation of $A \rtimes G$, which is faithful on A, weakly contains the regular representation of (A, G). (To see this use the fact that the surjection $\pi^A\colon A \rtimes G \to A \rtimes_r G$ is just the identity map on $C_c(G, A)$.)

Proposition 3.2.7. *Let (A, G) be a C*-dynamical system with G discrete. Then the following properties are equivalent*

(i) For every $J \in \mathcal{I}(A \rtimes_r G)$ we have that $J \supseteq \mathrm{Ideal}_{A \rtimes_r G}[E(J)]$.

(ii) For every $x \in (A \rtimes_r G)^+$ we have that $E(x) \in \mathrm{Ideal}_{A \rtimes_r G}[x]$.

(iii) The action satisfies the residual intersection property and for every $J \in \mathcal{I}(A \rtimes_r G)$ the intersection $(\rho \rtimes_r \mathrm{id})(J) \cap A/I$ is zero, where $I := J \cap A$ and $\rho \rtimes_r \mathrm{id}$ comes from the diagram (1.2.2).

Proof. $(i) \Rightarrow (iii)$: First we show the residual intersection property. Take $I \in \mathcal{I}(A)^G$ and $J \in \mathcal{I}(A/I \rtimes_r G)$ and assume that $J \cap A/I = 0$. We show that $E_{A/I}(J) = 0$, which is equivalent to $J = 0$. Set $J_1 := (\rho \rtimes_r \mathrm{id})^{-1}(J)$. As $J_1 \in \mathcal{I}(A \rtimes_r G)$ it follows from (i) that $E_A(J_1) \subseteq J_1 \cap A$ and hence

$$E_{A/I}(J) = E_{A/I}((\rho \rtimes_r \mathrm{id})(J_1)) = \rho(E_A(J_1)) \subseteq \rho(J_1 \cap A) \subseteq J \cap A/I = 0.$$

For the second part take $J \in \mathcal{I}(A \rtimes_r G)$ and $x \in (\rho \rtimes_r \mathrm{id})(J) \cap A/I$ where $I := J \cap A$. We show $x = 0$. Find $j \in J$ such that $x = (\rho \rtimes_r \mathrm{id})(j) \in A/I$. Using (i) we have that $E_A(J) \subseteq J \cap A$ and hence

$$x = E_{A/I}(x) = E_{A/I}((\rho \rtimes_r \mathrm{id})(j)) = \rho(E_A(j)) \in \rho(E_A(J)) \subseteq \rho(J \cap A) = 0.$$

(iii) $\Rightarrow$ *(ii)* Let $x \in (A \rtimes_r G)^+$. Set $J := \mathrm{Ideal}[x]$ and $I := J \cap A$. Using *(iii)* on $J \in \mathcal{I}(A \rtimes_r G)$ it follows that

$$(\rho \rtimes_r \mathrm{id})(J) \cap A/I = 0,$$

with the surjection $\rho \rtimes_r \mathrm{id}$ coming from (1.2.2). The residual intersection property implies that the ideal $(\rho \rtimes_r \mathrm{id})(J) = 0$. Using the diagram (1.2.2) we now have that $\rho(E_A(J)) = E_{A/I}(\rho \rtimes_r \mathrm{id}(J)) = 0$ and hence

$$E_A(x) \in E_A(J) \subseteq \ker \rho = J \cap A \subseteq J = \mathrm{Ideal}[x].$$

(ii) $\Rightarrow$ *(i)* Let $J \in \mathcal{I}(A \rtimes_r G)$. Then

$$E_A(J^+) = \bigcup_{x \in J^+} \{E_A(x)\} \subseteq \bigcup_{x \in J^+} \mathrm{Ideal}[x] \subseteq J.$$

As every element in J is a linear combination of positive elements in J and E_A is linear, it follows that $E_A(J) \subseteq J$ and hence also $\mathrm{Ideal}[E_A(J)] \subseteq J$. $\square$

Using an additional observation, contained in Lemma 3.2.8 below, we obtain a new characterization of when A separates the ideals in $A \rtimes_r G$.

Lemma 3.2.8. *Let (A, G) be a C*-dynamical system with G discrete. Suppose the action of G on A is exact. Then for every $J \in \mathcal{I}(A \rtimes_r G)$ the intersection $(\rho \rtimes_r \mathrm{id})(J) \cap A/I$ is zero, where $I := J \cap A$ and $\rho \rtimes_r \mathrm{id}$ comes from the diagram (1.2.2).*

Proof. For a given $J \in \mathcal{I}(A \rtimes_r G)$ set $I := J \cap A$. Using that the action is exact we have the short exact sequence

$$0 \longrightarrow I \rtimes_r G \xrightarrow{\iota \rtimes_r \mathrm{id}} A \rtimes_r G \xrightarrow{\rho \rtimes_r \mathrm{id}} A/I \rtimes_r G \longrightarrow 0$$

and can therefore identify $(\rho \rtimes_r \mathrm{id})(A \rtimes_r G)$ with $A \rtimes_r G/L$, where $L := I \rtimes_r G$. We obtain the identities

$$J \cap A = I = L \cap A,$$

$$(\rho \rtimes_r \mathrm{id})(J) = J/L, \quad A/I = A/(L \cap A) = (A + L)/L.$$

Assume $(\rho \rtimes_r \mathrm{id})(J) \cap A/I \neq 0$. Then there exists $j \in J$ and $a \in A$ such that

$$j + L = a + L \neq L.$$

As $L \subseteq J$ it follows that $a \in J$ and hence $a \in J \cap A = I \subseteq L$. But this implies $a + L = L$ and we get a contradiction. Hence $(\rho \rtimes_r \mathrm{id})(J) \cap A/I = 0$. $\square$

Theorem 3.2.9. *Let (A, G) be a C^*-dynamical system with G discrete. Then the following properties are equivalent.*

(i) A separates the ideals in $A \rtimes_r G$.

(ii) The action is exact and for every $x \in (A \rtimes_r G)^+ : E(x) \in \mathrm{Ideal}_{A\rtimes_r G}[x]$.

(iii) The action is exact and satisfies the residual intersection property.

Proof. Combine Proposition 3.1.2, Proposition 3.2.1 and Proposition 3.2.7 with Lemma 3.2.8. □

Using Proposition 3.2.2 one can slightly improve the last part of Theorem 3.2.9 in the following sense:

Corollary 3.2.10. *Let (A, G) be a C^*-dynamical system with G discrete. Then the following properties are equivalent.*

(i) A separates the ideals in $A \rtimes_r G$.

(ii) The action is exact and the intersection $J \cap A/I$ is non-zero for every $I \in \mathcal{I}(A)^G$ and for every non-zero primitive ideal $J \in \mathcal{I}(A/I \rtimes_r G)$.

Proof. $(ii) \Rightarrow (i)$ We show property (iii) in Proposition 3.1.2, i.e. that the map $\mathcal{I}(A)^G \to \mathcal{I}(A \rtimes_r G) : I \mapsto I \rtimes_r G$ is surjective. Fix $J \in \mathcal{I}(A \rtimes_r G)$. Find a family of primitive ideals (J_i) in $\mathcal{I}(A \rtimes_r G)$ together with irreducible representations $\pi_i \colon A \rtimes_r G \to B(H_i)$, such that

$$J = \cap_i J_i, \quad \ker \pi_i = J_i$$

Set $I_i := J_i \cap A$. From exactness of the action we get the canonical isomorphism, for every i, defined by

$$\iota_i \colon A \rtimes_r G / I_i \rtimes_r G \to (A/I_i) \rtimes_r G : \ a u_s + I_i \rtimes_r G \mapsto (a + I_i) u_s,$$

and the well-defined (as $\pi_i(I_i \rtimes_r G) = 0$) representation

$$\pi^i \colon A \rtimes_r G / I_i \rtimes_r G \to B(H_i) : \ a u_s + I_i \rtimes_r G \mapsto \pi_i(a u_s).$$

The map

$$\pi^i \circ \iota_i^{-1} \colon (A/I_i) \rtimes_r G \to B(H_i)$$

is an irreducible representation of $(A/I_i) \rtimes_r G$. With $J^{(i)} := \ker \pi^i \circ \iota_i^{-1}$ we have that $J^{(i)} \cap A/I_i = 0$ from the following argument: With $a + I_i \in J^{(i)} \cap A/I_i$

$$0 = \pi^i \circ \iota_i^{-1}(a + I_i) = \pi^i(a + I_i \rtimes_r G) = \pi_i(a).$$

Hence $a \in \ker \pi_i \cap A = I_i$ giving that $J^{(i)} \cap A/I_i = 0$. Using (ii) we obtain that $J^{(i)} = 0$ and hence also $\ker \pi^i = 0$. With

$$\rho_i \colon A \rtimes_r G \to A \rtimes_r G/I_i \rtimes_r G,$$

fix $b \in J_i$. As $\pi^i(\rho_i(b)) = \pi_i(b) = 0$ it follows that $\rho_i(b) = 0$. Hence

$$J_i \subseteq \ker \rho_i = I_i \rtimes_r G \subseteq J_i, \quad J = \cap_i J_i = \cap_i (I_i \rtimes_r G) = (\cap_i I_i) \rtimes_r G.$$

The last equality uses Proposition 3.2.2. The map $I \to I \rtimes_r G$ is surjective. □

3.3 Exactness and essential freeness

It was implicitly stated in the work by Renault, cf. [45], that "essential freeness" of G acting on $\widehat{A}$ might be enough to ensure that A separates the ideals in $A \rtimes_r G$. Recall that the action of G on $\widehat{A}$ is called *essentially free* provided that for every closed invariant subset $Y \subseteq \widehat{A}$, the subset of points in Y with trivial isotropy is dense in Y, cf. [46]. In the context of crossed products the claim of Renault can be phrased as follows:

Conjecture 3.3.1. *Let (A, G) be a C*-dynamical system with G discrete. If the action of G on $\widehat{A}$ is essentially free then A separates the ideals in $A \rtimes_r G$.*

We will now present a way to ensure that A separates the ideals in $A \rtimes_r G$ is by extending the result of Archbold and Spielberg in [3].

Let (A, G) be a C*-dynamical system with G discrete. Recall that the action of G on $\widehat{A}$ is defined by $t.[\pi] := [\pi \circ (t^{-1}.)]$ for every $t \in G, [\pi] \in \widehat{A}$. Recall that the action on A is called topologically free if the set $\cap_{t \in F}\{x \in \widehat{A} : t.x \neq x\}$ is dense in $\widehat{A}$ for any finite subset $F \subseteq G \setminus \{e\}$, cf. [3, Definition 1].

Let G be a discrete group acting on a topological space X. Recall that the isotropy group of $x \in X$ is the set of all elements in G that fix x. The action of G on X is *topologically free* provided that the points in X with trivial isotropy are dense in X, cf. [6].

Remark 3.3.2. We have the trivial implications for the action of G on X

$$\text{freness} \;\Rightarrow\; \text{essential freness} \;\Rightarrow\; \text{topological freness}$$

I general there tree properties are different. However is some cases they all coincide. This is for example the case when the integers act by a minimal action on a T_1-space X, cf. Lemma 3.3.3.

Lemma 3.3.3. *Given a transformation group* $(\mathbb{Z}, X)$ *with* X *a* T_1*-space. If the action is minimal and topologically free then it is also free.*

Proof. Since the action is topologically free there exist at least one point $y \in X$ with trivial isotropy. Hence the orbit $\mathcal{O}_y$ is infinite. We conclude that X is infinite.

Fix an arbitrary $x \in X$. Assume that the isotropy group of x is non-trivial. This implies the existence of a smallest positive integer n such that $(n+1).x = x$. We obtain that the set

$$X = \overline{\mathcal{O}_x} = \overline{\{-n.x, \ldots, n.x\}} = \{-n.x, \ldots, n.x\}$$

is finite giving a contradiction. Hence the isotropy group of x is trivial. □

Remark 3.3.4. Let (A, G) be a C*-dynamical system with G discrete. Archbold and Spielberg define topological freeness of G acting on A slightly weaker than topological freeness of G acting on $\widehat{A}$. If $\widehat{A}$ is Hausdorff (e.g. when A is abelian) and G is countable, then the two notions of topologically free agree because $\widehat{A}$ is always a Baire space.

Using the natural inclusion $\widehat{A/I} \subseteq \widehat{A}$, cf. Lemma 1.4.3, we remind how one can obtain the following result

Lemma 3.3.5. *Let* (A, G) *be a C*-dynamical system with G discrete. If the action on* $\widehat{A}$ *is essentially free then for every* $I \in \mathcal{I}(A)^G$ *the action of* G *on* A/I *is topologically free.*

Proof. Let $I \in \mathcal{I}(A)^G$. By definition of topological freeness (of the action on A/I) it suffices to show that that action of G on $\widehat{A/I}$ is topologically free.

Suppose G acts on topological spaces X and Y and $f : X \to Y$ is an open mapping preserving non-trivial isotropy. If the action on G nn Y is topologically free, then also the action on X is topologically free. The inclusion map $\psi \colon \widehat{A/I} \to Y := \{[\pi] \in \widehat{A} \colon I \subseteq \ker \pi\}$ form Lemma 1.4.3 is an equivariant homeomorphism. Hence we only need to show that the action of G on $Y \subseteq \widehat{A}$ is topologically free.

By definition of essential freeness the action of G on any closed invariant set of $\widehat{A}$ is topologically free. This reduces the proof to showing that Y is closed and invariant. This follows from Lemma 1.4.2 together with the fact that I is invariant. □

We will not make a weakening of the notion of essential freeness, but we use sometimes the following concept:
Let $\mathcal{P}$ denote a property for dynamical systems (A, G). If this property holds

(or is required) for all quotients $(A/I, G)$ with $I \in \mathcal{I}(A)^G$, then we say that (A, G) is *residually* $\mathcal{P}$. E.g. a conceptional name for the topological freeness for all quotients A/I should be *"residual" topological freeness* of the action on A.

