K_1-INJECTIVITY OF C*-ALGEBRAS

Ph.D. thesis by

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Abstract

We investigate if a unital $C(X)$-algebra is properly infinite when all its fibres are properly infinite. We show that this question can be rephrased in several different ways, including the question of whether every unital properly infinite C^*-algebra is K_1-injective. We provide partial answers to these questions, and we show that the general question on proper infiniteness of $C(X)$-algebras can be reduced to establishing proper infiniteness of a specific $C([0,1])$-algebra with properly infinite fibres.
We are interested in whether every unital purely infinite non-simple C^*-algebra is K_1-injective. The question is not answered but we give several conditions that imply K_1-injectivity. It is proved among other things that a unital purely infinite C^*-algebra is K_1-injective, if its maximal ideal space is a finite dimensional compact Hausdorff space that is closed in the primitive ideal space, with respect to the hull-kernel topology.
K_1-injectivity is considered for unital approximately divisible C^*-algebras and for $\mathcal{Z}$-stable C^*-algebras. It is proved that every unital approximately divisible C^*-algebra and every C^*-algebra that tensorially absorbs an approximately divisible C^*-algebra is K_1-injective. Also a $\mathcal{Z}$-stable C^*-algebra is K_1-injective if it is unital and properly infinite. Moreover we give a condition that will imply K_1-injectivity of a strongly self-absorbing C^*-algebra.

Danish abstract

Vi undersøger om en unital $C(X)$-algebra er properly infinite, hvis alle dens fibre er properly infinite. Vi viser, at dette spørgsmål på flere måder kan omformuleres, herunder til spørgsmålet om enhver unital properly infinite C^*-algebra er K_1-injektiv. Vi giver delvise svar til disse spørgsmål, og vi viser, at det generelle spørgsmål vedrørende proper infiniteness af $C(X)$-algebraer kan reduceres til at afgøre proper infiniteness af en konkret $C([0,1])$-algebra med properly infinite fibre.
Vi er interesserede i, om enhver unital purely infinite ikke-simpel C^*-algebra er K_1-injektiv. Spørgsmålet besvares ikke, men vi giver flere betingelser, der medfører K_1-injektivitet. Det vises bl.a. at en unital purely infinite C^*-algebra er K_1-injektiv, hvis dens maksimal ideal rum er et endeligt dimensionalt kompakt Hausdorffrum, der er afsluttet i primidealrummet mht. hylster-kerne topologien.
K_1-injektivitet betragtes for en unital approksimativ divisibel C^*-algebra og for $\mathcal{Z}$-stabile C^*-algebraer. Det vises, at enhver unital approksimativ divisibel C^*-algebra og enhver C^*-algebra, der tensorisk absorberer en approksimativ divisibel C^*-algebra, er K_1-injektive. En $\mathcal{Z}$-stabil C^*-algebra er også K_1-injektiv, hvis den er unital og properly infinite. Endvidere giver vi en betingelse, som vil medføre K_1-injektivitet af en stærkt selv-absorberende C^*-algebra.

Contents

Chapter 1

Preface

This thesis is based on my work as a Ph.D. student at the Department of Mathematics and Computer Science (IMADA), University of Southern Denmark from April 2006 until Marts 2009. It can be seen as a survey on K_1-injectivity of C^*-algebras.

Cuntz studied purely infinite - and in the process also properly infinite C^*-algebras. He was mainly interested in calculating the K-theory of his algebras $\mathcal{O}_n$, but among many other things he also proved that any unital properly infinite C^*-algebra A is K_1-surjective, i.e., the mapping $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ is surjective, and that any unital purely infinite simple C^*-algebra is K_1-injective, i.e., the mapping above is injective (and hence an isomorphism). The latter result is used throughout the thesis, and in the first section of Chapter 3 we give the proof of the result.
Chapter 3 also includes some of the work done by Rieffel. He is in his papers [25] and [26] considering different kinds of "ranks" of a Banach algebra, where we are mainly interested in his proof of K_1-injectivity of every unital C^*-algebra having stable rank one. This property is also used several times in this thesis, but the proof itself is also interesting since it is used as an inspiration for a construction in Chapter 4.

Chapter 4 is based on the paper [7] which is a joint work with Etienne Blanchard and Mikael Rørdam. Although Cuntz proved K_1-injectivity of unital simple purely infinite C^*-algebras, he did not address the question of whether any unital properly infinite C^*-algebra is K_1-injective. To our knowledge this question has not been raised before we did it here. Another question we are concerned with is whether any unital $C(X)$-algebra with properly infinite fibres is itself properly infinite. We do not answer those two questions, but we actually prove that they are equivalent, i.e., every unital properly infinite C^*-algebra is K_1-injective if and only if every $C(X)$-algebra with properly infinite fibres itself is properly infinite.

Cuntz's work also gave inspiration to study unital purely infinite C^*-algebras in the non-simple case. In Chapter 5 we raise the question whether any unital purely infinite C^*-algebra is K_1-injective. We do not come with an answer to the question, but at least we are able to prove K_1-injectivity if certain conditions are satisfied. For example, it is proved that a unital purely infinite C^*-algebra that is an extension of K_1-injective C^*-algebras is itself K_1-injective. Moreover K_1-injectivity holds for a unital purely infinite C^*-algebra A if $\mathrm{Max}(A)$ is a finite dimensional compact Hausdorff space that is closed in $\mathrm{Prim}(A)$ with respect to the hull-kernel topology. These results are obtained by using some of the ideas and techniques from Brown and Pedersen's paper [10]. Therefore their results (together with proofs) are included in the beginning of Chapter 5. Furthermore for the survey of K_1-injectivity, their result about K_1-injectivity of an extremally rich C^*-algebra with weak cancellation is also interesting.

In Chapter 6 we are considering approximately divisible C^*-algebras. Blackadar, Kumjian and Rørdam proved that every simple approximately divisible C^*-algebra is K_1-injective, but here we show that it also holds in the unital non-simple case. Moreover we obtain results about $\mathcal{Z}$-stable C^*-algebras. In particular it is proved that every unital $\mathcal{Z}$-stable C^*-algebra is K_1-injective if it is properly infinite.

In Toms' and Winter's paper [37], K_1-injectivity has to be assumed for the strongly self-absorbing C^*-algebras. So in Chapter 7 we raise the question whether K_1-injectivity holds for every strongly self-absorbing C^*-algebra. The question is not answered, but we give conditions that will imply K_1-injectivity.

To become familiar with definitions and notation in the thesis, Chapter 2 is dealing with back ground material.

During three years as a Ph.D. student, I will first of all thank my supervisor Mikael Rørdam, without whom I could not have made this thesis. I am grateful that he was willing to supervise me and spend time in Odense also after he moved to Copenhagen.
Next, I will thank Eduard Ortega for always having time and being very helpful.
Last but not least, thanks to the employees and students for making IMADA such a nice place to work.

Randi Rohde
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Chapter 2

Preliminaries

2.1 Properly infinite C^*-algebras

Two projections p and q in a C^*-algebra A are Murray-von Neumann equivalent, written $p \sim q$, if $p = v^*v$ and $q = vv^*$ for a partial isometry $v \in A$, and p is subequivalent to q, written $p \precsim q$, if p is equivalent to a subprojection of q.

A projection p in a C^*-algebra is said to be infinite if it is equivalent to a proper subprojection of itself, i.e., if there is a projection q in A such that $p \sim q < p$. If p is not infinite, then p is said to be finite.

A unital C^*-algebra A is said to be finite if its unit 1_A is a finite projection. Otherwise A is called infinite. If $M_n(A)$ is finite for all natural numbers n, then A is stably finite.

Definition 2.1.1. A non-zero projection p in a C^*-algebra A is said to be properly infinite if there are mutually orthogonal projections e, f in A such that $e \leq p$, $f \leq p$ and $p \sim e \sim f$.

A unital C^*-algebra is called properly infinite if its unit 1_A is a properly infinite projection.

If p, q are projections in a C^*-algebra A, let $p \oplus q$ denote the projection in $M_2(A)$ given by

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

Then it follows from [31, Proposition 1.1.2] that a non-zero projection p in a C^*-algebra A is properly infinite if and only if

$$p \oplus p \precsim p \oplus 0.$$

An element in a C^*-algebra A is said to be full if it is not contained in any proper closed two-sided ideal in A.

It is well known (see for example [35, Exercise 4.9]) that if p is a properly infinite and full projection in a C^*-algebra A, then $e \precsim p$ for every projection $e \in A$.

2.2 The Murray-von Neumann- and the Cuntz semigroups

To a C^*-algebra A we can associate the Murray-von Neumann semigroup $V(A)$ that consists of Murray-von Neumann equivalence classes of projections in $M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)$. This makes sense, since whenever it is convenient we will identify $M_n(A)$ with its image in the upper left-hand corner of $M_{n+k}(A)$ under the mapping:

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in M_n(A).$$

Hence a projection $p \in M_n(A)$ can be identified with $p \oplus 0_k$ for any $k \in \mathbb{N}$. So $V(A)$ is defined by

$$V(A) = \mathcal{P}_\infty(A)/\sim = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)/\sim,$$

where $\mathcal{P}_n(A)$ is the projections in $M_n(A)$.

Similarly one can associate the Cuntz semigroup $W(A)$ that consists of equivalence classes of positive elements in $M_\infty(A)$. The equivalence relation is defined as follows, due to Cuntz:

Definition 2.2.1. [34] Let $a \in M_n(A)^+$ and $b \in M_m(A)^+$. Then $a \precsim b$ if there is a sequence $(x_k)_{k=1}^{\infty} \subseteq M_{m,n}(A)$ such that

$$\lim_{k\to\infty} x_k^* b x_k = a.$$

We define a to be equivalent to b, written $a \approx b$, if and only if $a \precsim b$ and $b \precsim a$. This defines an equivalence relation on the positive elements in $M_\infty(A)$, and

$$W(A) = M_\infty(A)^+/\approx .$$

The sets $V(A)$ and $W(A)$ become ordered abelian semigroups when equipped with the relations:

$$\langle a \rangle + \langle b \rangle = \left\langle \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\rangle$$

and

$$\langle a \rangle \leq \langle b \rangle \iff a \precsim b.$$

The ordering on $V(A)$ coincides with the algebraic ordering: $x \leq y$ if and only if there is a z such that $y = x + z$.

Both $V(A)$ and $W(A)$ are positive in the sense that they have a zero-element that is the smallest element in the semigroup.

Like we defined properly infinite projections we also define proper infiniteness of positive elements:

Definition 2.2.2. [22, Definition 3.2] A non-zero positive element in a C^*-algebra is called properly infinite if

$$a \oplus a \precsim a.$$

If a is a properly infinite element in a C^*-algebra A, it follows from [22, Proposition 3.5] that $b \precsim a$ for every positive element b in the closed two-sided ideal $\overline{AaA}$, generated by a.

For each $\varepsilon > 0$, let $h_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ be the continuous function defined by

$$h_\varepsilon(t) = \max\{t - \varepsilon, 0\}.$$

Following the standard convention, we will for each positive element $a \in A$ and every $\varepsilon > 0$, by $(a - \varepsilon)_+$ denote the positive element $h_\varepsilon(a) \in A$.

From [29, Section 2] we recall some facts about comparison of two positive elements a, b in a C^*-algebra A:

(i) $a \precsim b$ if and only if $(a - \varepsilon)_+ \precsim b$ for every $\varepsilon > 0$.

(ii) $a \precsim b$ if and only if for each $\varepsilon > 0$ there is $\delta > 0$ and $x \in A$ such that $x^*(b - \delta)_+ x = (a - \varepsilon)_+$.

(iii) If $\|a - b\| < \varepsilon$, then $(a - \varepsilon)_+ \precsim b$.

(iv) $((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$.

2.3 Purely infinite C^*-algebras

There are several equivalent definitions for a simple C^*-algebra to be purely infinite. The following is one of them:

Definition 2.3.1. [34, Definition 2.3] *A* simple C^*-algebra A is said to be purely infinite if every non-zero hereditary sub-C^*-algebra of A contains an infinite projection.

Other equivalent definitions can be seen in [31, Proposition 4.1.1].

In general if A is a (non-simple) C^*-algebra we have the following definition of purely infiniteness, which is equivalent to Definition 2.3.1 if A is simple:

Definition 2.3.2. [31] A C^*-algebra A is said to be purely infinite if A has no non-zero abelian quotients and if for every pair of positive elements a, b in A, where b belongs to $\overline{AaA}$, the closed two-sided ideal generated by a, we have that $b \precsim a$.

Moreover, from [27] there is a result about projections in a unital, simple and purely infinite C^*-algebra which shall be used later on in the thesis:

Lemma 2.3.3. [27, Lemma A.3.7] *Let A be a unital, simple and purely infinite C^*-algebra. Every non-zero projection in A is properly infinite.*

2.4 The K_1-group of a C^*-algebra

For a unital C^*-algebra we let $\mathcal{U}(A)$ denote the group of unitary elements in A and we let $\mathcal{U}_n(A)$ be equal to $\mathcal{U}(M_n(A))$. Set

$$\mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$$

and define for $u \in \mathcal{U}_n(A)$, $v \in \mathcal{U}_m(A)$

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(A).$$

We write $u \sim_1 v$ if there exists a natural number $k \geq \max\{n, m\}$ such that

$$u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m} \quad \text{in} \quad \mathcal{U}_k(A)$$

where $\sim_h$ is the homotopy equivalence.

In [35] it is shown that $\sim_1$ is an equivalence relation on $\mathcal{U}_\infty(A)$ and we have the following definition for the K_1-group of a C^*-algebra:

Definition 2.4.1. For a C^*-algebra A we define

$$K_1(A) = \mathcal{U}_\infty(\widetilde{A}) / \sim_1 .$$

For a unitary $u \in \mathcal{U}_\infty(A)$ we let $[u]_1$ denote the equivalence class in $K_1(A)$ containing u, and we define

$$[u]_1 + [v]_1 = [u \oplus v]_1, \quad u, v \in \mathcal{U}(\widetilde{A}).$$

One can prove that $(K_1(A), +)$ is an abelian group with $-[u]_1 = [u^*]_1$ and zero-element $[1]_1$.

The following proposition is called the standard picture of K_1 which is a restatement of the definition.

Proposition 2.4.2. [35, Proposition 8.1.4] *Let A be a C^*-algebra. Then*

$$K_1(A) = \left\{ [u]_1 \ : \ u \in \mathcal{U}_\infty(\widetilde{A}) \right\}$$

and the map $[\cdot]_1 : \mathcal{U}_\infty(\widetilde{A}) \to K_1(A)$ has the following properties:

(i) $[u \oplus v]_1 = [u]_1 + [v]_1$.

(ii) *If $u, v \in \mathcal{U}_n(\widetilde{A})$ and $u \sim_h v$, then $[u]_1 = [v]_1$.*

(iii) *If $u, v \in \mathcal{U}_n(\widetilde{A})$, then $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$.*

(iv) *For $u, v \in \mathcal{U}_\infty(\widetilde{A})$, $[u]_1 = [v]_1$ if and only if $u \sim_1 v$.*

If A is a unital C^*-algebra, then $\widetilde{A} = A + \mathbb{C}f$ where $f = 1_{\widetilde{A}} - 1_A$. Let $\mu : \widetilde{A} \to A$ be given by

$$\mu(x + \lambda f) = x, \quad x \in A, \lambda \in \mathbb{C}.$$

Then μ is a unital *-homomorphism that can be extended to a unital *-homomorphism $M_n(\widetilde{A}) \to M_n(A)$ for each $n \in \mathbb{N}$. Thereby we get a map $\mu : \mathcal{U}_\infty(\widetilde{A}) \to \mathcal{U}_\infty(A)$.

If A is a unital C^*-algebra the following proposition will imply that we shall often identify $K_1(A)$ with $\mathcal{U}_\infty(A)/\sim_1$.

Proposition 2.4.3. [35, Proposition 8.1.6] *Let A be a unital C^*-algebra. Then there is an isomorphism $\rho : K_1(A) \to \mathcal{U}_\infty(A)/\sim_1$ making the diagram*

$$\begin{tikzcd} \mathcal{U}_\infty(\widetilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"'] & \mathcal{U}_\infty(A) \arrow[d] \\ K_1(A) \arrow[r, "\rho"] & \mathcal{U}_\infty(A)/\sim_1 . \end{tikzcd}$$

commutative.

Note that Proposition 2.4.3 implies that $K_1(A) \cong K_1(\widetilde{A})$ for any C^*-algebra A.

Proposition 2.4.4. [35, Proposition 2.1.8] *Let A be a unital C^*-algebra and let $\mathrm{GL}(A)$ be the invertible elements in A.*

(i) *If $x \in A$ is invertible then $|x| = (x^*x)^{\frac{1}{2}}$ is invertible and $\omega(x) = x|x|^{-1}$ belongs to $\mathcal{U}(A)$.*

(ii) *The map $\omega : \mathrm{GL}(A) \to \mathcal{U}(A)$ defined in* (i) *is continuous and $\omega(x) \sim_h x$ in $\mathrm{GL}(A)$ for every $x \in \mathrm{GL}(A)$.*

If we for a C^*-algebra A let $\mathrm{GL}_n(\widetilde{A}) = \mathrm{GL}(M_n(\widetilde{A}))$ and $\mathrm{GL}_\infty(\widetilde{A}) = \bigcup_{n=1}^\infty \mathrm{GL}_n(\widetilde{A})$ it follows from (ii) that the map $[\cdot]_1 : \mathcal{U}_\infty(\widetilde{A}) \to K_1(A)$ can be extended to a map

$$[\cdot]_1 : \mathrm{GL}_\infty(\widetilde{A}) \to K_1(A),$$

namely by replacing $[x]_1$ with $[\omega(x)]_1$ for every $x \in \mathrm{GL}_\infty(\widetilde{A})$.
Similarly we have an extension of the map $[\cdot]_1 : \mathcal{U}_\infty(A) \to K_1(A)$ to a map

$$[\cdot]_1 : \mathrm{GL}_\infty(A) \to K_1(A)$$

if A is unital.

Below we give examples of K_1-groups for some C^*-algebras. See [35, Table of K-groups] for more examples.

(i) $K_1(\mathbb{C}) = K_1(M_n(\mathbb{C})) = K_1(B(H)) = 0$.

(ii) If D is a UHF-algebra, then $K_1(D) = 0$.

(iii) $K_1(\mathcal{O}_n) = 0$ for $2 \leq n \leq \infty$, where $\mathcal{O}_n$ is the Cuntz-algebra generated by n isometries.

(iv) $K_1(C_0(\mathbb{R})) = \mathbb{Z}$.

To a C^*-algebra A one can also associate the K_0-group, $K_0(A)$ which is defined by equivalence of projections instead of equivalence of unitaries. The K_0-group will not be used as frequently as the K_1-group in this thesis, so we will not introduce it here. But the definition and properties of the K_0-group can be seen in [35].

2.5 Dimension functions

In the following we will give the definition of a dimension function on a C^*-algebra, which will be used in later chapters of the thesis. These functions can in some cases be described by quasi-traces or traces.

Definition 2.5.1. A trace on a C^*-algebra A is a linear function $\tau : A \to \mathbb{C}$ satisfying

$$0 \leq \tau(x^*x) = \tau(xx^*), \quad x \in A.$$

If $\|\tau\| = 1$, then τ is called a tracial state.
Note that if τ is a trace on A, then $\tau_n : M_n(A) \to \mathbb{C}$ defined by

$$\tau_n((a_{ij})_{i=1}^n) = \sum_{i=1}^n \tau(a_{ii}), \quad a_{ij} \in A$$

is a trace on $M_n(A)$, and if τ is a tracial state, then $\frac{1}{n}\tau_n$ is a tracial state on $M_n(A)$. In the following the extension of a trace τ on a C^*-algebra A to a trace on $M_n(A)$ will also be denoted by τ.

Definition 2.5.2. [29, Definition 4.2] A quasi-trace on a C^*-algebra A is a function $\tau : A \to \mathbb{C}$ satisfying

(i) τ is linear on abelian sub-C^*-algebras of A

(ii) $\tau(a + ib) = \tau(a) + i\tau(b)$ if a, b are self-adjoint elements

(iii) $0 \leq \tau(x^*x) = \tau(xx^*), \quad x \in A$

(iv) τ extends to a function from $M_n(A) \to \mathbb{C}$ satisfying (i)-(iii).

A linear quasi-trace is a trace, but it is an open problem whether all quasi-traces are traces.

Definition 2.5.3. Let A be a C^*-algebra and let $d : M_\infty(A)^+ \to [0, \infty]$ be a function that satisfies

$$\begin{aligned} d(a \oplus b) &= d(a) + d(b) \\ d(a) &\leq d(b) \quad \text{if} \quad a \precsim b \end{aligned}$$

for all $a, b \in M_\infty(A)^+$. Then d is called a *dimension function* on A. It is said to be *lower semi-continuous* if

$$d(a) \leq \liminf_n d(a_n) \quad \text{whenever} \quad \lim_{n\to\infty} a_n = a.$$

From [29, Proposition 4.1] it follows that d is lower semi-continuous if $d = \bar{d}$, where

$$\bar{d}(a) = \lim_{\varepsilon\to 0^+} d((a-\varepsilon)_+), \quad a \in M_\infty(A)^+.$$

Furthermore $\bar{d}$ is a lower semi-continuous dimension function on A for every dimension function d on A.

If τ is a quasi-trace on a C^*-algebra A, Blackadar and Handelman showed in [2] that

$$d_\tau(a) = \lim_{\varepsilon\to 0^+} \tau(f_\varepsilon(a)) = \lim_{n\to\infty} \tau\left(a^{\frac{1}{n}}\right), \quad a \in M_\infty(A)^+$$

where $f_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $f_\varepsilon(t) = \min\{\varepsilon^{-1}t, 1\}$, defines a lower semi-continuous dimension function on A. Moreover it is proved that a lower semi-continuous dimension function on A is of the form d_τ for some quasi-trace τ on A. The function is called the dimension function induced by τ.
On the other hand Haagerup [15] and Kirchberg [20] proved that quasi-traces on an exact C^*-algebra are traces. Thus, a lower semi-continuous dimension function on an exact C^*-algebra A is of the form d_τ for some trace τ on A.

2.6 Strongly self-absorbing C^*-algebras

The definition of a *strongly self-absorbing* C^*-algebra is introduced in [37] by Toms and Winter, and is given by unitarily equivalence of *-homomorphisms.

Definition 2.6.1. Let A and B be separable C^*-algebras. Two *-homomorphisms $\varphi, \psi : A \to B$ are called approximately unitarily equivalent if there is a sequence $(u_n)_{n=1}^\infty$ of unitaries in $\mathcal{M}(B)$ such that

$$\lim_{n\to\infty} \|u_n \varphi(a) u_n^* - \psi(a)\| = 0$$

for every $a \in A$.

Definition 2.6.2. Let D be a unital separable C^*-algebra. Then D is strongly self-absorbing if $D \ncong \mathbb{C}$ and there is a *-isomorphism $\varphi : D \to D \otimes D$ such that φ is approximately unitarily equivalent with $\psi = \mathrm{id}_D \otimes 1$, where id_D is the identity map on D.

Definition 2.6.3. Let A be a unital separable C^*-algebra. Then A is said to have approximately inner half flip if the *-homomorphisms $\varphi, \psi : A \to A \otimes A$ given by

$$\varphi(a) = a \otimes 1 \quad \text{and} \quad \psi(a) = 1 \otimes a, \quad a \in A$$

are approximately unitarily equivalent.

By Kirchberg and Phillips [21] it follows that a unital separable C^*-algebra with approximately inner half flip is simple and nuclear. Furthermore one can easily prove that it has at most one tracial state (see [27]).

We shall now recall some facts about strongly self-absorbing C^*-algebras that was given in [37]:

If D is a unital separable strongly self-absorbing C^*-algebra, then

(a) D has approximately inner half flip.

(b) D is nuclear, simple and has at most one tracial state.

(c) $D \cong \bigotimes_{i=1}^{k} D \cong \bigotimes_{i=1}^{\infty} D$ for any $k \in \mathbb{N}$.

Examples of strongly self-absorbing C^*-algebras are:

(i) The Cuntz-algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$.

(ii) UHF-algebras of infinite type, i.e., UHF-algebras with associated super natural number $(n_j)_{j=1}^{\infty}$ satisfying that $n_j \in \{0, \infty\}$ for every $j \in \mathbb{N}$ and $n_j \neq 0$ for at least one $j \in \mathbb{N}$.

(iii) The Jiang-Su algebra $\mathcal{Z}$.

(iv) All finite and infinite combinations of tensor products of the form $A \otimes B$ where A and B are one of the examples in (i)-(iii).

The Jiang-Su algebra will be introduced in the next section, and we remind the reader about that the examples from above are the only known examples of strongly self-absorbing C^*-algebras.

2.7 The Jiang-Su algebra

In [18] Jiang and Su are constructing a C^*-algebra from an inductive limit of dimension drop algebras. This C^*-algebra has later frequently been studied regarding classification theory and will also be considered in this thesis. We shall now give some facts about the construction and properties of the Jiang-Su algebra.

For natural numbers n, m we have that $M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \cong M_{mn}(\mathbb{C})$, where the isomorphism $\varphi : M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \mapsto M_{mn}(\mathbb{C})$ is given by

$$\varphi(a \otimes b) = \begin{pmatrix} b_{11}a & b_{12}a & \dots & b_{1n}a \\ b_{21}a & b_{22}a & \dots & b_{2n}a \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}a & b_{n2}a & \dots & b_{nn}a \end{pmatrix},$$

for $a \in M_m(\mathbb{C})$ and $b = (b_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$.
Let 1_n be the identity in $M_n(\mathbb{C})$ og let

$$M_m(\mathbb{C}) \otimes 1_n = \{a \otimes 1_n : a \in M_m(\mathbb{C})\} \quad \text{and} \quad 1_m \otimes M_n(\mathbb{C}) = \{1_m \otimes b : b \in M_n(\mathbb{C})\}.$$

Definition 2.7.1. A dimension drop algebra is a C^*-algebra of the form

$$I[m_0, m, m_1] = \left\{ f \in C([0,1], M_m(\mathbb{C})) : f(0) \in M_{m_0}(\mathbb{C}) \otimes 1_{\frac{m}{m_0}}, f(1) \in 1_{\frac{m}{m_1}} \otimes M_{m_1}(\mathbb{C}) \right\},$$

where m_0, m_1 and m are natural numbers such that m_0 and m_1 both divide m.

If m_0 og m_1 are relatively prime and $m = m_0 m_1$, then $I[m_0, m, m_1]$ is called a prime dimension drop algebra, and is often written as $I_{m_0 m_1}$. Jiang and Su proved that $I[m_0, m, m_1]$ has no non-trivial projections if and only if $I[m_0, m, m_1]$ is a prime dimension drop algebra. In this case

$$K_0(I[m_0, m, m_1]) \cong \mathbb{Z} \quad \text{and} \quad K_1(I[m_0, m, m_1]) = 0.$$

There exists an inductive sequence of prime dimension drop algebras $(A_n)_{n \in \mathbb{N}}$, where the connecting *-homomorphisms $\varphi_n : A_n \to A_{n+1}$ are unital and injective such that the inductive limit of

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

is a unital simple C^*-algebra with a unique tracial state.

