B Appendix

Her følger konstruktionen af figurene i afsnit 9, heriblandt Hofstadters sommerfugl.

Hofstadters sommerfugl

Først introduceres $H_{\theta,u,v}$:

```
In[1]:= Hθuv[p_, q_, θ1_, θ2_] := Module[{H},
          H = DiagonalMatrix[
            Table[2 Cos[2 (i - 1) p π / q + θ2], {i, q}]];
          Do[H[[i, i + 1]] = ⅇ^(ⅈ θ1), {i, q - 1}];
          Do[H[[i + 1, i]] = ⅇ^(-ⅈ θ1), {i, q - 1}];
          H[[q, 1]] = H[[q, 1]] + ⅇ^(ⅈ θ1);
          H[[1, q]] = H[[1, q]] + ⅇ^(-ⅈ θ1);
          H
         ];
```

For $\theta = \frac{1}{3}$ bliver $H_{\theta,u,v}$:

```
In[2]:= MatrixForm[Hθuv[1, 3, θ1, θ2]]
```

Out[2]//MatrixForm=

$$\begin{pmatrix} 2\,\mathrm{Cos}[\theta 2] & e^{i\,\theta 1} & e^{-i\,\theta 1} \\ e^{-i\,\theta 1} & 2\,\mathrm{Cos}[\frac{2\pi}{3} + \theta 2] & e^{i\,\theta 1} \\ e^{i\,\theta 1} & e^{-i\,\theta 1} & 2\,\mathrm{Cos}[\frac{4\pi}{3} + \theta 2] \end{pmatrix}$$

Nu findes $P_{\min}(\theta, \lambda)$. Der regnes med en nøjagtighed på 50 cifre:

```
In[3]:= P[{p_, q_}] := Module[{Hx, Polx},
          Hx = Chop[N[λ IdentityMatrix[q] - Hθuv[p, q, 0, 0], 50]];
          Polx = Chop[Expand[Det[Hx]]]
         ];
```

Med dette kan $P_{\min}(1/3, \lambda)$ findes til:

```
In[4]:= P[{1, 3}]

Out[4]= -4.0000000000000000000000000000000000000000000000000-
         6.0000000000000000000000000000000000000000000000000λ + λ^3
```

Proceduren TegnLinie finder rødderne til de to ekstremunspolynomier for et givet θ, og benytter dette til at finde $\sigma(H_\theta) = \bigcup_{u,v\in\mathbb{T}} \sigma(H_{\theta,u,v})$.

```
In[5]:= TegnLinie[{p_, q_}] := Module[{Pmin, Pmax, sol1, sol2},
          Pmin = P[{p, q}];
          Pmax = Pmin + 8;
          sol1 = λ /. Solve[Pmin == 0, λ];
          sol2 = λ /. Solve[Pmax == 0, λ];
          Table[Line[{{sol1[[i]], p / q}, {sol2[[i]], p / q}}], {i, q}]
        ];
```

Nedenfor ses spektert for H_θ med $\theta = \frac{1}{3}$:

```
In[6]:= Show[Graphics[TegnLinie[{1, 3}]], ImageSize → 300]
```



```
Out[6]= - Graphics -
```

MakeList er en simpel konstruktion til at finde irreducible brøker med nævner op til *length*.