Archbold and Spielberg considered when the reduced crossed product is simple. The key result was the following

Theorem 3.3.6 (Archbold, Spielberg [3, Theorem 1])**.** *Let (A, G) be a C*-dynamical system with G discrete and let π be the canonical surjection $A \rtimes G \to A \rtimes_r G$. If the action of G on A is topologically free then*

$$\forall J \in \mathcal{I}(A \rtimes G) : J \cap A = 0 \Rightarrow \pi(J) = 0.$$

Using the above result together with Theorem 3.2.9 we get the following generalization.

Theorem 3.3.7. *Let (A, G) be a C*-dynamical system with G discrete. If the group G (or the action) is exact and the action of G on $\widehat{A}$ is essentially free then the algebra A separates the ideals in $A \rtimes_r G$.*

Proof. Fix $I \in \mathcal{I}(A)^G$ and $J_1 \in \mathcal{I}((A/I) \rtimes_r G)$ and assume $J_1 \cap (A/I) = 0$. Using Theorem 3.2.9.(*iii*) it is enough to show that $J_1 = 0$. Let $\pi^{A/I}$ be the surjection $(A/I) \rtimes G \to (A/I) \rtimes_r G$ and set $J := (\pi^{A/I})^{-1}(J_1)$. Using that $\pi^{A/I}$ is just the identity on A/I we get that

$$J \cap (A/I) = J_1 \cap (A/I) = 0$$

By Remark 3.3.4 the action of G on A/I is topologically free. Using Theorem 3.3.6 we obtain that $\pi^{A/I}(J) = 0$. Hence $J_1 = 0$. ☐

Remark 3.3.8. Gromov showed the existence of a finitely presented non-exact discrete group, cf. [21]. Hence there exist a C*-dynamical system (A, G) with a finitely presented discrete group G and a non-exact action of G on A. By Theorem 3.2.9 A does not separates the ideals in $A \rtimes_r G$.

If such (A, G) can be found, such that the action is essentially free, then the Conjecture 3.3.1 fails. But this is unknown if the action is essentially free, and the Conjecture 3.3.1 remains an open problem. In fact, these considerations show that the Conjecture of Renault is equivalent to the question of whether essentially free actions are exact.

Remark 3.3.9. Let (A, G) be a C*-dynamical system with G discrete. If the action is essentially free then the properties (*i*)–(*iii*) in Proposition 3.2.7 are all fulfilled. (For a proof, use Theorem 3.3.6 for A/I, the automatic analogue of Lemma 3.2.8 for the full crossed product and use that $(\rho \rtimes_r \mathrm{id}) \circ \pi^A = \pi^{A/I} \circ (\rho \rtimes \mathrm{id})$ for the canonical maps $\rho\colon A \to A/I$, $\pi^B\colon B \rtimes G \to B \rtimes_r G$.)

Corollary 3.3.10. *Let (A, G) be a C*-dynamical system with G discrete. Assume that the group G (or the action) is exact and that for every $x \in \widehat{A}$ the points in $\overline{G.x}$ with trivial isotropy are dense in $\overline{G.x}$. Then A separates the ideals in $A \rtimes_r G$.*

Proof. The action is exact and essentially free. $\square$

Corollary 3.3.11 (cf. [46, Corollary 4.6])**.** *Let (A, G) be a C*-dynamical with G discrete. Suppose the action is minimal (i.e. A contains no non-trivial invariant ideals) and there exists an element in the spectrum of A with trivial isotropy. Then $A \rtimes_r G$ is simple.*

Proof. The action is exact and essentially free. $\square$

Remark 3.3.12. When considering the canonical action of $\mathbb{Z}$ on $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ (the right translation, fixing ∞) it follows that topological freeness of $\mathbb{Z}$ acting on $\mathbb{T}$ is not enough to ensure that the action satisfy the residual intersection property. (This follows from Remark 2.1.7. For a more direct proof consider the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}) \rtimes_r \mathbb{Z} \longrightarrow C(\mathbb{T}) \rtimes_r \mathbb{Z} \longrightarrow \mathbb{C} \rtimes_r \mathbb{Z} \longrightarrow 0$$

If the action of $\mathbb{Z}$ on $C(\mathbb{T})$ had the residual intersection property then the (trivial) action of $\mathbb{Z}$ on $\mathbb{C}$ would have the intersection property. But $\mathbb{C} \rtimes_r \mathbb{Z} \cong C^*(\mathbb{Z}) \cong C(\mathbb{T})$ has the property that all non-zero proper ideals (of which there are many) have zero intersection with the complex numbers $\mathbb{C}$.)

Remark 3.3.13. In many cases, including when a countable group acts by an amenable action on a unital and abelian C*-algebra, essential freeness and the residual intersection property are equivalent. (For a proof examine Corollary 3.5.6.) For this class of examples, the essential freeness and exactness of the action are together necessary and sufficient to ensure that A separates the ideals in $A \rtimes_r G$ (by Theorem 3.2.9 and Theorem 3.3.7).

Remark 3.3.14. Essential freeness and the residual intersection property are in general different conditions. (For a proof consider any simple crossed product where the action is not essentially free. For example the free group of two generators acting trivially on $\mathbb{C}$.)

3.4 The residual Rokhlin* property

One application of Theorem 3.2.9 is an easy proof of the fact that the algebra separates the ideals in the reduced crossed product provided the action has the "Rokhlin property".

In the following set $A_\infty := l^\infty(A)/c_0(A)$, where $l^\infty(A)$ is the C*-algebra of all bounded functions from $\mathbb{N}$ into A and where $c_0(A)$ is the ideal in $l^\infty(A)$ consisting of sequences $(a_n)_{n=1}^\infty$ for which $\|a_n\| \to 0$. We will suppress the canonical inclusion map $A \subseteq A_\infty$. The induced action of G on A_∞ is defined entry-wise.

Let (A, G) be a C*-dynamical system with A unital and G finite. We say that the action (of G on A) satisfies the *Rokhlin property*, cf. [25], provided there exists a projection $p_e \in A' \cap A_\infty$ such that

(*i*) $p_e \perp t.p_e, \quad t \neq e$

(*ii*) $\sum_{t\in G} t.p_e = 1_{A_\infty}$

Theorem 3.4.1. *Let (A, G) be a C*-dynamical system with A unital and G finite. Suppose that the action satisfies the Rokhlin property. Then A separates the ideals in $A \rtimes_r G$.*

Proof. Let $p_e \in A' \cap A_\infty$ be the projection ensuring the action satisfies the Rokhlin property and set $p_s = s.p_e$ for $s \in G$. As G is finite the action is exact. Fix $x := \sum_{t\in G} a_t u_t \in (A \rtimes_r G)^+$. Using Theorem 3.2.9.(*ii*) it is enough to show that $E(x) \in \mathrm{Ideal}[x]$.

We can take the implementing unitaries for the action of G on A_∞ to be the same as those for A. We have the commuting triangle with three canonical inclusions

$$\begin{tikzcd} A \rtimes_r G \arrow[r] \arrow[dr] & A_\infty \rtimes_r G \arrow[d] \\ & (A \rtimes_r G)_\infty \end{tikzcd}$$

giving the identity

$$\mathrm{Ideal}_{A_\infty \rtimes_r G}[x] \cap A \rtimes_r G = \mathrm{Ideal}_{A \rtimes_r G}[x].$$

For the completeness of the proof let us recall how the equality is obtained. With $B := A \rtimes_r G$ fix $b \in \mathrm{Ideal}_{B_\infty}[x] \cap B$ and $\epsilon > 0$. Denote the quotient map $l^\infty(B) \to B_\infty$ by π_∞. Find $a = \sum_{j=1}^n t_j x s_j \in \mathrm{Ideal}_{B_\infty}[x]$ such that $t_j = \pi_\infty(t_j^{(i)}), s_j = \pi_\infty(s_j^{(i)})$ and $\|a - b\| < \varepsilon$. From $\|\pi_\infty(\sum_{j=1}^n t_j^{(i)} x s_j^{(i)} - b)\| = \limsup_i \|\sum_{j=1}^n t_j^{(i)} x s_j^{(i)} - b\| < \varepsilon$ is follows that $b \in \mathrm{Ideal}_B[x]$. Since $\mathrm{Ideal}_B[x] \subseteq \mathrm{Ideal}_{A_\infty \rtimes_r G}[x] \subseteq \mathrm{Ideal}_{B_\infty}[x]$ and $\mathrm{Ideal}_{B_\infty}[x] \cap B \subseteq \mathrm{Ideal}_B[x]$ we are done.

We only need to verify that $E(x) \in \text{Ideal}_{A_\infty \rtimes_r G}[x]$. This follows from the calculation below ($\delta_{e,t}$ is the Kronecker delta)

$$\begin{aligned} p_s(au_t)p_s &= \delta_{e,t}p_s(au_t) = p_sE_A(au_t), \quad s,t \in G \; a \in A, \\ \sum_{s\in G} p_sxp_s &= \sum_{s\in G} p_sE_A(x) = E_A(x) \in \text{Ideal}_{A_\infty \rtimes_r G}[x]. \end{aligned}$$

□

Remark 3.4.2. Theorem 3.4.1 only applies when the group G is finite. It is natural to consider if there is a way to extend this result. As we will see that is indeed the case.

The Rokhlin property is a quite restrictive property. We present here a weaker condition, called "residual Rokhlin* property", ensuring that A separates the ideals in $A \rtimes_r G$, when considering a discrete exact group G acting on a C*-algebra A.

For a C*-dynamical system (A, G) the induced action of G on A^* (resp. on A^{**}) is given by $(t.\varphi)(a) := \varphi(t^{-1}.a)$ for $a \in A, \varphi \in A^*$ (resp. for $a \in A^*, \varphi \in A^{**}$). We write $(A, G) \cong (B, G)$ when the isomorphism between A and B is equivariant, i.e. action preserving.

Definition 3.4.3. Let (A, G) be a C*-dynamical system with G discrete. We say that the action (of G on A) has the *Rokhlin* property* provided that there exist a projection $p_e \in A' \cap (A_\infty)^{**}$, such that

(*i*) $p_e \perp t.p_e, \quad t \neq e$

(*ii*) For every $a \in A$ with $a \neq 0$ there exist $t \in G$ such that $a(t.p_e) \neq 0$,

If the induced action of G on A/I has the Rokhlin* property for every invariant ideal I in A, we say that the action (of G on A) has the *residual Rokhlin* property*.

Remark 3.4.4. Note that the residual Rokhlin* property is weaker than the Rokhlin property. This follows from the fact that the property (*ii*) in Definition 3.4.3 is equivalent to the condition that $\|a \sum_{t\in G} t.p_e\| = \|a\|$ for every $a \in A$ and the fact that the Rokhlin property automatically implies residual Rokhlin property. (Let us verify the second of the two mentioned properties. Suppose $p_e \in A' \cap A_\infty$ is the projection ensuring that the action of G on A has the Rokhlin property. Fix $I \in \mathcal{I}(A)^G$ and set $p_e^I = \varphi(p_e)$ using the canonical equivariant map $\varphi\colon A_\infty \to (A/I)_\infty$. We obtain that $p_e^I \perp t.p_e^I$ for $t \neq e$ and $\sum_{t\in G} t.p_e^I = \sum_{t\in G} \varphi(t.p_e) = \varphi(1_{A_\infty}) = 1_{(A/I)_\infty}$).

Example 3.4.5. Before giving a proof of Theorem 2.5 we present a concrete C*-dynamical system (A, G) with the residual Rokhlin* property. We consider the well-known example, where $A := M_{n^\infty} \otimes \mathcal{K}$, the stabilized UHF-algebra of type n^∞, where $G := \mathbb{Z}$, and where $\mathbb{Z}$ acts on $M_{n^\infty} \otimes \mathcal{K}$ via an automorphism $\bar{\lambda}$ that scales the trace by a factor $1/n$. In this case we have that $(M_{n^\infty} \otimes \mathcal{K}) \rtimes_r \mathbb{Z}$ is isomorphic to $\mathcal{O}_n \otimes \mathcal{K}$, cf. [13].

More precisely let $(A, \mu_m \colon M_{n^\infty} \to A)$, cf. [48, Definition 6.2.2], be the inductive limit of the sequence

$$M_{n^\infty} \xrightarrow{\lambda} M_{n^\infty} \xrightarrow{\lambda} M_{n^\infty} \xrightarrow{\lambda} \cdots$$

with the connecting map λ defined by $a \mapsto e_{11} \otimes a$, where e_{11} is the projection $\mathrm{diag}(1, 0, \dots, 0)$ in M_n. It is well known that λ induces an automorphism $\bar{\lambda}$ on A fulfilling that $\mu_m \circ \lambda = \bar{\lambda} \circ \mu_m$, cf. [48, Proposition 6.2.4]. Using the identification $A \rtimes_r G \cong \mathcal{O}_n \otimes \mathcal{K}$ and the simplicity of A we can give a new proof of the well known result, that the Cuntz algebra $\mathcal{O}_n$ is simple, by proving that the action of G on A satisfies the Rokhlin* property. This follows using the projection

$$p_e := (q_1, q_2, q_3 \dots) \in A_\infty,$$

where

$$q_k := \mu_k(\underbrace{1 \otimes \cdots \otimes 1}_{2k} \otimes (1 - e_{11}) \otimes \underbrace{e_{11} \otimes \cdots \otimes e_{11}}_{2k} \otimes 1 \otimes \dots),$$

for $k \in \mathbb{N}$. Let us first show that $p_e \in A' \cap (A_\infty)^{**}$ and property (ii). Fix $m, l \in \mathbb{N}$ and $a \in M_{n^m}$. Using the canonical inclusion $M_{n^m} \subseteq M_{n^\infty}$, together with the identities $M_{n^\infty} = \overline{\bigcup_{i=1}^\infty M_{n^i}}$, $A = \overline{\bigcup_{i=1}^\infty \mu_i(M_{n^\infty})}$, it is enough to show that $p_e \mu_l(a) = \mu_l(a) p_e$ and $\|\mu_l(a)\| = \|\mu_l(a) p_e\|$. But this follows from a simple calculation. To see (i) we simply use that the projections $(t.q_k)_{t \in \mathbb{N}}$ are pairwise orthogonal for a fixed $k \in \mathbb{N}$.