In particular, among all inductive limits of dimension drop algebras there is a unique C^*-algebra $\mathcal{Z}$ satisfying that $\mathcal{Z}$ is a unital, separable, simple, infinite dimensional and nuclear C^*-algebra with a unique tracial state such that $K_0(\mathcal{Z}) \cong \mathbb{Z}$ and $K_1(\mathcal{Z}) = 0$. This C^*-algebra is called the *Jiang-Su* algebra.

Jiang og Su proved that $\mathcal{Z}$ has approximately inner half flip and that

$$\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \quad \text{and} \quad \mathcal{Z} \cong \bigotimes_{i=1}^{\infty} \mathcal{Z}.$$

In the context of [37] it follows that $\mathcal{Z}$ is strongly self-absorbing.
Moreover Jiang and Su proved the existence of C^*-algebras that tensorially absorbs $\mathcal{Z}$:

Theorem 2.7.2. *Let A be a unital, purely infinite, simple, and nuclear C^*-algebra. Then $A \cong A \otimes \mathcal{Z}$.*

Theorem 2.7.3. *Let A be a unital, infinite dimensional, and simple* AF*-algebra. Then $A \cong A \otimes \mathcal{Z}$.*

Chapter 3

K_1-injectivity

In this chapter we will concentrate on two already known results about K_1-injectivity that will be used later on in the thesis. Namely that a unital purely infinite and simple C^*-algebra is K_1-injective (proved by Cuntz) and that a unital C^*-algebra with stable rank one is K_1-injective (proved by Rieffel). We shall give the proof of both results, and furthermore it should be mentioned that a C^*-algebra with real rank zero is also K_1-injective. This was proved by Lin in [23] but we will not go into the proof since the result is not used in this thesis.

For a unital C^*-algebra A, we let $\mathcal{U}(A)$ denote the group of unitary elements in A, $\mathcal{U}^0(A)$ denotes its connected component containing the unit of A, and $\mathcal{U}_n(A)$ and $\mathcal{U}_n^0(A)$ are equal to $\mathcal{U}(M_n(A))$ and $\mathcal{U}^0(M_n(A))$ respectively.

By the First Homomorphism Theorem there is a group homomorphism

$$\omega : \mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$$

making the following diagram commutative:

$$\begin{tikzcd} \mathcal{U}(A) \arrow[d] \arrow[dr, "{[\cdot]_1}"] & \\ \mathcal{U}(A)/\mathcal{U}^0(A) \arrow[r, dashed, "\omega"] & K_1(A). \end{tikzcd}$$

The unital C^*-algebra is called K_1-injective (K_1-surjective) if ω is injective (surjective). In other words, if A is K_1-injective, and u is a unitary element in A, then $u \sim_h 1$ in $\mathcal{U}(A)$ if (and only if) $[u]_1 = 0$ in $K_1(A)$.

One could argue that K_1-injectivity should entail that the natural mappings

$$\mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A)$$

be injective for every natural number n. However there seems to be an agreement for defining K_1-injectivity as above. As we shall see later, in Proposition 4.5.2, if A is properly infinite, then the two definitions agree.

If A is a non-unital C^*-algebra, then A is called K_1-injective (surjective) if $\widetilde{A}$ is K_1-injective (surjective).

3.1 Unital, simple and purely infinite C^*-algebras

As mentioned in the preface, Cuntz was in his paper [12] mainly interested in calculating the K-theory of the Cuntz-algebras $\mathcal{O}_n$. But in the process he also proved that every unital properly infinite C^*-algebra is K_1-surjective and that every unital simple purely infinite C^*-algebra is K_1-injective. In this section we shall give Cuntz's proof of K_1-injectivity of unital simple purely infinite C^*-algebras.

Definition 3.1.1. A C^*-algebra A is said to have property (SP) (small projections) if every non-zero hereditary sub-C^*-algebra of A contains a non-zero projection.

Note that a simple purely infinite C^*-algebra has property (SP). So in the proof of Theorem 3.1.3 we shall use the following Lemma:

Lemma 3.1.2. *Let A be a unital C^*-algebra with property* (SP). *For every unitary u in A there is a non-zero projection $p \in A$ and a unitary $u_0 \in (1-p)A(1-p)$ such that $u \sim_h p + u_0$ in $\mathcal{U}(A)$.*

Proof. If $1 \notin \mathrm{sp}(u)$, then $u \sim_h 1$ in $\mathcal{U}(A)$. Since A has property (SP) there is a projection $p \neq 0$ in A, and $u_0 = 1-p$ is a unitary in $(1-p)A(1-p)$. Since $1 = p + u_0$, it follows that $u \sim_h p + u_0$ in $\mathcal{U}(A)$.

Suppose now that $1 \in \mathrm{sp}(u)$. Let $\varepsilon > 0$ and choose continuous functions $f, g : \mathrm{sp}(u) \to [0,1]$ satisfying that

$$g(1) = 1, \quad f(t)g(t) = g(t),\ t \in \mathrm{sp}(u) \quad \text{and} \quad f(t) = 0 \quad \text{when} \quad |t-1| > \varepsilon.$$

Since the continuous functional calculus is an isometry, $g(u)$ is a non-zero positive element in A. Moreover $\overline{g(u)Ag(u)}$ is a hereditary sub-C^*-algebra of A which implies that there is a non-zero projection $p \in \overline{g(u)Ag(u)}$. Let

$$x = (1-p)u(1-p).$$

First we want to show that $\|(1-p) - x^*x\| \leq \varepsilon^2$:
Since $f(t)g(t) = g(t) = g(t)f(t)$ for $t \in \mathrm{sp}(u)$, it holds that $f(u)g(u) = g(u) = g(u)f(u)$. Hence $f(u)a = a = af(u)$ for every $a \in \overline{g(u)Ag(u)}$ and in particular

$$f(u)p = p = pf(u). \tag{3.1}$$

Consider $t \in \mathrm{sp}(u)$. If $|t-1| > \varepsilon$, then $|f(t)t - f(t)| = 0$ and if $|t-1| \leq \varepsilon$ then

$$|f(t)t - f(t)| \leq |f(t)||t-1| \leq \varepsilon.$$

I.e., $|f(t)t - f(t)| \leq \varepsilon$ for every $t \in \mathrm{sp}(u)$. Thereby,

$$\|f(u)u - f(u)\| \leq \varepsilon. \tag{3.2}$$

It follows by (3.1) and (3.2) that

$$\|pu - p\| = \|pf(u)u - pf(u)\| \leq \|p\|\|f(u)u - f(u)\| \leq \varepsilon \tag{3.3}$$

and similarly

$$\|up - p\| \leq \varepsilon. \tag{3.4}$$

Let $z = pu(1-p)$. We shall prove that $(1-p) - x^*x = z^*z$ and $\|z\| \leq \varepsilon$:
Actually,

$$(pu-p)(1-p) = pu - pup - p + p^2 = pu - pup = z$$

so by (3.3)

$$\|z\| = \|(pu-p)(1-p)\| \leq \|pu-p\|\|1-p\| \leq \varepsilon. \tag{3.5}$$

Furthermore,

$$\begin{aligned}
x^*x &= (1-p)u^*(1-p)(1-p)u(1-p) \\
&= (1-p)u^*(1-p)u(1-p) \\
&= (1-p)u^*u(1-p) - (1-p)u^*pu(1-p) \\
&= (1-p) - z^*z.
\end{aligned}$$

I.e.,

$$(1-p) - x^*x = z^*z \tag{3.6}$$

and therefore

$$\|(1-p) - x^*x\| = \|z^*z\| \leq \varepsilon^2. \tag{3.7}$$

A similar argument gives that

$$\|(1-p) - xx^*\| \leq \varepsilon^2. \tag{3.8}$$

If we choose ε sufficiently small ($< \frac{1}{3}$) it follows from (3.7) and (3.8) that there is a unitary $u_0 \in (1-p)A(1-p)$ such that $\|x - u_0\| < \frac{1}{3}$ (see Exercise 2.8 [35]).
From (3.4) and (3.5) we get

$$\begin{aligned}
\|u - (p+x)\| &= \|u - p - (1-p)u(1-p)\| \\
&= \|up - p + pu - pup\| \\
&\leq \|up - p\| + \|pu(1-p)\| \\
&= \|up - p\| + \|z\| \\
&< \frac{2}{3}.
\end{aligned}$$

Clearly, $p + u_0$ is a unitary in A and

$$\|u - (p + u_0)\| \leq \|u - (p + x)\| + \|x - u_0\| < \frac{2}{3} + \frac{1}{3} = 1.$$

Thus, $u \sim_h p + u_0$. □

Theorem 3.1.3. *Let A be a unital simple purely infinite C^*-algebra. Then A is K_1-injective.*

Proof. Let $u \in \mathcal{U}(A)$ with $[u]_1 = 0$. Since A is simple and purely infinite, A has property (SP). Lemma 3.1.2 gives the existence of a non-zero projection $p \in A$ and a unitary $u_0 \in (1-p)A(1-p)$ such that $u \sim_h p + u_0$ in $\mathcal{U}(A)$. Thus, $[p + u_0]_1 = 0$ and there is a natural number $n \in \mathbb{N}$ such that $(p + u_0) \oplus 1_n \sim_h 1_{n+1}$ in $\mathcal{U}_{n+1}(A)$. We can then find a continuous path $t \to w_t$ of unitaries in $\mathcal{U}_{n+1}(A)$ with $w_0 = 1_{n+1}$ and $w_1 = (p + u_0) \oplus 1_n$. It follows that p is a properly infinite and full projection since A is purely infinite and simple. This implies that $p \oplus 1_n \precsim p$ so there is a partial isometry $v_0 \in M_{1,n+1}(A)$ such that $v_0^* v_0 = p \oplus 1_n$ and $v_0 v_0^* \leq p$. Let

$$v = (1 - p, 0, \ldots, 0) + v_0 \in M_{1,n+1}(A)$$

and

$$v_1 = (1 - p, 0, \ldots, 0) \in M_{1,n+1}(A).$$

Then v_1 is a partial isometry with $v_1^* v_1 = (1-p) \oplus 0_n$ and $v_1 v_1^* = 1 - p$. Since $v_0^* v_0 \perp v_1^* v_1$ and $v_0 v_0^* \perp v_1 v_1^*$, v is a partial isometry with

$$v^* v = v_1^* v_1 + v_0^* v_0 = 1_{n+1}$$

and

$$v v^* = v_1 v_1^* + v_0 v_0^* \leq 1 - p + p = 1.$$

Let

$$z_t = v w_t v^* + (1 - v v^*), \quad t \in [0, 1].$$

Hence

$$\begin{aligned}
z_t^* z_t &= (v w_t^* v^* + (1 - v v^*))(v w_t v^* + (1 - v v^*)) \\
&= v w_t^* v^* v w_t v^* + v w_t^* v^* - v w_t^* v^* v v^* + v w_t v^* + 1 - v v^* - v v^* v w_t v^* - v v^* + v v^* v v^* \\
&= v v^* + v w_t^* v^* - v w_t^* v^* + v w_t v^* + 1 - v v^* - v w_t v^* - v v^* + v v^* \\
&= 1
\end{aligned}$$

and similarly $z_t z_t^* = 1$ so $t \to z_t$ is a continuous path of unitaries in A. Moreover,

$$z_0 = v w_0 v^* + (1 - v v^*) = v 1_{n+1} v^* + 1 - v v^* = 1$$

and

$$\begin{aligned}
z_1 &= vw_1v^* + 1 - vv^* \\
&= v((p+u_0) \oplus 1_n)v^* + 1 - vv^* \\
&= ((1-p, 0, \ldots, 0) + v_0)((p+u_0) \oplus 1_n)((1-p, 0, \ldots, 0)^T + v_0^*) + 1 - vv^* \\
&= u_0 + v_0((p+u_0) \oplus 1_n)v_0^* + 1 - vv^* \\
&= u_0 + v_0(v_0^*v_0)((p+u_0) \oplus 1_n)(v_0^*v_0)v_0^* + 1 - vv^* \\
&= u_0 + v_0(p \oplus 1_n)v_0^* + 1 - vv^* \\
&= u_0 + v_0v_0^* + 1 - v_1v_1^* - v_0v_0^* \\
&= p + u_0.
\end{aligned}$$

Therefore $u \sim_h 1$ in $\mathcal{U}(A)$. □

Example 3.1.4. In [11] it is proved that the Cuntz algebras $\mathcal{O}_n$ are unital, simple and purely infinite C^*-algebras for $2 \leq n \leq \infty$. This means that these C^*-algebras are K_1-injective.

As mentioned before, Lin proved that a C^*-algebra with real rank zero (i.e., every self-adjoint element is in the norm limit of self-adjoint elements of finite spectrum) is K_1-injective. Moreover Zhang proved in [38] that a purely infinite simple C^*-algebra has real rank zero. Thereby Lin's result become a generalization of Cuntz's result.

3.2 A C^*-algebra with stable rank one

Rieffel is in his papers [25] and [26] considering different kinds of "stable ranks" of a Banach algebra; for instance we can mention general stable rank, connected stable rank, and stable rank. We are in particular interested in the stable rank of a C^*-algebra, and Rieffel actually proves that a unital C^*-algebra A is K_1-injective (and K_1-surjective) if it has stable rank one (i.e., the invertible elements in A are dense in A). The result will be used throughout the thesis but the proof itself is also interesting for this thesis. In particular it is used as an inspiration for constructing a specific C^*-algebra $\mathcal{B}$ in Example 4.4.5. This C^*-algebra is important when we are constructing equivalent conditions for a unital properly infinite C^*-algebra being K_1-injective (see Chapter 4).

Let A be a unital C^*-algebra and denote by $\mathrm{GL}_n(A)$ the invertible elements in $M_n(A)$ and by $\mathrm{GL}_n^0(A)$ its connected component containing the unit of $M_n(A)$. Let e_n denote the last standard basis vector in the A-module A^n, and let

$$\mathrm{Lc}_n(A) = \{xe_n : x \in \mathrm{GL}_n(A)\},$$

i.e., $\mathrm{Lc}_n(A)$ consists of the last columns of the matrices in $\mathrm{GL}_n(A)$. Define the subset $\mathrm{Lg}_n(A) \subseteq A^n$ by

$$\mathrm{Lg}_n(A) = \left\{ (a_1, \ldots, a_n) \in A^n \mid \exists\, (b_1, \ldots, b_n) \in A^n : \sum_{i=1}^{n} b_i a_i = 1 \right\}.$$

It is easily seen that for every $y = (a_i) \in \mathrm{Lg}_n(A)$ and every $x \in \mathrm{GL}_n(A)$, then $xy \in \mathrm{Lg}_n(A)$. Since $e_n \in \mathrm{Lg}_n(A)$, we get that $\mathrm{Lc}_n(A) \subseteq \mathrm{Lg}_n(A)$.
For some natural numbers $n \in \mathbb{N}$ we may have that $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$, which we will consider in the following.

Definition 3.2.1. [25, Definition 10.1] For a unital C^*-algebra A, the general stable rank of A, $\mathrm{gsr}(A)$ is defined to be the smallest integer m such that $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$ for all $n \geq m$. I.e.,

$$\forall\, y_1, y_2 \in \mathrm{Lg}_n(A)\, \exists\, x \in \mathrm{GL}_n(A) \; : \; xy_1 = y_2.$$

Lemma 3.2.2. *Let A be a unital C^*-algebra. For each $n \in \mathbb{N}$, $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$ if and only if $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$.*

Proof. Suppose $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$ and let $y_1, y_2 \in \mathrm{Lg}_n(A)$. There exist $x_1, x_2 \in \mathrm{GL}_n(A)$ such that $x_1 e_n = y_1$ and $x_2 e_n = y_2$. Thus, $x_2 x_1^{-1} y_1 = y_2$ and $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$.
Suppose now that $\mathrm{GL}_n(A)$ acts transitively on $\mathrm{Lg}_n(A)$ and let $y \in \mathrm{Lg}_n(A)$. Since $e_n \in \mathrm{Lg}_n(A)$ we can find $x \in \mathrm{GL}_n(A)$ such that $y = xe_n \in \mathrm{Lc}_n(A)$. □

Thereby we get the following result:

Remark 3.2.3. Let A be a unital C^*-algebra. Then $\mathrm{gsr}(A)$ is the smallest integer m such that $\mathrm{Lc}_n(A) = \mathrm{Lg}_n(A)$ for all $n \geq m$.

Definition 3.2.4. [25, Definition 4.7] Let A be a unital C^*-algebra. The connected stable rank of A, $\mathrm{csr}(A)$ is the smallest integer m such that $\mathrm{GL}_n^0(A)$ acts transitively on $\mathrm{Lg}_n(A)$ for all $n \geq m$.

Note by the definition above, we have that $\mathrm{gsr}(A) \leq \mathrm{crs}(A)$.

Definition 3.2.5. [25, Definition 1.4] Let A be a unital C^*-algebra. The stable rank of A, $\mathrm{sr}(A)$ is the smallest integer n such that $\mathrm{Lg}_n(A)$ is dense in A^n. If no such integer exists, we set $\mathrm{sr}(A) = \infty$.

The following proposition gives an easier interpretation of the notion stable rank one, which is almost considered as the definition of a unital C^*-algebra having stable rank one.

Proposition 3.2.6. [25, Proposition 3.1] *Let A be a unital C^*-algebra. Then $\mathrm{sr}(A) = 1$ if and only if $\mathrm{GL}(A)$ is dense in A.*

From [25, Corollary 4.10 and Corollary 8.6] we have the following results which we shall use in the proof of Theorem 3.2.10.

Theorem 3.2.7. *Let A be a unital C^*-algebra. Then*

$$\begin{aligned} \operatorname{csr}(A) &\leq \operatorname{sr}(A)+1 \quad \textit{and} \\ \operatorname{csr}(C(\mathbb{T},A)) &\leq \operatorname{sr}(A)+1. \end{aligned}$$

Let $\varphi_{n-1} : \mathrm{GL}_{n-1}(A) \to \mathrm{GL}_n(A)$ be the group homomorphism given by

$$\varphi_{n-1}(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

and let $\pi_n : \mathrm{GL}_n(A) \to \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ be the quotient map. By the First Homomorphism Theorem there is a unique group homomorphism

$$\psi_{n-1} : \mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \to \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$$

such that the diagram

$$\begin{tikzcd} \mathrm{GL}_{n-1}(A) \arrow[r, "\varphi_{n-1}"] \arrow[d, "\pi_{n-1}"] & \mathrm{GL}_n(A) \arrow[d, "\pi_n"] \\ \mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \arrow[r, dashed, "\psi_{n-1}"] & \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A). \end{tikzcd} \tag{3.9}$$

commutes.

Proposition 3.2.8. [26, Proposition 2.6] *Let A be a unital C^*-algebra. For all $n \geq \operatorname{csr}(A)$ the group homomorphism $\pi_n \circ \varphi_{n-1} : \mathrm{GL}_{n-1}(A) \to \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is surjective.*

Theorem 3.2.9. [26, Theorem 2.9] *Let A be a unital C^*-algebra and let*

$$r = \max(\operatorname{csr}(A), \operatorname{gsr}(C(\mathbb{T},A))).$$

Then for $n \geq r$ the group homomorphism $\psi_{n-1} : \mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \to \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is an isomorphism, and in particular $\mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \cong K_1(A)$.

Proof. Let $n \geq r$ and let $y \in \ker(\psi_{n-1}) \subseteq \mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A)$. There exists $x \in \mathrm{GL}_{n-1}(A)$ such that $y = \pi_{n-1}(x)$ and since

$$\psi_{n-1}(y) = \pi_n(\varphi_{n-1}(x)) = \pi_n\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right),$$

this implies that $\operatorname{diag}(x,1) \sim_h 1$ in $\mathrm{GL}_n(A)$. So we can find a continuous path $t \mapsto u(t)$ of elements in $\mathrm{GL}_n(A)$ such that $u(0) = 1$ and $u(1) = \operatorname{diag}(x,1)$. Define

$$\gamma(t) = u(t)e_n \in \mathrm{Lc}_n(A) \subseteq \mathrm{Lg}_n(A).$$

Hence $\gamma(0) = \gamma(1) = e_n$ and since we associate $C(\mathbb{T}, A)$ with $\{f \in C([0,1], A) : f(0) = f(1)\}$, then $\gamma \in C(\mathbb{T}, A)^n$. In fact $\gamma \in \mathrm{Lg}_n(C(\mathbb{T}, A))$ since $\gamma(t) \in \mathrm{Lg}_n(A)$. Furthermore $\mathrm{Lc}_n(C(\mathbb{T}, A)) = \mathrm{Lg}_n(C(\mathbb{T}, A))$ because $n \geq \mathrm{gsr}(C(\mathbb{T}, A))$, so there is a $v \in \mathrm{GL}_n(C(\mathbb{T}, A))$ such that

$$ve_n = \gamma,$$

where we consider $v : [0,1] \to \mathrm{GL}_n(A)$ to be a continuous function with $v(0) = v(1)$ and $v(t)e_n = \gamma(t)$.
Let

$$w(t) = v(t)v(0)^{-1}.$$

Then $t \mapsto w(t)$ is a continuous path in $\mathrm{GL}_n(A)$ and

$$w(t)^{-1}u(t)e_n = v(0)v(t)^{-1}u(t)e_n = v(0)v(t)^{-1}\gamma(t) = v(0)e_n = \gamma(0) = e_n.$$

Thus, $w(t)^{-1}u(t)$ is on the form

$$w(t)^{-1}u(t) = \begin{pmatrix} z(t) & 0 \\ c(t) & 1 \end{pmatrix}$$

where $t \mapsto z(t)$ is a continuous path in $\mathrm{GL}_{n-1}(A)$. We have that

$$w(0)^{-1}u(0) = v(0)v(0)^{-1}u(0) = 1,$$

which implies that $z(0) = 1$, and

$$\begin{aligned} w(1)^{-1}u(1) &= v(0)v(1)^{-1}u(1) \\ &= v(1)v(1)^{-1}u(1) \\ &= \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

I.e., $z(1) = x$.
Thus $x \sim_h 1$ in $\mathrm{GL}_{n-1}(A)$, and therefore ψ_{n-1} is injective since $y = \pi_{n-1}(x) = 0$.

By Proposition 3.2.8 the map $\pi_n \circ \varphi_{n-1}$ is surjective. This implies that ψ_{n-1} is surjective since the diagram (3.9) commutes.

$K_1(A)$ is isomorphic to the inductive limit $(K_1(A), \{\mu_k\}, \{\psi_k\})$ of the inductive sequence

$$\mathrm{GL}(A)/\mathrm{GL}^0(A) \xrightarrow{\psi_1} \mathrm{GL}_2(A)/\mathrm{GL}_2^0(A) \xrightarrow{\psi_2} \mathrm{GL}_3(A)/\mathrm{GL}_3^0(A) \xrightarrow{\psi_3} \cdots \longrightarrow K_1(A).$$

Since $\psi_{n-1} : \mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \to \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A)$ is an isomorphism for $n \geq r$, it follows that $\mu_{n-1} : \mathrm{GL}_{n-1}(A)/\mathrm{GL}_{n-1}^0(A) \to K_1(A)$ is an isomorphism. □

Theorem 3.2.10. *Let A be a unital C^*-algebra. For all $n \geq \mathrm{sr}(A)$ the group homomorphism $\psi_n : \mathrm{GL}_n(A)/\mathrm{GL}_n^0(A) \to \mathrm{GL}_{n+1}(A)/\mathrm{GL}_{n+1}^0(A)$ is an isomorphism, and in particular $\mathrm{GL}_n(A)/\mathrm{GL}_n^0(A) \cong K_1(A)$.*

Proof. From Theorem 3.2.7 we have that

$$\mathrm{csr}(A) \le \mathrm{sr}(A) + 1$$

and

$$\mathrm{gsr}(C(\mathbb{T}, A)) \le \mathrm{csr}(C(\mathbb{T}, A)) \le \mathrm{sr}(A) + 1.$$

The result now follows from Theorem 3.2.9. □

Theorem 3.2.11. *Let A be a unital C^*-algebra with* $\mathrm{sr}(A) = 1$. *Then* $\mathcal{U}(A)/\mathcal{U}^0(A) \cong K_1(A)$.

Proof. Define a unital group homomorphism $\omega : \mathrm{GL}(A) \to \mathcal{U}(A)$ by

$$\omega(x) = x|x|^{-1}.$$

Note that $\omega(x)$ is the unitary part of the polar decomposition of the invertible element x. Furthermore ω is surjective since $\omega(u) = u$ for a unitary $u \in \mathcal{U}(A)$.
By the First Homomorphism Theorem there is a unital surjective group homomorphism $\rho : \mathrm{GL}(A)/\mathrm{GL}^0(A) \to \mathcal{U}(A)/\mathcal{U}^0(A)$ such that the diagram

$$\begin{tikzcd}
\mathrm{GL}(A) \arrow[r, "\omega"] \arrow[d, "\pi_1"] & \mathcal{U}(A) \arrow[d, "\pi"] \\
\mathrm{GL}(A)/\mathrm{GL}^0(A) \arrow[r, dashed, "\rho"] & \mathcal{U}(A)/\mathcal{U}^0(A)
\end{tikzcd}$$

commutes, where $\pi : \mathcal{U}(A) \to \mathcal{U}(A)/\mathcal{U}^0(A)$ is the quotient map.
We shall show that ρ is also injective:

Let $y \in \ker(\rho) \subseteq \mathrm{GL}(A)/\mathrm{GL}^0(A)$, and find an $x \in \mathrm{GL}(A)$ such that $y = \pi_1(x)$. I.e., $\pi(\omega(x)) = \rho(y) = 0$, and therefore $\omega(x) \sim_h 1$ in $\mathcal{U}(A)$.
But $\omega(x) \sim_h x$ in $\mathrm{GL}(A)$ (c.f. [35, Proposition 2.1.8]), so $x \sim_h 1$ in $\mathrm{GL}(A)$. Hence $y = 0$, and ρ is injective.

Therefore $\mathcal{U}(A)/\mathcal{U}^0(A) \cong \mathrm{GL}(A)/\mathrm{GL}^0(A)$, and by Theorem 3.2.10 $\mathrm{GL}(A)/\mathrm{GL}^0(A) \cong K_1(A)$. □

In [25] Rieffel gives examples of C^*-algebras with stable rank one, and therefore we get examples of K_1-injective (and K_1-surjective) C^*-algebras:

3.2.1 Examples of K_1-injective C^*-algebras with stable rank one

Invertible elements are dense in $M_n(\mathbb{C})$ for every $n \in \mathbb{N}$, so $\mathrm{sr}(M_n(\mathbb{C})) = 1$. Hence every direct sum of matrix algebras have stable rank one, which implies that $\mathrm{sr}(A) = 1$, when A is a finite dimensional C^*-algebra.

Moreover it is shown that stable rank one persists under inductive limits, and thereby AF-algebras have stable rank one.

From [33, Theorem 6.7] it follows that every simple unital finite $\mathcal{Z}$-absorbing C^*-algebra has stable rank one. In particular $\mathrm{sr}(\mathcal{Z}) = 1$.

By Theorem 3.2.11 it then follows that finite dimensional C^*-algebras, AF-algebras and $\mathcal{Z}$ are examples of K_1-injective C^*-algebras with stable rank one.