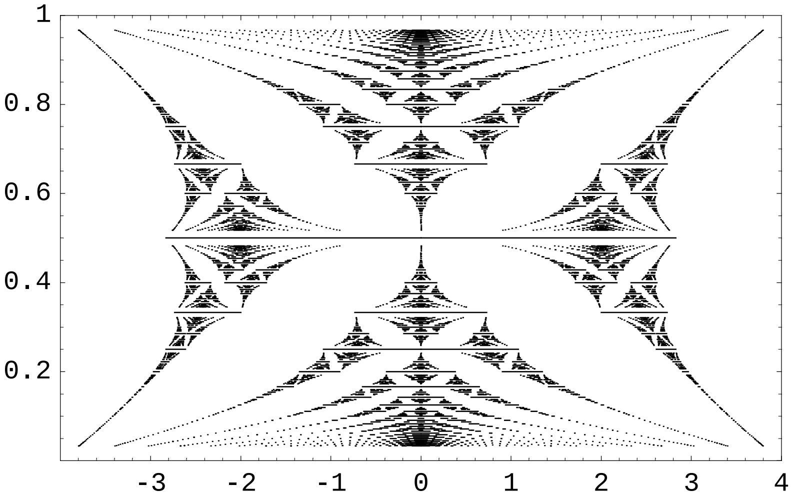
```
In[7]:= MakeList[length_Integer] := Module[{},
          Union[Flatten[Table[If[GCD[i, j] == 1, {j, i}, {1, 2}], {i, length}, {j, i - 1}], 1]]];
```

```
In[8]:= MakeList[6]
```

```
Out[8]= {{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {2, 3}, {2, 5}, {3, 4}, {3, 5}, {4, 5}, {5, 6}}
```

Ved at tegne spektret for en række rationale θ fremkommer Hofstadters sommerfugl. Her indtegnes spektert for $H_{p/q}$ med q gående op til 30:

```
In[9]:= figur =
  Show[Graphics[Table[TegnLinie[MakeList[30][[i]]], {i, Length[MakeList[30]]}]],
   PlotRange → {{-4, 4}, {0, 1}}, Frame → True, ImageSize → 300]
```




```
Out[9]= - Graphics -
```

Dette kan evt. eksporteres med kommandoen:

```
In[10]:= Export["C:\Documents and Settings\adam\Desktop\sommerfugl.EPS", figur]

Out[10]= C:\Documents and Settings\adam\Desktop\sommerfugl.EPS
```

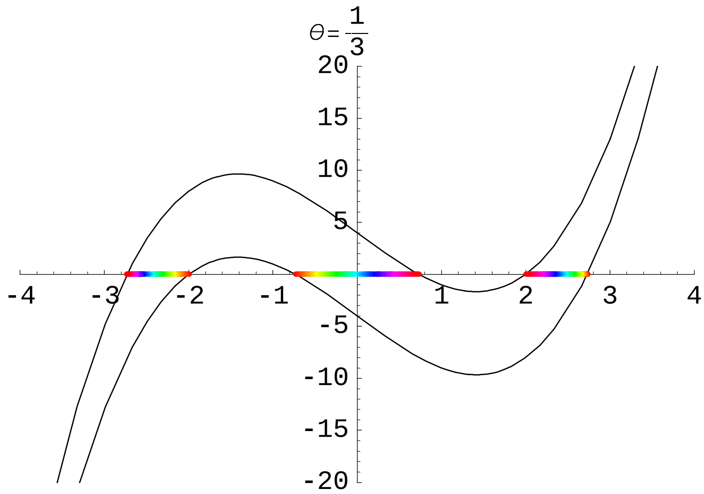
Disjunkte intervaller

Ved at forskyde $P_{\min}$ i invervaller af 0,05 mod $P_{\max}$ og finde røderne, fremkommer en forsimplet grafisk præsentation af $\sigma(H_\theta)$. Funktionen Tegnprikker finder disse rødder, og hver ef dem markeres med en prik. Farverne af prikkerne ændres, så man tydelig man se, hvordan $P_{\min}$ føres over i $P_{\max}$. Tegnprikker indtegener desuden selve $P_{\min}$ og $P_{\max}$.

```
In[11]:= TegnPrikker[{p_, q_}] := Module[{Pmin, Graf1, Pmax, Graf2, Prikker = {}},
    Pmin = P[{p, q}];
    Graf1 = Plot[Pmin, {λ, -4, 4}, DisplayFunction → Identity,
      AxesLabel → "θ=    ", PlotRange → {{-4, 4}, {-20, 20}}, PlotLabel → p / q];
    Pmax = Pmin + 8;
    Graf2 = Plot[Pmax, {λ, -4, 4}, DisplayFunction → Identity, PlotRange → All];
    For[j = 0.0, j ≤ 8.0, sol = λ /. Solve[Pmin + j == 0, λ]; Prikker =
      {Prikker, Show[Graphics[{Hue[j / 8.0], Table[Point[{sol[[i]], 0}], {i, q}]}],
        DisplayFunction → Identity]}; j = j + 0.05];
    Show[{Graf1, Graf2}, Prikker, DisplayFunction → $DisplayFunction,
     ImageSize → 300]
   ];
```