We now return to proving the fact that A separates the ideals in $A \rtimes_r G$ provided a discrete exact group G acts on A by an action with the residual Rokhlin* property. The key idea is the following Lemma

Lemma 3.4.6. *Fix a discrete group G and von Neumann algebras $N \subseteq M$. Suppose there exists an action of G on N such that $(N, G) \cong (l_\infty(G), G)$ (as usual G acts on $l_\infty(G)$ by left translation) and a group homomorphism $U \colon G \to \mathcal{U}(M)$ such that*

$$U(t) f U(t)^* = t.f, \quad t \in G, f \in N.$$

Then (B, G), given by

$$B := N' \cap M, \quad t.b := U(t) b U(t)^*, \quad t \in G, b \in B,$$

is a C^*-dynamical system. Further the representation

$$\mathrm{id}\times U\colon C_c(G,B)\to M : bu_s \mapsto bU(s), \quad s\in G, b\in B$$

is isometric with respect to the reduced norm.

Proof. We show that $\mathrm{id}\times U$ is isometric. The first statement is straightforward. We want to use Fell's result [9, Proposition 4.1.7]. Hence we make the identification $l_\infty(G) = N$ and $M \subseteq B(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Note that $(\mathrm{id}, U, \mathscr{H})$ is a normal covariant representation of $(l_\infty(G), G)$ in the von Neumann algebra sense ([1]). It is a general fact, cf. [61, Example 2.51], that any covariant representation $(\tilde{\pi}, \tilde{U}, \tilde{\mathscr{H}})$ of $(l_\infty(G), G)$, with $\tilde{\pi}\colon l_\infty(G) \to B(\tilde{\mathscr{H}})$ unital and normal, is unitarily equivalent to a tensor multiple of the standard covariant representation

$$l_\infty(G) \subseteq B(l_2(G)), \quad t\mapsto \lambda_t \in B(l_2(G)), \quad t\in G,$$

where the inclusion corresponds to multiplication operators $M_f, f\in l_\infty(G)$ and where $\lambda\colon G\to \mathcal{U}(l_2(G))$ is the left regular representation. Thus, there exist a Hilbert space H and a unitary $W\in B(l_2(G)\otimes H, \mathscr{H})$, such that

$$W^*U(t)W = \lambda_t\otimes \mathrm{id}_H, \quad W^*\,\mathrm{id}(f)W = M_f\otimes \mathrm{id}_H, \quad t\in G, f\in l_\infty(G).$$

For the reader unfamiliar with results on covariant representations of $(l_\infty(G), G)$ we include here a small argument on how to find the unitary W: Fix a covariant representation $(\rho, U, \mathscr{H})$ of $(l_\infty(G), G)$ with ρ unital and normal. Using universality (and non-degeneracy of ρ) we obtain the corresponding (non-degenerate) representation π of $l_\infty(G)\rtimes G$ on $\mathscr{H}$. By extension we have a normal representation $\pi\colon (l_\infty(G)\rtimes G)''\to B(\mathscr{H})$, cf. [43, Theorem 3.7.7]. Set

$$H := \pi(E_{e,e})\mathscr{H}, \quad V(h) := \sum_{r\in G}\delta_r\otimes \pi(E_{e,r})h, \quad h\in\mathscr{H},$$

where $(\delta_r)_{r\in G}$ is the basis for $l^2(G)$, where $(E_{s,t})_{s,t\in G}$ are the elementary matrices generating $\mathcal{K}(l^2(G))$ and where we make the identification $\mathcal{K}(l^2(G)) = c_0(G)\rtimes G$, cf. [61, p. 133], induced by the map $fu_t\mapsto \sum_{s\in G} f(s)E_{s,t^{-1}s}$. Using that ρ is unital and normal (implying $\|V(h)\|^2 = \langle \sum_{r\in G}\rho(E_{r,r})h, h\rangle =$

[1] A normal covariant representation of a discrete W*-dynamical system is a covariant representation in the usual C*-algebra sense with the additional assumption that the representation of the von Neumann algebra is normal.

$\langle \rho(1_{l_\infty(G)})h, h\rangle = \|h\|^2$, $h \in \mathscr{H}$) we obtain that $V \in B(\mathscr{H}, l^2(G) \otimes H)$ is isometric. An easy calculation shows that V is surjective and hence a unitary and fulfills

$$V\pi(u_t)V^* = \lambda_t \otimes \mathrm{id}_H, \quad V\pi(f)V^* = V\rho(f)V^* = M_f \otimes \mathrm{id}_H, \quad t \in G, f \in c_0(G).$$

The last equality extends to $l_\infty(G)$ by normality of ρ. Finally set $W := V^*$.([2])

With the canonical bijection $\varepsilon\colon B(l_2(G))\bar{\otimes}B(H) \to B(l_2(G) \otimes H)$ and the unitary $V \in B(l_2(G) \otimes l_2(G) \otimes H)$, defined by $\delta_s \otimes \delta_t \otimes h \mapsto \delta_{t^{-1}s} \otimes \delta_t \otimes h$ where $(\delta_r)_{r \in G}$ is the basis for $l^2(G)$, consider the commuting diagram

$$\begin{tikzcd}
(B, G) \arrow[r, "(\mathrm{id},U)"] \arrow[d, "(\mathrm{id},\mathrm{id})"'] & M \arrow[d, "\mathrm{id}"] \\
(l_\infty(G)' \cap B(\mathscr{H}), G) \arrow[r, "(\mathrm{id},U)"] \arrow[d, "(W^*\cdot W,\mathrm{id})"'] & B(\mathscr{H}) \arrow[d, "W^*\cdot W"] \\
(\varepsilon(l_\infty(G)\bar{\otimes}B(H)), G) \arrow[r, "(\mathrm{id},\lambda\otimes 1)"] \arrow[d, "(1\otimes\cdot,\mathrm{id})"'] & B(l_2(G) \otimes H) \arrow[d, "\delta_s\otimes\delta_t\otimes h\mapsto\delta_s\otimes\cdot(\delta_t\otimes h)"] \\
(1 \otimes \varepsilon(l_\infty(G)\bar{\otimes}B(H)), G) \arrow[r, "(1\otimes\mathrm{id},1\otimes\lambda\otimes 1)"] \arrow[d, "(\mathrm{id},\mathrm{id})"'] & B(l_2(G) \otimes l_2(G) \otimes H) \arrow[d, "V^*\cdot V"] \\
(1 \otimes \varepsilon(l_\infty(G)\bar{\otimes}B(H)), G) \arrow[r, "(1\otimes\mathrm{id},\lambda\otimes\lambda\otimes 1)"] & B(l_2(G) \otimes l_2(G) \otimes H)
\end{tikzcd}$$

(For a C*-algebra $A \subseteq B(\mathscr{K})$ the representation $1 \otimes \mathrm{id}\colon 1 \otimes A \to B(l^2(G) \otimes \mathscr{K})$ sends $1 \otimes a$ into the map defined by $\delta_s \otimes k \mapsto \delta_s \otimes ak$.) For completeness we show

$$W^*(l_\infty(G)' \cap B(\mathscr{H}))W \subseteq \varepsilon(l_\infty(G)\bar{\otimes}B(H)).$$

We have that $(l_\infty(G)\bar{\otimes}\mathbb{C}1_{B(H)})' = l_\infty(G)'\bar{\otimes}B(H) = l_\infty(G)\bar{\otimes}B(H)$ where the first equality is a special case of the commutation theorem. Since both ε and $W^* \cdot W$ map commutants to commutants it follows that

$$\begin{aligned} W^*(l_\infty(G)' \cap B(\mathscr{H}))W &= \varepsilon\big(l_\infty(G)\bar{\otimes}\mathbb{C}1_{B(H)}\big)' \\ &= \varepsilon\big((l_\infty(G)\bar{\otimes}\mathbb{C}1_{B(H)})'\big) = \varepsilon(l_\infty(G)\bar{\otimes}B(H)). \end{aligned}$$

Using Fells Absorption Principle II [9, Proposition 4.1.7] the map $(1 \otimes \mathrm{id}) \times (\lambda \otimes \lambda \otimes 1)$ is unitarily equivalent to the regular representation and hence

[2]Remark: Do not think that π could be extended to a normal representation of $(l_\infty(G) \rtimes G)^{**}$ and only the normality of π and not ρ could be used. This fails as the sequence of finite sums of $E_{r,r}$ converges only weakly in $l_\infty(G)$ to the unit. We do however not know if the convergence also holds weakly in $(l_\infty(G) \rtimes G)^{**}$.

is isometric with respect to the reduced norm $\|\cdot\|_\lambda$. From the commuting diagram it follows that $\mathrm{id}\times U$ is isometric with respect to $\|\cdot\|_\lambda$. □

Theorem 3.4.7. *Let (A, G) be a C^*-dynamical system with G discrete. Suppose the action is exact and satisfies the residual Rokhlin* property. Then A separates the ideals in $A \rtimes_r G$.*

Proof. By Theorem 3.2.9 it is enough to show that the action of G on A has the intersection property provided there exists a projection $p_e \in A' \cap (A_\infty)^{**}$ such that

(i) $p_e \perp t.p_e, \quad t \neq e$

(ii) For every $a \neq 0$ in A there exist $t \in G$ such that $a(t.p_e) \neq 0$.

From Remark 3.2.6 the intersection property is automatically fulfilled if every representation of $A \rtimes G$, which is faithful on A, weakly contains the regular representation of (A, G). Fix a covariant representation (π, u, H) of (A, G) with π faithful. As every representation of $A \rtimes G$ comes from a covariant representation of (A, G), cf. [61, Proposition 2.40], it is sufficient to show that $\pi \times u$ weakly contains the regular representation, i.e.

$$\|a\|_\lambda \leq \|\pi \times u(a)\|, \quad a \in C_c(G, A).$$

Using the commuting diagram

$$\begin{tikzcd}
 & A \arrow[r, hook] \arrow[d, hook, "\pi"] & A_\infty \arrow[r, hook] \arrow[d, hook, "\pi_\infty"] & (A_\infty)^{**} \arrow[d, hook, "\pi_\infty^{**}"] & \\
G \arrow[r, "u"] & B(H) \arrow[r, hook] & B(H)_\infty \arrow[r, hook] & (B(H)_\infty)^{**} \arrow[r, hook] & B(\mathscr{H}),
\end{tikzcd}$$

where $(B(H)_\infty)^{**}$ is faithfully represented on a Hilbert space $\mathscr{H}$ and letting u_∞^{**} be the composition of the maps in the lower part of the diagram, we get a covariant representation $(\pi_\infty^{**}, u_\infty^{**}, \mathscr{H})$ of $((A_\infty)^{**}, G)$, with π_∞^{**} faithful, such that

$$\|\pi_\infty^{**} \times u_\infty^{**}(a)\| \quad = \quad \|\pi \times u(a)\|, \quad a \in C_c(G, A). \tag{3.4.1}$$

Let R be the weak* closed algebra generated by $\{p_t \colon t \in G\}$ in $(A_\infty)^{**}$ with the identity p, where $p_t := t.p_e$, $t \in G$. Define von Neumann algebras $N \subseteq M$ and an action of G on N by

$$N := \pi_\infty^{**}(R), \quad M := \pi_\infty^{**}(p)(B(H)_\infty)^{**}\pi_\infty^{**}(p),$$

$$s.\pi_\infty^{**}(a) := \pi_\infty^{**}(s.a), \quad a \in R, \quad s \in G.$$

Note that $(N, G) \cong (l_\infty(G), G)$, where G acts on $l_\infty(G)$ by left translation. With
$$U(t) := \pi_\infty^{**}(p)u_\infty^{**}(t)\pi_\infty^{**}(p), \quad t \in G,$$
we get a group homomorphism $U\colon G \to \mathcal{U}(M)$ fulfilling that
$$U(t)fU(t)^* = t.f, \quad t \in G, f \in N.$$
Set $B := N' \cap M$. By Lemma 3.4.6, the representation
$$\mathrm{id} \times U\colon C_c(G, B) \to M : bu_s \mapsto bU(s), \quad s \in G, b \in B,$$
is isometric with respect to the reduced norm on $C_c(G, B)$, and hence extends isometrically to
$$\mathrm{id} \rtimes_r U\colon B \rtimes_r G \to M.$$
Now consider the homomorphism $\epsilon\colon A \to M, a \mapsto \pi_\infty^{**}(ap)$. For every $a \in A$ we have that ap commutes with the projection p_t (since $(ap)p_t = ap_t = p_t a = p_t(pa) = p_t(ap)$) for $t \in G$. This implies that ap commutes with the elements in R, and so $\epsilon(a)$ commutes with elements in N, i.e. $\epsilon(a) \in N' \cap M$. We conclude that ϵ has image contained in B.

By property (ii) the map $a \mapsto ap$ on A is faithful. As π_∞^{**} is faithful and the composition of faithful homomorphisms is faithful we conclude that ϵ is faithful. Furthermore $\epsilon\colon A \to B$ is equivariant and hence the canonical homomorphism $\epsilon \rtimes_r \mathrm{id}\colon A \rtimes_r G \to B \rtimes_r G$ is isometric.

When composing the two norm preserving maps $\mathrm{id} \rtimes_r U$ and $\epsilon \rtimes_r \mathrm{id}$ we obtain the following:
$$A \rtimes_r G \longrightarrow B \rtimes_r G \longrightarrow M$$
$$a \quad \mapsto \quad \pi_\infty^{**} \times u_\infty^{**}(a)\pi_\infty^{**}(p) \qquad a \in C_c(G, A)$$
It now follows that
$$\|a\|_\lambda = \|\pi_\infty^{**} \times u_\infty^{**}(a)\pi_\infty^{**}(p)\| \leq \|\pi \times u(a)\|, \quad a \in C_c(G, A),$$
where the last inequality comes from (3.4.1). This completes the proof. □

Remark 3.4.8. The ideas in Theorem 3.4.7 can be used to make more general results concerning C*-dynamical systems with locally compact groups.