Chapter 4

Properly infinite $C(X)$-algebras and K_1-injectivity

This chapter is based on the paper [7] which is a joint work with Mikael Rørdam and Etienne Blanchard. The problem that we mainly are concerned with is whether any unital $C(X)$-algebra with properly infinite fibres is itself properly infinite. An analogous study was carried out in the recent paper [17] where it was decided when $C(X)$-algebras, whose fibres are either stable or absorb tensorially a given strongly self-absorbing C^*-algebra, themselves have the same property. This was answered in the affirmative in [17] under the crucial assumption that the dimension of the space X is finite, and counterexamples were given in the infinite dimensional case.

Along similar lines, Dadarlat, [13], recently proved that $C(X)$-algebras, whose fibres are Cuntz algebras, are trivial under some K-theoretical conditions provided that the space X is finite dimensional.

The property of being properly infinite turns out to behave very differently than the property of being stable or of absorbing a strongly self-absorbing C^*-algebra. It is relatively easy to see (Lemma 4.2.9) that if a fibre A_x of a $C(X)$-algebra A is properly infinite, then A_F is properly infinite for some closed neighborhood F of x. The (possible) obstruction to proper infiniteness of the $C(X)$-algebra is hence not local. Such an obstruction is also not related to the possible complicated structure of the space X, as we can show that a counterexample, if it exists, can be taken to be a (specific) $C([0,1])$-algebra (Example 4.4.1 and Theorem 4.5.5). The problem appears to be related with some rather subtle internal structure properties of properly infinite C^*-algebras.

In this chapter we also raise the question whether any properly infinite C^*-algebra is K_1-injective. We show that every properly infinite C^*-algebra is K_1-injective if and only if every $C(X)$-algebra with properly infinite fibres itself is properly infinite. We also show that a *matrix algebra* over any such $C(X)$-algebra is properly infinite. Examples of unital C^*-algebras A, where $M_n(A)$ is properly infinite for some natural number $n \geq 2$ but where $M_{n-1}(A)$ is not properly infinite, are known, see [30] and [32], but still quite exotic.

We relate the question of whether a given properly infinite C^*-algebra is K_1-injective to questions regarding homotopy of projections (Proposition 4.5.1). In particular we show that our main questions are equivalent to the following question: is any non-trivial projection in the first copy of $\mathcal{O}_\infty$ in the full unital universal free product $\mathcal{O}_\infty * \mathcal{O}_\infty$ homotopic to any (non-trivial) projection in the second copy of $\mathcal{O}_\infty$? The specific $C([0,1])$-algebra, mentioned above, is perhaps not surprisingly a sub-algebra of $C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$.

Using ideas implicit in Rieffel's paper, [26], we construct in Section 4.4 a $C(\mathbb{T})$-algebra $\mathcal{B}$ for each C^*-algebra A and for each unitary $u \in A$ for which $\operatorname{diag}(u, 1)$ is homotopic to $1_{M_2(A)}$; and $\mathcal{B}$ is non-trivial if u is not homotopic to 1_A. In this way we relate our question about proper infiniteness of $C(X)$-algebras to a question about K_1-injectivity.

4.1 Introduction to $C(X)$-algebras

Let X be a compact Hausdorff space and let $C(X)$ be the C^*-algebra of continuous functions on X with values in the complex field $\mathbb{C}$.

Definition 4.1.1. A $C(X)$-algebra is a C^*-algebra A endowed with a unital *-homomorphism from $C(X)$ to the center of the multiplier C^*-algebra $\mathcal{M}(A)$ of A.

If A is as above and $Y \subseteq X$ is a closed subset, then we put $I_Y = C_0(X \setminus Y)A$, which is a closed two-sided ideal in A. We set $A_Y = A/I_Y$ and denote the quotient map by π_Y.

For an element $a \in A$ we put $a_Y = \pi_Y(a)$, and if Y consists of a single point x, we will write A_x, I_x, π_x and a_x in the place of $A_{\{x\}}, I_{\{x\}}, \pi_{\{x\}}$ and $a_{\{x\}}$, respectively. We say that A_x is the *fibre* of A at x.

The function

$$x \mapsto \|a_x\| = \inf\{\| [1 - f + f(x)]a\| : f \in C(X)\}$$

is upper semi-continuous for all $a \in A$ (as one can see using the right-hand side identity above). A $C(X)$-algebra A is said to be *continuous* (or to be a *continuous C^*-bundle over X*) if the function $x \mapsto \|a_x\|$ is actually continuous for all elements a in A.

Example 4.1.2. Let X be a compact Hausdorff space and let D be a unital C^*-algebra. Then $A = C(X, D)$ is a $C(X)$-algebra with fibres $A_x = D$ for all $x \in X$.
The $C(X)$-algebra A is called a trivial $C(X)$-algebra.

Proof. Since $A \cong C(X) \otimes D$ we define

$$\mu : C(X) \to C(X) \otimes D$$

by

$$\mu(f) = f \otimes 1.$$

Since $C(X)$ is a commutative C^*-algebra, $\mu(C(X)) \subseteq Z(C(X) \otimes D)$, so A is a $C(X)$-algebra.

Let $x \in X$ and let $\varphi : A \to D$ be the surjective *-homomorphism given by

$$\varphi(f) = f(x).$$

By the First Homomorphism Theorem, there is a *-isomorphism $\psi : A_x \to D$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi_x"'] & D \\ A_x \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

commutes. $\square$

Example 4.1.3. Let X be a compact Hausdorff space and let D be a unital C^*-algebra. For every projection $p \in C(X, D)$, then $A = pC(X, D)p$ is a $C(X)$-algebra with fibres $A_x = p(x)Dp(x)$ for $x \in X$.

Proof. From Example 4.1.2 there is a unital *-homomorphism $\mu : C(X) \to Z(C(X, D))$, so define $\bar{\mu} : C(X) \to A$ by

$$\bar{\mu}(f) = p\mu(f)p.$$

Since p is a projection and $\mu(C(X)) \subseteq Z(C(X, D))$ it follows that $\bar{\mu}(C(X)) \subseteq Z(A)$. Hence A is a $C(X)$-algebra.

Let $x \in X$ and let $\varphi : A \to p(x)Dp(x)$ be given by

$$\varphi(pfp) = p(x)f(x)p(x), \quad f \in C(X, D).$$

Then φ is surjective and by the First Homomorphism Theorem, it holds that $A_x \cong p(x)Dp(x)$. $\square$

As a motivation for the next section, we shall now consider an example of a $C(X)$-algebra where proper infiniteness of the fibres ensures proper infiniteness of the $C(X)$-algebra itself:

Example 4.1.4. Let X be a compact Hausdorff space and let D be a unital C^*-algebra. If $A = C(X, D)$ is a trivial $C(X)$-algebra, then A is properly infinite if the fibres $A_x = D$, $x \in X$, are properly infinite.

Proof. Since A is isomorphic to $C(X) \otimes D$, and the tensor product of two unital C^*-algebras is properly infinite if one of the two C^*-algebras is properly infinite, then A is properly infinite if and only if D is properly infinite. $\square$

4.2 $C(X)$-algebras with properly infinite fibres

In this section we will study stability properties of proper infiniteness under upper semi-continuous deformations using the *Cuntz-Toeplitz algebra* which is defined as follows. For all integers $n \geq 2$ the Cuntz-Toeplitz algebra $\mathcal{T}_n$ is the universal C^*-algebra generated by n isometries $s_1, \dots, s_n$ satisfying the relation

$$s_1 s_1^* + \cdots + s_n s_n^* \leq 1.$$

Remark 4.2.1. A unital C^*-algebra A is properly infinite if and only if $\mathcal{T}_n$ embeds unitally into A for some $n \geq 2$, in which case $\mathcal{T}_n$ embeds unitally into A for all $n \geq 2$.

A powerful tool in the classification of C^*-algebras is the study of their projections.We state below more formally three more or less well-known results that will be used frequently throughout this chapter, the first of which is due to Cuntz, [12].

Proposition 4.2.2 (Cuntz). *Let A be a C^*-algebra which contains at least one properly infinite, full projection.*

(i) *Let p and q be properly infinite, full projections in A. Then $[p]_0 = [q]_0$ in $K_0(A)$ if and only if $p \sim q$.*

(ii) *For each element $g \in K_0(A)$ there is a properly infinite, full projection $p \in A$ such that $g = [p]_0$.*

The second statement is a variation of the Whitehead lemma.

Lemma 4.2.3. *Let A be a unital C^*-algebra.*

(i) *Let v be a partial isometry in A such that $1 - vv^*$ and $1 - v^*v$ are properly infinite and full projections. Then there is a unitary element u in A such that $[u]_1 = 0$ in $K_1(A)$ and $v = uv^*v$, i.e., u extends v.*

(ii) *Let u be a unitary element A such that $[u]_1 = 0$ in $K_1(A)$. Suppose there exists a projection $p \in A$ such that $\|up - pu\| < 1$ and p and $1 - p$ are properly infinite and full. Then u belongs to $\mathcal{U}^0(A)$.*

Proof. (i). It follows from Proposition 4.2.2 (i) that $1 - v^*v \sim 1 - vv^*$, so there is a partial isometry w such that $1 - v^*v = w^*w$ and $1 - vv^* = ww^*$. Now, $z = v + w$ is a unitary element in A with $zv^*v = v$. The projection $1 - v^*v$ is properly infinite and full, so $1 \precsim 1 - v^*v$, which implies that there is an isometry s in A with $ss^* \leq 1 - v^*v$. As $-[z]_1 = [z^*]_1 = [sz^*s^* + (1 - ss^*)]_1$ in $K_1(A)$ (see eg. [35, Exercise 8.9 (i)]), we see that $u = z(sz^*s^* + (1 - ss^*))$ is as desired.

(ii). Put $x = pup + (1-p)u(1-p)$ and note that $\|u - x\| < 1$. It follows that x is invertible in A and that $u \sim_h x$ in $\mathrm{GL}(A)$. Let $x = v|x|$ be the polar decomposition of x, where

$|x| = (x^*x)^{1/2}$ and $v = x|x|^{-1}$ is unitary. Then $u \sim_h v$ in $\mathcal{U}(A)$ (c.f. Proposition 2.4.4), and $pv = vp$. We proceed to show that v belongs to $\mathcal{U}^0(A)$ (which will entail that u belongs to $\mathcal{U}^0(A)$).

Write $v = v_1v_2$, where

$$v_1 = pvp + (1-p), \qquad v_2 = p + (1-p)v(1-p).$$

As $1-p \precsim p$ we can find a symmetry t in A such that $t(1-p)t \leq p$. As t belongs to $\mathcal{U}^0(A)$ (being a symmetry), we conclude that $v_2 \sim_h tv_2t$, and one checks that tv_2t is of the form $w + (1-p)$ for some unitary w in pAp. It follows that v is homotopic to a unitary of the form $v_0 + (1-p)$, where v_0 is a unitary in pAp. We can now apply eg. [35, Exercise 8.11] to conclude that $v \sim_h 1$ in $\mathcal{U}(A)$. □

We remind the reader that if p, q are projections in a unital C^*-algebra A, then p and q are homotopic, in symbols $p \sim_h q$, (meaning that they can be connected by a continuous path of projections in A) if and only if $q = upu^*$ for some $u \in \mathcal{U}^0(A)$, eg. cf. [35, Proposition 2.2.6].

Proposition 4.2.4. *Let A be a unital C^*-algebra. Let p and q be two properly infinite, full projections in A such that $p \sim q$. Suppose that there exists a properly infinite, full projection $r \in A$ such that $p \perp r$ and $q \perp r$. Then $p \sim_h q$.*

Proof. Take a partial isometry $v_0 \in A$ such that $v_0^*v_0 = p$ and $v_0v_0^* = q$. Take a subprojection r_0 of r such that r_0 and $r - r_0$ both are properly infinite and full. Put $v = v_0 + r_0$. Then $vpv^* = q$ and $vr_0 = r_0 = r_0v$. Note that $1 - v^*v$ and $1 - vv^*$ are properly infinite and full (because they dominate the properly infinite, full projection $r - r_0$). Use Lemma 4.2.3 (i) to extend v to a unitary $u \in A$ with $[u]_1 = 0$ in $K_1(A)$. Now, $upu^* = q$ and $ur_0 = vr_0 = r_0 = r_0v = r_0u$. Hence $u \in \mathcal{U}^0(A)$ by Lemma 4.2.3 (ii), and so $p \sim_h q$ as desired. □

Let A_1, A_2, and B be C^*-algebras, let $\pi_1 : A_1 \to B$ and $\pi_2 : A_2 \to B$ be *-homomorphisms. We seek a C^*-algebra A and *-homomorphisms $\varphi_1 : A \to A_1$ and $\varphi_2 : A \to A_2$ making the following diagram commutative:

$$\begin{tikzcd} & A \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

and which is universal in the sense that if C is any C^*-algebra and $\omega_1 : C \to A_1$, $\omega_2 : C \to A_2$ are *-homomorphisms satisfying that $\pi_1 \circ \omega_1 = \pi_2 \circ \omega_2$, then there is a unique *-homomorphism $\theta : C \to A$ such that $\omega_i = \varphi_i \circ \theta$, $i = 1, 2$.

Any such A is unique up to isomorphism. One way of constructing A is as

$$\{(a_1, a_2) : \pi_1(a_1) = \pi_2(a_2)\} \subseteq A_1 \oplus A_2.$$

Definition 4.2.5. [1, Definition II.8.4.9] With the notation from above, A is called the pull-back of (A_1, A_2) along (π_1, π_2).

Proposition 4.2.6. *Let A be a unital C^*-algebra that is the pull-back of two unital, properly infinite C^*-algebras A_1 and A_2 along the *-epimorphisms $\pi_1\colon A_1 \to B$ and $\pi_2\colon A_2 \to B$:*

$$\begin{tikzcd} & A \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

Then $M_2(A)$ is properly infinite. Moreover, if B is K_1-injective, then A itself is properly infinite.

Proof. Take unital embeddings $\sigma_i\colon \mathcal{T}_3 \to A_i$ for $i = 1, 2$, where $\mathcal{T}_3$ is the Cuntz-Toeplitz algebra (defined earlier), and put

$$v = \sum_{j=1}^{2} (\pi_1 \circ \sigma_1)(t_j)(\pi_2 \circ \sigma_2)(t_j^*),$$

where t_1, t_2, t_3 are the canonical generators of $\mathcal{T}_3$. Note that v is a partial isometry with $(\pi_1 \circ \sigma_1)(t_j) = v(\pi_2 \circ \sigma_2)(t_j)$ for $j = 1, 2$. As $(\pi_1 \circ \sigma_1)(t_3 t_3^*) \leq 1 - vv^*$ and $(\pi_2 \circ \sigma_2)(t_3 t_3^*) \leq 1 - v^*v$, Lemma 4.2.3 (i) yields a unitary $u \in B$ with $[u]_1 = 0$ in $K_1(B)$ and with $(\pi_1 \circ \sigma_1)(t_j) = u(\pi_2 \circ \sigma_2)(t_j)$ for $j = 1, 2$.

If B is K_1-injective, then u belongs to $\mathcal{U}^0(B)$, whence u lifts to a unitary $v \in A_2$. Define $\widetilde{\sigma}_2\colon \mathcal{T}_2 \to A_2$ by $\widetilde{\sigma}_2(t_j) = v\sigma_2(t_j)$ for $j = 1, 2$ (observing that t_1, t_2 generate $\mathcal{T}_2$). Then $\pi_1 \circ \sigma_1 = \pi_2 \circ \widetilde{\sigma}_2$, which by the universal property of the pull-back implies that σ_1 and $\widetilde{\sigma}_2$ lift to a (necessarily unital) embedding $\sigma\colon \mathcal{T}_2 \to A$, thus forcing A to be properly infinite.

In the general case (where B is not necessarily K_1-injective) u may not lift to a unitary element in A_2, but $\operatorname{diag}(u, u)$ does lift to a unitary element v in $M_2(A_2)$ by Lemma 4.2.3 (ii) (applied with $p = \operatorname{diag}(1, 0)$). Define unital embeddings $\widetilde{\sigma}_i\colon \mathcal{T}_2 \to M_2(A_i)$, $i = 1, 2$, by

$$\widetilde{\sigma}_1(t_j) = \begin{pmatrix} \sigma_1(t_j) & 0 \\ 0 & \sigma_1(t_j) \end{pmatrix}, \qquad \widetilde{\sigma}_2(t_j) = v \begin{pmatrix} \sigma_2(t_j) & 0 \\ 0 & \sigma_2(t_j) \end{pmatrix},$$

for $j = 1, 2$. As $(\pi_1 \otimes \mathrm{id}_{M_2}) \circ \widetilde{\sigma}_1 = (\pi_2 \otimes \mathrm{id}_{M_2}) \circ \widetilde{\sigma}_2$, the unital embeddings $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ lift to a (necessarily unital) embedding of $\mathcal{T}_2$ into $M_2(A)$, thus completing the proof. $\square$

Question 4.2.7. *Is the pull-back of any two properly infinite unital C^*-algebras again properly infinite?*

As mentioned in the introduction, one cannot in general conclude that A is properly infinite if one knows that $M_n(A)$ is properly infinite for some $n \geq 2$.

One obvious way of obtaining an answer to Question 4.2.7, in the light of the last statement in Proposition 4.2.6, is to answer the question below in the affirmative:

Question 4.2.8. *Is every properly infinite unital C^*-algebra K_1-injective?*

We shall see later, in Section 4.5, that the two questions above in fact are equivalent.

The lemma below, which shall be used several times in this chapter, shows that one can lift proper infiniteness from a fibre of a $C(X)$-algebra to a whole neighborhood of that fibre.

Lemma 4.2.9. *Let X be a compact Hausdorff space, let A be a unital $C(X)$-algebra, let $x \in X$, and suppose that the fibre A_x is properly infinite. Then A_F is properly infinite for some closed neighborhood F of x.*

Proof. Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a decreasing net of closed neighborhoods of $x \in X$, fulfilling that $\bigcap_{\lambda\in\Lambda} F_\lambda = \{x\}$, and set $I_\lambda = C_0(X\setminus F_\lambda)A$. Then $\{I_\lambda\}_{\lambda\in\Lambda}$ is an increasing net of ideals in A, $A_{F_\lambda} = A/I_\lambda$, $I := \overline{\bigcup_{\lambda\in\Lambda} I_\lambda} = C_0(X\setminus\{x\})A$, and $A_x = A/I$.

By the assumption that A_x is properly infinite there is a unital *-homomorphism $\psi\colon \mathcal{T}_2 \to A_x$, and since $\mathcal{T}_2$ is semi-projective there is a $\lambda_0 \in \Lambda$ and a unital *-homomorphism $\varphi\colon \mathcal{T}_2 \to A_{F_{\lambda_0}}$ making the diagram

$$\begin{tikzcd} & A_{F_{\lambda_0}} \arrow[d, "\pi_x"] \\ \mathcal{T}_2 \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & A_x \end{tikzcd}$$

commutative. We can thus take F to be F_{λ_0}. $\square$

Theorem 4.2.10. *Let A be a unital $C(X)$-algebra where X is a compact Hausdorff space. If all fibres A_x, $x \in X$, are properly infinite, then some matrix algebra over A is properly infinite.*

Proof. By Lemma 4.2.9, X can be covered by finitely many closed sets $F_1, F_2, \ldots, F_n$ such that A_{F_j} is properly infinite for each j. Put $G_j = F_1 \cup F_2 \cup \cdots \cup F_j$. For each $j = 1, 2, \ldots, n-1$ we have a pull-back diagram

$$\begin{tikzcd} & A_{G_{j+1}} \arrow[dl] \arrow[dr] & \\ A_{G_j} \arrow[dr] & & A_{F_{j+1}} \arrow[dl] \\ & A_{G_j\cap F_{j+1}} & \end{tikzcd}$$

We know that $M_{2^{j-1}}(A_{G_j})$ is properly infinite when $j = 1$. Proposition 4.2.6 (applied to the diagram above tensored with $M_{2^{j-1}}(\mathbb{C})$) tells us that $M_{2^j}(A_{G_{j+1}})$ is properly infinite if $M_{2^{j-1}}(A_{G_j})$ is properly infinite. Hence $M_{2^{n-1}}(A)$ is properly infinite. □

Remark 4.2.11. Uffe Haagerup has suggested another way to prove Theorem 4.2.10: If no matrix-algebra over A is properly infinite, then there exists a bounded non-zero lower semi-continuous quasi-trace on A, see [16] and [2, page 327], and hence also an extremal quasi-trace. Now, if A is also a $C(X)$-algebra for some compact Hausdorff space X, this implies that there is a bounded non-zero lower semi-continuous quasitrace on A_x for (at least) one point $x \in X$ (see eg. [17, Proposition 3.7]). But then the fibre A_x cannot be properly infinite.

Question 4.2.12. *Is any unital $C(X)$-algebra A properly infinite if all its fibres A_x, $x \in X$, are properly infinite?*

We shall show in Section 4.5 that the question above is equivalent to Question 4.2.7 which again is equivalent to Question 4.2.8.

4.3 Lower semi-continuous fields of properly infinite C^*-algebras

Let us briefly discuss whether the results from Section 4.2 can be extended to lower semi-continuous C^*-bundles $(A, \{\sigma_x\})$ over a compact Hausdorff space X. Recall that any such *separable* lower semi-continuous C^*-bundle admits a faithful $C(X)$-linear representation on a Hilbert $C(X)$-module E such that, for all $x \in X$, the fibre $\sigma_x(A)$ is isomorphic to the induced image of A in $\mathcal{L}(E_x)$, [4]. Thus, the problem boils down to the following: Given a separable Hilbert $C(X)$-module E with infinite dimensional fibres E_x, such that the unit p of the C^*-algebra $\mathcal{L}_{C(X)}(E)$ of bounded adjointable $C(X)$-linear operators acting on E has a properly infinite image in $\mathcal{L}(E_x)$ for all $x \in X$. Is the projection p itself properly infinite in $\mathcal{L}_{C(X)}(E)$?

Dixmier and Douady proved that this is always the case if the space X has finite topological dimension, [14]. But it does not hold anymore in the infinite dimensional case, see [14, §16, Corollaire 1] and [32], even if X is contractible, [5, Corollary 3.7].

4.4 Two examples

We describe here two examples of continuous fields; the first is over the interval and the second (which really is a class of examples) is over the circle.

Example 4.4.1. Let $(\mathcal{O}_\infty * \mathcal{O}_\infty, (\iota_1, \iota_2))$ be the universal unital free product of two copies of $\mathcal{O}_\infty$, and let $\mathcal{A}$ be the unital sub-C^*-algebra of $C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$ given by

$$\mathcal{A} = \{f \in C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty) : f(0) \in \iota_1(\mathcal{O}_\infty),\ f(1) \in \iota_2(\mathcal{O}_\infty)\}.$$

Observe that $\mathcal{A}$ (in a canonical way) is a $C([0,1])$-algebra with fibres

$$\mathcal{A}_t = \begin{cases} \iota_1(\mathcal{O}_\infty), & t = 0, \\ \mathcal{O}_\infty * \mathcal{O}_\infty, & 0 < t < 1, \\ \iota_2(\mathcal{O}_\infty), & t = 1 \end{cases} \cong \begin{cases} \mathcal{O}_\infty, & t = 0, \\ \mathcal{O}_\infty * \mathcal{O}_\infty, & 0 < t < 1, \\ \mathcal{O}_\infty, & t = 1. \end{cases}$$

In particular, all fibres of $\mathcal{A}$ are properly infinite.

One claim to fame of the example above is that the question below is equivalent to Question 4.2.12 above. Hence, to answer Question 4.2.12 in the affirmative (or in the negative) we need only consider the case where $X = [0,1]$, and we need only worry about this one particular $C([0,1])$-algebra (which of course is bad enough!).

Question 4.4.2. *Is the $C([0,1])$-algebra $\mathcal{A}$ from Example 4.4.1 above properly infinite?*

The three equivalent statements in the proposition below will in Section 4.5 be shown to be equivalent to Question 4.4.2.

Proposition 4.4.3. *The following three statements concerning the $C([0,1])$-algebra $\mathcal{A}$ and the C^*-algebra $(\mathcal{O}_\infty * \mathcal{O}_\infty, (\iota_1, \iota_2))$ defined above are equivalent:*

(i) *$\mathcal{A}$ contains a non-trivial projection (i.e., a projection other than 0 and 1).*

(ii) *There are non-zero projections $p, q \in \mathcal{O}_\infty$ such that $p \neq 1$, $q \neq 1$, and $\iota_1(p) \sim_h \iota_2(q)$.*

(iii) *Let s be any isometry in $\mathcal{O}_\infty$. Then $\iota_1(ss^*) \sim_h \iota_2(ss^*)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$.*

We warn the reader that all three statements above could be false.

Proof. (i) $\Rightarrow$ (ii). Let e be a non-trivial projection in $\mathcal{A}$. Let $\pi_t \colon \mathcal{A} \to \mathcal{A}_t$, $t \in [0,1]$, denote the fibre map. As $\mathcal{A} \subseteq C([0,1], \mathcal{O}_\infty * \mathcal{O}_\infty)$, the mapping $t \mapsto \pi_t(e) \in \mathcal{O}_\infty * \mathcal{O}_\infty$ is continuous, so in particular, $\pi_0(e) \sim_h \pi_1(e)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$. The mappings ι_1 and ι_2 are injective, so there are projections $p, q \in \mathcal{O}_\infty$ such that $\pi_0(e) = \iota_1(p)$ and $\pi_1(e) = \iota_2(q)$. The projections p and q are non-zero because the mapping $t \mapsto \|\pi_t(e)\|$ is continuous and not constant equal to 0. Similarily, $1 - p$ and $1 - q$ are non-zero because $1 - e$ is non-zero.

(ii) $\Rightarrow$ (iii). Take non-trivial projections $p, q \in \mathcal{O}_\infty$ such that $\iota_1(p) \sim_h \iota_2(q)$. Take a unitary v in $\mathcal{U}^0(\mathcal{O}_\infty * \mathcal{O}_\infty)$ with $\iota_2(q) = v\iota_1(p)v^*$. Let $s \in \mathcal{O}_\infty$ be an isometry. If s is unitary, then $\iota_1(ss^*) = 1 = \iota_2(ss^*)$ and there is nothing to prove. Suppose that s is non-unitary. Then ss^* is homotopic to a subprojection p_0 of p and to a subprojection q_0 of q (use that p and

q are properly infinite and full, then Lemma 4.2.3 (i), and last the fact that the unitary group of $\mathcal{O}_\infty$ is connected). Hence $\iota_1(ss^*) \sim_h \iota_1(p_0) \sim_h v\iota_1(p_0)v^*$ and $\iota_2(ss^*) \sim_h \iota_2(q_0)$, so we need only show that $v\iota_1(p_0)v^* \sim_h \iota_2(q_0)$. But this follows from Proposition 4.2.4 with $r = 1 - \iota_2(q) = \iota_2(1-q)$, as we note that $p_0 \sim 1 \sim q_0$ in $\mathcal{O}_\infty$, whence

$$\iota_2(q_0) \sim \iota_2(1) = 1 = \iota_1(1) \sim \iota_1(p_0) \sim v\iota_1(p_0)v^*.$$

(iii) $\Rightarrow$ (i). Take a non-unitary isometry $s \in \mathcal{O}_\infty$. Then $\iota_1(ss^*) \sim_h \iota_2(ss^*)$, and so there is a continuous function $e\colon [0,1] \to \mathcal{O}_\infty * \mathcal{O}_\infty$ such that $e(t)$ is a projection for all $t \in [0,1]$, $e(0) = \iota_1(ss^*)$ and $e(1) = \iota_2(ss^*)$. But then e is a non-trivial projection in $\mathcal{A}$. $\square$

It follows from Theorem 4.2.10 that some matrix algebra over $\mathcal{A}$ (from Example 4.4.1) is properly infinite. We can sharpen that statement as follows:

Proposition 4.4.4. *$M_2(\mathcal{A})$ is properly infinite; and if $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective, then $\mathcal{A}$ itself is properly infinite.*

It follows from Theorem 4.5.5 below that $\mathcal{A}$ is properly infinite if *and only if* $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective.