Evaluering af Tegnprikker for $\theta = \frac{1}{3}$ giver:

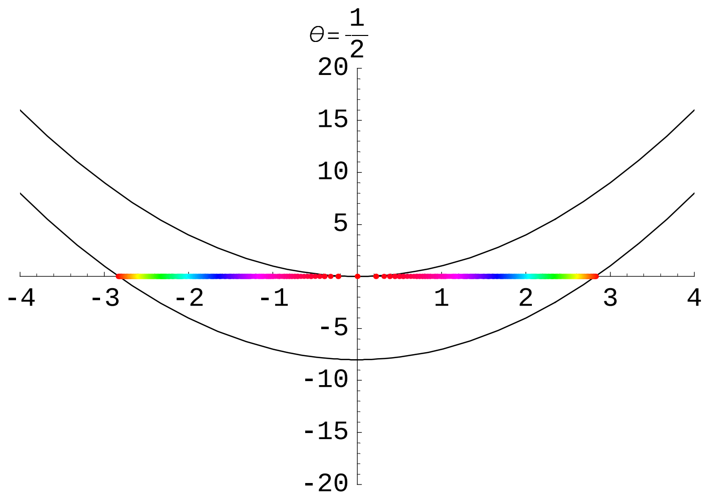
```
In[12]:= figur = TegnPrikker[{1, 3}]
```