Corollary 3.4.9. *Let (A, G) be a C*-dynamical system with G discrete. Suppose the action is exact and there exists a projection $p_e \in A' \cap (A_\infty)^{**}$ such that* (*i*) $p_e \perp t.p_e, \quad t \neq e$

(*ii*) *For every $a \in A$ and every closed invariant projection q in the center of A^{**} with $aq \neq 0$ there exist $t \in G$ such that $aq(t.p_e) \neq 0$*

Then A separates the ideals in $A \rtimes_r G$

Proof. It is a general fact that, for every C*-dynamical system (B, G) with G discrete and $I \in \mathcal{I}(B)^G$, we have the canonical equivariant inclusions $B \subseteq B_\infty$ and $B \subseteq B^{**}$. Further one has the decomposition $B^{**} = (B/I)^{**} + I^{**}$ with $(B/I)^{**} \cong B^{**}q_B^I$, where q_B^I is the biggest central projection in B^{**} orthogonal to I (q_B^I is the orthogonal complement to the supporting open central projection of I). Hence q_B^I is a closed invariant projection in the center of B^{**}.

Fix $I \in \mathcal{I}(A)^G$ and set $B := A/I$. Using the identification $B_\infty = A_\infty/I_\infty$ we have the commuting diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & A^{**} \arrow[r, hook] \arrow[d, "\cdot q_A^I"] & (A_\infty)^{**} \arrow[d, "\cdot q_{A_\infty}^{I_\infty}"] \\ B \arrow[r, hook] & B^{**} \arrow[r, hook] & (B_\infty)^{**} \end{tikzcd}$$

Note that $q_A^I \leq q_{A_\infty}^{I_\infty}$ as elements in $(A_\infty)^{**}$. This follows from the fact that q_A^I is central in A^{**} (and hence also central in $(A_\infty)^{**}$) and orthogonal to I (and hence also orthogonal to I_∞).

Define $p_e^I := p_e q_{A_\infty}^{I_\infty} \in B' \cap (B_\infty)^{**}$. We now show that the action satisfies the residual Rokhlin* property by verifying

(i') $p_e^I \perp t.p_e^I, \quad t \neq e$

(ii') For every $b \in B$ with $b \neq 0$ there exist $t \in G$ such that $b(t.p_e^I) \neq 0$.

Property (i') follows easily from (i). To show (ii') fix $b \in B$ with $b \neq 0$. By (ii) and the left part of the diagram we have elements $a \in A$ and $t \in G$ such that $b = a + I$ and $aq_A^I(t.p_e) \neq 0$. From the right part of the diagram $b = aq_{A_\infty}^{I_\infty}$ in $(B_\infty)^{**}$. As $q_A^I \leq q_{A_\infty}^{I_\infty}$ we get that

$$\begin{aligned} b(t.p_e^I) &= a(t.p_e)q_{A_\infty}^{I_\infty} = 0 \text{ in } (B_\infty)^{**} \Rightarrow a(t.p_e)q_{A_\infty}^{I_\infty} = 0 \text{ in } (A_\infty)^{**} \\ &\Rightarrow aq_A^I(t.p_e) = (a(t.p_e)q_{A_\infty}^{I_\infty})q_A^I = 0 \text{ in } (A_\infty)^{**}, \end{aligned}$$

showing that $b(t.p_e^I) \neq 0$ and hence (ii'). □

Remark 3.4.10. The condition (ii) of Corollary 3.4.9 can be replaced with a weaker condition without changing the conclusion. It is sufficient to consider only the closed invariant projections in the center of A^{**} obtained as complement of support projections of invariant ideals in A.

3.5 Properly outer actions

We have shown in Theorem 3.3.7 and Theorem 3.4.7 that essential freeness and the residual Rokhlin*-property are sufficient to ensure that A separates the ideals in $A \rtimes_r G$ for a discrete exact group G acting on A.

We will now show a connection between essential freeness, residual Rokhlin*-property and proper outerness.

Lemma 3.5.1. *Let (A, G) be a C*-dynamical system with A abelian and G discrete. If φ_1, φ_2 are orthogonal states on A and $\epsilon > 0$ then there exist open sets $U_1, U_2 \subseteq X := \mathrm{Prim}(A) \cong \widehat{A}$ such that*

$$U_1 \cap U_2 = \emptyset, \quad 1 - \epsilon \le \mu_i(U_i),$$

where μ_i is the unique regular Borel measure on X corresponding to φ_i.

Proof. By [53, Definition 1.14.1] we have that $\|\varphi_1 - \varphi_2\| = 2$. Find a selfadjoint element $h \in A$ such that $\|h\| \le 1$ and $2 - \epsilon \le (\varphi_1 - \varphi_2)(h)$. With

$$h := h_+ - h_-, \quad h_+, h_- \ge 0, \quad h_+ h_- = 0$$

it follows that $1 - \epsilon \le \varphi_1(h_+) \le \sup_n \varphi_1(h_+^{1/n}) = \mu_1(\{x : h_+(x) > 0\})$. Similarly we get that $1 - \epsilon \le \mu_2(\{x : h_-(x) > 0\})$. □

Lemma 3.5.2. *Let (A, G) be a C*-dynamical system with A abelian, G discrete and let φ be a state on A. For every $t \in G$ such that φ is orthogonal to $t.\varphi$, every $\epsilon > 0$ and every open set $U \subseteq X := \mathrm{Prim}(A) \cong \widehat{A}$ there exists an open set $U' \subseteq U$ such that*

$$U' \cap t.U' = \emptyset, \quad \mu(U') \ge \mu(U) - \epsilon,$$

where μ is the unique regular Borel measure on X corresponding to φ.

Proof. Assume having $t \in G, \epsilon > 0$ and $U \subseteq X$ open with the property that φ is orthogonal to $t.\varphi$. Using Lemma 3.5.1 find open sets $U_1, U_2 \subseteq X$ such that

$$U_1 \cap U_2 = \emptyset, \quad 1 - \frac{\epsilon}{2} \le \mu_i(U_i),$$

where $\mu_1 := \mu$ and $\mu_2 := t.\mu = \mu(t^{-1}.(\,\cdot\,))$. Define

$$U'' := U_1 \cap t^{-1} U_2, \quad U' := U \cap U''.$$

Note that $U' \subseteq U$ is open and disjoint from $t.U'$. Using

$$\begin{aligned} \mu(U) &= \mu(U \cap U'') + \mu(U \cap U''^c) \le \mu(U') + \mu(U''^c) \\ \mu(U''^c) &= \mu((U_1 \cap t^{-1}.U_2)^c) \le \mu_1(U_1^c) + \mu_2(U_2^c) \le \epsilon \end{aligned}$$

one get the desired property $\mu(U) \le \mu(U') + \epsilon$. □

Theorem 3.5.3. *Let (A, G) be a C^*-dynamical system with A abelian and G countable. Then the following are equivalent*

(*i*) *The action of G on $\widehat{A}$ is topologically free.*

(*ii*) *There exists a projection $p_e \in A' \cap A^{**}$ $(= Z(A^{**}))$ such that*

$$\|a \sum_{t \in G} (t.p_e)\| = \|a\|, \quad p_e \perp s.p_e, \quad s \in G \setminus \{e\},\ a \in A.$$

In particular if the action is essentially free we obtain the residual Rokhlin property.*

Proof. (*i*) $\Rightarrow$ (*ii*): This direction works also in general for (A, G) with discrete G (and A not necessary abelian and G not necessary countable): Let $F \subseteq \widehat{A}$ denote the set of points in $\widehat{A}$ with trivial isotropy. By axiom of choice we can find a subset $H \subseteq F$ such that $\{t.H : t \in G\}$ is a partition of F. Recall that each von Neumann algebra canonically splits in the direct sum $N_{discrete} \oplus N_{continuous}$ by the maximal central projection C_d such that $N_{discrete} := NC_d$ is a discrete Type I von Neumann algebra. In particular, for $N := A^{**}$, the minimal projections in the center $Z(N_{discrete})$ of $N_{discrete}$ corresponds to the unitary equivalence classes of irreducible representation of A. In this way we get G-equivariant bijection between $\widehat{A}$ and the set of minimal projections in $Z(N_{discrete})$, i.e. the center of $A^{**}_{discrete}$ is naturally isomorphic to $l_\infty(\widehat{A}) = c_0(\widehat{A})^{**}$, the bounded functions on $\widehat{A}$ equipped with the discrete topology. In particular, this gives a one-to one correspondence between subsets $S \subseteq \widehat{A}$ of $\widehat{A}$ and projections q_S in the center of A^{**}, such that $t.q_S = q_{t.S}$ for $t \in G$. The projection

$$p_e := q_H$$

in the discrete part of A^{**} satisfies the desired conditions giving (*ii*), because the projection $\bigvee_{t \in G} t.q_H$ is the same as the projection q_F corresponding to $F \subseteq \widehat{A}$, and $\|aq_F\| = \|a\|$ for all $a \in A$ (because the irreducible representations in F are separating for A).

(*ii*) $\Rightarrow$ (*i*): Identify $A = C_0(X)$ for $X := \mathrm{Prim}(A) \cong \widehat{A}$ and let $F \subseteq X$ be the subset of points in X with trivial isotropy, i.e. elements that are fixed only by $e \in G$.

Set $U := X \setminus \bar{F}$ and $p_e^U := p_e q$, where $q \in C_0(X)^{**}$ is the open invariant projection corresponding to the invariant ideal $C_0(U)$ in $C_0(X)$ such that

$$C_0(U)^{**} \cong C_0(X)^{**} q, \quad aq = a, \quad a \in C_0(U).$$

Note that $U = \emptyset$ implies (i). Assuming $U \neq \emptyset$ we show U contains elements with trivial isotropy, giving a contradiction. First we show $p_e^U \neq 0$. Using that A is weakly dense in A^{**} find an increasing net (a_i) of positive norm one elements in $C_0(U)$ weakly converging to $q \in A^{**}$. With $p := \sum_{t \in G} t.p_e$ it follows from (ii) that

$$1 = \|a_i\| = \|a_i p\| = \|a_i q p\|.$$

Since $p_e^U = 0 \Rightarrow qp = 0 \Rightarrow a_i qp = 0$, we get that $p_e^U \neq 0$.

Find a normal state φ^{**} on $C_0(U)^{**}$ with $\varphi^{**}(p_e^U) = 1$ and let φ be the restriction to $C_0(U)$. Now let μ be the regular Borel measure on U corresponding to φ. Fix $\{t_1, t_2, \dots\} := G \setminus \{e\}$ and $\epsilon_n = 2^{-(n+1)}, n \in \mathbb{N}$. We now show, using induction, the existence of open sets $U_n \subseteq \dots \subseteq U_1 \subseteq U$ fulfilling that

$$U_n \cap t_j.U_n = \emptyset,\ j = 1, \dots, n \quad \mu(U_n) \geq 1 - (\epsilon_1 + \dots + \epsilon_n).$$

For any $t \neq e$ the state φ is orthogonal to $t.\varphi$ (since $\varphi^{**}(p_e^U) = 1$ the projection p_e^U is larger that the support projection of φ and as $p_e^U \perp t.p_e^U$ the support projections of φ and $t.\varphi$ are orthogonal).

For $n = 1$ we note that φ is orthogonal to $t_1.\varphi$. Using Lemma 3.5.2 on $t_1 \in G$, $\epsilon_1 > 0$ and $U \subseteq X$ there is an open set $U_1 \subseteq U$ such that

$$U_1 \cap t_1.U_1 = \emptyset, \quad \mu(U_1) \geq \mu(U) - \epsilon_1 = 1 - \epsilon_1.$$

For $n \geq 1$ we assume having the desired sets $U_1, \dots, U_n$. Using Lemma 3.5.2 on $t_{n+1} \in G$ and $\epsilon_{n+1} > 0$ and $U_n \subseteq X$ one gets an open set $U_{n+1} \subseteq U_n$ with

$$U_{n+1} \cap t_{n+1}.U_{n+1} = \emptyset, \quad \mu(U_{n+1}) \geq \mu(U_n) - \epsilon_{n+1}.$$

This implies the desired properties for U_{n+1}. As $\mu(\cap U_n) = \lim_n \mu(U_n) > 0$ the set $\cap U_n$ is a non-empty subset in U consisting of elements with trivial isotropy. $\square$

Remark 3.5.4. The property (ii) in Theorem 3.5.3 is different from either the Rokhlin or Rokhlin* statements. However it is clear that property (ii) in Theorem 3.5.3 implies the Rokhlin* property.

Remark 3.5.5. We have the following more general result:
If (A, G) is a C^*-dynamical system with G discrete and if the action of G on $\widehat{A}$ is essentially free, then (A, G) satisfy the residual Rokhlin property.* See the proof of the direction $(i) \Rightarrow (ii)$ in the proof of Theorem 3.5.3.

As a corollary we get a particularly nice reformulation of when A separates the ideals in $A \rtimes_r G$ in the case where the action is amenable. Recall that for a transformation group (X, G) with X compact and G discrete the action is called *amenable* if there exist a net of continuous maps $m_i \colon X \to \mathrm{Prob}(G)$ such that for each $s \in G$,

$$\lim_{i\to\infty} \Big(\sup_{x\in X} \|s.m_i^x - m_i^{s.x}\|_1 \Big) = 0,$$

where $s.m_i^x(t) = m_i^x(s^{-1}t)$.

Corollary 3.5.6. *Let (A, G) be a C*-dynamical system with A unital, abelian and G countable, discrete. Suppose the action is amenable. Then A separates the ideals in $A \rtimes_r G$ if and only if the action satisfy the residual Rokhlin* property.*

Proof. Assume A separates the ideals in $A \rtimes_r G$. By Theorem 3.2.9 the action of G of A satisfies the residual intersection property. By Lemma 3.2.5 this can be reformulated in the following way. For every $I \in \mathcal{I}(A)^G$ we have that

$$\forall J \in \mathcal{I}(A/I \rtimes G) : \pi^{-1}(\pi(J)) = J, J \cap A = 0 \Rightarrow \pi(J) = 0,$$

where π is the canonical surjection $A/I \rtimes G \to A/I \rtimes_r G$. By Proposition 2.2.4 π is an isomorphism. Using [3, Theorem 2] the action of G on A/I is topologically free for every $I \in \mathcal{I}(A)^G$. Since G is countable, we obtain from Remark 3.3.4 that the action of G on $\widehat{A}$ is essentially free. Using Theorem 3.5.3 we get the residual Rokhlin* property.