Proof. We have a pull-back diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl] \arrow[dr] & \\ \mathcal{A}_{[0,\frac{1}{2}]} \arrow[dr, "\pi_{1/2}"'] & & \mathcal{A}_{[\frac{1}{2},1]} \arrow[dl, "\pi_{1/2}"] \\ & \mathcal{O}_\infty * \mathcal{O}_\infty & \end{tikzcd}$$

One can unitally embed $\mathcal{O}_\infty$ into $\mathcal{A}_{[0,\frac{1}{2}]}$ via ι_1, so $\mathcal{A}_{[0,\frac{1}{2}]}$ is properly infinite, and a similar argument shows that $\mathcal{A}_{[\frac{1}{2},1]}$ is properly infinite. The two statements now follow from Proposition 4.2.6. $\square$

The example below, which will be the focus of the rest of this section, and in parts also of Section 4.5, is inspired by arguments from Rieffel's paper [26].

Example 4.4.5. Let A be a unital C^*-algebra, and let v be a unitary element in A such that

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in} \quad \mathcal{U}_2(A).$$

Let $t \mapsto u_t$ be a continuous path of unitaries in $\mathcal{U}_2(A)$ such that $u_0 = 1$ and $u_1 = \operatorname{diag}(v, 1)$. Put

$$p(t) = u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_t^* \in M_2(A),$$

and note that $p(0) = p(1)$. Identifying, for each C^*-algebra D, $C(\mathbb{T}, D)$ with the algebra of all continuous functions $f\colon [0,1] \to D$ such that $f(1) = f(0)$, we see that p belongs to $C(\mathbb{T}, M_2(A))$. Put

$$\mathcal{B} = pC(\mathbb{T}, M_2(A))p,$$

and note that $\mathcal{B}$ is a unital (sub-trivial) $C(\mathbb{T})$-algebra, being a corner of the trivial $C(\mathbb{T})$-algebra $C(\mathbb{T}, M_2(A))$. The fibres of $\mathcal{B}$ are

$$\mathcal{B}_t = p(t)M_2(A)p(t) \cong A$$

for all $t \in \mathbb{T}$.

Summing up, for each unital C^*-algebra A, for each unitary v in A for which $\operatorname{diag}(v,1) \sim_h 1$ in $\mathcal{U}_2(A)$, and for each path $t \mapsto u_t \in \mathcal{U}_2(A)$ implementing this homotopy we get a $C(\mathbb{T})$-algebra $\mathcal{B}$ with fibres $\mathcal{B}_t \cong A$. We shall investigate this class of $C(\mathbb{T})$-algebras below.

Lemma 4.4.6. *In the notation of Example 4.4.5,*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - p \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{in} \quad C(\mathbb{T}, M_2(A)).$$

In particular, p is stably equivalent to $\operatorname{diag}(1,0)$.

Proof. Put

$$v_t = u_t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,1].$$

Then

$$v_0 = u_0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_1 = u_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so v belongs to $C(\mathbb{T}, M_2(A))$. It is easy to see that $v_t^*v_t = \operatorname{diag}(0,1)$ and $v_tv_t^* = 1 - p(t)$, and so the lemma is proved. $\square$

Proposition 4.4.7. *Let A, $v \in \mathcal{U}(A)$, and $\mathcal{B}$ be as in Example 4.4.5. Conditions* (i) *and* (ii) *below are equivalent for any unital C^*-algebra A, and all three conditions are equivalent if A in addition is assumed to be properly infinite.*

(i) $v \sim_h 1$ *in* $\mathcal{U}(A)$.

(ii) $p \sim \operatorname{diag}(1_A, 0)$ *in* $C(\mathbb{T}, M_2(A))$.

(iii) *The $C(\mathbb{T})$-algebra $\mathcal{B}$ is properly infinite.*

Proof. (ii) $\Rightarrow$ (i). Suppose that $p \sim \mathrm{diag}(1,0)$ in $C(\mathbb{T}, M_2(A))$. Then there is a $w \in C(\mathbb{T}, M_2(A))$ such that

$$w_t w_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad w_t^* w_t = p_t$$

for all $t \in [0,1]$ and $w_1 = w_0$ (as we identify $C(\mathbb{T}, M_2(A))$ with the set of continuous functions $f\colon [0,1] \to M_2(A)$ with $f(1) = f(0)$). Upon replacing w_t with $w_0^* w_t$ we can assume that $w_1 = w_0 = \mathrm{diag}(1,0)$. Now, with $t \mapsto u_t$ as in Example 4.4.5,

$$w_t u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_t & 0 \\ 0 & 0 \end{pmatrix},$$

where $t \mapsto a_t$ is a continuous path of unitaries in A. Because $u_0 = \mathrm{diag}(1,1)$ and $u_1 = \mathrm{diag}(v,1)$ we see that $a_0 = 1$ and $a_1 = v$, whence $v \sim_h 1$ in $\mathcal{U}(A)$.

(i) $\Rightarrow$ (ii). Suppose conversely that $v \sim_h 1$ in $\mathcal{U}(A)$. Then we can find a continuous path $t \mapsto v_t \in \mathcal{U}(A)$, $t \in [1-\varepsilon, 1]$, such that $v_{1-\varepsilon} = v$ and $v_1 = 1$ for an $\varepsilon > 0$ (to be determined below). Again with $t \mapsto u_t$ as in Example 4.4.5, define

$$\widetilde{u}_t = \begin{cases} u_{(1-\varepsilon)^{-1}t}, & 0 \le t \le 1-\varepsilon, \\ \mathrm{diag}(v_t, 1), & 1-\varepsilon \le t \le 1. \end{cases}$$

Then $t \mapsto \widetilde{u}_t$ is a continuous path of unitaries in $\mathcal{U}_2(A)$ such that $\widetilde{u}_{1-\varepsilon} = u_1 = \mathrm{diag}(v,1)$ and $\widetilde{u}_0 = \widetilde{u}_1 = 1$. It follows that $\widetilde{u}$ belongs to $C(\mathbb{T}, M_2(A))$. Provided that $\varepsilon > 0$ is chosen small enough we obtain the following inequality:

$$\left\| \widetilde{u}_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \widetilde{u}_t^* - p(t) \right\| = \left\| \widetilde{u}_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \widetilde{u}_t^* - u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_t^* \right\| < 1$$

for all $t \in [0,1]$, whence $p \sim \widetilde{u}\,\mathrm{diag}(1,0)\,\widetilde{u}^* \sim \mathrm{diag}(1,0)$ as desired.

(iii) $\Rightarrow$ (ii). Suppose that $\mathcal{B}$ is properly infinite. From Lemma 4.4.6 we know that $[p]_0 = [\mathrm{diag}(1_A, 0)]_0$ in $K_0(C(\mathbb{T}, A))$. Because $\mathcal{B}$ and A are properly infinite, it follows that p and $\mathrm{diag}(1_A, 0)$ are properly infinite (and full) projections, and hence they are equivalent by Proposition 4.2.2 (i).

(ii) $\Rightarrow$ (iii). Since A is properly infinite, $\mathrm{diag}(1_A, 0)$ and hence p (being equivalent to $\mathrm{diag}(1_A, 0)$) are properly infinite (and full) projections, whence $\mathcal{B}$ is properly infinite. $\square$

We will now use (the ideas behind) Lemma 4.4.6 and Proposition 4.4.7 to prove the following general statement about C^*-algebras.

Corollary 4.4.8. *Let A be a unital C^*-algebra such that $C(\mathbb{T}, A)$ has the cancellation property. Then A is K_1-injective.*

Proof. It suffices to show that the natural maps $\mathcal{U}_{n-1}(A)/\mathcal{U}_{n-1}^0(A) \to \mathcal{U}_n(A)/\mathcal{U}_n^0(A)$ are injective for all $n \geq 2$. Let $v \in \mathcal{U}_{n-1}(A)$ be such that $\mathrm{diag}(v, 1_A) \in \mathcal{U}_n^0(A)$ and find a continuous path of unitaries $t \mapsto u_t$ in $\mathcal{U}_n(A)$ such that

$$u_0 = 1_{M_n(A)} = \begin{pmatrix} 1_{M_{n-1}(A)} & 0 \\ 0 & 1_A \end{pmatrix} \quad \text{and} \quad u_1 = \begin{pmatrix} v & 0 \\ 0 & 1_A \end{pmatrix}.$$

Put

$$p_t = u_t \begin{pmatrix} 1_{M_{n-1}(A)} & 0 \\ 0 & 0 \end{pmatrix} u_t^*, \quad t \in [0,1],$$

and note that $p_0 = p_1$ so that p defines a projection in $C(\mathbb{T}, M_n(A))$. Repeating the proof of Lemma 4.4.6 we find that $1_{M_n(A)} - p \sim \mathrm{diag}(0, 1_A)$ in $C(\mathbb{T}, M_n(A))$, whence $p \sim \mathrm{diag}(1_{M_{n-1}(A)}, 0)$ by the cancellation property of $C(\mathbb{T}, A)$, where we identify projections in $M_n(A)$ with constant projections in $C(\mathbb{T}, M_n(A))$. The arguments going into the proof of Proposition 4.4.7 show that $v \sim_h 1_{M_{n-1}(A)}$ in $\mathcal{U}_{n-1}(A)$ if (and only if) $p \sim \mathrm{diag}(1_{M_{n-1}(A)}, 0)$. Hence v belongs to $\mathcal{U}_{n-1}^0(A)$ as desired. □

4.5 K_1-injectivity of properly infinite C^*-algebras

In this section we prove our main result that relates K_1-injectivity of arbitrary unital properly infinite C^*-algebras to proper infiniteness of $C(X)$-algebras and pull-back C^*-algebras. More specifically we shall show that Question 4.2.8, Question 4.2.12, Question 4.2.7, and Question 4.4.2 are equivalent.

First we reformulate in two different ways the question if a given properly infinite unital C^*-algebra is K_1-injective.

Proposition 4.5.1. *The following conditions are equivalent for any unital properly infinite C^*-algebra A:*

(i) *A is K_1-injective.*

(ii) *Let p, q be projections in A such that $p \sim q$ and p, q, $1-p$, $1-q$ are properly infinite and full. Then $p \sim_h q$.*

(iii) *Let p and q be properly infinite, full projections in A. There exist properly infinite, full projections $p_0, q_0 \in A$ such that $p_0 \leq p$, $q_0 \leq q$, and $p_0 \sim_h q_0$.*

Proof. (i) ⇒ (ii). Let p, q be properly infinite, full projections in A with $p \sim q$ such that $1-p, 1-q$ are properly infinite and full. Then by Lemma 4.2.3 (i) there is a unitary $v \in A$ such that $vpv^* = q$ and $[v]_1 = 0$ in $K_1(A)$. By the assumption in (i), $v \in \mathcal{U}^0(A)$, whence $p \sim_h q$.

(ii) $\Rightarrow$ (i). Let $u \in \mathcal{U}(A)$ be such that $[u]_1 = 0$ in $K_1(A)$. Take, as we can, a projection p in A such that p and $1-p$ are properly infinite and full. Set $q = upu^*$. Then $p \sim_h q$ by (ii), and so there exists a unitary $v \in \mathcal{U}^0(A)$ with $p = vqv^*$. It follows that

$$pvu = vqv^*vu = v(upu^*)v^*vu = vup.$$

Therefore $vu \in \mathcal{U}^0(A)$ by Lemma 4.2.3 (ii), which in turn implies that $u \in \mathcal{U}^0(A)$.

(ii) $\Rightarrow$ (iii). Let p, q be properly infinite and full projections in A. There exist mutually orthogonal projections e_1, f_1 such that $e_1 \leq p$, $f_1 \leq p$ and $e_1 \sim p \sim f_1$, and mutually orthogonal projections e_2, f_2 such that $e_2 \leq q$, $f_2 \leq q$ and $e_2 \sim q \sim f_2$. Being equivalent to either p or q, the projections e_1, e_2, f_1 and f_2 are properly infinite and full. There are properly infinite, full projections $p_0 \leq e_1$ and $q_0 \leq e_2$ such that $[p_0]_0 = [q_0]_0 = 0$ in $K_0(A)$ and $p_0 \sim q_0$ (cf. Proposition 4.2.2). As $f_1 \leq 1 - p_0$ and $f_2 \leq 1 - q_0$, we see that $1 - p_0$ and $1 - q_0$ are properly infinite and full, and so we get $p_0 \sim_h q_0$ by (ii).

(iii) $\Rightarrow$ (ii). Let p, q be equivalent properly infinite, full projections in A such that $1-p, 1-q$ are properly infinite and full. From (iii) we get properly infinite and full projections $p_0 \leq p$, $q_0 \leq q$ which satisfy $p_0 \sim_h q_0$. Thus there is a unitary $v \in \mathcal{U}^0(A)$ such that $vp_0v^* = q_0$. Upon replacing p by vpv^* (as we may do because $p \sim_h vpv^*$) we can assume that $q_0 \leq p$ and $q_0 \leq q$. Now, q_0 is orthogonal to $1-p$ and to $1-q$, and so $1 - p \sim_h 1 - q$ by Proposition 4.2.4, whence $p \sim_h q$. □

Proposition 4.5.2. *Let A be a unital properly infinite C^*-algebra. The following conditions are equivalent:*

(i) *A is K_1-injective, ie., the natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ is injective.*

(ii) *The natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$ is injective.*

(iii) *The natural maps $\mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A)$ are injective for each natural number n.*

Proof. (i) $\Rightarrow$ (ii) holds because the map $\mathcal{U}(A)/\mathcal{U}^0(A) \to K_1(A)$ factors through the map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$.

(ii) $\Rightarrow$ (i). Take $u \in \mathcal{U}(A)$ and suppose that $[u]_1 = 0$ in $K_1(A)$. Then $\mathrm{diag}(u, 1_A) \in \mathcal{U}_2^0(A)$ by Lemma 4.2.3 (ii) (with $p = \mathrm{diag}(1_A, 0)$). Hence $u \in \mathcal{U}^0(A)$ by injectivity of the map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$.

(i) $\Rightarrow$ (iii). Let $n \geq 1$ be given and consider the natural maps

$$\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_n(A)/\mathcal{U}_n^0(A) \to K_1(A).$$

The first map is onto, as proved by Cuntz in [12], see also [35, Exercise 8.9], and the composition of the two maps is injective by assumption, hence the second map is injective.

(iii) $\Rightarrow$ (i) is trivial. □

We give below another application of K_1-injectivity for properly infinite C^*-algebras. First we need a lemma:

Lemma 4.5.3. *Let A be a unital, properly infinite C^*-algebra, and let $\varphi, \psi \colon \mathcal{O}_\infty \to A$ be unital embeddings. Then ψ is homotopic to a unital embedding $\psi' \colon \mathcal{O}_\infty \to A$ for which there is a unitary $u \in A$ with $[u]_1 = 0$ in $K_1(A)$ and for which $\psi'(s_j) = u\varphi(s_j)$ for all j (where $s_1, s_2, \dots$ are the canonical generators of $\mathcal{O}_\infty$).*

Proof. For each n set

$$v_n = \sum_{j=1}^{n} \psi(s_j)\varphi(s_j)^* \in A, \qquad e_n = \sum_{j=1}^{n} s_j s_j^* \in \mathcal{O}_\infty.$$

Then v_n is a partial isometry in A with $v_n v_n^* = \psi(e_n)$, $v_n^* v_n = \varphi(e_n)$, and $\psi(s_j) = v_n\varphi(s_j)$ for $j = 1, 2, \dots, n$. Since $1 - e_n$ is full and properly infinite it follows from Lemma 4.2.3 that each v_n extends to a unitary $u_n \in A$ with $[u_n]_1 = 0$ in $K_1(A)$. In particular, $\psi(s_j) = u_n\varphi(s_j)$ for $j = 1, 2, \dots, n$.

We proceed to show that $n \mapsto u_n$ extends to a continuous path of unitaries $t \mapsto u_t$, for $t \in [2, \infty)$, such that $u_t\varphi(e_n) = u_n\varphi(e_n)$ for $t \geq n + 1$. Fix $n \geq 2$. To this end it suffices to show that we can find a continuous path $t \mapsto z_t$, $t \in [0, 1]$, of unitaries in A such that $z_0 = 1$, $z_1 = u_n^* u_{n+1}$, and $z_t\varphi(e_{n-1}) = \varphi(e_{n-1})$ (as we then can set u_t to be $u_n z_{t-n}$ for $t \in [n, n + 1]$).

Observe that

$$u_{n+1}\varphi(e_n) = v_{n+1}\varphi(e_n) = v_n = u_n\varphi(e_n).$$

Set $A_0 = (1 - \varphi(e_{n-1}))A(1 - \varphi(e_{n-1}))$, and set $y = u_n^* u_{n+1}(1 - \varphi(e_{n-1}))$. Then y is a unitary element in A_0 and $[y]_{K_1(A_0)} = 0$ in $K_1(A_0)$. Moreover, y commutes with the properly infinite full projection $\varphi(e_n) - \varphi(e_{n-1}) \in A_0$. We can therefore use Lemma 4.2.3 to find a continuous path $t \mapsto y_t$ of unitaries in A_0 such that $y_0 = 1_{A_0} = 1 - \varphi(e_{n-1})$ and $y_1 = y$. The continuous path $t \mapsto z_t = y_t + \varphi(e_{n-1})$ is then as desired.

For each $t \geq 2$ let $\psi_t \colon \mathcal{O}_\infty \to A$ be the *-homomorphism given by $\psi_t(s_j) = u_t\varphi(s_j)$. Then $\psi_t(s_j) = \psi(s_j)$ for all $t \geq j + 1$, and so it follows that

$$\lim_{t \to \infty} \psi_t(x) = \psi(x)$$

for all $x \in \mathcal{O}_\infty$. Hence ψ_2 is homotopic to ψ, and so we can take ψ' to be ψ_2. □

Proposition 4.5.4. *Any two unital *-homomorphisms from $\mathcal{O}_\infty$ into a unital K_1-injective (properly infinite) C^*-algebra are homotopic.*

Proof. In the light of Lemma 4.5.3 it suffices to show that if $\varphi, \psi \colon \mathcal{O}_\infty \to A$ are unital *-homomorphisms such that, for some unitary $u \in A$ with $[u]_1 = 0$ in $K_1(A)$, $\psi(s_j) = u\varphi(s_j)$ for all j, then $\psi \sim_h \varphi$. By assumption, $u \sim_h 1$, so there is a continuous path $t \mapsto u_t$ of unitaries in A such that $u_0 = 1$ and $u_1 = u$. Letting $\varphi_t \colon \mathcal{O}_\infty \to A$ be the *-homomorphism given by $\varphi_t(s_j) = u_t\varphi(s_j)$ for all j, we get $t \mapsto \varphi_t$ is a continuous path of *-homomorphisms connecting $\varphi_0 = \varphi$ to $\varphi_1 = \psi$. □

Our main theorem below, which in particular implies that Question 4.2.8, Question 4.2.12, Question 4.2.7 and Question 4.4.2 all are equivalent, also give a special converse to Proposition 4.5.4: Indeed, with $\iota_1, \iota_2 \colon \mathcal{O}_\infty \to \mathcal{O}_\infty * \mathcal{O}_\infty$ the two canonical inclusions, if $\iota_1 \sim_h \iota_2$, then condition (iv) below holds, whence $\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective, which again implies that all unital properly infinite C^*-algebras are K_1-injective. Below we retain the convention that $\mathcal{O}_\infty * \mathcal{O}_\infty$ is the universal *unital* free product of two copies of $\mathcal{O}_\infty$ and that ι_1 and ι_2 are the two natural inclusions of $\mathcal{O}_\infty$ into $\mathcal{O}_\infty * \mathcal{O}_\infty$.

Theorem 4.5.5. *The following statements are equivalent:*

(i) *Every unital, properly infinite C^*-algebra is K_1-injective.*

(ii) *For every compact Hausdorff space X, every unital $C(X)$-algebra A, for which A_x is properly infinite for all $x \in X$, is properly infinite.*

(iii) *Every unital C^*-algebra A, that is the pull-back of two unital, properly infinite C^*-algebras A_1 and A_2 along *-epimorphisms $\pi_1 \colon A_1 \to B$, $\pi_2 \colon A_2 \to B$:*

$$\begin{tikzcd} & A \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ A_1 \arrow[dr, "\pi_1"'] & & A_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

is properly infinite.

(iv) *There exist non-zero projections $p, q \in \mathcal{O}_\infty$ such that $p \neq 1$, $q \neq 1$, and $\iota_1(p) \sim_h \iota_2(q)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$.*

(v) *The specific $C([0,1])$-algebra $\mathcal{A}$ considered in Example 4.4.1 (and whose fibres are properly infinite) is properly infinite.*

(vi) *$\mathcal{O}_\infty * \mathcal{O}_\infty$ is K_1-injective.*

Note that statement (i) is reformulated in Propositions 4.5.1, 4.5.2, and 4.5.4; and that statement (iv) is reformulated in Proposition 4.4.3. The reader should be warned that all these statements may turn out to be false (in which case, of course, there will be counterexamples to all of them).

Proof. (i) $\Rightarrow$ (iii) follows from Proposition 4.2.6.

(iii) $\Rightarrow$ (ii). This follows from Lemma 4.2.9 as in the proof of Theorem 4.2.10, except that one does not need to pass to matrix algebras.

(ii) $\Rightarrow$ (i). Suppose that A is unital and properly infinite. Take a unitary $v \in \mathcal{U}(A)$ such that $\operatorname{diag}(v,1) \in \mathcal{U}_2^0(A)$. Let $\mathcal{B}$ be the $C(\mathbb{T})$-algebra constructed in Example 4.4.5 from A, v, and a path of unitaries $t \mapsto u_t$ connecting $1_{M_2(A)}$ to $\operatorname{diag}(v,1)$. Then $\mathcal{B}_t \cong A$ for all $t \in \mathbb{T}$, so all fibres of $\mathcal{B}$ are properly infinite. Assuming (ii), we can conclude that $\mathcal{B}$ is properly infinite. Proposition 4.4.7 then yields that $v \in \mathcal{U}^0(A)$. It follows that the natural map $\mathcal{U}(A)/\mathcal{U}_0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$ is injective, whence A is K_1-injective by Proposition 4.5.2.

(ii) $\Rightarrow$ (v) is trivial (because $\mathcal{A}$ is a $C([0,1])$-algebra with properly infinite fibres).

(v) $\Rightarrow$ (iv) follows from Proposition 4.4.3.

(iv) $\Rightarrow$ (i). We show that Condition (iii) of Proposition 4.4.3 implies Condition (iii) of Proposition 4.5.1.

Let A be a properly infinite C^*-algebra and let p, q be properly infinite, full projections in A. Then there exist (properly infinite, full) projections $p_0 \leq p$ and $q_0 \leq q$ such that $p_0 \sim 1 \sim q_0$ and such that $1 - p_0$ and $1 - q_0$ are properly infinite and full, cf. Propositions 4.2.2. Take isometries $t_1, r_1 \in A$ with $t_1 t_1^* = p_0$ and $r_1 r_1^* = q_0$; use the fact that $1 \precsim 1 - p_0$ and $1 \precsim 1 - q_0$ to find sequences of isometries $t_2, t_3, t_4, \dots$ and $r_2, r_3, r_4, \dots$ in A such that each of the two sequences $\{t_j t_j^*\}_{j=1}^\infty$ and $\{r_j r_j^*\}_{j=1}^\infty$ consist of pairwise orthogonal projections.

By the universal property of $\mathcal{O}_\infty$ there are unital *-homomorphisms $\varphi_j \colon \mathcal{O}_\infty \to A$, $j = 1, 2$, such that $\varphi_1(s_j) = t_j$ and $\varphi_2(s_j) = r_j$, where $s_1, s_2, s_3, \dots$ are the canonical generators of $\mathcal{O}_\infty$. In particular,

$$\varphi_1(s_1 s_1^*) = p_0 \quad \text{and} \quad \varphi_2(s_1 s_1^*) = q_0.$$

By the property of the universal unital free products of C^*-algebras, there is a unique unital *-homomorphism $\varphi \colon \mathcal{O}_\infty * \mathcal{O}_\infty \to A$ making the diagram

$$\begin{tikzcd}
& \mathcal{O}_\infty * \mathcal{O}_\infty \arrow[dd, "\varphi"] & \\
\mathcal{O}_\infty \arrow[ur, "\iota_1"] \arrow[dr, "\varphi_1"'] & & \mathcal{O}_\infty \arrow[ul, "\iota_2"'] \arrow[dl, "\varphi_2"] \\
& A &
\end{tikzcd}$$

commutative. It follows that $p_0 = \varphi(\iota_1(s_1 s_1^*))$ and $q_0 = \varphi(\iota_2(s_1 s_1^*))$. By Condition (iii) of Proposition 4.4.3, $\iota_1(s_1 s_1^*) \sim_h \iota_2(s_1 s_1^*)$ in $\mathcal{O}_\infty * \mathcal{O}_\infty$, whence $p_0 \sim_h q_0$ as desired.

(i) $\Rightarrow$ (vi) is trivial.

(vi) $\Rightarrow$ (v) follows from Proposition 4.4.4. $\square$

4.6 Concluding remarks

We do not know if all unital properly infinite C^*-algebras are K_1-injective, but we observe that K_1-injectivity is assured in the presence of certain central sequences:

Proposition 4.6.1. *Let A be a unital properly infinite C^*-algebra that contains an asymptotically central sequence $\{p_n\}_{n=1}^{\infty}$, where p_n and $1-p_n$ are properly infinite, full projections for all n. Then A is K_1-injective*

Proof. This follows immediately from Lemma 4.2.3 (ii). ☐

It remains open if arbitrary $C(X)$-algebras with properly infinite fibres must be properly infinite. If this fails, then we already have a counterexample of the form $\mathcal{B} = pC(\mathbb{T}, M_2(A))p$, cf. Example 4.4.5, for some unital properly infinite C^*-algebra A and for some projection $p \in C(\mathbb{T}, M_2(A))$. (The C^*-algebra $\mathcal{B}$ is a $C(\mathbb{T})$-algebra with fibres $\mathcal{B}_t \cong A$.)