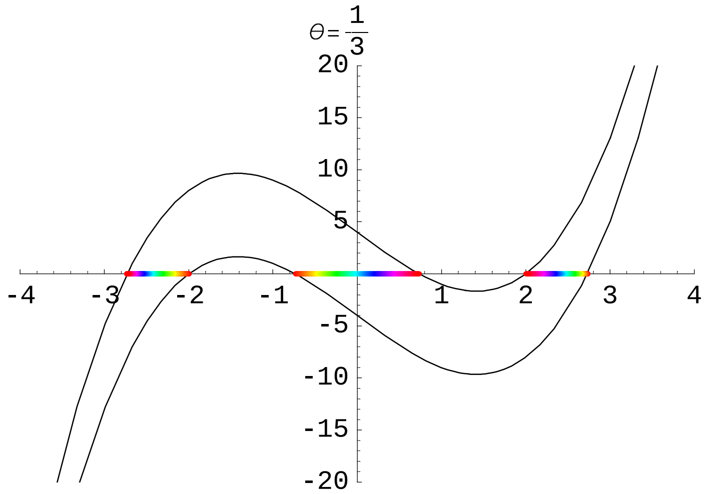


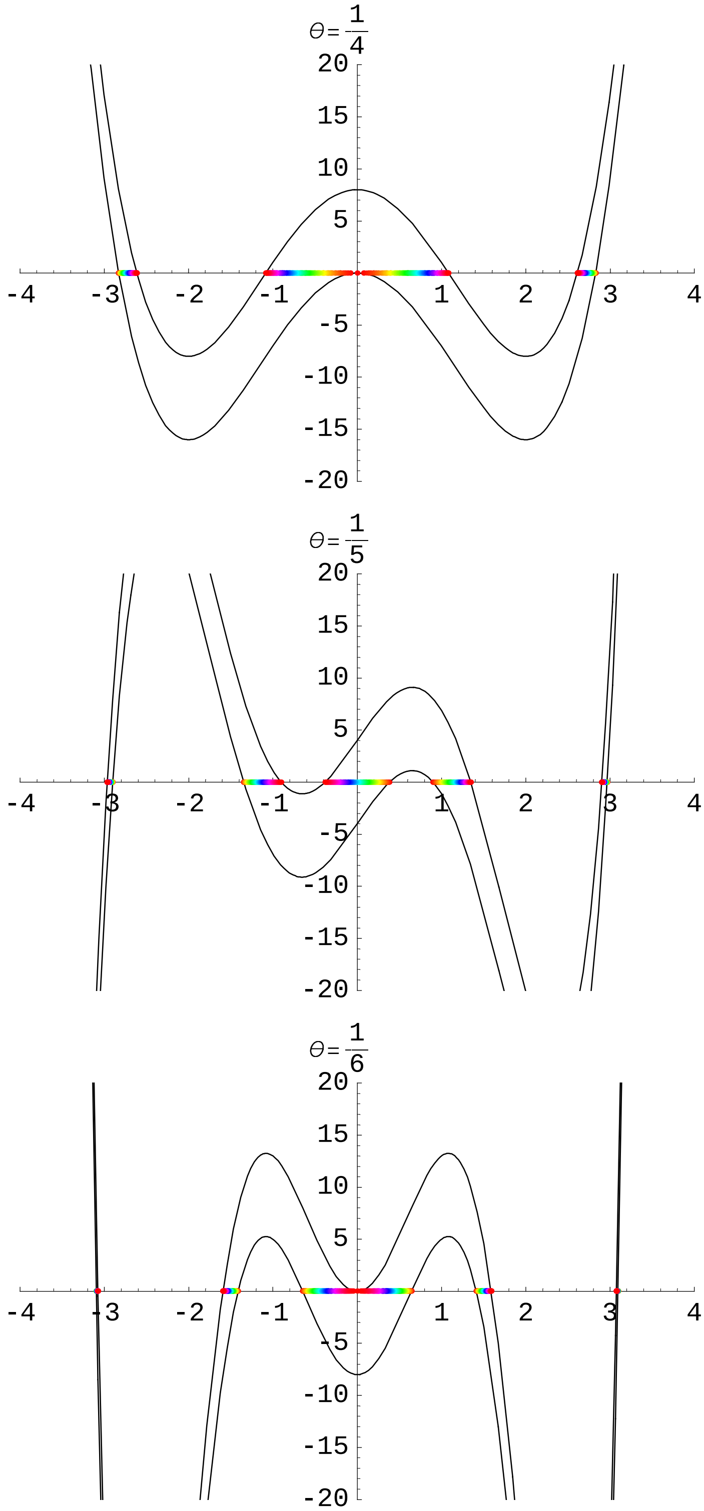

```
Out[12]= - Graphics -
```

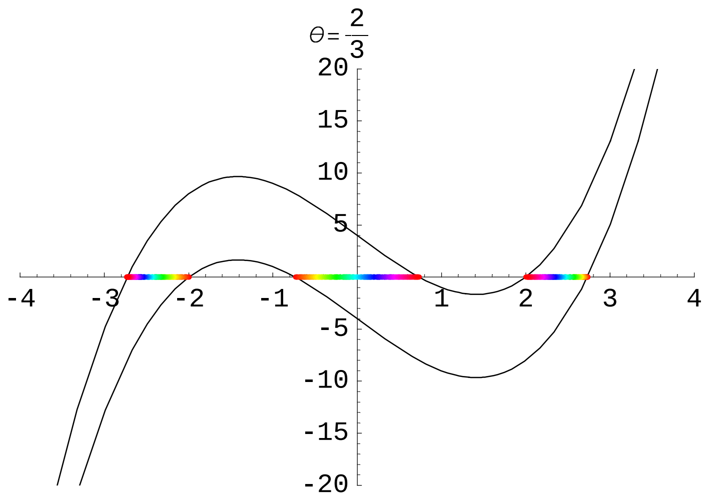
Ovenstående procedure gentages for en række forskellige brøker:

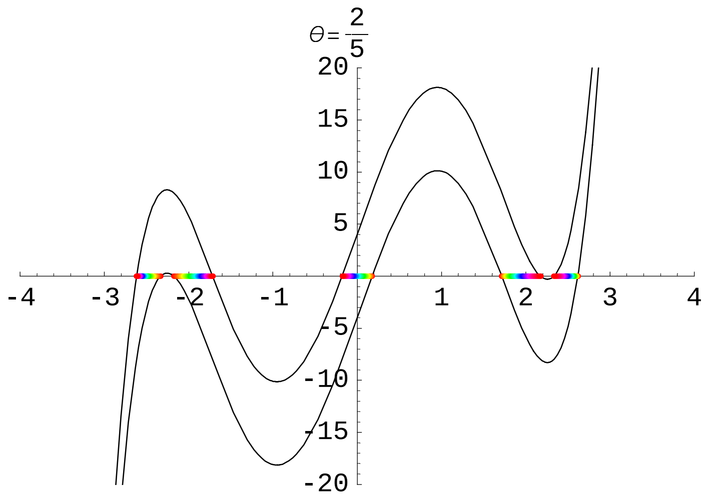
```
In[13]:= Map[TegnPrikker, MakeList[6]]
```

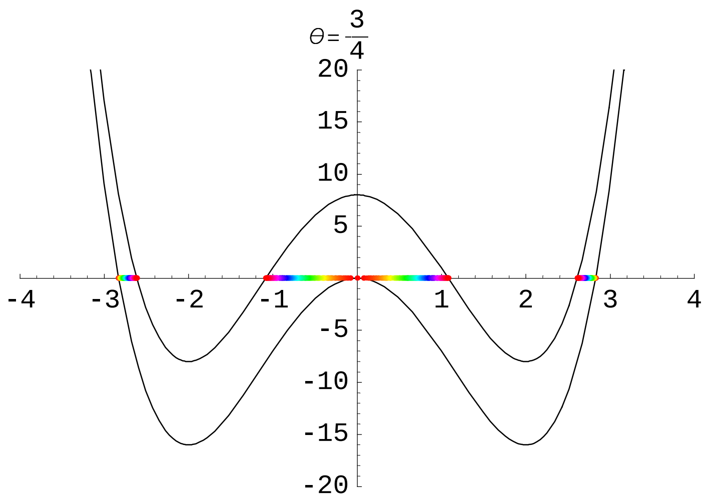


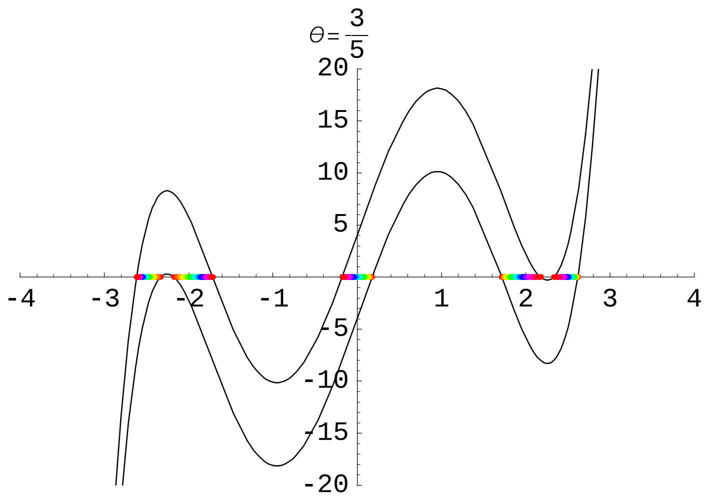


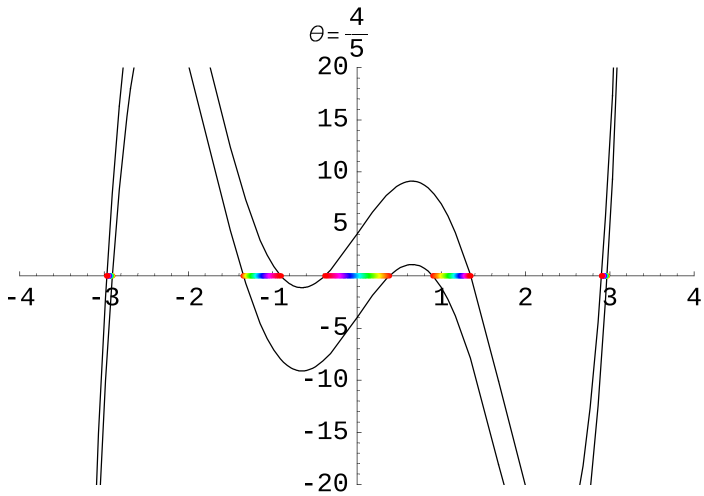

$\Theta=\frac{1}{4}$
20
15
10
5
-5
-10
-15
-20
-4 -3 -2 -1 1 2 3 4
$\Theta=\frac{1}{5}$
20
15
10
5
-5
-10
-15
-20
-4 -3 -2 -1 1 2 3 4
$\Theta=\frac{1}{6}$
20
15
10
5
-5
-10
-15
-20
-4 -3 -2 -1 1 2 3 4

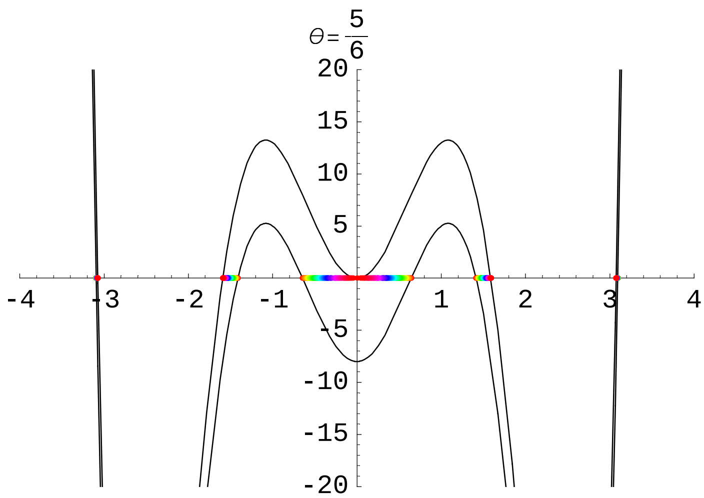

$\Theta = \frac{2}{3}$
20
15
10
5
-5
-10
-15
-20
-4
-3
-2
-1
1
2
3
4


$\Theta = \frac{2}{5}$
20
15
10
5
-5
-10
-15
-20
-4
-3
-2
-1
1
2
3
4


$\Theta = \frac{3}{4}$
20
15
10
5
-5
-10
-15
-20
-4
-3
-2
-1
1
2
3
4








```
Out[13]=  {- Graphics -, - Graphics -, - Graphics -, - Graphics -, - Graphics -,
           - Graphics -, - Graphics -, - Graphics -, - Graphics -, - Graphics -, - Graphics -}
```

Man ser for q ulige netop q disjunkte intervaller. For q lige ses, at enten $P_{\min}$ (for $q = 0 \bmod 4$) eller $P_{\max}$ (for $q = 0 \bmod 2$) har en dobbelrod, hvormed antallet af disjunkte intervaller reduceres til $q - 1$. I [1, 4.7] bevises, at dette gælder for vilkårlig irreducibel rational $\theta = \frac{p}{q}$.