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Quasidiagonality of Nuclear C*-Algebras

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Author:
Mikkel Munkholm

Advisor:
Mikael Rørdam

Department of Mathematical Sciences

Abstract

The thesis investigates the impact of quasidiagonality on the classification programme, the programme being the attempt to classify nuclear separable simple unital C*-algebras fulfilling the UCT-condition. The programme achieved prominent progress during the last couple of decades, and finally culminated into the classification theorem in the finite nuclear dimensional case. The theorem, however, assumed quasidiagonality of all traces on the aforementioned type of C*-algebras. The thesis specifically pursues the proof of this particular assumption being automatic, a deep theorem due to Aaron Tikuisis, Stuart White and Wilhelm Winter in 2015. This includes an in depth analysis of quasidiagonality in the nuclear separable framework in terms of ultrapowers, von Neumann algebras induced from traces of such C*-algebras, KK-theory of nuclear C*-algebras and comparison theory. Towards the end, the connections between the Tikuisis-White-Winter theorem and the classification programme, the Blackadar-Kirchberg problem alongside the Rosenberg conjecture are provided.

Abstract

Denne afhandling undersøger den indflydelse som quasidiagonalitet har haft på klassifikationsprogrammet. Programmet er et forsøg på at klassifisere nukleære, separable, simple, C*-algebraer med enhed, der opfylder UCT-kriteriet. Indenfor de seneste årtier har man opnået et gennembrud og endeligt fuldendt klassifikationen under yderligere forudsætninger. Den ekstra antagelse vedrører quasidiagonalitet af samtlige spor. Afhandlingens primære formål omhandler at påvise overflødeligheden af denne forudsætning. Sætningen der giver anledning til dette blev udledt af Aaron Tikusis, Stuart White og Wilhelm Winter i 2015. Hertil inkluderes en analyse af quasidiagonalitet for nukleære separable C*-algebraer beskrevet i termer af ultraprodukter, von Neumann algebraer som afstammer fra spor, KK-teori af nukleære C*-algebraer samt strikt-sammenligningsteori. Imod slutningen angives sammenhængen mellem den førnævnte hovedsætning og klassifikationsprogrammet, Blackadar-Kirchberg problemet og Rosenberg-formodningen.

Contents

Introduction

Classification theory of C*-algebras is central to the thesis and serves as the primary motivation. Classification theory attempts to convey complete invariants attached to some prescribed class of C*-algebras. Such invariants are frequently described in terms of K-theoretic data or modifications thereof. One of the earliest and ramifying accomplishments of such kin is the complete classification of UHF - and AF-algebras, which originates back the Glimm. Glimm succeeded in characterizing UHF-algebras solely through supernatural numbers. Thereafter, Elliott achieved a classification of AF-algebras using the (ordered) K$_0$-group associated to C*-algebras. It was further revealed that the aforementioned classification of UHF-algebras may be encapsulated in the K$_0$-group.

These pioneering classifications, alongside Elliott's classification of $A\mathbb{T}$-algebras in 1989, sparked embers towards searching for a classification theory of nuclear separable C*-algebras. Nuclear C*-algebras have dominated the C*-algebraic scene for decades. Elliott himself conjectured a certain subclass of such C*-algebras to be classified via K-theoretic data, although the conjecture was thwarted for the nuclear separable C*-algebras. In an attempt to remedy the loss, added restrictive properties were posed upon the C*-algebras. The additional features required were in particular simplicity and the UCT-condition. With no counterexamples, seemingly, lurking around, this spurred motivations of answering the following question: Let $\mathcal{N}$ represent the class of nuclear, simple, separable, unital C*-algebras fulfilling the UCT-condition;

can we classify members of $\mathcal{N}$ in terms of K-theoretic data?

After decades filled with devoted attempts and partial answers, the greatest marvel achieved is probably Elliott, Gong, Niu and Lin's classification in 2015. It engages the original question raised above with two major modifications. Firstly, nuclearity was insufficient and finite nuclear dimension, a stronger property, took its place. The second major modification was the entry of quasidiagonality of every bounded tracial state.

Quasidiagonality originates back to Paul Halmos in the seventies, who introduced it as an approximation alteration to block-diagonality of bounded operators acting on Hilbert spaces. Following Voiculescu's compelling work on the matter, an abstract characterization attached to C*-algebras was established and the notion of quasidiagonal traces was hereby spawned. Later on, quasidiagonality emerged in the classification programme. Ergo, determining the scenario in which quasidiagonality (of C*-algebras and their traces) became a pivotal task. Prior to the Tikuisis-White-Winter theorem, the following question remained unanswered:

Are traces of members in $\mathcal{N}$ automatically quasidiagonal?