From Example 4.1.4 we have that any trivial $C(X)$-algebra $C(X, D)$ with constant fibre D is properly infinite if its fibre(s) D is unital and properly infinite. We extend this observation in the following easy proposition:

Proposition 4.6.2. *Let X be a compact Hausdorff space, let $p \in C(X, D)$ be a projection, and consider the sub-trivial $C(X)$-algebra $pC(X, D)p$ whose fibre at x is equal to $p(x)Dp(x)$.*

If p is Murray-von Neumann equivalent to a constant projection $x \mapsto q$, then $pC(X, D)p$ is $C(X)$-isomorphic to the trivial $C(X)$-algebra $C(X, D_0)$, where $D_0 = qDq$. In this case, $pC(X, D)p$ is properly infinite if and only if D_0 is properly infinite.

In particular, if X is contractible, then $pC(X, D)p$ is $C(X)$-isomorphic to a trivial $C(X)$-algebra for any projection $p \in C(X, D)$ and for any C^*-algebra D.

Proof. Suppose that $p = v^*v$ and $q = vv^*$ for some partial isometry $v \in C(X, D)$. The map $f \mapsto vfv^*$ defines a $C(X)$-isomorphism from $pC(X, D)p$ onto $qC(X, D)q$, and $qC(X, D)q = C(X, D_0)$.

If X is contractible, then any projection $p \in C(X, D)$ is homotopic, and hence equivalent, to the constant projection $x \mapsto p(x_0)$ for any fixed $x_0 \in X$. ☐

Remark 4.6.3. One can elaborate a little more on the construction considered above. Take a unital C^*-algebra D such that for some natural number $n \geq 2$, $M_n(D)$ is properly infinite, but $M_{n-1}(D)$ is not properly infinite (see [30] or [32] for such examples). Take any space X, preferably one with highly non-trivial topology, eg. $X = S^n$, and take, for some $k \geq n$, a sufficiently non-trivial n-dimensional projection p in $C(X, M_k(D))$ such that $p(x)$ is equivalent to the trivial n dimensional projection $1_{M_n(D)}$ for all x (if X is connected we need only assume that this holds for one $x \in X$). The $C(X)$-algebra

$$\mathcal{A} = p\, C(X, M_k(D))\, p,$$

then has properly infinite fibres $\mathcal{A}_x = p(x)M_k(D)p(x) \cong M_n(D)$. Is A always properly infinite? We guess that a possible counterexample to the questions posed in this chapter could be of this form (for suitable D, X, and p).

Let us end this chapter by remarking that the answer to Question 4.2.12, which asks if any $C(X)$-algebra with properly infinite fibres is itself properly infinite, does not depend (very much) on X. If it fails, then it fails already for $X = [0,1]$ (cf. Theorem 4.5.5), and $[0,1]$ is a contractible space of low dimension. However, if we make the dimension of X even lower than the dimension of $[0,1]$, then we do get a positive answer to our question:

Proposition 4.6.4. *Let X be a totally disconnected space, and let A be a $C(X)$-algebra such that all fibres A_x, $x \in X$, of A are properly infinite. Then A is properly infinite.*

Proof. Using Lemma 4.2.9 and the fact that X is totally disconnected we can write X as the disjoint union of clopen sets $F_1, F_2, \dots, F_n$ such that A_{F_j} is properly infinite for all j. As

$$A = A_{F_1} \oplus A_{F_2} \oplus \cdots \oplus A_{F_n},$$

the claim is proved. □

Chapter 5

Extremally rich C^*-algebras with weak cancellation. Purely infinite C^*-algebras

In the first section of this chapter we will show that an extremally rich C^*-algebra with weak cancellation is K_1-injective. The result was proved Brown and Pedersen in [10]. It is not only the result itself that is interesting for this thesis. Also the proof is useful, when we in the next section are using some of their ideas and techniques, when we are considering purely infinite C^*-algebras.

Cuntz proved that unital, purely infinite, and simple C^*-algebras are K_1-injective, but here we are also interested in whether every unital purely infinite non-simple C^*-algebra is K_1-injective. We do not come with an answer to the question, but we give conditions that imply that K_1-injectivity holds. We prove that a unital purely infinite C^*-algebra that is an extension of K_1-injective C^*-algebras is itself K_1-injective, which also implies that a unital, separable, and purely infinite C^*-algebra with a finite ideal lattice is K_1-injective. By considering an example from Section 4.4 it is shown that if X is a finite dimensional compact Hausdorff space, then every unital continuous $C(X)$-algebra is K_1-injective if all the fibres are purely infinite and simple. Moreover we prove that a unital and purely infinite C^*-algebra is K_1-injective if its maximal ideal space has nice properties.

5.1 Extremally rich C^*-algebras with weak cancellation

Although they do not come with an answer, Brown and Pedersen are in [10] focusing on answering the question whether every extremally rich C^*-algebra has weak cancellation. In the process they are also giving results about K_1-injectivity of C^*-algebras that has relevance for the survey of K_1-injectivity that is given in this thesis. Namely it is proved that an extremally rich C^*-algebra with weak cancellation is K_1-injective. Furthermore we are using some of their results and techniques in Section 5.2, where we will try to answer the question whether every unital, purely infinite C^*-algebra is K_1-injective. In particular, we shall use Proposition 5.1.14 but also Lemma 5.1.7 and Lemma 5.1.8 are used. In the paper of Brown and Pedersen, Lemma 5.1.6 - Lemma 5.1.9 are not stated directly. But the technique can be extracted from their proof of Lemma 5.1.13.

First we will recall some facts about extreme points and quasi invertible elements in a C^*-algebra.

Theorem 5.1.1. [19, Theorem 1] *Let A be a unital C^*-algebra and let $(A)_1$ be the closed unit ball of A and $\mathcal{E}(A)$ the set of extreme points in the convex set $(A)_1$. The elements in $\mathcal{E}(A)$ is exactly the partial isometries $x \in A$ such that*

$$(1 - xx^*)A(1 - x^*x) = (0).$$

The projections $1 - xx^*$ and $1 - x^*x$ are called defect projections of x. For a unital C^*-algebra A, the *defect ideal* of A, denoted $\mathcal{D}(A)$, is the ideal generated by all defect projections of elements in $\mathcal{E}(A)$.
If A is non-unital, $\mathcal{D}(A)$ is generated by the defect projections of elements in $\mathcal{E}(\widetilde{A})$.

Definition 5.1.2. [1, Definition II.3.2.21] An element in a unital C^*-algebra A of the form yxz, where $y, z \in \mathrm{GL}(A)$ and $x \in \mathcal{E}(A)$, is called a quasi-invertible element. The set of quasi-invertible elements in A is denoted A_q.

Clearly, an extreme point of $(A)_1$ is quasi-invertible, but it also follows that every right- or left invertible element is quasi-invertible (c.f. [8]).

Definition 5.1.3. [1, Definition V.3.2.18] A unital C^*-algebra A is called extremally rich if A_q is dense in A.
If A is non-unital, A is called extremally rich if $\widetilde{A}$ is extremally rich.

If $\mathrm{sr}(A) = 1$, then A is extremally rich (see [9]). But in general, extremal richness is weaker than stable rank one. For example from [24] it follows that a unital simple purely infinite C^*-algebra is extremally rich.

Definition 5.1.4. [10, Section 1] A C^*-algebra A has weak cancellation if whenever p and q are projections in A that generate the same closed two-sided ideal I (i.e., $I = ApA = AqA$) and $[p]_{K_0(I)} = [q]_{K_0(I)}$, then $p \sim q$.

In [10] Brown and Pedersen are introducing the notion of a C^*-algebra being *weakly K_0-surjective*. This property has to be assumed in several of their results, and it is also important in Section 5.2.

Definition 5.1.5. A C^*-algebra A is called weakly K_0-surjective, if the suspension of A

$$SA = \{f \in C([0,1], A) \;:\; f(0) = f(1) = 0\},$$

is K_1-surjective.

Lemma 5.1.6. *Let A be a unital C^*-algebra, let $u \in \mathcal{U}(A)$ and let I be a closed two-sided ideal in A such that $u + I \sim_h 1$ in $\mathcal{U}(A/I)$. Then there exists $v \in \mathcal{U}(\widetilde{I})$ such that $u \sim_h v$ in $\mathcal{U}(A)$.*

Proof. Since $u + I \sim_h 1$ in $\mathcal{U}(A/I)$, there is a unitary $w \in \mathcal{U}^0(A)$ such that $u + I = w + I$. Then $v = w^* u$ is a unitary in $\widetilde{I}$, and $u \sim_h v$ in $\mathcal{U}(A)$. □

Lemma 5.1.7. *Let A be a C^*-algebra and let $u \in \mathcal{U}(\widetilde{A})$ with $[u]_{K_1(A)} = 0$. If $u \nsim_h 1$ in $\mathcal{U}(\widetilde{A})$, then there is a closed two-sided ideal I in A, which is maximal with respect to the property that $u + I \nsim_h 1$ in $\mathcal{U}(\widetilde{A}/I)$.*

Proof. Let
$$E = \{I \lhd A : u + I \nsim_h 1 \quad \text{in} \quad \mathcal{U}(\widetilde{A}/I)\}$$
be partially ordered by inclusion. Let (I_i) be a totally ordered subset of E and let $J = \overline{\bigcup_i I_i}$. Then J is a closed two-sided ideal in A. Suppose as a contradiction that $u + J \sim_h 1$ in $\mathcal{U}(\widetilde{A}/J)$. By Lemma 5.1.6 there is a unitary $v \in \widetilde{J}$ such that $u \sim_h v$ in $\mathcal{U}(\widetilde{A})$. After multiplying v with a complex number, we can assume that v can be written on the form $v = 1 + x, x \in J$. Thus, $\|x + I_i\| < 1$ for some i, and $\|(v-1) + I_i\| = \|x + I_i\| < 1$, which implies that $v + I_i \sim_h 1$ in $\mathcal{U}(\widetilde{A}/I_i)$. Therefore, as a contradiction we have that $u + I_i \sim_h 1$ in $\mathcal{U}(\widetilde{A}/I_i)$. Hence $J \in E$, and by Zorn's lemma E has a maximal element. □

Lemma 5.1.8. *Let A be a unital C^*-algebra, and let I be a closed two-sided ideal in A such that A/I is weakly K_0-surjective. Let $u \in \mathcal{U}(A)$ with $[u]_{K_1(A)} = 0$. If $u + I \sim_h 1$ in $\mathcal{U}(A/I)$, then there is a unitary $v \in \widetilde{I}$ such that $[v]_{K_1(I)} = 0$ and $u \sim_h v$ in $\mathcal{U}(A)$.*

Proof. Lemma 5.1.6 gives the existence of a unitary $u_1 \in \mathcal{U}(\widetilde{I})$ such that $u \sim_h u_1$ in $\mathcal{U}(A)$. Let $\delta_0 : K_1(S(A/I)) \to K_1(I)$ be the index map. Then there exists $\beta \in K_1(S(A/I))$ such that $[u_1]_{K_1(I)} = \delta_0(\beta)$ because $[u_1]_{K_1(A)} = 0$. But $S(A/I)$ is K_1-surjective, so $\beta = [\widetilde{u}]_{K_1(S(A/I))}$ for some $\widetilde{u} \in \mathcal{U}(\widetilde{S(A/I)})$. Thus, $\widetilde{u}$ is given by a continuous function $f\colon [0,1] \to \mathcal{U}(A/I)$ such that $f(0) = f(1) = 1$. The function f can be lifted to a continuous function $g : [0,1] \to \mathcal{U}(A)$ such that $g(0) = 1$, $g(1) \in \mathcal{U}(\widetilde{I})$ and $[u_1]_{K_1(I)} = \delta_0(\beta) = [g(1)]_{K_1(I)}$. Let $w = g(1)$, i.e $w \sim_h 1$ in $\mathcal{U}(A)$. Hence $v = w^* u_1$ is a unitary in $\widetilde{I}$ such that $v \sim_h u$ in $\mathcal{U}(A)$ and $[v]_{K_1(I)} = [w^*]_{K_1(I)} + [u_1]_{K_1(I)} = 0$. □

Going through the proof of the Lemma it is clearly seen that the result also holds in the non-unital case. Therefore we have the following Lemma, which shall be used in the proof of Lemma 5.1.13.

Lemma 5.1.9. *Let A be a non-unital C^*-algebra, and let I be a closed two-sided ideal in A such that A/I is weakly K_0-surjective. Let $u \in \mathcal{U}(\widetilde{A})$ with $[u]_{K_1(A)} = 0$. If $u + I \sim_h 1$ in $\mathcal{U}(\widetilde{A}/I)$, then there is a unitary $v \in \widetilde{I}$ such that $[v]_{K_1(I)} = 0$ and $u \sim_h v$ in $\mathcal{U}(\widetilde{A})$.*

Before we can give the proof of Lemma 5.1.13 we will need the following results, where $\mathcal{K}$ is the compact operators on a separable Hilbert space.

Proposition 5.1.10. [10, Proposition 3.15 ii)] *Let A be an extremally rich C^*-algebra with weak cancellation. If p is a projection in $\mathcal{D}(A) \otimes \mathcal{K}$ then there is a sequence $(p_n)_{n=1}^{\infty}$ of mutually orthogonal projections in $\mathcal{D}(A)$ such that $p_n \sim p$ for every $n \in \mathbb{N}$.*

Proposition 5.1.11. [10, Proposition 6.3] *If A is an extremally rich C^*-algebra, then $\mathcal{D}(A)$ is weakly K_0-surjective.*

Lemma 5.1.12. [10, Lemma 6.4] *Let A be a C^*-algebra and let B be a σ-unital hereditary sub-C^*-algebra of A such that $B^{\perp}$ contains a sequence $(p_n)_{n=1}^{\infty}$ of mutually orthogonal and equivalent projections which are full in A. Then there is a full hereditary sub-C^*-algebra B' of A such that $B \subseteq B'$ and such that $B' \cong B'' \otimes \mathcal{K}$ for a unital C^*-algebra B''. In particular B' has an approximate identity, $(e_n)_{n=1}^{\infty}$, consisting of full projections, such that for each $m, n \in \mathbb{N}$, $m\langle e_n \rangle \leq \langle 1_{\widetilde{A}} - e_n \rangle$.*

We are now able to prove the following lemma which is the main part of the proof of K_1-injectivity of an extremally rich C^*-algebra with weak cancellation (Theorem 5.1.15).

Lemma 5.1.13. [10, Lemma 6.5] *If A is an extremally rich C^*-algebra with weak cancellation, then $\mathcal{D}(A)$ is K_1-injective.*

Proof. Let $D = \mathcal{D}(A)$ and let $u \in \widetilde{D}$ with $[u]_1 = 0$, and assume that $u \nsim_h 1$ in $\mathcal{U}(\widetilde{D})$. By Lemma 5.1.7 there is a closed two-sided ideal J in D which is maximal with respect to the property that $u + J \nsim_h 1$ in $\mathcal{U}(\widetilde{D}/J)$. By [8, Theorem 6.1] extremal partial isometries lift from $\widetilde{A}/J$ to $\widetilde{A}$, and thus $\mathcal{D}(A/J) = D/J$. So we can assume (with A replaced with A/J and D replaced with D/J) that $u \in \widetilde{D}$ and $u + I \sim_h 1$ in $\mathcal{U}(\widetilde{D}/I)$ for every non-zero closed two-sided ideal I in D.

Choose a continuous function $f : \mathbb{T} \to [0, \infty[$ such that

$$\{z : f(z) \neq 0\} = \left\{ e^{i\theta} : \frac{2\pi}{3} < \theta < \frac{4\pi}{3} \right\},$$

and let $B_1 = \overline{f(u)\widetilde{D}f(u)}$. In fact $B_1 \subseteq D$ since $f(1) = 0$. Moreover B_1 is non-zero, since otherwise $f(\mathrm{sp}(u)) = \mathrm{sp}(f(u)) = 0$, which implies that $u \sim_h 1$, because $\mathrm{sp}(u) \subsetneq \mathbb{T}$.

Suppose now that $\mathrm{sr}(B_1) = 1$ and let I be the closed two-sided ideal generated by B_1. By construction, $u + I \sim_h 1$ in $\mathcal{U}(\widetilde{D}/I) = \mathcal{U}((D/I)^{\sim})$. From [10, Section 2] it follows that extremal richness passes to quotients, so A/I is also extremally rich. Thereby $\mathcal{D}(A/I) = D/I$ is weakly K_0-surjective, (c.f. Proposition 5.1.11). Lemma 5.1.9 gives the existence of a unitary $u_1 \in \mathcal{U}(\widetilde{I})$ such that $u \sim_h u_1$ in $\mathcal{U}(\widetilde{D})$ and $[u_1]_{K_1(I)} = 0$. But I is a full hereditary sub-C^*-algebra of B_1 which implies that $I \otimes \mathcal{K} \cong B_1 \otimes \mathcal{K}$. Hence $\mathrm{sr}(I) = 1$ and by Theorem 3.2.11

$$u \sim_h u_1 \sim_h 1 \quad \text{in} \quad \mathcal{U}(\widetilde{D}).$$

Suppose now that $\mathrm{sr}(B_1) > 1$, and let p be a non-zero defect projection of an element in $\mathcal{E}(\widetilde{B_1})$. Then $1-p$ is a full projection by [10, Lemma 3.3]. Let I be the ideal generated by p, and let $\pi : \widetilde{D} \to \widetilde{D}/I$ be the quotient map. First we shall show that $p \sim_h pup$ in $\mathrm{GL}(p\widetilde{D}p)$:
Since $f \neq 0$ on $\{e^{i\theta} : \frac{2\pi}{3} < \theta < \frac{4\pi}{3}\}$, it follows that $f(u)uf(u) \leq -\frac{1}{2}f(u)^2$. Thus, $bub^* \leq -\frac{1}{2}bb^*$ for $b \in B_1$ and thereby $pup \leq -\frac{1}{2}p$. I.e., $\mathrm{sp}(pup) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and therefore $\|p - pup\| \leq \frac{1}{2} < 1$.
Hence pup is invertible in $p\widetilde{D}p$ and $p \sim_h pup$ in $\mathrm{GL}(p\widetilde{D}p)$.

Next we shall show that $u \sim_h p + u_2$ in $\mathcal{U}(\widetilde{D})$ for some $u_2 \in \mathcal{U}((1-p)\widetilde{D}(1-p))$ with $\pi(u_2) = \pi(u)$:
Let $a = pup$, $b = (1-p)u$, $c = u(1-p)$, and $d = (1-p)u(1-p)$. Since a is invertible in $p\widetilde{D}p$ we can decompose u as

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ba^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ba^{-1}c \end{pmatrix} \begin{pmatrix} 1 & a^{-1}c \\ 0 & 1 \end{pmatrix}.$$

The two matrices

$$\begin{pmatrix} 1 & 0 \\ ba^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a^{-1}c \\ 0 & 1 \end{pmatrix}$$

are invertible and homotopic to 1 which implies that $\mathrm{diag}(a, d - ba^{-1}c)$ is invertible and homotopic to u. Hence $u' = d - ba^{-1}c$ is invertible in $(1-p)\widetilde{D}(1-p)$ and

$$u \sim_h pup + u' \sim_h p + u' \quad \text{in} \quad \mathrm{GL}(\widetilde{D}).$$

Let v be the unitary part of the polar decomposition of $p + u'$. I.e., $p + u' = v|p + u'|$, where $v = p + u_2$ for a unitary $u_2 \in (1-p)\widetilde{D}(1-p)$. It follows that

$$u \sim_h v = p + u_2 \quad \text{in} \quad \mathcal{U}(\widetilde{D}).$$

Furthermore

$$\pi(p + u') = \pi(u') = \pi(d) - \pi(ba^{-1}c) = \pi(d) = \pi(u),$$

which implies that $\pi(p + u')$ is unitary. Hence $\pi(|p + u'|) = 1$, and therefore

$$\pi(u_2) = \pi(v) = \pi(p + u') = \pi(u).$$

Since $\pi(u_2) = \pi(u)$ we get by construction that $\pi(u_2) \sim_h 1$ in $\mathcal{U}(\widetilde{D}/I)$, and since $\pi(\widetilde{D}) = \pi((1-p)\widetilde{D}(1-p))$ every element of $\mathcal{U}^0(\widetilde{D}/I)$ lifts to $\mathcal{U}^0((1-p)\widetilde{D}(1-p))$. Since extremal richness passes to quotients and hereditary sub-C^*-algebras, it follows by Proposition 5.1.11 that $(1-p)D(1-p)/(1-p)I(1-p)$ is weakly K_0-surjective. So we get a unitary $u_3 \in \mathcal{U}((1-p)\widetilde{I}(1-p))$ such that $[u_3]_{K_1(I)} = 0$ and $u_2 \sim_h u_3$ in $\mathcal{U}((1-p)\widetilde{D}(1-p))$.
Then

$$u \sim_h p + u_2 \sim_h p + u_3 \quad \text{in} \quad \mathcal{U}(\widetilde{D})$$

and $[p+u_3]_{K_1(I)} = 0$.

Let $\rho : \widetilde{I} \to \widetilde{I}/\mathcal{D}(I)$ be the quotient map. Since $(1-p)I(1-p)$ is a full hereditary sub-C^*-algebra of I, it follows that $[u_3]_{K_1((1-p)I(1-p))} = [u_3]_{K_1(I)} = 0$. Furthermore $\mathcal{D}(I)$ is the smallest ideal such that $\mathrm{sr}(\widetilde{I}/\mathcal{D}(I)) = 1$. (c.f. [10, Section 2.4]). So by Theorem 3.2.11 $\rho(u_3) \sim_h 1$ in $\mathcal{U}(\rho((1-p)\widetilde{I}(1-p)))$ since stable rank one passes to hereditary sub-C^*-algebras.
By [10, Section 6.2] $\widetilde{I}/\mathcal{D}(I)$ is weakly K_0-surjective since $\mathrm{sr}(\widetilde{I}/\mathcal{D}(I)) = 1$. Hence we can find a unitary $u_4 \in \mathcal{U}((1-p)\widetilde{\mathcal{D}(I)}(1-p))$ such that $u_3 \sim_h u_4$ in $\mathcal{U}((1-p)\widetilde{I}(1-p))$ and $[u_4]_{K_1(\mathcal{D}(I))} = 0$.

Let $(I_j)_{j=1}^{\infty}$ be an increasing sequence of ideals generated by finitely many defect projections of elements in $\mathcal{E}(\widetilde{I})$. Then $\mathcal{D}(I) = \overline{\bigcup_{j=1}^{\infty} I_j}$ and it follows that $\mathcal{D}(I)$ is isomorphic to the inductive limit of the inductive sequence

$$I_1 \xrightarrow{\iota_1} I_2 \xrightarrow{\iota_2} I_3 \xrightarrow{\iota_3} \cdots \longrightarrow \mathcal{D}(I)$$

where $\iota_j : I_j \to I_{j+1}$ are inclusion maps. Therefore there exists a $j \in \mathbb{N}$ and a unitary $u_5 \in \mathcal{U}((1-p)\widetilde{I}_j(1-p))$ such that $u_4 \sim_h u_5$ in $\mathcal{U}((1-p)\widetilde{I}_j(1-p))$ and because of continuity of K_1 we may assume that $[u_5]_{K_1(I_j)} = 0$.

Since pIp is a full hereditary sub-C^*-algebra of I and I is extremally rich (since extremally richness passes to ideals), it follows that $\mathcal{D}(pIp) = pIp \cap \mathcal{D}(I)$ (c.f. [10, Section 2.4]) which is a full hereditary sub-C^*-algebra of $\mathcal{D}(I)$. Thus every projection in $\mathcal{D}(I)$ is equivalent to a projection in $\mathcal{D}(pIp) \otimes \mathcal{K}$.

There exists a full projection q in I_j since I_j is generated by finitely many defect projections. Thus q is equivalent to a projection $\widetilde{q}$ in $\mathcal{D}(pIp) \otimes \mathcal{K}$, and by Proposition 5.1.10 there is a sequence $(q_n)_{n=1}^{\infty}$ of mutually orthogonal projections in $\mathcal{D}(pIp) \subseteq pIp$ such that $q \sim q_n$ for every $n \in \mathbb{N}$. Hence q_n is a full projection in I_j for every $n \in \mathbb{N}$.

We shall now apply Lemma 5.1.12 with A replaced by I_j and $B = \overline{(u_5 - 1 + p)I_j(u_5 - 1 + p)}$. Note that $u_5 - 1 + p \in I_j$ since after multiplying with a complex number we can assume that $u_5 = 1 - p \pmod{I_j}$. Moreover $q_n \in B^{\perp}$ since $q_n \in pI_jp$.
Let B' and $(e_n)_{n=1}^{\infty}$ be as in the lemma, and let $t_n = 1 + e_n(u_5 - 1 + p)e_n \in e_nI_je_n + \mathbb{C}(1-e_n)$. Then $t_n \to p + u_5$, so for a sufficiently large n, $\|(p+u_5) - t_n\| < 1$, which implies that t_n is invertible and $p + u_5 \sim_h t_n$ in $\mathrm{GL}(\widetilde{I}_j)$. Let $\bar{w}_n \in e_nI_je_n + \mathbb{C}(1-e_n)$ be the unitary part of the polar decomposition of t_n. Then $p + u_5 \sim_h \bar{w}_n$ in $\mathcal{U}(\widetilde{I}_j)$ and after multiplying with a complex number, $\bar{w}_n$ can be written on the form $\bar{w}_n = w_n + 1 - e_n$ for a unitary $w_n \in \mathcal{U}(e_nI_je_n)$. Hence

$$p + u_5 \sim_h w_n + 1 - e_n \quad \text{in} \quad \mathcal{U}(\widetilde{I}_j).$$

We have that

$$[p+u_5]_{K_1(I_j)} = [p+u_4]_{K_1(I)} = [p+u_3]_{K_1(I)} = 0,$$

and e_n is full in I_j, so $K_1(e_n I_j e_n) \cong K_1(I_j)$. Hence $[w_n]_{K_1(e_n I_j e_n)} = 0$. Thereby there exists a natural number m such that

$$w_n \oplus (e_n)_m \sim_h (e_n)_{m+1} \quad \text{in} \quad \mathcal{U}(M_{m+1}(e_n I_j e_n)),$$

where $(e_n)_m = e_n \oplus \cdots \oplus e_n$ with m summands. But by Lemma 5.1.12, $(e_n)_m \precsim 1 - e_n$. Thus $w_n + 1 - e_n \sim_h e_n + 1 - e_n = 1$ in $\mathcal{U}(\widetilde{I_j})$. I.e.

$$u \sim_h p + u_3 \sim_h p + u_4 \sim_h p + u_5 \sim_h w_n + 1 - e_n \sim_h 1 \quad \text{in} \quad \mathcal{U}(\widetilde{D}).$$

□

Under the assumption of the quotient being weakly K_0-surjective, the following Proposition shows that K_1-injectivity persists under taking extensions. Together with Lemma 5.1.13 this result will prove Theorem 5.1.15.