Conversely assume that the action satisfy the residual Rokhlin* property. An amenable action is automatically exact, cf. Proposition 2.2.4. Hence A separates the ideals in $A \rtimes_r G$ by Theorem 3.4.7. □

Let (A, G) be a C*-dynamical system with G discrete. Recall that the action, say α, of G on A is properly outer if α_t is properly outer for every $t \neq e$, cf. [40]. Let $\mathcal{L}(A, A)$ be the set of linear and bounded maps from A into A. We have a natural *isometric and linear* map from $\mathcal{L}(A, A)$ to $\mathcal{L}(A_\infty, A_\infty)$ given by

$$\big(a \mapsto Ta\big) \to \big((a_1, a_2, \cdots) + c_0(A) \mapsto (Ta_1, Ta_2, \cdots) + c_0(A)\big)$$

In particular, we get a natural unital monomorphism $\psi \colon \mathcal{M}(A) \to \mathcal{M}(A_\infty)$. The $\sigma((A_\infty)^{**}, (A_\infty)^*)$-closure of A in $(A_\infty)^{**}$ is naturally W*-isomorphic to A^{**} via a natural (not necessarily unital) monomorphism $\kappa \colon A^{**} \to (A_\infty)^{**}$.

For a unitary $u \in \mathcal{M}(A)$ we let $\mathrm{Ad}(u)_\infty \colon A_\infty \to A_\infty$ denote the natural automorphism $(a_1, a_2, \cdots) + c_0(A) \mapsto (ua_1u^*, ua_2u^*, \cdots) + c_0(A)$. Note that

$\mathrm{Ad}(u)_\infty$ is given by the inner automorphism $\mathrm{Ad}(U)\colon A_\infty \to A_\infty, b \mapsto UbU^*$ for the unitary $U := \psi(u)$. We identify $\mathcal{M}(A_\infty)$ naturally with its image in $(A_\infty)^{**}$ inducing $\mathrm{Ad}(U)^{**}\colon (A_\infty)^{**} \to (A_\infty)^{**}, b \mapsto UbU^*$.

Lemma 3.5.7. *Fix a unitary $u \in \mathcal{M}(A)$ and an automorphism α of A. With $q := \kappa(1_{A^{**}})$, $U := \psi(u)$ and $\beta := \alpha_\infty$ we have the following identities*

$$\beta^{**}(q) = q \quad \text{and} \quad Ad(U)^{**}(b) = b \text{ for all } b \in (A' \cap (A_\infty)^{**})q.$$

Proof. The proof is a just an application of the two diagrams below.

$$\begin{tikzcd} A^{**} \arrow[r, "\alpha^{**}"] \arrow[d, "\kappa"'] & A^{**} \arrow[d, "\kappa"'] \\ (A_\infty)^{**} \arrow[r, "\beta^{**}"] & (A_\infty)^{**} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}(A) \arrow[r] \arrow[d, "\psi"'] & A^{**} \arrow[d, "\kappa"'] \\ \mathcal{M}(A_\infty) \arrow[r, "\cdot q"] & (A_\infty)^{**} \end{tikzcd}$$

For an element $b \in A' \cap (A_\infty)^{**} = \kappa(A^{**})' \cap (A_\infty)^{**}$ we obtain that

$$\mathrm{Ad}(U)^{**}(bq) = (Uq)b(Uq)^* = \kappa(u)b\kappa(u^*) = bq.$$

□

Lemma 3.5.8. *Let (A, G) be a C^*-dynamical system with G discrete. Assume the action of G on A has the Rokhlin* property. Then for every subgroup H in G and every H-invariant ideal I in A the restricted action of H on I has the Rokhlin* property.*

Proof. Using the equivalence relation $s \sim_H t \Leftrightarrow \exists h \in H\colon t = hs$ on G let F be a subset of G with one element from each equivalent class. Further let p be the supporting open central projection of I. Using the strong convergent sum $q_e := \kappa(p) \sum_{t \in F} p_t$ in $I' \cap (I_\infty)^{**}$ it follows that the action of H on I has the Rokhlin* property. □

Theorem 3.5.9. *Let (A, G) be a C^*-dynamical system with G discrete. If the action has the Rokhlin* property then it is automatically properly outer.*

Proof. Assume the action α of G on A is not properly outer. Find $t \neq e$, an $\langle \alpha_t \rangle$-invariant ideal I in A and $u \in \mathcal{M}(I)$ such that

$$\|\alpha_t|_I - \mathrm{Ad}(u)\| < 2.$$

By Lemma 3.5.8 it is sufficient to show that the action of $\langle \alpha_t \rangle$ on I does not have the Rokhlin* property. Hence we can assume $G = \langle \alpha_t \rangle$ and $A = I$ and show that the action of G on A does not have the Rokhlin* property. With $\alpha := \alpha_t$ we have an automorphism α of A and a unitary $u \in \mathcal{M}(A)$ such that

$$\|\alpha - \mathrm{Ad}(u)\| < 2.$$

Define $q := \kappa(1_{A^{**}})$, $U := \psi(u)$ and $\beta := \alpha_\infty$ as in Lemma 3.5.7. The linear map $\beta^{**} - \mathrm{Ad}(U)^{**}$ is equal to the natural extension $(\alpha - \mathrm{Ad}(u))^{**}_\infty$ of $\alpha - \mathrm{Ad}(u) \in \mathcal{L}(A, A)$ to a bounded linear operator on $(A_\infty)^{**}$. In particular, we get $\|\beta^{**} - \mathrm{Ad}(U)^{**}\| < 2$.

Assume there exist a projection $p_e \in A' \cap (A_\infty)^{**}$ such that $p_e \perp \beta^{**}(p_e)$. With $p := p_e q$ it follows from Lemma 3.5.7 that

$$\mathrm{Ad}(U)^{**}(q) = q = \beta^{**}(q), \quad \mathrm{Ad}(U)^{**}(p) = p.$$

Since $pq = qp = p$ we have that $(2p - q)^*(2p - q) = q$ and hence

$$2\|\beta^{**}(p) - p\| = \|\beta^{**}(2p - q) - (2p - q)\| < 2.$$

Note that $p \perp \beta^{**}(p)$. This implies that $\|\beta^{**}(p) - p\|^2 = \|\beta^{**}(p) + p\| \in \{0, 1\}$. It follows that $p = 0$. By definition of q and Lemma 3.5.7 we obtain

$$at.p_e = t.(p_e t^{-1}.a) = t.(p_e t^{-1}.(qa)) = t.(p_e q t^{-1}.a) = 0, \quad a \in A, t \in G.$$

The action of G on A does not have the Rokhlin* property. □

Corollary 3.5.10. *Let (A, G) be a C*-dynamical system with G discrete. The properties*

(i) The action of G on $\widehat{A}$ is topologically free.

(ii) The action of G on A has the Rokhlin property.*

(iii) The action of G on A is properly outer.

fulfills the implications (i) ⇒ (ii) ⇒ (iii). In addition if A is abelian and G countable then we obtain the implication (iii) ⇒ (i) making all the conditions equivalent.

Proof. (*i*) ⇒ (*ii*). We refer to the proof of Theorem 3.5.3 ((*i*) ⇒ (*ii*)).

(*ii*) ⇒ (*iii*). See Theorem 3.5.9.

(*iii*) ⇒ (*i*). Suppose a countable discrete group G acts by α on an abelian C*-algebra A. Set $X := \widehat{A}$ and $U_t := \{x \in X : t.x \neq x\}$. If the action on X is not topologically free there exist $t \neq e$ such that U_t is not dense in X (using that G is countable). Hence there exist an open non-empty subset V in U_t^c. Note that $I := C_0(V)$ is in a natural way a α_t-invariant ideal in A fulfilling that $t.x = x$ for all $x \in V$. In particular $\|\alpha_t|_I - id_I\| = 0$. Hence the action is not properly outer. □

Remark 3.5.11. We know from the paper of Archbold and Spielberg [3] that topological freeness of G on $\widehat{A}$ implies the action is properly outer, which is also contained in Corollary 3.5.10.

If follows from the paper of Olesen and Pedersen [40, Lemma 7.1] (or of Kishimoto [34]), that for a properly outer action on a *separable* C*-algebra A we obtain that for $b \in (A \rtimes G)^+$ and $\varepsilon > 0$ there exist $x \in A^+$ with $\|x\| = 1$, $\|xbx - xE(b)x\| < \varepsilon$ and $\|xE(b)x\| > \|E(b)\| - \varepsilon$.

An inspection of the proof of their Theorem 7.2 gives that the latter observation implies that any closed ideal J of $A \rtimes G$ with $J \cap A = \{0\}$ must be contained in the kernel of $\pi \colon A \rtimes G \to A \rtimes_r G$ (in particular the intersection property holds).

We can apply this in a similar way to all quotients, and get the following result: *Let G be a discrete group acting by α on a separable C*-algebra A. If $[\alpha_t]_I : A/I \to A/I$ is properly outer for every G-invariant closed ideal $I \neq A$ and every $t \in G \setminus \{e\}$, then we obtain the residual intersection property. Thus, if in addition the action of G on A is exact, then A separates the ideals in $A \rtimes_r G$, by Theorem 3.2.9.*

Chapter 4

The non-simple case

This chapter contains the work on non-simple purely infinite crossed products. Using the work on ideal structure of reduced crossed products we are able to extend some of the previous the work on locally contracting actions and n-filling actions introduced in [36] and [27].

4.1 Cuntz relation and conditional expectation

Let (A, G) be a C*-dynamical system with G discrete. Recall that the action is exact if every invariant ideal in A induces a short exact sequence at the level of reduced crossed products, cf. Definition 2.2.3, and that A separates the ideals in $A \rtimes_r G$ if the map $J \mapsto J \cap A, J \in \mathcal{I}(A \rtimes_r G)$ is injective, cf. Definition 3.2.4.

Proposition 4.1.1. *Let (A, G) be a C*-dynamical system with G discrete. Suppose that A separates the ideals in $A \rtimes_r G$. Then $A \rtimes_r G$ is purely infinite if and only if the non-zero elements in A^+ are properly infinite (viewed as elements in $A \rtimes_r G$) and $E(a) \precsim a$ for every $a \in (A \rtimes_r G)^+$.*

Proof. Suppose that $A \rtimes_r G$ is purely infinite. Then every non-zero element in A^+ is properly infinite cf. [31, Theorem 4.16]. Fix now a non-zero element $a \in (A \rtimes_r G)^+$. By Theorem 3.2.9 we have that $E(a) = \text{Ideal}[a]$. Using A is purely infinte we obtain that $E(a) \precsim a$.

Now suppose that every non-zero elements in A^+ is properly infinite and $E(a) \precsim a$ for every $a \in (A \rtimes_r G)^+$. Since the action of G on A is exact it follows that $a \in \text{Ideal}[E(a)]$ for $a \in (A \rtimes_r G)^+$, cf. Proposition 3.2.1. Using [31, Proposition 3.5 (ii)] we obtain that $a \precsim E(a)$ for $a \in (A \rtimes_r G)^+$. By the inequalities

$$a \oplus a \precsim E(a) \oplus E(a) \precsim E(a) \precsim a$$

in $A \rtimes_r G$, we conclude that $A \rtimes_r G$ is purely infinite, cf. [31, Theorem 4.16]. $\square$

Remark 4.1.2. If follows from the proof of Proposition 4.1.1 that when the group G is exact and every non-zero element in A^+ is properly infinite (with respect to $A \rtimes_r G$) then $A \rtimes_r G$ is purely infinite provided that

$$E(a) \precsim a, \quad a \in (A \rtimes_r G)^+.$$

This last condition is automatic in many cases of interest (cf. Theorem 4.2.4).

Remark 4.1.3. In the proof of Proposition 4.1.1 exactness of the action of G on A is used to obtain the property that

$$a \precsim E(a), \quad a \in (A \rtimes_r G)^+.$$

It is unclear when this property is valid. A partial answer is presented below:

Proposition 4.1.4. *Let (A, G) be a C*-dynamical system with G discrete. For every $a \in C_c(G, A)$ the hereditary sub-C*-algebra in $A \rtimes_r G$ generated by $E(a^*a)$ contains a^*a. In particular $a^*a \precsim E(a^*a)$.*

Proof. Fix an element $a = \sum_{t \in G} a_t u_t$ in $C_c(G, A)$. We have the identities

$$aa^* = \sum_{t,s \in G} (a_t u_t)(a_s u_s)^*, \quad E(aa^*) = \sum_{t \in G} a_t a_t^*.$$

Let B be the hereditary sub-C*-algebra in $A \rtimes_r G$ generated by $E(aa^*)$ and let (e_λ) be an approximative identity for B. Using that $0 \le a_t a_t^* \le E(aa^*) \in B$ for $t \in G$ we have that $a_t a_t^* \in B$ and hence we have the convergence $e_\lambda a_t \to a_t$ (in norm) for $t \in G$.

Fix $t, s \in F$. Form the inequality

$$\|e_\lambda (a_t u_t)(a_s u_s)^* e_\lambda - (a_t u_t)(a_s u_s)^*\| \le \|a_t\| \|e_\lambda a_s - a_s\| + \|a_s\| \|e_\lambda a_t - a_t\|,$$

we obtain that $(a_t u_t)(a_s u_s)^* \in B$. Hence $aa^* \in B$. From [31, Proposition 2.7 (ii)] we conclude $aa^* \precsim E(aa^*)$. $\square$

4.2 Sufficient conditions

A C*-algebra A has *the ideal property*, abbreviated (IP), if projections in A separate ideals in A, i.e., whenever I, J are ideals in A such that $I \nsubseteq J$, then there is a projection in $I \setminus (I \cap J)$, cf. [41, Remark 2.1].

Let (A, G) be a C*-dynamical system with G discrete, exact such that the action of G on $\widehat{A}$ is essentially free. As we have seen in Theorem 3.3.7 these properties imply that A separates the ideals in $A \rtimes_r G$. If A furthermore is separable and has property (IP) we will show that $A \rtimes_r G$ is purely infinite if and only if the non-zero elements in A^+ are properly infinite viewed as elements in $A \rtimes_r G$.