Indeed, answering this question provides both a connection between $\mathcal{N}$ and quasidiagonality together with simplifying the classification result into the original designation, vis-a-vis the intriguing class $\mathcal{N}$. The Tikuisis-White-Winter theorem provides an affirmative answer to the question. In fact, it answers an old conjecture of Rosenberg as well. Rosenberg proved that if the reduced C*-algebra associated to a discrete group G is quasidiagonal, then G must be amenable. He further conjectured the converse statement to be valid, albeit the answer has remained unanswered until 2015 with a partial answer for the elementary amenable groups due to Rørdam, Sato and Ozawa being present since 2013. A full-fledged answer, in the affirmative, of Rosenberg's conjecture was granted by the Tikuisis-White-Winter theorem.

There are other various applications of the theorem. The thesis does not exert itself to include a survey of every existing one, but Rosenberg's conjecture alongside the classification programme will be explored. As another problem to be encountered in detail will be the Blackadar-Kirchberg problem. The conjecture asks whether the converse in, the nuclear separable setting, of the valid statement that *quasidiagonal C*-algebras are stably finite* holds. The converse is generally false, without nuclear counterexamples to be found currently. In the thesis, we supply a partial answer by throwing the UCT-condition alongside simplicity into the mix of assumptions.

The final long-standing conjecture, to be discussed in brevity during the thesis, is the Toms-Winter conjecture. It predicts the equivalence of the most frequently occurring invariants for the class $\mathcal{N}$, including finite decomposition rank, finite nuclear dimension, strict comparison and $\mathcal{Z}$-stability. It remains to be verified in its full shape currently. Fortunately, solace is to be found in the monotracial case through a plethora of results derived by several participants in conjunction with the theorem of Tikuisis-White-Winter theorem. We shall present an overview of how this may be deduced towards the end of the thesis.

A vague overview of how the thesis pursues the proof of Tikuisis, White and Winter's theorem is found beneath. This includes the list of primary sources exploited during the process and the structure of the thesis. Some crucial results are omitted in the thesis, however, references and their roles are supplied in an attempt to ensure a sense of completeness.

- *Chapter* 1: The first chapter is devoted to establishing the necessary background knowledge, conventions and notation. It will feature some proofs of the theory that was hitherto unknown to the author. The primary sources used in the chapter are [31], [9], [30] and [29].

- *Chapter* 2: The second chapter carries an exposition of ultrapowers and von Neumann algebras associated to C*-algebras admitting traces. It serves the purpose of developing the setup required to build the skeleton of the main theorem in the shape of three maps. The primary sources used here are [41], [35] and [9]

- *Chapter* 3: Here the three aforementioned maps are built after having discussed quasidiagonality in the nuclear separable setting. The majority of results is based on the main article [42] and [30]. Furthermore, a fake proof of the main theorem is granted for emphasis on the general idea.

- *Chapter* 4: This chapter takes the KK-theoretic aspects into account, including a survey regarding the entrance of the UCT-condition. The issue solved here concerns the so-called stable uniqueness result of Dardalat and Eilers. The primary sources are [35], [16] and [18].

- *Chapter* 5: The final chapter conveys a full-fledged proof of the main theorem in [42] and applications. It is solely based on [42] together with a vast amount of older classification natured results used during the applications.

Prerequisites: A solid understanding of basics in C*-algebras, and deeper concepts thereof including but not limited to nuclearity, exactness, quasidiagonality etc., is absolutely imperative. Finally, K-theoretic knowledge corresponding to the first seven chapters in [29] will be highly helpful.

Acknowledgement. I wish to express my sincere gratitude to my advisor Mikael Rørdam, without whom I would not have been capable of grasping the theory used throughout the thesis. Nor would my former project, which paved the way for the thesis, have taken place. Your patience and uncanny ability to connect the theory into a greater perspective continues to surprise me. The subject has been quite challenging and the ordeal has often reminded me of Sysiphus' struggle. To my fortune, you have been supportive throughout the process and by the mesmerizing words of Albert Camus': ,,I can only imagine Sysiphus a happy man".

Comment. The thesis leans heavily on my former project [30], so the reader is highly encouraged to have a copy nearby if the background knowledge exhibited therein seems unfamiliar. With these comments and the introduction given; let us set sail and start, ab initio.

Chapter 1

Preliminaries

1.1 Setup

We commence our voyage towards quasidiagonality of nuclear C*-algebras by establishing various conventions. First and foremost, C*-algebras will not be unital nor separable unless specified otherwise. In continuation hereof, ∗-homomorphisms are never unital without mentioning either. We mainly use the capital letters A, B, C and D to represent C*-algebras. We prioritize using the symbols $\pi, \varrho, \Lambda, \sigma$ and γ for ∗-homomorphisms while reserving φ, ψ for linear maps. Ideals of C*-algebras will by default be two-sided closed involutive algebraic ideals. Hilbert spaces are a priori non-separable, although we blatantly assume these to be infinite dimensional and reserve $\mathcal{H}, \mathcal{K}$ to symbolically represent these. Furthermore, in a normed space we write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$.