Proposition 5.1.14. [10, Proposition 6.6] *Let I be a closed two-sided ideal in a C^*-algebra A. If I and A/I are K_1-injective and A/I is weakly K_0-surjective, then A is K_1-injective.*

Proof. Let $u \in \mathcal{U}(\widetilde{A})$ with $[u]_1 = 0$. The K_1-injectivity of A/I implies that $u + I \sim_h 1$ in $\mathcal{U}(\widetilde{A/I})$. Lemma 5.1.8 now gives the existence of a unitary $v \in \mathcal{U}(\widetilde{I})$ such that $u \sim_h v$ in $\mathcal{U}(\widetilde{A})$ and $[v]_{K_1(I)} = 0$. By K_1-injectivity of I we get that $u \sim_h v \sim_h 1$ in $\mathcal{U}(\widetilde{A})$. □

Theorem 5.1.15. [10, Theorem 6.7] *Let A be an extremally rich C^*-algebra with weak cancellation. Then A is K_1-injective.*

Proof. By Lemma 5.1.13, $\mathcal{D}(A)$ is K_1-injective. Moreover $\mathcal{D}(A)$ is the smallest ideal such that $\mathrm{sr}(A/\mathcal{D}(A)) = 1$, which implies that $A/\mathcal{D}(A)$ is K_1-injective and weakly K_0-surjective, (c.f. [10, Section 6.2]). The result now follows from Proposition 5.1.14. □

5.2 Purely infinite C^*-algebras

5.2.1 Extensions

We will now try to give a condition that implies K_1-injectivity of a unital non-simple purely infinite C^*-algebra. We prove that K_1-injectivity is obtained if every quotient of the C^*-algebra has a stable ideal. Moreover it is proved that stable C^*-algebras, and purely infinite C^*-algebras that are extensions of K_1-injective C^*-algebras, are K_1-injective.

Lemma 5.2.1. *Let A be a C^*-algebra. Consider for each $n \in \mathbb{N}$, $M_n(A)^\sim$ as a unital sub-C^*-algebra of $M_n(\widetilde{A})$. Let u be a unitary in $M_n(A)^\sim$ such that $u \sim_h 1$ in $\mathcal{U}(M_n(\widetilde{A}))$. Then $u \sim_h 1$ in $\mathcal{U}(M_n(A)^\sim)$.*

Proof. Let $\pi : M_n(\widetilde{A}) \to M_n(\mathbb{C})$ be the *-homomorphism given by

$$\pi((a_{ij} + \lambda_{ij}1)_{i,j=1}^n) = (\lambda_{ij})_{i,j=1}^n, \quad a_{ij} \in A,\ \lambda_{i,j} \in \mathbb{C}.$$

After multiplying u by a complex number, we can assume that u is on the form, $u = x+1$, where $x \in M_n(A)$. So without loss of generality, it can be assumed that $\pi(u) = 1_n$. Let $v : [0,1] \to \mathcal{U}(M_n(\widetilde{A}))$ be a continuous path of unitaries, such that $v_0 = u$ and $v_1 = 1$. Then $z_t = \pi(v_t)^* v_t$ is a continuous path of unitaries in $M_n(A)^\sim$ with $z_0 = u$ and $z_1 = 1$. Hence $u \sim_h 1$ in $\mathcal{U}(M_n(A)^\sim)$. $\square$

We shall now prove that stable C^*-algebras are K_1-injective. The proposition will also be used in some of the other results in this section.

Proposition 5.2.2. *Any stable C^*-algebra is K_1-injective.*

Proof. If A is a stable C^*-algebra, there is a C^*-algebra A_0 such that $A \cong A_0 \otimes \mathcal{K}$. It follows that $(A_0 \otimes \mathcal{K})^\sim$ is isomorphic to the inductive limit of the inductive sequence:

$$\widetilde{A_0} \xrightarrow{\varphi_1} M_2(A_0)^\sim \xrightarrow{\varphi_2} M_3(A_0)^\sim \xrightarrow{\varphi_3} \cdots \longrightarrow (A_0 \otimes \mathcal{K})^\sim$$

where

$$\varphi_k(a + \lambda 1_{M_k}) = \begin{pmatrix} a + \lambda 1_{M_k} & 0 \\ 0 & \lambda 1 \end{pmatrix}, \quad a \in M_k(A_0),\ \lambda \in \mathbb{C}.$$

Let $\mu_n : M_n(A_0)^\sim \to (A_0 \otimes \mathcal{K})^\sim$ be unital *-homomorphisms such that

$$(A_0 \otimes \mathcal{K})^\sim = \overline{\bigcup_{n=1}^{\infty} \mu_n\left(M_n(A_0)^\sim\right)},$$

and let $u \in (A_0 \otimes \mathcal{K})^\sim$ be a unitary with $[u]_{K_1(A_0 \otimes \mathcal{K})} = 0$. Then there is $n \in \mathbb{N}$ and $v \in \mathcal{U}(M_n(A_0)^\sim)$ such that $\|u - \mu_n(v)\| < 1$, which implies that $u \sim_h \mu_n(v)$ in $\mathcal{U}((A_0 \otimes \mathcal{K})^\sim)$. It follows that $[v]_{K_1(M_n(A_0))} = 0$, since $K_1(\mu_n) : K_1(M_n(A_0)) \to K_1(A_0 \otimes \mathcal{K})$ are isomorphisms for all $n \in \mathbb{N}$ by stability of K_1. Therefore, there exists a $k \in \mathbb{N}$ such that

$$\begin{pmatrix} v & 0 \\ 0 & 1_{kn} \end{pmatrix} \sim_h 1_{n(k+1)} \quad \text{in} \quad \mathcal{U}(M_{k+1}(M_n(A_0)^\sim)) \subseteq \mathcal{U}(M_{n(k+1)}(\widetilde{A_0})).$$

We can assume that v is of the form $v = x+1$, for some $x \in M_n(A_0)$. Thus, $\operatorname{diag}(v, 1_{kn}) \in M_{n(k+1)}(A_0)^\sim$, and by Lemma 5.2.1 $\operatorname{diag}(v, 1_{kn}) \sim_h 1_{n(k+1)}$ in $\mathcal{U}(M_{n(k+1)}(A_0)^\sim)$. Moreover, $\mu_n(v) \sim_h 1$ in $\mathcal{U}((A_0 \otimes \mathcal{K})^\sim)$ since

$$\mu_n(v) = \mu_{n(k+1)}((\varphi_{n(k+1)-1} \circ \cdots \circ \varphi_n)(v)) = \mu_{n(k+1)}(\operatorname{diag}(v, 1_{kn})) \sim_h 1 \quad \text{in} \quad \mathcal{U}((A_0 \otimes \mathcal{K})^\sim).$$

Thereby, $u \sim_h 1$ in $\mathcal{U}((A_0 \otimes \mathcal{K})^\sim)$. $\square$

Lemma 5.2.3. *Let A be a unital C^*-algebra. If for any closed two-sided ideal I in A, there is a stable closed two-sided ideal J in A/I such that $(A/I)/J$ is weakly K_0-surjective, then A is K_1-injective.*

Proof. Suppose as a contradiction that $u \in \mathcal{U}(A)$ with $[u]_{K_1(A)} = 0$ and $u \nsim_h 1$ in $\mathcal{U}(A)$. By Lemma 5.1.7 there is an ideal I in A which is maximal with respect to the property that $u + I \nsim_h 1$ in $\mathcal{U}(A/I)$. Let $\pi\colon A \to A/I$ be the quotient map, and let J be a stable closed two-sided ideal in A/I. Then $\pi(u) + J \sim_h 1$ in $\mathcal{U}((A/I)/J)$. Lemma 5.1.8 gives the existence of a unitary $v \in \widetilde{J}$ such that $[v]_{K_1(J)} = 0$ and $\pi(u) \sim_h v$ in $\mathcal{U}(A/I)$. Since J is stable, Lemma 5.2.2 implies that $u + I \sim_h v \sim_h 1$ in $\mathcal{U}(A/I)$, which is a contradiction. □

In some of the results by Brown and Pedersen, we had to assume weakly K_0-surjectivity of the C^*-algebras. This property is automatically satisfied for purely infinite C^*-algebras, which is proved in Proposition 5.2.6. For the proof we need the following lemmas.

Lemma 5.2.4. *If A is a unital, properly infinite C^*-algebra, then the multiplier algebra $\mathcal{M}(SA)$, of the suspension of A, is properly infinite.*

Proof. Since $SA = C_0(\mathbb{R}) \otimes A$ it follows that $\mathcal{M}(SA) \supseteq \mathcal{M}(C_0(\mathbb{R})) \otimes A$. There exist mutually orthogonal projections p and q in A such that $p \sim 1_A \sim q$. Therefore $(1_{\mathcal{M}(C_0(\mathbb{R}))} \otimes p)$ and $(1_{\mathcal{M}(C_0(\mathbb{R}))} \otimes q)$ are orthogonal projections in $\mathcal{M}(SA)$ with $(1_{\mathcal{M}(C_0(\mathbb{R}))} \otimes p) \sim 1_{\mathcal{M}(C_0(\mathbb{R}))} \otimes 1_A \sim (1_{\mathcal{M}(C_0(\mathbb{R}))} \otimes q)$. Hence $\mathcal{M}(SA)$ is properly infinite. □

Lemma 5.2.5. *Let A be a C^*-algebra, let u be a unitary in $\widetilde{A}$ and let v be a unitary in $\mathcal{M}(A)$. Then $vuv^* \sim_1 u$ in $\mathcal{U}(\widetilde{A})$ (and $vuv^* \in \widetilde{A}$).*

Proof. From Whitehead's Lemma we have that $\mathrm{diag}(v, 1) \sim_h \mathrm{diag}(1, v)$ in $\mathcal{U}(M_2(\mathcal{M}(A)))$. Then there exists a continuous path of unitaries $w_t : [0, 1] \to \mathcal{U}(M_2(\mathcal{M}(A)))$ such that $w_0 = \mathrm{diag}(v, 1)$ and $w_1 = \mathrm{diag}(1, v)$. It follows that $\bar{w}_t = w_t \mathrm{diag}(u, 1) w_t^*$ is a continuous path of unitaries in $\mathcal{U}(M_2(\widetilde{A}))$ such that $\bar{w}_0 = \mathrm{diag}(vuv^*, 1)$ and $\bar{w}_1 = \mathrm{diag}(u, 1)$. Hence $vuv^* \sim_1 u$. □

Proposition 5.2.6. *Let A be a unital purely infinite C^*-algebra. Then A is weakly K_0-surjective.*

Proof. Let $z \in K_1(SA)$ and find $n \in \mathbb{N}$ and a unitary $u \in M_n(\widetilde{SA})$ such that $z = [u]_1$. We shall prove that $u \sim_1 1 + y$ for some $y \in SA$ for which $1 + y$ is unitary.

We can assume that u can be written on the form

$$\begin{aligned} u &= 1_{M_n(\mathbb{C})} + x, \quad x \in M_n(SA) \\ &= \begin{pmatrix} 1 + x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & 1 + x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & 1 + x_{nn} \end{pmatrix}, \quad x_{ij} \in SA. \end{aligned}$$

This can be seen if we let $\pi : M_n(\widetilde{SA}) \to M_n(\mathbb{C})$ be given by

$$\pi((a_{ij} + \lambda_{ij})_{i,j=1}^n) = (\lambda_{ij})_{i,j=1}^n, \quad a_{ij} \in SA,\ \lambda_{ij} \in \mathbb{C}.$$

Then $\pi(u)$ is a unitary in $M_n(\mathbb{C})$ and $u \sim_h \pi(u)^* u$, where $\pi(u)^* u$ is of the form

$$\pi(u)^* u = 1_{M_n(\mathbb{C})} + x$$

for some $x \in M_n(SA)$.

Since A is purely infinite, $\mathcal{M}(SA)$ is properly infinite by Lemma 5.2.4. Therefore there is an embedding of $\mathcal{O}_\infty$ into $\mathcal{M}(SA)$, and we can find isometries $s_1, \ldots, s_n \in \mathcal{M}(SA)$ such that $s_i s_i^*$ and $s_j s_j^*$ are orthogonal for $i \neq j$. Let

$$p = 1 - s_1 s_1^* - \cdots - s_n s_n^*,$$

and define a *-homomorphism $\varphi : M_n(\widetilde{SA}) \to (1-p)\widetilde{SA}(1-p)$ by

$$\varphi\left((a_{ij})_{i,j=1}^n\right) = \sum_{i,j=1}^n s_i a_{ij} s_j^*, \quad a_{ij} \in \widetilde{SA}.$$

Then

$$\varphi(u) = (1-p) + y, \quad y = \sum_{i,j=1}^n s_i x_{ij} s_j^* \in SA.$$

Since $1 + y = (1-p) + y + p$, and $(1-p) + y$ is unitary in $(1-p)\widetilde{SA}(1-p)$, it follows that $1 + y$ is a unitary in $\widetilde{SA}$. Let

$$v = \begin{pmatrix} s_1 & s_2 & \cdots & s_n & p \\ 0 & 0 & \cdots & 0 & s_1^* \\ 0 & 0 & \cdots & 0 & s_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & s_n^* \end{pmatrix} \in M_{n+1}(\mathcal{M}(SA)) = \mathcal{M}(M_{n+1}(SA)).$$

Then v is unitary, and $v\mathrm{diag}(u,1)v^* = \mathrm{diag}(y+1, 1_{M_n(\mathcal{M}(SA))})$. By Lemma 5.2.5 we get:

$$[u]_1 = [v\mathrm{diag}(u,1)v^*]_1 = [\mathrm{diag}(y+1, 1_{M_n(\mathcal{M}(SA))})]_1 = [y+1]_1.$$

□

Cuntz proved that any unital simple purely infinite C^*-algebra is K_1-injective. In the next theorem we give a condition that implies K_1-injectivity of a unital purely infinite C^*-algebra in the non-simple case.

Theorem 5.2.7. *Let A be a unital purely infinite C^*-algebra. If for any closed two-sided ideal I in A, there exists a stable closed two-sided ideal J in A/I, then A is K_1-injective.*

Proof. Since A is unital and purely infinite, $(A/I)/J$ is also a unital purely infinite C^*-algebra. It follows from Proposition 5.2.6 that $(A/I)/J$ is weakly K_0-surjective, which implies that A is K_1-injective by Lemma 5.2.3. $\square$

If I is a closed two-sided ideal in a unital purely infinite C^*-algebra A, then A/I is purely infinite and thereby weakly K_0-surjective. So from Proposition 5.1.14 we get that an extension of K_1-injective C^*-algebras is K_1-injective, if it is purely infinite.

Proposition 5.2.8. *Let A be a unital purely infinite C^*-algebra and let I be a closed two-sided ideal in A. If I and A/I are K_1-injective, then A is K_1-injective.*

From Proposition 5.2.8 we get the following result:

Proposition 5.2.9. *Let A be a unital separable purely infinite C^*-algebra with a finite ideal lattice. Then A is K_1-injective.*

Proof. Since the ideal lattice of A is finite, there is a finite number of ideals

$$0 = I_0 \lhd I_1 \lhd I_2 \cdots \lhd I_{n-1} \lhd I_n = A$$

such that I_{j+1}/I_j is simple for $j = 1, \dots, n-1$. We also have that I_j and I_{j+1}/I_j are purely infinite since A is purely infinite.

We shall now prove that A/I_{n-1} is K_1-injective. But $A/I_{n-1} = I_n/I_{n-1}$, so it is unital, purely infinite and simple and therefore K_1-injective by Theorem 3.1.3.

Suppose now that A/I_{n-k} is also K_1-injective for a natural number $k < n$. The ideal $I_{n-k}/I_{n-(k+1)} \lhd A/I_{n-(k+1)}$ is a separable, purely infinite and simple C^*-algebra. Hence it is either unital or stable. I.e., $I_{n-k}/I_{n-(k+1)}$ is K_1-injective by either Theorem 3.1.3 or Proposition 5.2.2 respectively. Thus $A/I_{n-(k+1)}$ is K_1-injective by Proposition 5.2.8 since the quotient A/I_{n-k} is by assumption also K_1-injective. Therefore by induction, $A = A/I_0$ is K_1-injective. $\square$

5.2.2 A unital $C(X)$-algebra with purely infinite fibres

We shall once again consider an example from Section 4.4. This time it will be used to prove that a unital continuous $C(X)$-algebra is K_1-injective if all the fibres are simple and purely infinite, and X is a finite dimensional compact Hausdorff space.

Let A be a unital C^*-algebra and let $v \in \mathcal{U}(A)$ be a unitary in A satisfying

$$\begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in} \quad \mathcal{U}_2(A).$$

Let $t \mapsto u_t$ be a continuous path of unitaries in $\mathcal{U}_2(A)$ such that $u_0 = 1$ and $u_1 = \operatorname{diag}(v, 1)$. Let

$$p(t) = u_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_t^* \in M_2(A).$$

When we identify $C(\mathbb{T}, M_2(A))$ with the C^*-algebra $\{f \in C([0,1], M_2(A)) : f(0) = f(1)\}$, it follows that $p \in C(\mathbb{T}, M_2(A))$. Put

$$\mathcal{B} = pC(\mathbb{T}, M_2(A))p,$$

and we get that $\mathcal{B}$ is a $C(\mathbb{T})$-algebra with fibres

$$\mathcal{B}_t = p(t)M_2(A)p(t) \cong A, \quad t \in \mathbb{T}.$$

First we shall prove that if X is a compact Hausdorff space and A is a unital $C(X)$-algebra, then $\mathcal{B}$ is also a $C(X \times \mathbb{T})$-algebra.

Lemma 5.2.10. *Let X be a compact Hausdorff space, let A be a unital $C(X)$-algebra, and let $\mathcal{B}$ be the C^*-algebra defined above. Then $\mathcal{B}$ is a $C(X \times \mathbb{T})$-algebra with fibres $\mathcal{B}_{(x,t)} \cong A_x$ for all $x \in X, t \in \mathbb{T}$.*

Proof. Since $\mathcal{B}$ is a $C(\mathbb{T})$-algebra and A is a $C(X)$-algebra there exist a unital *-homomorphism $\mu : C(\mathbb{T}) \to Z(\mathcal{B})$ and a unital *-homomorphism $\gamma : C(X) \to Z(A)$.

Define $\varphi : Z(A) \to \mathcal{B}$ by

$$\varphi(a)(t) = p(t) \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} p(t),$$

where p is the projection defined above.

If $a \in Z(A)$, then $\mathrm{diag}(a, a)$ and $p(t)$ commute, which implies that φ is a *-homomorphism with $\varphi(Z(A)) \subseteq Z(\mathcal{B})$. Thereby $\rho = \varphi \circ \gamma$ is a unital *-homomorphism from $C(X)$ into $Z(\mathcal{B})$. So there exists a unique unital *-homomorphism $\rho \otimes \mu : C(X) \otimes C(\mathbb{T}) \to Z(\mathcal{B})$, and since $C(X) \otimes C(\mathbb{T}) \cong C(X \times \mathbb{T})$, we have that $\mathcal{B}$ is a $C(X \times \mathbb{T})$-algebra.

It is now left to show that the fibres $\mathcal{B}_{(x,t)} \cong A_x$ for all $x \in X, t \in \mathbb{T}$. This means that we shall find a unital *-isomorphism $\psi : \mathcal{B}_{(x,t)} \to A_x$ such that the diagram

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\rho_x \circ \pi_t"] \arrow[dr, "\nu_{(x,t)}"'] & & A_x \\ & \mathcal{B}_{(x,t)} \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

commutes, where $\pi_t : \mathcal{B} \to A$, $\rho_x : A \to A_x$, and $\nu_{(x,t)} : \mathcal{B} \to \mathcal{B}_{(x,t)}$ are the quotient maps.

Let $g \in C(X)$ and $f \in C(\mathbb{T})$. Then

$$\nu_{(x,t)}((\rho \otimes \mu)(g \otimes f)1_{\mathcal{B}}) = (g \otimes f)(x,t)1_{\mathcal{B}_{(x,t)}} = g(x)f(t)1_{\mathcal{B}_{(x,t)}}$$

and

$$\begin{aligned} \pi_t((\rho \otimes \mu)(g \otimes f)1_{\mathcal{B}}) &= \pi_t((\rho(g)\mu(f)1_{\mathcal{B}}) \\ &= \pi_t(\rho(g))\pi_t(\mu(f))1_{\mathcal{B}_t} \\ &= \pi_t(\varphi(\gamma(g)))\pi_t(\mu(f))1_{\mathcal{B}_t} \\ &= \varphi(\gamma(g))(t)f(t)1_{\mathcal{B}_t} \\ &= \gamma(g)f(t)1_A \end{aligned}$$

where the last equality holds because of the isomorphism $\mathcal{B}_t = p(t)M_2(A)p(t) \cong A$. Hence

$$(\rho_x \circ \pi_t)((\rho \otimes \mu)(g \otimes f)1_{\mathcal{B}}) = \rho_x(\gamma(g)f(t)1_A) = g(x)f(t)1_{A_x}.$$

Thus, $\ker(\nu_{(x,t)} \circ (\rho \otimes \mu)) = \ker(\rho_x \circ \pi_t \circ (\rho \otimes \mu))$ which implies that $\ker(\nu_{(x,t)}) = \ker(\rho_x \circ \pi_t)$ for $x \in X, t \in \mathbb{T}$. The First Homomorphism Theorem gives the existence of an injective unital *-homomorphism $\psi : \mathcal{B}_{(x,t)} \to A_x$ such that the diagram above is commuting, and since the quotient maps are surjective, ψ becomes an isomorphism. □

Theorem 5.2.11. *Let X be a finite dimensional compact Hausdorff space and let A be a unital continuous $C(X)$-algebra, such that A_x is purely infinite and simple for all $x \in X$. Then A is K_1-injective.*

Proof. Let $v \in \mathcal{U}(A)$ such that $\mathrm{diag}(v, 1) \in \mathcal{U}_2^0(A)$, and let $\mathcal{B}$ be the C^*-algebra constructed above. By Lemma 5.2.10, $\mathcal{B}$ is a $C(X \times \mathbb{T})$-algebra with fibres $\mathcal{B}_{(x,t)} \cong A_x$ for all $(x,t) \in X \times \mathbb{T}$. I.e., $\mathcal{B}_{(x,t)}$ is purely infinite and simple for all $(x,t) \in X \times \mathbb{T}$. Then $\mathcal{B}$ has the Global Glimm halving property by [6, Theorem 2.7], and it follows from [6, Proposition 5.2] that $\mathcal{B}$ is purely infinite. Hence, $v \sim_h 1$ in $\mathcal{U}(A)$ by 4.4.7. Therefore the natural map $\mathcal{U}(A)/\mathcal{U}^0(A) \to \mathcal{U}_2(A)/\mathcal{U}_2^0(A)$ is injective, and A is K_1-injective. □

5.2.3 Maximal ideals

We are going to study the primitive ideal space and the maximal ideal space of a C^*-algebra in this section. Unfortunately these studies give for every unital purely infinite C^*-algebra, the existence of a quotient without stable ideals. This means that the conditions we gave in Theorem 5.2.7 for a unital purely infinite C^*-algebra being K_1-injective are useless. On the other hand we give a new condition that imply K_1-injectivity, which depends on the maximal ideal space.

Definition 5.2.12. [1, Definition II.6.5.1] Let A be a C^*-algebra and let I be a closed two-sided ideal in A. Then I is called a primitive ideal, if I is the kernel of an irreducible representation of A.
The primitive ideal space of A, $\mathrm{Prim}(A)$, is the set of primitive ideals of A.

The topology on $\mathrm{Prim}(A)$ is the hull-kernel topology. I.e., for a subset $M \subseteq \mathrm{Prim}(A)$ and a primitive ideal $J \in \mathrm{Prim}(A)$, then

$$J \in \overline{M} \iff J \supseteq \bigcap_{I \in M} I.$$

Let A be a C^*-algebra. If $x \in A$ we define a function $\check{x} : \mathrm{Prim}(A) \to \mathbb{R}^+$ by

$$\check{x}(J) = \|x + J\|, \quad J \in \mathrm{Prim}(A).$$

In general $\check{x}$ is lower semi-continuous ([1, Proposition II.6.5.6]), but if x is in the center of A, the function is actually continuous. This is part of the important theorem below that is known as the Dauns-Hoffmann Theorem:

Theorem 5.2.13 (Dauns-Hoffmann). [1, Theorem II.6.5.10] *Let A be a unital C^*-algebra, and $Z(A)$ its center. Then for each $x \in Z(A)$, the function $\check{x}$ is continuous, and $x \mapsto \check{x}$ ($x \in Z(A)^+$) extends to an isomorphism from $Z(A)$ onto $C(Prim(A))$.*

For a C^*-algebra A, we define $\mathrm{Max}(A)$ to be the set of maximal ideals in A. Note that $\mathrm{Max}(A) \subseteq \mathrm{Prim}(A)$ and as the following theorem shows, $\mathrm{Max}(A)$ is dense in $\mathrm{Prim}(A)$ in some cases.

Theorem 5.2.14. *Let A be a unital C^*-algebra. Then the following conditions are equivalent:*

(i) *Every non-zero ideal in A has a unital quotient.*

(ii) $\bigcap_{I \in \mathrm{Max}(A)} I = (0)$.

(iii) $\mathrm{Max}(A)$ *is dense in* $\mathrm{Prim}(A)$.

Proof. (ii) $\Rightarrow$ (i): Let $I \in \mathrm{Max}(A)$ and let J be a closed two-sided ideal in A. Let $\pi_I : A \to A/I$ be the quotient map. Then $\pi_I(J)$ is a closed two-sided ideal in A/I and since A/I is simple, then $\pi_I(J) = A/I$ or $\pi_I(J) = (0)$. But A/I is unital, so if $\pi_I(J) = A/I$, then J has a unital quotient.

On the other hand, if $\pi_I(J) = (0)$ for every $I \in \mathrm{Max}(A)$, then $J \subseteq \bigcap_{I \in \mathrm{Max}(A)} I = (0)$ which implies that $J = (0)$.

(i) $\Rightarrow$ (ii): Let $I_0 = \bigcap_{I \in \mathrm{Max}(A)} I \lhd A$, and let by (i) $I_0^{'}$ be an ideal in I_0 such that $I_0/I_0^{'}$ is unital. Let e be a unit in $I_0/I_0^{'}$ and let $\varphi : A/I_0^{'} \to I_0/I_0^{'}$ be given by

$$\varphi(x) = xe.$$

Since $I_0/I_0^{'}$ is an ideal in $A/I_0^{'}$, φ is a surjective *-homomorphism.

Every ideal in $I_0/I_0^{'}$ is of the form $I_0^{''}/I_0^{'}$ for some ideal $I_0^{''} \lhd I_0$, where $I_0^{'} \subseteq I_0^{''}$. So by the Third Isomorphism Theorem, a quotient in $I_0/I_0^{'}$ is on the form $I_0/I_0^{''}$. We choose $I_0^{''}$ such that $I_0/I_0^{''}$ is simple (this can be done since $I_0/I_0^{'}$ is unital).

Consider the maps

$$A \xrightarrow{\pi_{I_0^{'}}} A/I_0^{'} \xrightarrow{\varphi} I_0/I_0^{'} \xrightarrow{\pi_{I_0^{''}/I_0^{'}}} I_0/I_0^{''}$$

which are all surjective.