Lemma 4.2.1. *Let (A, G) be a C*-dynamical system with G discrete. Assume that A separates ideals in $A \rtimes_r G$ and that A has property (IP). Then the crossed product $A \rtimes_r G$ has property (IP).*

Proof. Let K, L be ideals in $A \rtimes_r G$ such that $K \nsubseteq L$. We have that $K = I \rtimes_r G$ and $L = J \rtimes_r G$ for $I := K \cap A$, $J := L \cap A$, cf. Proposition 3.1.2. Note that $I \nsubseteq J$ (since $I \subseteq J \Rightarrow K = \mathrm{Ideal}[I] \subseteq \mathrm{Ideal}[J] = L$). By property (IP) there is a projection p in $I \setminus (I \cap J)$. We have that $p \in I \subseteq K$ but also $p \notin L$ (since $p \in L \Rightarrow p \in L \cap A = J$). Therefore $p \in K \setminus (K \cap L)$. $\square$

Lemma 4.2.2. *Let (A, G) be a C*-dynamical system with G discrete. Assume that every non-zero element in A^+ is properly infinite with respect to $A \rtimes_r G$ and that A has property (IP). Then for every $I \in \mathcal{I}(A)^G$ and every non-zero $a \in (A/I)^+$ there exists a projection $q \in A$ such that $q + I \neq I$ and $q + I \precsim a$ with respect to $A/I \rtimes_r G$.*

Proof. Fix an arbitrary $I \in \mathcal{I}(A)^G$ and write $a = f + I \in (A/I)^+$ for some $f \in A^+$. With $J = \mathrm{Ideal}_A[f]$ we have that $J \nsubseteq I$. As A has property (IP) there is a projection $q \in A$ such that $q \in J \setminus (J \cap I)$. We obtain that $q + I \neq I$.

Using that $f \in A^+$ is a properly infinite positive element (with respect to $A \rtimes_r G$) and $q \in J \subseteq \mathrm{Ideal}_{A \rtimes_r G}[f]$ it follows that $q \precsim f$ with respect to $A \rtimes_r G$. This relation is preserved under the canonical homomorphisms $\rho \rtimes_r \mathrm{id}\colon A \rtimes_r G \to A/I \rtimes_r G$ implying the relation $q + I \precsim a$ with respect to $A/I \rtimes_r G$, cf. [31, p. 643]. $\square$

Lemma 4.2.3. *Let (A, G) be a C*-dynamical system with A separable and G discrete. If the action of G on $\widehat{A}$ is essentially free, then for every $I \in \mathcal{I}(A)^G$ and every non-zero $b \in (A/I \rtimes_r G)^+$ there exists a non-zero element $a \in (A/I)^+$ such that $a \precsim b$.*

Proof. Fix an arbitrary $I \in \mathcal{I}(A)^G, I \neq A$ and non-zero $b \in (A/I \rtimes_r G)^+$. By Remark 3.5.11 essential freeness of the action on $\widehat{A}$ implies that the induced action of G on A/I is *properly outer*. Define $c = b/\|E(b)\|$ and $\epsilon = 1/4$. By Remark 3.5.11 (using that A is separable) there exist $x \in (A/I)^+$ satisfying that

$$\|x\| = 1, \quad \|xE(c)x - xcx\| \leq \epsilon, \quad \|xE(c)x\| \geq \|E(c)\| - \epsilon. \tag{4.2.1}$$

With $a := f_{2\epsilon}(xE(c)x) = (xE(c)x - 2\epsilon)_+$ we claim that $0 \neq a \precsim xcx$. The element a is non-zero because $\|xE(c)x\| > 2\epsilon$ and $a \precsim xcx$ since $\|xE(c)x - xcx\| < 2\epsilon$, cf. [49, Proposition 2.2]. As $c \sim b$ we obtain that $xcx \sim xbx \precsim b$ with respect to $A/I \rtimes_r G$ implying that $a \precsim b$. $\square$

Theorem 4.2.4. *Let (A, G) be a C*-dynamical system G discrete and exact. Suppose that the action of G on $\widehat{A}$ is essentially free and A is separable and has the property (IP). Then the following statements are equivalent*

(i) The non-zero elements in A^+ are properly infinite with respect to the crossed product $A \rtimes_r G$.

(ii) The C-algebra $A \rtimes_r G$ is purely infinite.*

(iii) Every non-zero hereditary sub-C-algebra in any quotient of $A \rtimes_r G$ contains an infinite projection.*

Proof. $(ii) \Leftrightarrow (iii)$. By Lemma 4.2.1 the reduced crossed product $A \rtimes_r G$ has property (IP). The equivalence is obtained by [41, Proposition 2.1].

$(ii) \Rightarrow (i)$. Proposition 4.1.1 (or [31, Theorem 4.16]).

$(i) \Rightarrow (iii)$. Fix a non-zero hereditary sub-C*-algebra B in the quotient of $A \rtimes_r G$ by some ideal J in $A \rtimes_r G$. We will show that B contains an infinite projection.

By essential freeness of the action on $\widehat{A}$ and exactness of G we have the identification

$$(A \rtimes_r G)/J = (A/I) \rtimes_r G,$$

for $I := J \cap A$, cf. Proposition 3.3.7. Fix now a non-zero positive element b in B. By Lemma 4.2.3 there exist a non-zero element $a \in (A/I)^+$ such that $a \precsim b$ with respect to $A/I \rtimes_r G$. From Lemma 4.2.2 we can find a projection $q \in A$ such that $q + I \neq I$ and $q + I \precsim a$ with respect to $A/I \rtimes_r G$. By the transitivity of $\precsim$ if follows that $q + I \precsim b$ with respect to $A/I \rtimes_r G$. From the comment after [31, Proposition 2.6] we can find $z \in A/I \rtimes_r G$ such that $q + I = z^*bz$. With $v = b^{1/2}z \in A/I \rtimes_r G$ it follows that

$$v^*v = z^*bz = q + I, \quad p := vv^* = b^{1/2}zz^*b^{1/2} \in B$$

Now q is non-zero and properly infinite (in $A \rtimes_r G$), so q and hence p, are infinite (infiniteness of projections is preserved under von Neumann equivalence). $\square$

4.3 Actions on the Cantor set

Recall that a C*-algebra A has *real rank zero* provided that every self-adjoint element can be approximated arbitrarily well by a self-adjoint element with finite spectrum, cf. [8]. For an abelian C*-algebra $A = C_0(X)$ the following properties are equivalent:

(i) The space X has a basis of clopen sets.

(ii) The C*-algebra A has property (IP).

(iii) The C*-algebra A has real rank zero.

We immediately obtain

Lemma 4.3.1. *Let (A, G) be a C*-dynamical system with $A = C_0(X)$ abelian and G discrete. Assume that X has a basis of clopen sets. Then for every $I \in \mathcal{I}(A)^G$ and every non-zero $a \in (A/I)^+$ there exist a projection $q \in A$ such that $q + I \neq I$ and $q + I \precsim a$ with respect to $A/I \rtimes_r G$.*

Proof. By [8, Theorem 3.14] A has real rank zero if and only if I and A/I has real rank zero and every projection in A/I lift to a projection in A. $\square$

In the abelian case the result in Lemma 4.2.3 can be slightly strengthen.

Lemma 4.3.2. *Let (A, G) be a C*-dynamical system with $A = C_0(X)$ abelian and G discrete. If the action of G on X is essentially free, then for every $I \in \mathcal{I}(A)^G$ and every non-zero $b \in (A/I \rtimes_r G)^+$ there exists a non-zero element $a \in (A/I)^+$ such that $a \precsim b$.*

Proof. Follow the proof of Lemma 4.2.3. The separability requirement in the proof can be omitted by an application of [19, Proposition 2.4] instead of the use of Remark 3.5.11. $\square$

Theorem 4.3.3. *Let (A, G) be a C*-dynamical system with $A = C_0(X)$ and with G discrete and exact. Suppose that the action of G on X is essentially free and that X has a basis of clopen sets. Then the following statements are equivalent*

(i) Every non-zero projections in A/I is infinite (with respect to $A/I \rtimes_r G$) for every invariant ideal I in A.

(ii) Every non-zero projections in A is properly infinite (with respect to $A \rtimes_r G$).

(iii) The C-algebra $A \rtimes_r G$ is purely infinite.*

Proof. $(iii) \Rightarrow (ii)$. Every non-zero projection in any purely infinite C*-algebra is properly infinite.

$(ii) \Rightarrow (i)$. Use real rank zero of A to lift a projection in A/I to a projection in A, cf. [8, Theorem 3.14].

$(i) \Rightarrow (iii)$. Fix a non-zero hereditary sub-C*-algebra B in the quotient of $A \rtimes_r G$ by some ideal J in $A \rtimes_r G$. We remind that the implication $(iii) \Rightarrow (ii)$ of Theorem 4.2.4 is valid for any C*-algebra, cf. [31, Proposition 4.7]. Hence we only need to show that B contains an infinite projection.

By essential freeness of the action on X and exactness of G we have the identification

$$(A \rtimes_r G)/J = (A/I) \rtimes_r G,$$

for $I := J \cap A$, cf. Theorem 3.3.7. Fix now a non-zero positive element b in B. By Lemma 4.3.2 there exist a non-zero element $a \in (A/I)^+$ such that $a \precsim b$ with respect to $A/I \rtimes_r G$. From Lemma 4.3.1 we can find a projection $q \in A$ such that $q + I \neq I$ and $q + I \precsim a$ with respect to A/I. By the transitivity of $\precsim$ if follows that $q + I \precsim b$ with respect to $A/I \rtimes_r G$. From the comment after [31, Proposition 2.6] we can find $z \in A/I \rtimes_r G$ such that $q + I = z^*bz$. With $v = b^{1/2}z \in A/I \rtimes_r G$ it follows that

$$v^*v = z^*bz = q + I, \quad p := vv^* = b^{1/2}zz^*b^{1/2} \in B$$

We now only need to show that the projection $p \in B$ is infinite. As $q + I \neq I$ we have by assumption (i) that $q + I \in A/I$ is an infinite with respect to $A/I \rtimes_r G$. Since $p \sim q + I$ we conclude that p is infinite.

□

Corollary 4.3.4. *Let (A, G) be a C*-dynamical system with $A = C(X)$ abelian unital and with G discrete and exact. Suppose that the action of G on X is essentially free and X has a basis of clopen τ_X-paradoxical sets. Then $C(X) \rtimes_r G$ is purely infinite.*

Remark 4.3.5. Let (X, G) be a transformation group with G countable (exact)[1] and X the Cantor set. Then the conditions in [27, Theorem 1.2] automatically imply that the actions of G on X is essentially free and that X has a basis of clopen τ_X-paradoxical sets, cf. Remark 2.6.1. The main difference

[1]A small change of the formulation of Corollary 4.3.4 would allow us to remove the requirement "exact" in this paragraph.

is that the requirements in [27, Theorem 1.2] forces the crossed product to be simple. This is not the case in Corollary 4.3.4.

We conclude that for this special family of transformation groups (mentioned in the privies sentence) our work extends the result of Jolissaint and Robertson to non-simple reduced crossed products.

Remark 4.3.6. Let (X, G) be a transformation group with G discrete and X the Cantor set. Then the conditions in Proposition 2.6.3 automatically imply that for every closed invariant subset Y of X and every Y-clopen subset U of Y there exist Y-open subsets $U_1, U_2 \ldots, U_n \subseteq U$ and elements $t_1, t_2, \ldots, t_n \in G$ such that[2]

$$\bigcup_{i=1}^{n} U_i = U, \quad \bigsqcup_{i=1}^{n} t_i.U_i \subsetneq U, \quad t_k.U_k \cap t_l.U_l = \emptyset \ (k \neq l)$$

By Theorem 4.3.3 we obtain that $C(X) \rtimes_r G$ is purely infinite. We leave this as an exercise. Hence Theorem 4.3.3 generalizes Proposition 2.6.3 (that consider local boundary actions) for actions on the Cantor set.

Remark 4.3.7. Let (X, G) be a transformation group with G countable and X the Cantor set such that all non-zero projections in $C(X)$ are properly infinite viewed as elements in $C(X) \rtimes G$. One might expect that

$$C(X) \rtimes G \text{ simple } \Rightarrow C(X) \rtimes G \text{ purely infinite.}$$

This is indeed true. Using Corollary 4.3.4 (together with Proposition 2.2.4, [3, Theorem 2] and Remark 3.3.4) the implication above can be shown by verifying that

$$\text{exactness, regularity, } C(X) \text{ separates ideals } \Rightarrow C(X) \rtimes G \text{ purely infinite.}$$

We remind that exactness and regularity of the action of G on $C(X)$ and the property that $C(X)$ separates the ideals in $C(X) \rtimes G$ are all necessary conditions to obtain simplicity of $C(X) \rtimes G$, cf. Corollary 2.1.3

4.4 Necessary conditions

It seems a difficult task i general to find properties of a dynamical system necessary to ensure a crossed product is purely infinite. Here we present one necessary conditions valid for any transformation group (G, X) with G discrete.

[2]We use $\bigsqcup$ to emphasize the union is disjoint.