Let A be any C*-algebra. We will frequently employ the notation A_r to symbolically represent the closed ball in A of radius r. The dual space A^* of A is implicitly endowed with the weak*-topology. Elements in A^* will be represented by ω, mostly. The weak*-topological subspace of states, which is compact whenever A admits a unit, is represented by $S(A)$.

Throughout the entire thesis, a *trace* will be the placeholder for *bounded tracial state*. Adopting this convention, we keep the usual notation $T(A)$ for the trace simplex on A consisting of all traces acting on A. This is known to constitute a convex weak*-compact space in A^* whenever A is unital. We reserve τ to denote traces, commonly writing τ_A for emphasis on the ambient algebra.

- For each positive integer n, we denote the C*-algebra consisting of complex $n \times n$-matrices by $\mathbb{M}_n$. The standard (unnormalized) tracial functional will be denoted by Tr_n whereas the unique trace on $\mathbb{M}_n$, i.e., the normalization of Tr_n, will be denoted by τ_n.

- Fix some Hilbert space $\mathcal{H}$. The C*-algebra consisting of bounded operators acting on $\mathcal{H}$ is written as $\mathrm{B}(\mathcal{H})$. The sets $\mathrm{F}(\mathcal{H})$ and $\mathrm{K}(\mathcal{H})$ will represent the finite rank - and the compact operators acting on $\mathcal{H}$, respectively. In the separable scenario, we abbreviate these subspaces $\mathbb{F}$ and $\mathbb{K}$ instead. In fact, $\mathrm{K}(\mathcal{H})$ is known to be the sole ideal in $\mathrm{B}(\mathcal{H})$.

- Suppose Ω denotes a locally compact Hausdorff topological space. The space consisting of continuous functions $f \colon \Omega \longrightarrow \mathbb{C}$ vanishing at infinity will have the symbol $C_0(\Omega)$ attached. This is known to be unital if and only if Ω is compact, in which case $C_0(\Omega) = C(\Omega)$. A frequently exploited fact, referred to as *the Riez-Markov-Kakutani representation theorem*, concerns $C_0(\Omega)$ and its dual: A bounded *positive* linear functional ω hereon attains the form

$$\omega(f) = \int_\Omega f \, d\mu, \quad f \in C_0(\Omega),$$

for some unique regular Borel measure μ on Ω.

Due to the Gelfand-Naimark classification of commutative C*-algebras, we may and shall assume that commutative C*-algebras arise of the form $C_0(\Omega)$ for some suitable locally compact Hausdorff

space Ω. Recall that for a commutative C*-algebra, the character space $\mathrm{Spec}(A)$ defines a weak* locally compact space Hausdorff space.

Theorem 1.1.1 (Gelfand-Naimark)**.** *Let A be a commutative C*-algebra and write $\Omega = \mathrm{Spec}(A)$. Then there exists a *-isomorphism $\Gamma\colon A \longrightarrow C_0(\Omega)$, unital if A admits a unit. When A is separable, the space Ω becomes second countable.*

Including a complete classification of commutative C*-algebras, the Gelfand-Naimark theorem provides an enigmatic machinery, the *continuous functional calculus*. We require some terminology to aptly present it. Given a C*-algebra A together with some nonempty subset $M \subseteq A$, denote the C*-subalgebra of A generated by M via the symbol $C^*(M)$, abbreviating $C^*(a_1, a_2, \ldots, a_n)$ whenever $a_1, a_2, \ldots, a_n$ are elements belonging to A. Thus a normal element a in A generates a commutative C*-algebra $C^*(a)$. The second ingredient necessary is the spectrum. The *spectrum* of some element a in a unital C*-algebra A is the nonempty set

$$\sigma(a) = \big\{ z \in \mathbb{C} : z 1_A - a \notin \mathrm{GL}(A) \big\},$$

where $\mathrm{GL}(A)$ indicates the topological group of invertible elements in A. In the non-unital case, one regards a as an element in the unitization (see below).

Theorem 1.1.2 (The Continuous Functional Calculus)**.** *For every normal element a belonging to some C*-algebra A, the inverse isomorphism in theorem 1.1.1 given by $f \mapsto f(a)$ is an isometric *-epimorphism, which is unital should A admit a unit. Moreover,*

- *for any *-homomorphism $\pi\colon A \longrightarrow B$ between C*-algebras, one has $\pi(f(a)) = f(\pi(a))$;*

- *for any element f in $C_0(\mathrm{spec}(A))$, one has $\sigma(f(a)) = f(\sigma(a))$.*

The continuous functional calculus yields numerous indispensable tools, including the existence of positive square roots, but exhibiting each of these will deter from the overall theme. The fundamental properties exploited are assumed to be quaint, although certain ones may perhaps be mentioned during the thesis to serve as reminders. Let some C*-algebra A be momentarily fixed.