Hence there is a surjective map $\psi : A \to I_0/I_0^{''}$. This implies that $I_0/I_0^{''} \cong A/I$ for some maximal ideal I in A. Let $\pi_I : A \to A/I$ be the quotient map. Suppose as a contradiction that $I_0 \neq (0)$, and let $x \in I_0 \setminus I_0^{''}$. Then $\pi_I(x) \neq 0$, so $I_0 \nsubseteq I$ which is a contradiction.

(ii) $\Leftrightarrow$ (iii): We have that $\mathrm{Max}(A) \subseteq \mathrm{Prim}(A)$, so by definition if $J \in \mathrm{Prim}(A)$, then

$$J \in \overline{\mathrm{Max}(A)} \iff J \supseteq \bigcap_{I \in \mathrm{Max}(A)} I.$$

If $\bigcap_{I\in\mathrm{Max}(A)} I = (0)$, it follows that $J \in \overline{\mathrm{Max}(A)}$ for every $J \in \mathrm{Prim}(A)$, which implies that $\mathrm{Max}(A)$ is dense in $\mathrm{Prim}(A)$. On the other hand, since $\bigcap_{J\in\mathrm{Prim}(A)} J = (0)$, it follows that $\bigcap_{I\in\mathrm{Max}(A)} I = (0)$ if $\overline{\mathrm{Max}(A)} = \mathrm{Prim}(A)$. $\square$

Lemma 5.2.15. *Let A be a unital C^*-algebra and let*

$$I_0 = \bigcap_{I\in\mathrm{Max}(A)} I.$$

Then A/I_0 satisfies the conditions (i)-(iii) *from Theorem 5.2.14*
If J is a closed two-sided ideal in A, then A/J satisfies (i)-(iii) *from Theorem 5.2.14 if and only if $I_0 \subseteq J$.*

Proof. In Theorem 5.2.14, condition (ii) and thereby (i) and (iii) are clearly satisfied by A/I_0.
Let J be a closed two-sided ideal in A, and let $\pi : A \to A/J$ be the quotient map. Since $\{\pi^{-1}(I) : I \in \mathrm{Max}(A/J)\} = \{I \in \mathrm{Max}(A) : J \subseteq I\}$, it follows that

$$\pi^{-1}\left(\bigcap_{I\in\mathrm{Max}(A/J)} I\right) = \bigcap_{I\in\mathrm{Max}(A/J)} \pi^{-1}(I) = \bigcap_{I\in\mathrm{Max}(A)} \{J \subseteq I\}.$$

Hence $\bigcap_{I\in\mathrm{Max}(A)}\{J \subseteq I\} = J$ if $\bigcap_{I\in\mathrm{Max}(A/J)} I = (0)$, and since $I_0 \subseteq \bigcap_{I\in\mathrm{Max}(A)}\{J \subseteq I\}$, it follows that $I_0 \subseteq J$ if $\bigcap_{I\in\mathrm{Max}(A/J)} I = (0)$.
On the other hand we have to prove (i) for A/J if $I_0 \subseteq J$. Every ideal in A/J is on the form K/J for a closed to two-sided ideal K in A, where $J \subseteq K$. We also have that $K/I_0 \lhd A/I_0$ because $I_0 \subseteq J \subseteq K$.
Since A/I_0 satisfies (i), K/I_0 has a unital quotient. Therefore we can find a closed two-sided ideal L in K with $I_0 \subseteq L$ such that

$$\frac{K/I_0}{L/I_0} \cong \frac{K}{L}$$

is unital. We shall prove that K/J has a unital quotient. Let $L' = (L + J) \cap K$, which is a closed two-sided ideal in K with $J \subseteq L'$ and $L \subseteq L'$. Furthermore

$$\frac{K/J}{L'/J} \cong K/L' \cong \frac{K/L}{L'/L}$$

and thus K/J has a unital quotient, since every quotient of the unital C^*-algebra K/L is unital. $\square$

In Theorem 5.2.7 it was proved that a unital purely infinite C^*-algebra is K_1-injective if every quotient of it has a stable ideal. But we shall now use Kirchberg and Rørdams result about stability in a σ-unital purely infinite C^*-algebra to give a new formulation of Theorem 5.2.7.

Theorem 5.2.16. [22, Theorem 4.24] *Let A be a σ-unital purely infinite C^*-algebra. Then A is stable if and only if A has no unital quotient.*

Theorem 5.2.17. *Let A be a unital purely infinite C^*-algebra. If for any closed two-sided ideal I in A, there exists a closed two-sided ideal J in A/I with no unital quotient, then A is K_1-injective.*

Remark 5.2.18. Let A be a unital purely infinite C^*-algebra. If $I_0 = \bigcap_{I \in \mathrm{Max}(A)} I$, then by Lemma 5.2.15, every ideal J in A/I_0 has a unital quotient. So we cannot use Theorem 5.2.17 to conclude that A is K_1-injective.

Next we will prove that a unital purely infinite C^*-algebra is K_1-injective if A/I_0 is K_1-injective, where $I_0 = \bigcap_{I \in \mathrm{Max}(A)} I$. This result is used to show that a unital purely infinite C^*-algebra is automatically K_1-injective if the maximal ideal space has some nice properties. (Theorem 5.2.21)

Lemma 5.2.19. *Let A be a unital purely infinite C^*-algebra and let $I_0 = \bigcap_{I \in \mathrm{Max}(A)} I$. Then A is K_1-injective if A/I_0 is K_1-injective.*

Proof. Let $u \in \mathcal{U}(A)$ with $[u]_1 = 0$ and suppose as a contradiction that $u \nsim_h 1$ in $\mathcal{U}(A)$. Let I be a closed two-sided ideal in A that is maximal with respect to the property that $u + I \nsim_h 1$ in $\mathcal{U}(A/I)$. Since A/I_0 is K_1-injective, $u + I_0 \sim_h 1$ in $\mathcal{U}(A/I_0)$, which implies that $u + I \sim_h 1$ in $\mathcal{U}(A/I)$ if $I_0 \subseteq I$, giving a contradiction.
If $I_0 \nsubseteq I$ then A/I has a stable ideal by Theorem 5.2.16 and Lemma 5.2.15. The same argument as in the proof of Lemma 5.2.3 gives that $u + I \sim_h 1$ in $\mathcal{U}(A/I)$, contradicting the assumption. □

Lemma 5.2.20. *Let A be a unital C^*-algebra and let $I_0 = \bigcap_{I \in \mathrm{Max}(A)} I$. Then*

$$\mathrm{Prim}(A/I_0) \cong \overline{\mathrm{Max}(A)}.$$

Proof. Let $\pi : A \to A/I_0$ be the quotient map, and let J be a closed two-sided ideal in A/I_0. First we will show that $J \in \mathrm{Prim}(A/I_0)$ if and only if $\pi^{-1}(J) \in \mathrm{Prim}(A)$:
Suppose that $J \in \mathrm{Prim}(A/I_0)$ and let $\rho : A/I_0 \to B(H)$ be an irreducible representation with $\ker(\rho) = J$. Then $\rho \circ \pi : A \to B(H)$ is an irreducible representation with $\ker(\rho \circ \pi) = \pi^{-1}(J)$ which means that $\pi^{-1}(J) \in \mathrm{Prim}(A)$.

Suppose conversely that $\pi^{-1}(J) \in \mathrm{Prim}(A)$ and let $\varphi : A \to B(H)$ be an irreducible representation with $\ker(\varphi) = \pi^{-1}(J)$. Since $\pi^{-1}(0) = I_0$, it follows that $I_0 \subseteq \ker(\varphi)$ so there is an irreducible representation $\psi : A/I_0 \to B(H)$ making the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi"'] & B(H) \\ A/I_0 \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

Since $\ker(\varphi) = \pi^{-1}(J)$ we get that $\ker(\psi) = J$ and therefore $J \in \mathrm{Prim}(A/I_0)$. Thus,

$$\mathrm{Prim}(A/I_0) \cong \{I \in \mathrm{Prim}(A) : I_0 \subseteq I\} = \overline{\mathrm{Max}(A)},$$

where the last equation follows from the definition of the topology on $\mathrm{Prim}(A)$. $\square$

Theorem 5.2.21. *Let A be a unital purely infinite C^*-algebra. Then A is K_1-injective if* $\mathrm{Max}(A)$ *is a finite dimensional compact Hausdorff space that is closed in* $\mathrm{Prim}(A)$.

Proof. By Lemma 5.2.19 we have to show that A/I_0 is K_1-injective. Since

$$\mathrm{Max}(A) = \overline{\mathrm{Max}(A)} = \mathrm{Prim}(A/I_0)$$

it follows from Dauns-Hoffmann's theorem, that A/I_0 is a continuous $C(X)$-algebra with $X = \mathrm{Max}(A)$. For every $x \in \mathrm{Max}(A)$ the fibre $(A/I_0)_x$ is simple and purely infinite. Hence A/I_0 is K_1-injective by Theorem 5.2.11. $\square$

Chapter 6

Approximately divisible C^*-algebras and $\mathcal{Z}$-stability

Blackadar, Kumjian and Rørdam proved in [3] that a simple unital approximately divisible C^*-algebra is K_1-injective. In the first section of this chapter we will generalize this result to hold also for a unital and non-simple approximately divisible C^*-algebra. From this result we prove that a unital C^*-algebra is K_1-injective, if it absorbs tensorially a unital approximately divisible C^*-algebra.
In the next section we consider C^*-algebras that tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$. These C^*-algebras are called $\mathcal{Z}$-stable, and we will prove that a unital $\mathcal{Z}$-stable C^*-algebra is K_1-injective if it is properly infinite.

6.1 Approximately divisible C^*-algebras

Definition 6.1.1. [31, Definition 3.1.10] A C^*-algebra A is said to be approximately divisible if for every $n \in \mathbb{N}$ there is a sequence of unital *-homomorphisms

$$\varphi_k : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to \mathcal{M}(A)$$

such that

$$\|\varphi_k(x)a - a\varphi_k(x)\| \to 0$$

for all $x \in M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ and all $a \in A$.

Theorem 6.1.2. [3, Proposition 3.11] *Let A be a simple approximately divisible C^*-algebra. Then A is K_1-injective.*

The above theorem shall in the following be generalized to prove that every unital approximately divisible C^*-algebra is K_1-injective. We start by showing that the result holds for a C^*-algebra that is unital, properly infinite and approximately divisible. Moreover we shall prove that a unital C^*-algebra that tensorially absorbs an approximately divisible C^*-algebra, is K_1-injective.

We start with the following definition:

Definition 6.1.3. An ordered abelian positive semi group $(W, +, \leq)$ is said to be almost unperforated if

$$\forall\, n, m \in \mathbb{N}, n > m\, \forall\, x, y \in W : nx \leq my \;\Rightarrow\; x \leq y.$$

Theorem 6.1.4. *Let A be a unital, properly infinite, and approximately divisible C^*-algebra. Then A is K_1-injective.*

Proof. Let $\varphi_n : M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$ be a unital asymptotically central sequence of *-homomorphisms. For each $n \in \mathbb{N}$, let

$$p_n = \varphi_n \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Since $\mathrm{diag}(1, 0)$ and $\mathrm{diag}(1, 0_{M_2(\mathbb{C})})$ are full projections in $M_2(\mathbb{C})$ and $M_3(\mathbb{C})$ respectively, p_n is full in A. A similar argument gives that $1 - p_n$ is also a full projection. We shall now show that p_n and $1 - p_n$ are properly infinite:
Since p_n is full there is a $k \in \mathbb{N}$ such that

$$\langle 1 \rangle \leq k\langle p_n \rangle$$

(c.f. [35, Exercise 4.8]). But 1 is properly infinite and full, which implies that

$$\langle 1 \rangle \leq k\langle p_n \rangle \leq \langle 1 \rangle.$$

I.e., $p_n \oplus \cdots \oplus p_n$ (with k summands) is properly infinite because 1 is properly infinite. Thus,

$$k\langle p_n \oplus p_n \rangle \leq k\langle p_n \rangle$$

and it follows that

$$(k + 1)\langle p_n \oplus p_n \rangle \leq 2k\langle p_n \oplus p_n \rangle \leq 2k\langle p_n \rangle = k\langle p_n \oplus p_n \rangle \leq k\langle p_n \rangle.$$

From [3, Lemma 3.8] we get that $V(A)$ is almost unperforated when A is approximately divisible. Hence

$$p_n \oplus p_n \precsim p_n$$

and p_n is properly infinite. The same argument gives that $1 - p_n$ is properly infinite.

Since $(\varphi_n)_{n \in \mathbb{N}}$ is an asymptotically central sequence of *-homomorphisms, it follows that

$$\lim_{n \to \infty} \|p_n a - a p_n\| = 0$$

for all $a \in A$. Hence A is K_1-injective by Lemma 4.2.3. $\square$

We shall now prove that the above theorem also holds for a general approximately divisible C^*-algebra. But to prove this we need the following Lemmas:

Lemma 6.1.5. *Let A be a unital C^*-algebra and let $A_0 \subseteq A$ be a unital sub-C^*-algebra of A. Let $n \in \mathbb{N}$ and let $u \in \mathcal{U}(A_0)$ such that $u \oplus 1_{n-1} \sim_h 1_n$ in $\mathcal{U}(M_n(A_0))$. Suppose there is a unital *-homomorphism $\psi : M_n(\mathbb{C}) \to A \cap A_0'$. Then $u \sim_h 1$ in $\mathcal{U}(A)$.*

Proof. Since $u \oplus 1_{n-1} \sim_h 1$ in $\mathcal{U}(M_n(A_0))$ it follows that

$$\begin{pmatrix} u^n & 0 \\ 0 & 1_{n-1} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1_{n-1} \end{pmatrix}^n \sim_h 1_n \quad \text{in} \quad \mathcal{U}(M_n(A_0)),$$

and by Whitehead's Lemma $\operatorname{diag}(u, \ldots, u) \sim_h 1_n$ in $\mathcal{U}(M_n(A_0))$.

The C^*-algebras $A_0 \otimes M_n(\mathbb{C})$ and $M_n(A_0)$ are canonically isomorphic, which implies that $u \otimes 1_n \sim_h 1$ in $\mathcal{U}(A \otimes M_n(\mathbb{C}))$. Since A_0 and $\psi(M_n(\mathbb{C}))$ are commuting C^*-algebras and $M_n(\mathbb{C})$ is simple, we have that $\rho : A_0 \otimes M_n(\mathbb{C}) \to C^*(A_0, \psi(M_n(\mathbb{C}))$ given by

$$\rho(a \otimes x) = a\psi(x), \quad a \in A_0,\ x \in M_n(\mathbb{C})$$

is a *-isomorphism. Thus, $u \sim_h 1$ in $\mathcal{U}(C^*(A_0, \psi(M_n(\mathbb{C}))))$. □

Lemma 6.1.6. *Let A be a unital C^*-algebra and let $A_0 \subseteq A$ be a unital sub-C^*-algebra of A. Let $n \in \mathbb{N}$ and let $u \in \mathcal{U}(A_0)$ such that $u \oplus 1_{n-1} \sim_h 1_n$ in $\mathcal{U}(M_n(A_0))$. Suppose there is a unital *-homomorphism $\varphi : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to A \cap A_0'$. Then $u \sim_h 1$ in $\mathcal{U}(A)$.*

Proof. Let $p = (1, 0) \in M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C})$ and let $q = \varphi(p)$, and define *-homomorphisms $\rho : A \to qAq$ by

$$\rho(x) = qxq, \quad x \in A$$

and $\gamma : A \to (1-q)A(1-q)$ by

$$\gamma(x) = (1-q)x(1-q), \quad x \in A.$$

Since the projections q and $1-q$ commute with A_0, there are unital *-homomorphisms

$$\rho \circ \varphi|_{M_n} : M_n(\mathbb{C}) \to (A_0 q)' \cap qAq$$

and

$$\gamma \circ \varphi|_{M_{n+1}} : M_{n+1}(\mathbb{C}) \to (A_0(1-q))' \cap (1-q)A(1-q).$$

Furthermore

$$uq \oplus q_{n-1} = \rho_n(u \oplus 1_{n-1}) \sim_h \rho_n(1_n) = q_n \quad \text{in} \quad \mathcal{U}(M_n(A_0 q)).$$

Thus by Lemma 6.1.5, $uq \sim_h q$ in $\mathcal{U}(qAq)$, and similarly $u(1-q) \sim_h (1-q)$ in $\mathcal{U}((1-q)A(1-q))$. Therefore, $u = uq + u(1-q) \sim_h q + (1-q) = 1$ in $\mathcal{U}(A)$. □

Theorem 6.1.7. *Let A be a unital approximately divisible C^*-algebra. Then A is K_1-injective.*

Proof. Let $u \in \mathcal{U}(A)$ with $[u]_1 = 0$. By [3, Theorem 1.3] and by continuity of K_1, there is a unital sub-C^*-algebra $A_0 \subseteq A$ and a unitary $u_0 \in \mathcal{U}(A_0)$ such that $u \sim_h u_0$ and $[u_0]_{K_1(A_0)} = 0$, and such that for every $n \in \mathbb{N}$ there is a unital *-homomorphism $\varphi : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to A \cap A_0'$. From Lemma 6.1.6 it follows that $u \sim_h u_0 \sim_h 1$ in $\mathcal{U}(A)$. $\square$

We are going to prove that a unital C^*-algebra is K_1-injective if it tensorially absorbs an approximately divisible C^*-algebra. So first it is proved that a unital C^*-algebra A is approximately divisible if $A \cong A \otimes D$, and D is a unital and approximately divisible C^*-algebra.

Lemma 6.1.8. *Let A be a unital C^*-algebra and let D be a unital approximately divisible C^*-algebra. If $A \cong A \otimes D$, then A is approximately divisible.*

Proof. There is for every $n \in \mathbb{N}$ an asymptotically central sequence

$$\varphi_k : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to D$$

of unital *-homomorphisms. Let $\Phi_k : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to A \otimes D$ be given by

$$\Phi_k(x) = 1 \otimes \varphi_k(x).$$

Then $\Phi_k : M_n(\mathbb{C}) \oplus M_{n+1}(\mathbb{C}) \to A \otimes D$ is an asymptotically central sequence of unital *-homomorphisms, which implies that $A \otimes D$ is an approximately divisible C^*-algebra. $\square$

By Theorem 6.1.7 and Lemma 6.1.8 we get the following theorem:

Theorem 6.1.9. *Let D be a unital approximately divisible C^*-algebra, and let A be a unital C^*-algebra such that $A \cong A \otimes D$. Then A is K_1-injective.*

Since a UHF-algebra is approximately divisible (c.f [3, Proposition 4.1]), we thereby get the following class of examples of K_1-injective C^*-algebras:

Theorem 6.1.10. *Let A be a unital C^*-algebra and let D be a* UHF*-algebra. If $A \cong A \otimes D$, then A is K_1-injective.*

6.2 $\mathcal{Z}$-stable C^*-algebras

If A is a unital and simple C^*-algebra, then $A \otimes \mathcal{Z}$ is K_1-injective. This follows since $A \otimes \mathcal{Z}$ is either simple and purely infinite, or $A \otimes \mathcal{Z}$ has stable rank one (see [31, Theorem 4.1.10] and [33, Theorem 6.7]). In the following we shall prove that $A \otimes \mathcal{Z}$ is K_1-injective if A is unital and properly infinite.

First we need the following two lemmas:

Lemma 6.2.1. *Let A be a unital C^*-algebra such that $A \cong A \otimes \mathcal{Z}$. Then there is a sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$ of unital sub-C^*-algebras of A such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, and such that for every $n \in \mathbb{N}$ there is a unital *-homomorphism $\varphi_n : \mathcal{Z} \to A \cap A_n'$.*

Proof. Since $A \cong A \otimes \mathcal{Z}$ and $\mathcal{Z} \cong \bigotimes_{i=1}^{\infty} \mathcal{Z}$, A can be identified with $A = A_0 \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \ldots$ where $A_0 \cong A$. If we let $A_n = A_0 \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots \otimes \mathcal{Z} \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}} \otimes \ldots$ with n copies of $\mathcal{Z}$, and let $\varphi_n : \mathcal{Z} \to A$ be given by

$$\varphi_n(x) = 1_{A_0} \otimes 1_{\mathcal{Z}} \otimes 1_{\mathcal{Z}} \otimes \cdots \otimes x \otimes 1_{\mathcal{Z}} \otimes \ldots, \quad x \in \mathcal{Z}$$

with x in the $(n+1)$'th position, then the lemma is proved. □

Lemma 6.2.2. *Let A be a unital properly infinite C^*-algebra. Let $u \in \mathcal{U}(A)$ and let a be a non-zero positive, properly infinite and full element in A. Then there is a properly infinite and full projection $p \in \overline{aAa}$.*

In particular: if $aua = a^2$, then $u \sim_h 1$ if $[u]_1 = 0$.

Proof. Since a is properly infinite and full, $b \precsim a$ for every $b \in A^+$. In particular, $1 \precsim a$. Hence there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq A$ such that $x_n^* a x_n \to 1$. So for n sufficiently large, we have that $\|x_n^* a x_n - 1\| < 1$, and thereby $x_n^* a x_n$ is invertible. Thus,

$$(x_n^* a x_n)^{-\frac{1}{2}} (x_n^* a x_n)(x_n^* a x_n)^{-\frac{1}{2}} = 1.$$

It follows that there exists $x \in A$ such that $x^* a x = 1$. Let $v = a^{\frac{1}{2}} x$. I.e.,

$$v^* v = x^* a^{\frac{1}{2}} a^{\frac{1}{2}} x = 1.$$

Let $p = vv^*$. Hence $p \in \overline{aAa}$ and $p \sim 1$. It follows immediately that p is a full projection, and from the equivalence $p \sim 1$ and by proper infiniteness of A we also get,

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \precsim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore p is a full and properly infinite projection.

Moreover, if $aua = a^2$, then $pup = p$ and $pu^*p = p$. Hence

$$\begin{aligned} \|up - p\|^2 &= \|(up-p)^*(up-p)\| \\ &= \|(pu^* - p)(up - p)\| \\ &= \|p - pu^*p - pup + p\| \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}\|pu-p\|^2 &= \|(pu-p)(pu-p)^*\| \\ &= \|(pu-p)(u^*p-p)\| \\ &= \|p-pup-pu^*p+p\| \\ &= 0.\end{aligned}$$

Now, copying the proofs of Lemma 3.1.2 and Theorem 3.1.3, we get that $u \sim_h 1$ in $\mathcal{U}(A)$ if $[u]_1 = 0$. □

We are now ready to prove the main theorem of this section:

Theorem 6.2.3. *Let A be a unital properly infinite C^*-algebra such that $A \cong A \otimes \mathcal{Z}$. Then A is K_1-injective.*

Proof. Let $u \in \mathcal{U}(A)$ with $[u]_1 = 0$. By Lemma 6.2.1 and by continuity of K_1, there is a unital sub-C^*-algebra $A_0 \subseteq A$, a unital *-homomorphism $\varphi : \mathcal{Z} \to A \cap A_0'$, and a unitary $u_0 \in \mathcal{U}(A_0)$ such that $u \sim_h u_0$ and $[u_0]_{K_1(A_0)} = 0$.
If $\mathrm{sp}(u_0) \subsetneq \mathbb{T}$, then $u_0 \sim_h 1$ in $\mathcal{U}(A)$ and we are done. So suppose that $\mathrm{sp}(u_0) = \mathbb{T}$.

We can consider the cone $CM_2(\mathbb{C}) = C_0(]0,1], M_2(\mathbb{C}))$ to be a sub-C^*-algebra of $\mathcal{Z}$. Then $C^*(u_0) \cong C(\mathbb{T})$ and $\varphi(CM_2(\mathbb{C})^\sim)$ are commuting sub-C^*-algebras of A. Thus, by the universal property of the maximal tensor product, there is a unital surjective *-homomorphism

$$\psi : C(\mathbb{T}) \otimes \varphi(CM_2(\mathbb{C})^\sim) \to C^*(u_0, \varphi(CM_2(\mathbb{C})^\sim)) = C^*(u_0, \varphi(CM_2(\mathbb{C}))).$$

Since the C^*-algebras are nuclear, we have considered the minimal tensor product.
On the other hand, there is a unital surjective *-homomorphism from $C(\mathbb{T}) \otimes CM_2(\mathbb{C})^\sim$ onto $C(\mathbb{T}) \otimes \varphi(CM_2(\mathbb{C})^\sim)$, so the First Homomorphism Theorem gives that

$$C^*(u_0, \varphi(CM_2(\mathbb{C}))) \cong (C(\mathbb{T}) \otimes CM_2(\mathbb{C})^\sim)/I$$

for a closed two-sided ideal I in $C(\mathbb{T}) \otimes CM_2(\mathbb{C})^\sim$. And

$$C(\mathbb{T}) \otimes CM_2(\mathbb{C})^\sim \cong C(\mathbb{T}, CM_2(\mathbb{C})^\sim) \cong \{f \in C(\mathbb{T} \times [0,1], M_2(\mathbb{C})) : f(z,0) \in \mathbb{C}1_2\}$$

since $CM_2(\mathbb{C})^\sim \cong \{f \in C([0,1], M_2(\mathbb{C})) : f(0) \in \mathbb{C}1_2\}$. Hence

$$C^*(u_0, \varphi(CM_2(\mathbb{C}))) \cong B/I$$

where $B = \{f \in C(\mathbb{T} \times [0,1], M_2(\mathbb{C})) : f(z,0) \in \mathbb{C}1_2\}$.

Let $v \in \mathcal{U}(B)$ be the unitary that corresponds to u_0 under the natural identification of $C^*(u_0, \varphi(CM_2(\mathbb{C})))$ with B/I. I.e,

$$v(z,t) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix},$$

and let b be the full and positive element in B given by

$$b(z,t) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}.$$

From Whitehead's Lemma, there is for each $0 < \varepsilon < 1$ a continuous function $w : \mathbb{T} \times [0, \varepsilon] \to \mathcal{U}(M_2(\mathbb{C}))$ such that

$$w(z,0) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

and

$$w(z,\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix}.$$

If we define

$$w(z,t) = \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix}$$

for $\varepsilon \le t \le 1$, then w is a unitary in B. Moreover $v \sim_h w$ in $\mathcal{U}(B)$ by Whitehead's Lemma. For $t \ge \varepsilon$ it follows that

$$b(z,t)w(z,t)b(z,t) = b(z,t)^2$$

for all $z \in \mathbb{T}$. Thus

$$(b-\varepsilon)_+ w (b-\varepsilon)_+ = (b-\varepsilon)_+^2,$$

where the full and positive element $(b-\varepsilon)_+ \in B$ can be associated with the function

$$(z,t) \mapsto \begin{pmatrix} (t-\varepsilon)_+ & 0 \\ 0 & 0 \end{pmatrix}.$$

From the computations above in B, and because of the isomorphism

$$C^*(u_0, \varphi(CM_2(\mathbb{C}))) \cong B/I,$$

we can find a unitary $\bar{w} \in C^*(u_0, \varphi(CM_2(\mathbb{C})))$ such that $u_0 \sim_h \bar{w}$ and a positive and full element $a \in C^*(u_0, \varphi(CM_2(\mathbb{C})))$ such that $a\bar{w}a = a^2$. An argument similar to the one given in the proof of Theorem 6.1.4, implies that a is properly infinite, since $W(A)$ is almost unperforated (c.f. [33, Theorem 4.5]). Hence,

$$u \sim_h u_0 \sim_h \bar{w} \sim_h 1 \quad \text{in} \quad \mathcal{U}(A)$$

by Lemma 6.2.2. □

It would be nice if Theorem 6.2.3 could be generalized also to hold for every unital C^*-algebra that is $\mathcal{Z}$-stable, but not necessarily properly infinite. In Theorem 6.2.5 we will prove that a unitary in a $\mathcal{Z}$-stable C^*-algebra A, can always be homotoped to a unitary in a hereditary sub-C^*-algebra of A. After having studied the results by Cuntz (Section 3.1) and Brown and Pedersen (Section 5.1), we may raise the question if this condition implies K_1-injectivity of A?