Proposition 4.4.1. *Let (G,X) be a transformation group with G discrete. If $C_0(X)\rtimes_r G$ is purely infinite then for every non-empty clopen subset U of X there exist $t\neq e$ in G such that $U\cap t.U\neq\emptyset$.*

Proof. Fix $p:=1_U$ and $A:=C_0(X)$. Since $A\rtimes_r G$ is purely infinite if follows by [31, Theorem 4.16] that p is properly infinite and hence (cf. Lemma 2.3.7) we can find $x,y\in A\rtimes_r G$ such that

$$p=x^*x=y^*y,\quad xx^*\perp yy^*\leq p.$$

Find sequences $(x_n),(y_n)$ in $C_c(G,A)$ such that

$$x_n=px_np,\quad \|x_n^*x_n-p\|\leq 3/n,\quad \|x_n\|\leq 2,\quad \|x_n^*y_n\|\leq 3/n,$$

$$y_n=py_np,\quad \|y_n^*y_n-p\|\leq 3/n,\quad \|y_n\|\leq 2.$$

Simply find $x_n'\in C_c(G,A)$ such that $\|x_n'-x\|<1/n$, then set $x_n=px_n'p$. Similar method can be used to find y_n. Neither (x_n) or (y_n) can be a normal sequence. We can therefore assume that

$$x_n=\sum_{t\in F_n}f_{t,n}u_t\in C_c(G,A),\quad |F_n|>1,$$

for suitable non-zero functions $f_{t,n}\in A$. Since $x_n=px_np$ if follows that

$$\sum_{t\in F_n}(f_{t,n}-1_Uf_{t,n}1_{t.U})u_t=0.$$

We obtain that $0\neq f_{t,n}=1_{t.U\cap U}f_{t,n}$ and hence that $t.U\cap U\neq\emptyset$ for any $t\in F_n, n\in\mathbb{N}$. ∎

Remark 4.4.2. By Proposition 4.4.1 we have that there are no non-zero projection $p\in C_0(X)$ such that

$$t.p\perp p,\quad t\neq e,$$

for any purely infinite crossed product $C_0(X)\rtimes_r G$ with G discrete.

Chapter 5

Traceless C*-algebras

This chapter presents our work on the relation between traceless and purely infinite crossed products. The main result is an application of the previews results on non-simple purely infinite crossed products.

5.1 The monoid $S(X, G, \mathbb{E})$

A *semigroup* is a non-empty set S together with a commutative operation $+$ which satisfies closure and associative law. A semigroup with an *identity element*, i.e an element $0 \in S$ such that $0 + x = x + 0 = x$ for all $x \in S$ is called a *monoid*. A monoid $(S, \leq)$ endowed with a (reflexive and transitive) preorder $\leq$ is called a *preordered* monoid.

Let $(S, \leq)$ be a *preordered monoid*. A *state* f on S is a map $f\colon S \to [0, \infty]$ which respects $+$ and $\leq$ and fulfills that $f(0) = 0$. A state is called *non-trivial* if it takes a value different from 0 and ∞. The monoid S is *purely infinite* if $2x \leq x$ for every $x \in S$. The monoid S is *almost unperforated* if, whenever $x, y \in S$ and $n, m \in \mathbb{N}$ are such that $nx \leq my$ and $n > m$, one has that $x \leq y$.

Let (G, X) be a transformation group with G discrete. By a *subalgebra* we mean a collection of subsets of X containing $\emptyset, X$ and closed under finite union, intersection and complement. An *invariant* subalgebra $\mathbb{E} \subseteq \mathcal{P}(X)$ further fulfills that $t.E \in \mathbb{E}$ for every $E \in \mathbb{E}$ and $t \in G$. Let $\mathbb{E}$ be a invariant subalgebra of $\mathcal{P}(X)$. We write $S(X, G, \mathbb{E})$ for *type semigroup* of the induced action of G on $\mathbb{E}$, where only pieces in $\mathbb{E}$ can by used to witness the equidecomposability of sets in $\mathbb{E}$, cf. [60, p.116]. More precisely $S(X, G, \mathbb{E})$

is the set ([1])

$$\left\{\bigsqcup_{i=1}^{n} A_i \times \{i\} \colon A_i \in \mathbb{E}, n \in \mathbb{N}\right\} / \sim_S,$$

where the equivalence relation $\sim_S$ is defined as follows: Two arbitrary sets $A := \bigsqcup_{i=1}^{n} A_i \times \{i\}$ and $B := \bigsqcup_{j=1}^{m} B_j \times \{j\}$ are equivalent, denoted $A \sim_S B$ if there exist $l \in \mathbb{N}$ and elements $(A_k, t_k, n_k, m_k)_{k=1,\dots,l}$ such that

$$A = \bigsqcup_{k=1}^{l} A_k \times \{n_k\}, \quad B = \bigsqcup_{k=1}^{l} t_k.A_k \times \{m_k\}, \quad A_k \in \mathbb{E}, t_k \in G.$$

The equivalence class containing A is denoted $[A]$ and the addition is defined by the identity

$$[A] + [B] := [\bigsqcup_{k=1}^{n+m} A_k \times \{k\}], \quad A_{n+j} = B_j,\ j = 1, \dots, m.$$

The semigroup $S(X, G, \mathbb{E})$ has the identity $0_S := \emptyset$ and it is equipped with the *algebraic preorder* ($x \leq y$ if $y = x + z$ for some z). This makes $S(X, G, \mathbb{E})$ into a preordered monoid.

5.2 Constructing a trace

Here we show how a state on the monoid $S(X, G, \mathbb{E})$ can be lifted to an invariant measure on $(X, \mathbb{B}(X))$ and then to a "quasi-trace" on the crossed product $C(X) \rtimes_r G$.

Lemma 5.2.1. *Let (G, X) be a transformation group with G discrete and X compact. Let $\mathbb{E}$ denote the family of clopen subsets of X fulfilling that $\sigma(\mathbb{E}) = \mathbb{B}(X)$. Every non-trivial state on $S(X, G, \mathbb{E})$ lifts to an invariant measure on $(X, \mathbb{B}(X))$ such that $0 < \mu(F) < \infty$ for some $F \in \mathbb{E}$.*

Proof. Define $\mu \colon \mathbb{E} \to [0, \infty]$ by

$$\mu(F) := f([F]), \quad F \in \mathbb{E}.$$

We show that μ is a *premeasure* in the sense of [20, p.30]. We trivially have that $\mu(\emptyset) = f(0) = 0$. Let $(F_i)_1^\infty$ be a sequence in $\mathbb{E}$ of disjoint sets such that $F := \bigsqcup_1^\infty F_i \in \mathbb{E}$. Since F is compact there exist $n \in \mathbb{N}$ with the property that the sets $F_1, \dots, F_n$ from the sequence $(F_i)_1^\infty$ cover F. As the sets are

[1]We use $\bigsqcup$ to emphasize that the union is disjoint.

disjoint we have that $F_i = \emptyset$ and hence $\mu(F_i) = 0$ for every $i > n$. We obtain that

$$\mu(\bigsqcup_1^\infty F_i) = \mu(\bigsqcup_1^n F_i) = f([\bigsqcup_1^n F_i]) = f(\sum_1^n [F_i]) = \sum_1^n \mu(F_i) = \sum_1^\infty \mu(F_i).$$

By [20, Theorem 1.14], μ extends to a measure $\bar{\mu}$ on $\mathbb{B}(X)$ given by

$$\bar{\mu}(F) = \inf \left\{ \sum_1^\infty \mu(F_i) \colon (F_i)_1^\infty \subseteq \mathbb{E} \text{ and } F \subseteq \bigcup_1^\infty F_i \right\}, \quad F \in \mathbb{B}(X).$$

The existence of a clopen subset F of X such that $0 < \bar{\mu}(F) < \infty$ follows from the fact that f is non-trivial.

We now show that $\bar{\mu}$ does the trick. We only need to show that $\bar{\mu}$ is invariant. For $F \in \mathbb{E}$ and $t \in G$ we have that $\mu(F) = f([F]) = f([t.F]) = \mu(t.F)$. Now let $F \in \mathbb{B}(X)$ and $t \in G$. We claim that $\bar{\mu}(t.F) \leq \bar{\mu}(F)$. Let $(F_i)_1^\infty$ be a sequence in $\mathbb{E}$ such that $F \subseteq \bigcup_1^\infty F_i$. From

$$\begin{aligned} \bar{\mu}(t.F) &= \inf \left\{ \sum_1^\infty \mu(H_i) \colon (H_i)_1^\infty \subseteq \mathbb{E} \text{ and } t.F \subseteq \bigcup_1^\infty H_i \right\} \\ &\leq \sum_1^\infty \mu(t.F_i) = \sum_1^\infty \mu(F_i), \end{aligned}$$

we obtain the desired inequality $\bar{\mu}(t.F) \leq \bar{\mu}(F)$. Together with $\bar{\mu}(t^{-1}.(t.F)) \leq \bar{\mu}(t.F)$ we have that $\bar{\mu}(t.F) = \bar{\mu}(F)$. Hence $\bar{\mu}$ is invariant. □

Before we show the second step let us recall a few definitions. A *quasi-trace* on a C*-algebra A is a function $\tau \colon A^+ \to [0, \infty]$ which satisfies $\tau(d^*d) = \tau(dd^*)$ for all $d \in A$ and $\tau(a + b) = \tau(a) + \tau(b)$ for all commuting elements $a, b \in A^+$, cf. [5, Definition 2.22]. A quasi-trace τ is said to be:

- a *2-quasi-trace* if it extends to a quasi-trace τ_2 on $M_2(A)$ with $\tau_2(a \otimes e_{1,1}) = \tau(a)$ for all $a \in A^+$,
- *non-trivial* if it takes a value different from 0 and ∞,
- *lower semi-continuous* if $\tau(a) \leq \liminf_n \tau(a_n)$ for every sequence (a_n) in A^+ converging to a,

A C*-algebra is said to be *traceless* if it admits no non-trivial lower semi-continuous 2-quasi-trace. A purely infinite C*-algebra is always traceless, cf. [5, 31, 32].

Lemma 5.2.2. *Let (G,X) be a transformation group with G discrete, countable and X compact. Let μ be an invariant measure on $(X,\mathbb{B}(X))$ such that $0<\mu(F)<\infty$ for some clopen subset F of X. Then $\tau\colon a\mapsto \int E(a)d\mu, a\in (C(X)\rtimes_r G)^+$ is a non-trivial lower semi-continuous 2-quasi-trace on $C(X)\rtimes_r G$.*

Proof. Let $\pi := \sum_{\varphi\in S(A)} \oplus\pi_\varphi$ be the universal representation of the C*-algebra $A := C(X)$ on $H := \sum_{\varphi\in S(A)}\oplus H_\varphi$, cf. [28, p.281]. Recall that the regular representation of $C_c(G,A)$ on $\mathscr{H} = l^2(G,H)$, denoted $\tilde{\pi}\times\lambda$, is defined by

$$\tilde{\pi}(a)\delta_{t,\xi} = \delta_{t,\pi(t^{-1}.a)\xi},\quad \lambda(s)\delta_{t,\xi} = \delta_{st,\xi},\quad a\in A,\ s\in G,\ \xi\in H,$$

where $\delta_{t,\xi}\in\mathscr{H}$ is the map $s\mapsto \delta_{t,s}\xi\in H$ ($\delta_{t,s}$ is the Kronecker delta). Recall that the conditional expectation $E\colon A\rtimes_r G\to A$ can be defined using the contraction $\tilde{E}\colon B(\mathscr{H})\to B(H), T\mapsto V_e^*TV_e$, where we use the maps $V_s\colon H\to\mathscr{H}, \xi\mapsto\delta_{s,\xi}$ and $V_s^*\colon\mathscr{H}\to H, \delta_{t,\xi}\mapsto\delta_{t,s}\xi$. More precisely we have the identity

$$\pi(E(x)) = \tilde{E}(\tilde{\pi}\times\lambda(x)),\quad x\in A\rtimes_r G.$$

With $\tilde{E}_s\colon B(\mathscr{H})\to B(H), T\mapsto \tilde{E}(T\lambda(s^{-1}))$ we get (as $\lambda(s^{-1})V_e = V_{s^{-1}}$) that

$$\tilde{E}(TT^*) = \sum_{s\in G} V_e^*TV_{s^{-1}}V_{s^{-1}}^*T^*V_e = \sum_{s\in G}\tilde{E}_s(T)\tilde{E}_s(T)^*,\quad T\in B(\mathscr{H}),$$

where the convergence is in strong operator topology. With $x\in A\rtimes_r G$ we get that $\pi(E_s(x^*)) = \pi(s.E_{s^{-1}}(x))^*$ and hence

$$\pi(E(xx^*)) = \sum_{s\in G}\pi(|E_s(x)|^2),\quad \pi(E(x^*x)) = \sum_{s\in G}\pi(s^{-1}.|E_s(x)|^2).$$

Fix a state φ in the state space $S(A)$ on A and let $\xi\in H=\sum_{\varphi\in S(A)}\oplus H_\varphi$ be the element with only one non-zero entry x_φ, that is the cyclic vector for π_φ, at index φ. Applying $\langle\cdot\xi,\xi\rangle$ on the (strongly and hence) weakly convergent sum, we obtain the norm convergent sum

$$\varphi(E(xx^*)) = \sum_{s\in G}\varphi(|E_s(x)|^2),\quad \varphi(E(x^*x)) = \sum_{s\in G}\varphi(s^{-1}.|E_s(x)|^2).$$

Since the point evaluation is a state on A we have that the sum $\sum_{s\in G}|E_s(x)|^2$ is pointwise (by Dini's theorem also uniformly) convergent to $E(xx^*)$. Hence

$$\begin{aligned}\int E(xx^*)d\mu &= \lim_{F\subseteq G \text{ finite}}\int\sum_{s\in F}|E_s(x)|^2 d\mu\\ &= \lim_{F\subseteq G \text{ finite}}\int\sum_{s\in F}s^{-1}.|E_s(x)|^2 d\mu = \int E(x^*x)d\mu.\end{aligned}$$

Using the identity $0 < \mu(F) = \tau(1_F) < \infty$ we obtain that τ is a non-trivial quasi-trace. We now show that τ is *locally lower semi-continuous*, i.e.

$$\tau(a) = \sup_{t>0} \tau((a-t)_+), \quad a \in (A \rtimes_r G)^+.$$

Fix $a \in (A \rtimes_r G)^+$. Since $((a-1/n)_+)_n$ is an increasing sequence converging to a, the increasing sequence $(E((a-1/n)_+))_n$ converges to $E(a)$. We conclude that

$$\lim_n \tau((a-1/n)_+) = \lim_n \int E((a-1/n)_+) d\mu = \int E(a) d\mu = \tau(a).$$

It is easy to see that $\sup_{t>0} \tau((a-t)_+) = \sup_n \tau((a-1/n)_+)$ since τ is order preserving.