If A is a C^*-algebra and a is a non-zero positive element in A, we will by $a^\perp$ denote the hereditary sub-C^*-algebra of A, that consists of all elements in A that are orthogonal to a. This notation will be used in the following lemma:

Lemma 6.2.4. *Let A be a C^*-algebra, let $a \in A^+ \setminus \{0\}$ and let $g : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that $g(0) = 1$. Then $a^\perp \subseteq \overline{g(a)Ag(a)}$.*

Proof. Let $x \in a^\perp$ and let f be the continuous function defined by $f = 1 - g$. Then $1_{\tilde{A}} = g(a) + f(a)$.
Since $x \in a^\perp$, it follows that

$$xa = 0 = ax,$$

which implies that

$$xf(a) = 0 = f(a)x.$$

Hence

$$x = 1_{\tilde{A}} x 1_{\tilde{A}} = (g(a) + f(a))x(g(a) + f(a)) = g(a)xg(a).$$

□

Theorem 6.2.5. *Let A be a unital C^*-algebra such that $A \cong A \otimes \mathcal{Z}$ and let $u \in \mathcal{U}(A)$. Then there is a unital *-homomorphism $\varphi : \mathcal{Z} \to A$ such that for every $c \in \mathcal{Z}^+ \setminus \{0\}$ there is a unitary $u' \in \overline{\varphi(c)A\varphi(c)} + \mathbb{C}1$ such that $u \sim_h u'$ in $\mathcal{U}(A)$.*

Proof. Let $n \in \mathbb{N}$ and let $I_{n(n+1)}$ be the dimension drop algebra

$$I_{n(n+1)} = \{f \in C([0,1], M_n(\mathbb{C}) \otimes M_{n+1}(\mathbb{C})) : f(0) \in M_n(\mathbb{C}) \otimes 1_{n+1},\ f(1) \in 1_n \otimes M_{n+1}(\mathbb{C})\}.$$

By [33, Theorem 2.1] there exists an embedding $\iota : I_{n(n+1)} \to \mathcal{Z}$ such that

$$\tau(\iota(f)) = \int_0^1 \tau_{n(n+1)}(f(t))dt, \quad f \in I_{n(n+1)}$$

where τ is the unique tracial state on $\mathcal{Z}$ and $\tau_{n(n+1)}$ is the normalized trace on $M_{n(n+1)}(\mathbb{C})$.
Hence the dimension function induced by τ is given by

$$d_\tau(\iota(f)) = \lim_{k\to\infty} \tau((\iota(f))^{\frac{1}{k}}) = \int_0^1 d_{\tau_{n(n+1)}}(f(t))dt = \frac{1}{n(n+1)} \int_0^1 \mathrm{Rank}(f(t))dt$$

where $d_{\tau_{n(n+1)}}$ is the dimension function on $M_{n(n+1)}(\mathbb{C})$ induced by $\tau_{n(n+1)}$.
Consider the embedding $\gamma : CM_n(\mathbb{C}) \to I_{n(n+1)}$ given by

$$\gamma(f)(t) = f(1-t) \otimes 1_{n+1}.$$

We will copy the idea from the proof of the properly infinite case (Theorem 6.2.3). So given $u \in \mathcal{U}(A)$, there are a unital sub-C^*-algebra $A_0 \subseteq A$, a unitary $u_0 \in A_0$ such that

$u \sim_h u_0$ in $\mathcal{U}(A)$ and a unital *-homomorphism $\varphi : \mathcal{Z} \to A \cap A_0'$. If we let $\psi = \varphi \circ \iota \circ \gamma$, it follows as before that

$$C^*(u_0, \psi(CM_n(\mathbb{C}))) \cong B/I$$

where $B = \{f \in C(\mathbb{T} \times [0,1], M_n(\mathbb{C})) \ : f(z,0) = \mathbb{C}1_n\}$, and I is a closed two-sided ideal in B.

Let $v \in \mathcal{U}(B)$ be the unitary that corresponds to u_0 under the natural identification of $C^*(u_0, \psi(CM_n(\mathbb{C})))$ with B/I. I.e,

$$v(z,t) = \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z \end{pmatrix}$$

and for $j = 1, \ldots, n$, let $b_j(z,t)$ be the full and positive element in B given by

$$b_j(z,t) = te_{jj},$$

where e_{jj} is the j'th matrix unit.
From Whiteheads Lemma there is for each $0 < \varepsilon < 1$ a continuous function $w : \mathbb{T} \times [0, \varepsilon] \to \mathcal{U}(M_n(\mathbb{C}))$ such that

$$w(z,0) = \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & z & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z \end{pmatrix}$$

and

$$w(z, \varepsilon) = \mathrm{diag}(1_{n-1}, z^n).$$

If we define

$$w(z,t) = \mathrm{diag}(1_{n-1}, z^n)$$

for $\varepsilon \leq t \leq 1$, then w is a unitary in B and $v \sim_h w$ in $\mathcal{U}(B)$. For $t \geq \varepsilon$ and $j = 1, \ldots, n-1$ it follows that

$$b_j(z,t)w(z,t)b_j(z,t) = b_j(z,t)^2$$

for all $z \in \mathbb{T}$. Thus,

$$(b_j - \varepsilon)_+ w (b_j - \varepsilon)_+ = (b_j - \varepsilon)_+^2 \quad \text{for} \quad j = 1, \ldots, n-1.$$

Since $C^*(u_0, \psi(CM_n(\mathbb{C}))) \cong B/I$ there exist a unitary $\bar{w} \in C^*(u_0, \psi(CM_n(\mathbb{C})))$ such that $u \sim_h u_0 \sim_h \bar{w}$ in $\mathcal{U}(A)$ and positive elements $a_1, \ldots, a_n \in C^*(u_0, \psi(CM_n(\mathbb{C})))$ such that

$$a_j \bar{w} a_j = a_j^2, \quad j = 1, \ldots, n-1.$$

This implies that

$$a_j\bar{w} = \bar{w}a_j = a_j, \quad \text{for} \quad j = 1, \ldots, n-1$$

since

$$\begin{aligned}\|a_j\bar{w} - a_j\|^2 &= \|(a_j\bar{w} - a_j)(a_j\bar{w} - a_j)^*\| \\ &= \|a_j\bar{w}\bar{w}^*a_j - a_j\bar{w}a_j - a_j\bar{w}^*a_j + a_j^2\| \\ &= \|a_j^2 - a_j^2 - a_j^2 + a_j^2\| \\ &= 0.\end{aligned}$$

Similarly we have that $\|\bar{w}a_j - a_j\| = 0$.
Let $c_j \in I_{n(n+1)}$ be defined by

$$c_j(t) = (\gamma(b_j - \varepsilon)_+)(t) = (1 - \varepsilon - t)_+ e_{jj} \otimes 1_{n+1} \quad \text{for} \quad j = 1, \ldots, n.$$

This means that

$$a_j = \varphi(\iota(c_j)), \quad j = 1, \ldots, n.$$

Let $g_\delta : \mathbb{R}^+ \to [0, 1]$ be the continuous function given by

$$g_\delta(t) = \begin{cases} 1, & t = 0 \\ -\frac{1}{\delta}t + 1, & 0 < t < \delta \\ 0, & t \geq \delta. \end{cases}$$

Hence

$$\begin{aligned}&(g_\delta(c_1 + \cdots + c_{n-1}))(t) \\ &= \sum_{j=1}^{n-1} g_\delta((1 - \varepsilon - t)_+)e_{jj} \otimes 1_{n+1} \\ &= \begin{cases} e_{nn} \otimes 1_{n+1}, & 0 \leq t \leq 1 - \varepsilon - \delta \\ \sum_{j=1}^{n-1}(\frac{t}{\delta} + \frac{\delta - 1 + \varepsilon}{\delta})e_{jj} \otimes 1_{n+1} + e_{nn} \otimes 1_{n+1}, & 1 - \varepsilon - \delta < t < 1 - \varepsilon \\ 1_n \otimes 1_{n+1}, & 1 - \varepsilon \leq t \leq 1. \end{cases}\end{aligned}$$

Let

$$\bar{b} = \iota(g_\delta(c_1 + \cdots + c_{n-1})).$$

Since $\mathrm{Rank}(e_{nn} \otimes 1_{n+1}) = n + 1$ and $\mathrm{Rank}(1_n \otimes 1_{n+1}) = n(n + 1)$, it follows that

$$\begin{aligned}d_\tau(\bar{b}) &= \int_0^1 d_{\tau_{n(n+1)}}(g_\delta(c_1 + \cdots + c_{n-1}))(t)dt \\ &\leq \int_0^{1-\varepsilon-\delta} \left(\frac{n+1}{n(n+1)}\right) dt + \int_{1-\varepsilon-\delta}^{1-\varepsilon} 1dt + \int_{1-\varepsilon}^{1} 1dt \\ &= (1 - \varepsilon - \delta)\frac{1}{n} + \delta + \varepsilon \\ &< \frac{1}{n} + \delta + \varepsilon.\end{aligned}$$

Let $a = a_1 + \cdots + a_{n-1}$ and let $b = \varphi(\bar{b})$. From Lemma 6.2.4 it follows that

$$a^\perp \subseteq \overline{g_\delta(a)Ag_\delta(a)} = \overline{\varphi(\bar{b})A\varphi(\bar{b})} = \overline{bAb}.$$

Since

$$a\bar{w}a = \sum_{i,j=1}^{n-1} a_j\bar{w}a_i = \sum_{i,j=1}^{n-1} a_ja_i = a^2,$$

the same argument as before implies that $(\bar{w}-1)a = a(\bar{w}-1) = 0$. Hence $\bar{w} \in \overline{bAb} + \mathbb{C}1$.

Let $c \in \mathcal{Z}^+ \setminus \{0\}$. By choosing ε and δ sufficiently small and n sufficiently large, we can assume that

$$d_\tau(\bar{b}) < d_\tau(c).$$

Therefore $\bar{b} \precsim c$ because $W(\mathcal{Z})$ is almost unperforated by [33, Corollary 4.6 and Theorem 4.5]. Thereby for each $\eta > 0$ there exists a unitary $\bar{v} \in \mathcal{U}(\mathcal{Z})$ such that

$$\bar{v}(\bar{b}-\eta)_+\bar{v}^* \subseteq \overline{c\mathcal{Z}c}$$

since $\mathcal{Z}$ has stable rank one (c.f. [29, Proposition 2.4]). (Note that $\bar{v} \in \mathcal{U}^0(\mathcal{Z})$ since $\mathcal{Z}$ is K_1-injective and $K_1(\mathcal{Z}) = 0$).

Hence

$$\varphi(\bar{v})(b-\eta)_+\varphi(\bar{v})^* = \varphi(\bar{v})\varphi((\bar{b}-\eta)_+)\varphi(\bar{v})^* \in \overline{\varphi(c)A\varphi(c)}$$

for every $\eta > 0$.

We now choose $0 < \delta' < \delta$. There exists $\eta > 0$ such that

$$\overline{g_{\delta'}(a)Ag_{\delta'}(a)} \subseteq \overline{(b-\eta)_+A(b-\eta)_+},$$

and by Lemma 6.2.4, $a^\perp \subseteq \overline{g_{\delta'}(a)Ag_{\delta'}(a)}$. Thus $\bar{w} \in \overline{(b-\eta)_+A(b-\eta)_+} + \mathbb{C}1$ and thereby

$$\bar{w} \sim_h \varphi(\bar{v})\bar{w}\varphi(\bar{v})^* \quad \text{in} \quad \mathcal{U}\left(\overline{\varphi(c)A\varphi(c)} + \mathbb{C}1\right).$$

□

When Cuntz proved that a unital, simple and purely infinite C^*-algebra A is K_1-injective, he used that every unitary in A could be homotoped to $p + u_0$ for a some projection $p \in A$ and a unitary $u_0 \in (1-p)A(1-p)$. This property actually implied K_1-injectivity of A.

Moreover when Brown and Pedersen proved K_1-injectivity of an extremally rich C^*-algebra with weak cancellation, they first showed (see the proof of Lemma 5.1.13) that for every unitary $u \in \widetilde{\mathcal{D}(A)}$ with $[u]_1 = 0$, there exist a closed two-sided ideal I in $\mathcal{D}(A)$ and a unitary $v \in \widetilde{I}$ with $[v]_{K_1(I)} = 0$ such that $u \sim_h v$. This fact was used several times to prove that $\mathcal{D}(A)$ was K_1-injective, which also implied K_1-injectivity of A itself.

So our hope is that Theorem 6.2.5 (maybe together with other properties of $\mathcal{Z}$ and $\mathcal{Z}$-stable C^*-algebras) can be used to prove that every unital $\mathcal{Z}$-stable C^*-algebra is K_1-injective.

Chapter 7

Strongly self-absorbing C^*-algebras

In many of the results that Toms and Winter obtain about strongly self-absorbing C^*-algebras in [37], it has to be assumed that the strongly self-absorbing C^*-algebra is K_1-injective. For all known examples of strongly self-absorbing C^*algebras, K_1-injectivity actually holds, but in this chapter we shall consider conditions that imply K_1-injectivity for a general strongly self-absorbing C^*-algebra.

Let D be a unital strongly self-absorbing C^*-algebra and consider the two cases where D is either not stably finite or stably finite respectively.

7.1 Non-stably finite strongly self-absorbing C^*-algebras

If D is a unital strongly self-absorbing C^*-algebra, then

$$D \cong D \otimes D,$$

and D is a simple C^*-algebra.
There does not exist any $n \in \mathbb{N}$ such that $D \cong M_n(\mathbb{C})$, and D cannot be isomorphic to the compact operators $\mathcal{K}$, since $\mathcal{K}$ is not unital. So D is not of type I. Thus, if D is not stably finite, it is proved by Kirchberg that D is simple and purely infinite ([31, Theorem 4.1.10]) and thereby K_1-injective.

7.2 Stably finite strongly self-absorbing C^*-algebras

When we shall give conditions that imply K_1-injectivity of a strongly self-absorbing C^*-algebra, it follows from the discussion above that it is sufficient to consider the stably finite case, since in the non-stably finite case K_1-injectivity holds automatically.

In the following we shall consider a C^*-algebra of the form $A \otimes D$ where A is any unital simple finite C^*-algebra, and D is a stably finite strongly self-absorbing C^*-algebra. We shall give conditions that imply that $A \otimes D$ has stable rank one which means that $A \otimes D$ is K_1-injective. If these conditions can actually be satisfied for any unital simple finite C^*-algebra A, it follows in particular that D is K_1-injective.

It should be mentioned that in the conditions we give below, it is sufficient for D just to be finite and not stably finite. But from our point of view, it is only interesting in the stably finite case. Otherwise K_1-injectivity of D is already known.

Definition 7.2.1. [1, Definition II.5.4.4] A C^*-algebra A is called prime if, whenever J and K are closed two-sided ideals in A with $J \cap K = (0)$, then either $J = (0)$ or $K = (0)$.

It clearly follows that every simple C^*-algebra, is prime.

Lemma 7.2.2. [28, Lemma 3.5] *Let A be a unital prime C^*-algebra and let $\varepsilon > 0$. If $x \in A$ is not one-sided invertible then there exist $y \in A$, $v \in \mathcal{U}(A)$ and $a \in A^+ \setminus \{0\}$ such that $\|x - y\| < \varepsilon$ and $vy \perp a$.*

Lemma 7.2.3. *Let A be a unital simple finite C^*-algebra and let $\varepsilon > 0$. If $x \in A$ is a non-invertible element, then there exist $y \in A$, $v \in \mathcal{U}(A)$ and mutually orthogonal positive elements $e, f \in A^+ \setminus \{0\}$ such that:*

(i) $\|x - y\| < \varepsilon$,

(ii) $fvy = vyf = vy$ *and*

(iii) $(e + f)^\perp = (0)$.

Proof. Since A is finite, x cannot be one-sided invertible (see [35, Lemma 5.1.2]), so we can use Lemma 7.2.2 to find $y \in A$, $v \in \mathcal{U}(A)$ and $a \in A^+ \setminus \{0\}$ such that $\|x - y\| < \varepsilon$ and

$$vya = avy = 0,$$

which implies that

$$(1 - a)vy = vy = vy(1 - a).$$

Without loss of generality we can assume that $0 \leq 1 - a \leq 1$. Define for $0 < \lambda < 1$ functions $f_\lambda, g_\lambda \in C([0, 1])$ by

$$f_\lambda(t) = \begin{cases} -\frac{1}{\lambda}t + 1, & 0 \leq t \leq \lambda \\ 0, & \lambda < t \leq 1 \end{cases}$$

and

$$g_\lambda(t) = \begin{cases} 0, & 0 \leq t \leq \lambda \\ \frac{1}{1-\lambda}t + \frac{\lambda}{\lambda - 1}, & \lambda < t \leq 1 \end{cases}$$

Let $f = g_\lambda(1 - a)$. Then

$$fvy = vy = vyf$$

because $g_\lambda(1) = 1$ and $(1-a)vy - vy = 0$. Let $e = f_\lambda(1-a)$. Since $f_\lambda \perp g_\lambda$, it follows that e and f are orthogonal elements. We need to show that $(e+f)^\perp = (0)$, so let b be a positive element such that $b \perp (e+f)$.

Let τ be a faithful tracial state on $C([0,1])$. Then there exists a unique probability measure μ on $[0,1]$ such that

$$\tau(g) = \int_0^1 g(t)d\mu(t), \quad g \in C([0,1]).$$

Since τ is faithful, μ is non-zero on any open subset of $[0,1]$.
Let d_τ be the dimension function on $C([0,1])$ induced by τ. When g is a positive continuous function on $[0,1]$, then $\lim_{n\to\infty} g^{\frac{1}{n}} = \chi_{\{t\in[0,1]:g(t)\neq 0\}}$. I.e.,

$$d_\tau(g) = \lim_{n\to\infty} \tau(g^{\frac{1}{n}}) = \tau(\chi_{\{t\in[0,1]:g(t)\neq 0\}}) = \mu(\{t \in [0,1] : g(t) \neq 0\})$$

for $g \in C([0,1])^+$.

Choose $\lambda \in]0,1[$ such that $\mu(\{\lambda\}) = 0$. Hence

$$d_\tau(f_\lambda + g_\lambda) = d_\tau(f_\lambda) + d_\tau(g_\lambda) = \mu([0,\lambda[) + \mu(]\lambda,1]) = 1.$$

Because of the isomorphism $C^*(1, 1-a) \cong C([0,1])$, there is a faithful tracial state τ' on $C^*(1,1-a)$ which induces a dimension function $d_{\tau'}$ on $C^*(1,1-a)$ such that $d_{\tau'}(e+f) = 1$. Since b is orthogonal to $e+f$, it follows that $d_{\tau'}(b) = 0$. Thus, $b = 0$ because $d_{\tau'}$ is faithful. □

When we shall give conditions that imply that the stable rank of $A \otimes D$ is one, we are going to use Lemma 7.2.5, which is easily proved from the following lemma by Rørdam and Winter.

Lemma 7.2.4. [36, Lemma 6.4] *Let D be a finite strongly self-absorbing C^*-algebra and let τ be the unique tracial state on D. There are positive elements $b, c \in D$ such that $\langle b \rangle = \langle c \rangle$, $b \perp c$ and $d_\tau(b) = d_\tau(c) = \frac{1}{2}$, where d_τ is the dimension function induced by τ.*

Lemma 7.2.5. *Let D be a finite strongly self-absorbing C^*-algebra. For every $k \in \mathbb{N}$ there are non-zero positive elements $a_1, a_2, \ldots, a_{2^k} \in D$ such that $\langle a_i \rangle = \langle a_j \rangle$, $a_i \perp a_j$ for $i \neq j$ and $(a_1 + \cdots + a_{2^k})^\perp = (0)$.*

Proof. For each $k \in \mathbb{N}$, $D \cong \bigotimes_{i=1}^k D$, so using Lemma 7.2.4 we can for each $i \in \{1,\ldots,k\}$ find non-zero positive elements a_{i1}, a_{i2} in the i'th tensor of $\bigotimes_{i=1}^k D$ such that

$$\langle a_{i1} \rangle = \langle a_{i2} \rangle, \quad a_{i1} \perp a_{i2} \quad \text{and} \quad d_\tau(a_{i1}) = d_\tau(a_{i2}) = \frac{1}{2}.$$

Defining elements $a_1, \ldots, a_{2^k}$ on the form

$$a_{1j_1} \otimes a_{2j_2} \otimes \cdots \otimes a_{kj_k}$$

where $j_i \in \{1,2\}$, it follows that $a_1, \dots, a_{2^k}$ are non-zero positive elements in $\bigotimes_{i=1}^k D \cong D$ satisfying

$$\langle a_i \rangle = \langle a_j \rangle \quad \text{and} \quad a_i \perp a_j, \quad i \neq j.$$

Identifying D with $\bigotimes_{i=1}^k D$ we have that

$$d_\tau(b_1 \otimes \cdots \otimes b_k) = d_\tau(b_1) \cdots d_\tau(b_k)$$

for all $b_i \in D^+$. Hence

$$d_\tau(a_1 + \cdots + a_{2^k}) = 1.$$

If c is a positive element in $(a_1 + \cdots + a_{2^k})^\perp$, then

$$d_\tau(c + a_1 + \cdots + a_{2^k}) = d_\tau(c) + d_\tau(a_1 + \cdots + a_{2^k}) = d_\tau(c) + 1.$$

Thus, $d_\tau(c) = 0$ which implies that $c = 0$ since d_τ is faithful.
Therefore $(a_1 + \cdots + a_{2^k})^\perp = (0)$. □

Let A be a unital simple finite C^*-algebra and let D be a unital finite strongly self-absorbing C^*-algebra. As written above we shall give some conditions that imply that $\operatorname{sr}(A \otimes D) = 1$, and thereby that $A \otimes D$ is K_1-injective. But $A \otimes D$ is isomorphic to the inductive limit of the inductive sequence

$$A_0 \xrightarrow{\varphi_k} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \longrightarrow \cdots \longrightarrow A \otimes (\textstyle\bigotimes_{i=1}^\infty D),$$

where $A_k = A \otimes (\bigotimes_{i=1}^k D)$ and $\varphi_k(x) = x \otimes 1_D$. So $A \otimes D$ has stable rank one if for every $k \in \mathbb{N}$ and every $x \in A_k$, one has that $\varphi_k(x) \in \overline{\mathrm{GL}(A_{k+1})}$. Therefore, if for each $x_0 \in A$ we can prove that $x = x_0 \otimes 1_D \in \overline{\mathrm{GL}(A \otimes D)}$, we have the desired result.

We assume that x_0 is not invertible, otherwise we are already done.

From Lemma 7.2.3 we can for each $\varepsilon > 0$ find $y \in A$, $v \in \mathcal{U}(A)$ and mutually orthogonal positive elements $e, f \in A^+ \setminus \{0\}$ such that $\|x_0 - y\| < \varepsilon$, $fvy = vyf = vy$ and $(e+f)^\perp = (0)$. Let $\bar{x} = vy \otimes 1_D \in A \otimes D$.
Since A is simple there exists $k \in \mathbb{N}$ and elements $t_1, \dots, t_{2^k} \in A$ such that $\sum_{j=1}^{2^k} t_j^* e t_j = 1_A$. Use this k to find $a_1, \dots, a_{2^k} \in D$ that satisfies the conditions in Lemma 7.2.5.

Let $b_0 = e \otimes 1_D$ and $b_j = f \otimes a_j$ for $j = 1, \dots, 2^k$. Then

$$\langle b_1 \rangle = \langle b_2 \rangle = \cdots = \langle b_{2^k} \rangle, \quad b_j \bar{x} = \bar{x} b_j,\ j = 0, \dots, 2^k \quad \text{and} \quad (b_0 + \dots b_{2^k})^\perp = (0).$$

Theorem 7.2.6. *Let A be a unital simple finite C^*-algebra, and let D be a unital finite strongly self-absorbing C^*-algebra. With the notation from above, if there exists a unitary $u \in \mathcal{U}(A \otimes D)$ that satisfies*

(i) $u^* b_{j+1} u = b_j, \quad j = 1, \dots, 2^k - 1$

(ii) $u^*b_1u \in \overline{b_0(A \otimes D)b_0}$,

then $\mathrm{sr}(A \otimes D) = 1$.

Proof. Let $n = 2^k$, and using the notation from above, let $\bar{x} = vy \otimes 1_D \in A \otimes D$. Then

$$b_0(\bar{x}u)^{n+1} = (e \otimes 1_D)(\bar{x}u)^n = 0$$

since $e \perp f$ and $fvy = vy = vyf$. There is a $b_0' \in \overline{b_0(A \otimes D)b_0}$ such that $b_1u = ub_0'$. Thereby it follows that

$$b_1(\bar{x}u)^2 = \bar{x}b_1u\bar{x}u = \bar{x}ub_0'\bar{x}u = 0.$$

Furthermore,

$$b_2(\bar{x}u)^3 = \bar{x}b_2u(\bar{x}u)^2 = \bar{x}ub_1(\bar{x}u)^2 = 0$$

and it follows by induction that $b_j(\bar{x}u)^{n+1} = 0$ for $j = 0, \dots, 2^k$.
Hence

$$(b_0 + \cdots + b_{2^k})(\bar{x}u)^{n+1} = 0,$$

which implies that $(\bar{x}u)^{n+1} = 0$ since $(b_0 + \cdots + b_{2^k})^\perp = (0)$. Thus, $\mathrm{sp}(\bar{x}u) = \{0\}$ and therefore $\bar{x}u + \lambda 1 \in \mathrm{GL}(A \otimes D)$ for each $\lambda \neq 0$. Then $\bar{x} + \lambda u^* \in \mathrm{GL}(A \otimes D)$, which implies that $\bar{x} \in \overline{\mathrm{GL}(A \otimes D)}$.

Moreover,

$$\|\bar{x} - (v \otimes 1_D)x\| = \|vy \otimes 1_D - vx_0 \otimes 1_D\| = \|v(y - x_0) \otimes 1_D\| < \varepsilon.$$

Therefore $(v \otimes 1_D)x \in \overline{\mathrm{GL}(A \otimes D)}$, and since $v \otimes 1_D$ is unitary we get that $x \in \overline{\mathrm{GL}(A \otimes D)}$. $\square$

So the strategy to prove that a stably finite strongly self-absorbing C^*-algebra D is K_1-injective, is to consider $A \otimes D$ where A is any unital simple finite C^*-algebra. Then we shall construct a unitary $u \in A \otimes D$ that satisfies the conditions in the Theorem above. If this can be done, it follows that D is K_1-injective, since A can be replaced by D and $D \otimes D \cong D$.

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