By construction we have that $\tau(a+b) \leq 2(\tau(a) + \tau(b))$ for every $a, b \in (A \rtimes_r G)^+$. This implies, cf. [5, Proposition 2.24], that τ is a lower semi continuous 2-quasi-trace. ☐

Theorem 5.2.3. *Let $(C(X), G)$ be a C*-dynamical system with G discrete, countable, exact and X the Cantor set. Let $\mathbb{E}$ denote the family of clopen subsets of X. Suppose the action of G on X is essentially free. Consider the properties*

(i) The semigroup $S(X, G, \mathbb{E})$ is purely infinite.

(ii) Every clopen subset of X is τ_X-paradoxical.

(iii) The C-algebra $C(X) \rtimes_r G$ is purely infinite.*

(iv) The C-algebra $C(X) \rtimes_r G$ is traceless.*

(v) There are no non-trivial states on $S(X, G, \mathbb{E})$.

Then

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v).$$

Further if $S(X, G, \mathbb{E})$ is almost unperforated we have that $(v) \Rightarrow (i)$ making all the properties equivalent.

Proof. $(i) \Rightarrow (ii)$. Let $E \in \mathbb{E}$ be given. Using that $2[E] \leq [E]$ we have that $2[E] + [F] = [E]$ for some $[F] \in S(X, G, \mathbb{E})$. This implies that the two copies of E, say $E = \bigsqcup_{i=1}^n A_i = \bigsqcup_{i=n+1}^{n+m} A_i$, $A_i \in \mathbb{E}$, can be moved by the group (when applying possibly different group elements on each A_i, $i = 1, \dots, n+m$) to give a family of pairwise disjoint subsets of E. We obtain that E is $\mathbb{E}$-paradoxical. Since $\mathbb{E} \subseteq \tau_X$ the set E is τ_X-paradoxical.

$(ii) \Rightarrow (iii)$ Corollary 4.3.4.

$(iii) \Rightarrow (iv)$ We refer to [5, 31, 32].

$(iv) \Rightarrow (v)$ Suppose f is a non-trivial state on the algebraically preordered monoid $S(X, G, \mathbb{E})$. Since $\mathbb{E}$ is an invariant subalgebra of $\mathcal{P}(X)$ consisting of countably[2] many sets we obtain that $\mathbb{B}(X) = \sigma(\mathbb{E})$. To see this use the fact that every open subset of X is a countable union of clopen sets and hence belongs to $\sigma(\mathbb{E})$. By Lemma 5.2.1 there exists an invariant Borel measure μ on X and a clopen subset F of X such that $0 < \mu(F) < \infty$. Using Lemma 5.2.2 the map $\tau\colon a \mapsto \int E(a)d\mu$ is a non-trivial lower semi-continuous 2-quasi-trace on $C(X) \rtimes_r G$.

$(v) \Rightarrow (i)$. Suppose that the algebraically preordered monoid $S := S(X, G, \mathbb{E})$ is not purely infinite. Find an element $x \in S$ such that $2x \not\leq x$. We claim that $(n+1)x \not\leq nx$ for every $n \in \mathbb{N}$. This follows from the fact that if $(n+1)x \leq nx$ for some $n \in \mathbb{N}$ then $2nx \leq nx$. Using that S is almost unperforated this would imply $2x \leq x$ giving a contradiction. From [60, Theorem 9.1] we have a semigroup homomorphism $f\colon S \to [0, \infty]$ such that $f(x) = 1$. In particular f respects $+$ and $\leq$ and $f(0) = 0$. Hence f is a non-trivial state on S. $\square$

Remark 5.2.4. Let (X, G) be a transformation group with G discrete and X the Cantor set. Let $\mathbb{E}$ denote the family of clopen subsets of X. It would be interesting to known when $S(X, G, \mathbb{E})$ is almost unperforated. An example one might consider is the transformation group (X, G) constructed in the proof of Proposition 2.5.6.

When considering the algebraically preordered monoid $S(G, G, \mathcal{P}(G))$ Tarski made a remarkable observation: *Any discrete group G is non-amenable if and only if $S(G, G, \mathcal{P}(G))$ is purely infinite*, cf. [60, Theorem 9.1]. Hence $S(G, G, \mathcal{P}(G))$ is almost unperforated for every discrete non-amenable group.

[2]The cylinder subsets of $\{0, 1\}^{\mathbb{N}}$ form a countable family of clopen sets.

Bibliography

[1] Claire Anantharaman-Delaroche, *Purely infinite C^*-algebras arising from dynamical systems*, Bull. Soc. Math. France **125** (1997), no. 2, 199–225. MR MR1478030 (99i:46051)

[2] ______, *Amenability and exactness for dynamical systems and their C^*-algebras*, Trans. Amer. Math. Soc. **354** (2002), no. 10, 4153–4178 (electronic). MR MR1926869 (2004e:46082)

[3] R. J. Archbold and J. S. Spielberg, *Topologically free actions and ideals in discrete C^*-dynamical systems*, Proc. Edinburgh Math. Soc. (2) **37** (1994), no. 1, 119–124. MR MR1258035 (94m:46101)

[4] Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, vol. 91, Springer-Verlag, New York, 1995, Corrected reprint of the 1983 original. MR MR1393195 (97d:22011)

[5] Etienne Blanchard and Eberhard Kirchberg, *Non-simple purely infinite C^*-algebras: the Hausdorff case*, J. Funct. Anal. **207** (2004), no. 2, 461–513. MR MR2032998 (2005b:46136)

[6] Mike Boyle and Jun Tomiyama, *Bounded topological orbit equivalence and C^*-algebras*, J. Math. Soc. Japan **50** (1998), no. 2, 317–329. MR MR1613140 (99d:46088)

[7] L. E. J. Brouwer, *On the structure of perfect sets of points*, Proc. Akad. Amsterdam **12** (1910), 785–794.

[8] Lawrence G. Brown and Gert K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131–149. MR MR1120918 (92m:46086)

[9] Nathanial P. Brown and Narutaka Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. MR MR2391387

[10] John W. Bunce and James A. Deddens, *A family of simple C^*-algebras related to weighted shift operators*, J. Functional Analysis **19** (1975), 13–24. MR MR0365157 (51 #1410)

[11] Michel Coornaert and Athanase Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Lecture Notes in Mathematics, vol. 1539, Springer-Verlag, Berlin, 1993. MR MR1222644 (94d:58054)

[12] Joachim Cuntz, *C^*-algebras associated with the $ax + b$-semigroup over* $\mathbb{N}$, preprint.

[13] ______, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173–185. MR MR0467330 (57 #7189)

[14] ______, *K-theory for certain C^*-algebras*, Ann. of Math. (2) **113** (1981), no. 1, 181–197. MR MR604046 (84c:46058)

[15] Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR MR1402012 (97i:46095)

[16] Pierre de la Harpe and Georges Skandalis, *Powers' property and simple C^*-algebras*, Math. Ann. **273** (1986), no. 2, 241–250. MR MR817879 (87f:46124)

[17] Edward G. Effros and Frank Hahn, *Locally compact transformation groups and C^*- algebras*, Memoirs of the American Mathematical Society, No. 75, American Mathematical Society, Providence, R.I., 1967. MR MR0227310 (37 #2895)

[18] George A. Elliott, *Some simple C^*-algebras constructed as crossed products with discrete outer automorphism groups*, Publ. Res. Inst. Math. Sci. **16** (1980), no. 1, 299–311. MR MR574038 (81j:46102)

[19] Ruy Exel, Marcelo Laca, and John Quigg, *Partial dynamical systems and C^*-algebras generated by partial isometries*, J. Operator Theory **47** (2002), no. 1, 169–186. MR MR1905819 (2003f:46108)

[20] Gerald B. Folland, *Real analysis*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1984, Modern techniques and their applications, A Wiley-Interscience Publication. MR MR767633 (86k:28001)

[21] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003), no. 1, 73–146. MR MR1978492 (2004j:20088a)

[22] Uffe Haagerup, *Every quasi-trace on an exact C^*-algebra is a trace*, preprint, 1991.

[23] Jacob v. B. Hjelmborg, *Purely infinite and stable C^*-algebras of graphs and dynamical systems*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1789–1808. MR MR1869070 (2002h:46112)

[24] Birger Iversen, *Hyperbolic geometry*, London Mathematical Society Student Texts, vol. 25, Cambridge University Press, Cambridge, 1992. MR MR1205776 (94b:51023)

[25] Masaki Izumi, *Finite group actions on C^*-algebras with the Rohlin property. I*, Duke Math. J. **122** (2004), no. 2, 233–280. MR MR2053753 (2005a:46142)

[26] Ja A Jeong and Hiroyuki Osaka, *Extremally rich C^*-crossed products and the cancellation property*, J. Austral. Math. Soc. Ser. A **64** (1998), no. 3, 285–301. MR MR1623278 (99k:46097)

[27] Paul Jolissaint and Guyan Robertson, *Simple purely infinite C^*-algebras and n-filling actions*, J. Funct. Anal. **175** (2000), no. 1, 197–213. MR MR1774856 (2002j:46065)

[28] Richard V. Kadison and John R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I*, Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Providence, RI, 1997, Elementary theory, Reprint of the 1983 original. MR MR1468229 (98f:46001a)

[29] ———, *Fundamentals of the theory of operator algebras. Vol. II*, Graduate Studies in Mathematics, vol. 16, American Mathematical Society, Providence, RI, 1997, Advanced theory, Corrected reprint of the 1986 original. MR MR1468230 (98f:46001b)

[30] Shinzō Kawamura and Jun Tomiyama, *Properties of topological dynamical systems and corresponding C^*-algebras*, Tokyo J. Math. **13** (1990), no. 2, 251–257. MR MR1088230 (92e:46131)

[31] Eberhard Kirchberg and Mikael Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), no. 3, 637–666. MR MR1759891 (2001k:46088)

[32] ———, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebras O_∞*, Adv. Math. **167** (2002), no. 2, 195–264. MR MR1906257 (2003k:46080)

[33] Eberhard Kirchberg and Simon Wassermann, *Exact groups and continuous bundles of C^*-algebras*, Math. Ann. **315** (1999), no. 2, 169–203. MR MR1721796 (2000i:46050)

[34] Akitaka Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Comm. Math. Phys. **81** (1981), no. 3, 429–435. MR MR634163 (83c:46061)

[35] Akitaka Kishimoto and Alexander Kumjian, *Crossed products of Cuntz algebras by quasi-free automorphisms*, Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 173–192. MR MR1424962 (98h:46076)

[36] Marcelo Laca and Jack Spielberg, *Purely infinite C^*-algebras from boundary actions of discrete groups*, J. Reine Angew. Math. **480** (1996), 125–139. MR MR1420560 (98a:46085)

[37] P. J. Morandi, *The Smith Normal Form of a Matrix*, 2005.

[38] James R. Munkres, *Topology: a first course*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975. MR MR0464128 (57 #4063)

[39] Dorte Olesen and Gert K. Pedersen, *Applications of the Connes spectrum to C^*-dynamical systems*, J. Funct. Anal. **30** (1978), no. 2, 179–197. MR MR515224 (81i:46076a)

[40] ______, *Applications of the Connes spectrum to C^*-dynamical systems. III*, J. Funct. Anal. **45** (1982), no. 3, 357–390. MR MR650187 (83i:46080)

[41] Cornel Pasnicu and Mikael Rørdam, *Purely infinite C^*-algebras of real rank zero*, J. Reine Angew. Math. **613** (2007), 51–73. MR MR2377129 (2009b:46119)

[42] Alan L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics, vol. 170, Birkhäuser Boston Inc., Boston, MA, 1999. MR MR1724106 (2001a:22003)

[43] Gert K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979. MR MR548006 (81e:46037)

[44] Ciprian Pop, *Finite sums of commutators*, Proc. Amer. Math. Soc. **130** (2002), no. 10, 3039–3041 (electronic). MR MR1908928 (2003c:46070)

[45] Jean Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980. MR MR584266 (82h:46075)

[46] ———, *The ideal structure of groupoid crossed product C^*-algebras*, J. Operator Theory **25** (1991), no. 1, 3–36, With an appendix by Georges Skandalis. MR MR1191252 (94g:46074)

[47] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, Encyclopaedia Math. Sci., vol. 126, Springer, Berlin, 2002, pp. 1–145. MR MR1878882 (2003i:46060)

[48] M. Rørdam, F. Larsen, and N. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000. MR MR1783408 (2001g:46001)

[49] Mikael Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), no. 2, 255–269. MR MR1172023 (93f:46094)

[50] ———, *On sums of finite projections*, Operator algebras and operator theory (Shanghai, 1997), Contemp. Math., vol. 228, Amer. Math. Soc., Providence, RI, 1998, pp. 327–340. MR MR1667668 (2000a:46098)

[51] ———, *A simple C^*-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), no. 1, 109–142. MR MR2020420 (2005m:46096)

[52] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR MR924157 (88k:00002)

[53] Shôichirô Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag, New York, 1971, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. MR MR0442701 (56 #1082)

[54] Adam Sierakowski, *Transformartion groups and C^*-algebras, Status Report during PhD*, 2007.

[55] ———, *The ideal structure of reduced crossed products, to be published*, 2008.

[56] Adam Sierakowski and Mikael Rørdam, *Purely infinite reduced crossed products, preprint*, 2009.

[57] J. S. Spielberg, *Cuntz-Krieger algebras associated with Fuchsian groups*, Ergodic Theory Dynam. Systems **13** (1993), no. 3, 581–595. MR MR1245830 (94k:46143)

[58] John Spielberg, *Free-product groups, Cuntz-Krieger algebras, and covariant maps*, Internat. J. Math. **2** (1991), no. 4, 457–476. MR MR1113572 (92j:46120)

[59] Masamichi Takesaki, *On the cross-norm of the direct product of C^*-algebras*, Tôhoku Math. J. (2) **16** (1964), 111–122. MR MR0165384 (29 #2668)

[60] Stan Wagon, *The Banach-Tarski paradox*, Cambridge University Press, Cambridge, 1993, With a foreword by Jan Mycielski, Corrected reprint of the 1985 original. MR MR1251963 (94g:04005)

[61] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007. MR MR2288954 (2007m:46003)

[62] G. Zeller-Meier, *Produits croisés d'une C^*-algèbre par un groupe d'automorphismes*, J. Math. Pures Appl. (9) **47** (1968), 101–239. MR MR0241994 (39 #3329)