- The set of self-adjoint elements in A will be denoted by A_{sa} and the set of positive elements by A_+. To represent positivity of an element a, one further writes $a \geq 0$. The set of self-adjoints thereby admits a partial order $\leq$ defined by stipulating that $a \leq b$ if and only if $b - a \geq 0$. It is a non-trivial fact that A_+ under this relation determines a norm-closed cone. Additionally, we adopt the short-hand $|a| := (a^*a)^{1/2}$ throughout the thesis.

- The set of unitaries is denoted by $\mathcal{U}(A)$ whereas the set of projections is symbolically represented by $\mathrm{Proj}(A)$. Whenever we work within $\mathrm{B}(\mathcal{H})$, we abbreviate these $\mathcal{U}(\mathcal{H})$ and $\mathrm{Proj}(\mathcal{H})$, respectively. A basic result states that projections p, q in A fulfill $q \leq p$ if and only if $pq = qp = q$.

- Recall that an element v in A is called a *partial isometry* should v^*v and vv^* be projections or equivalently if $v = vv^*v$. One thereof declares that two projections p, q in A are *Murray - von Neumann equivalent*, written $p \sim q$ in symbols, if and only if there exists some partial isometry v inside A such that $v^*v = p$ together with $vv^* = q$ hold. One writes $q \preceq p$ if there exists some projection p_0 in A satisfying $q \sim p_0 \leq p$. Lastly, we call p and q *orthogonal* whenever $pq = 0$.

- There are two frequently used decomposition rules. Every element a belonging to A decomposes into a $\mathbb{C}$-linear combination of self-adjoint elements; one may write $a = \mathrm{Re}(a) + i \cdot \mathrm{Im}(a)$ for two self-adjoint elements $\mathrm{Re}(a), \mathrm{Im}(a)$ in A, called the *real* and *imaginary* part respectively. Moreover, any self-adjoint element a in A decomposes into a linear combination $a = a_+ - a_-$ consisting of positive elements, respectively called the *positive part* and *negative part*.

We proceed to displaying several features of the unitization; the unitization will emerge numerous times. Suppose A denotes a non-unital C*-algebra. The *unitization* A^+ associated to A is the involutive algebra $A \oplus \mathbb{C}$ endowed with the multiplication

$$(a + z1_{A^+})(a_0 + z_0 1_{A^+}) := aa_0 + za_0 + z_0 a + zz_0 1_{A^+},$$

and the remaining algebraic operations are the obvious component-wise ones. The unitization may be equipped with a norm turning it into a unital C*-algebra containing A as an ideal. We implicitly impose this structure on A throughout the thesis. The unitization therefore admits the following short-exact sequence:

$$0 \longrightarrow A \xrightarrow{\iota_A} A^+ \xrightarrow{q_A} A^+/A \cong \mathbb{C} \longrightarrow 0.$$

In the above, ι_A denotes the canonical *-monomorphism given by $a \mapsto a + 01_{A^+}$ and q_A is the quotient map. Through the short-exact sequence, one may easily deduce a functoriality property of A^+ described as follows. Suppose A, B are C*-algebras. Let $\pi \colon A \longrightarrow B$ be any bounded linear map (resp. *-homomorphism). Then there exists a unique unital bounded linear map $\pi_+ \colon A^+ \longrightarrow B$ (resp. unital *-homomorphism) making the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\iota_A} & A^+ & \xrightarrow{q_A} & \mathbb{C} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & B & \xrightarrow{\iota_B} & B^+ & \xrightarrow{q_B} & \mathbb{C} & \longrightarrow & 0
\end{array}
$$

commute with exact-rows. Here q_A is the *-epimorphism given by $a + s1_{A^+} \mapsto s1_E$ with q_B being the resembling one for B. Thus π_+ sends an element $a + s1_{A^+}$ in A to $\pi(a) + s1_{B^+}$.

For the sake of reference, a few versatile inequalities are supplied. Suppose one has self-adjoint elements a, b inside some *unital* C*-algebra A. Since the spectrum of a belongs to the closed ball of radius $\|a\|$ in A, some functional calculus applied to the continuous function $f \colon \sigma(a) \longrightarrow \mathbb{R}^+$ given by the assignment $t \mapsto \|a\| - t$ yields

$$a \le \|a\| 1_A. \tag{1.1}$$

Appealing to (1.1), the relation $0 \le a \le b$ in A easily entails that $\|a\| \le \|b\|$. Moreover, the partial ordering on A_{sa} is conjugation invariant, meaning $a \le b$ implies that $cac^* \le cbc^*$ for all c in A. Hence, in conjunction with (1.1), this observation provides the inequality

$$b^* a^* ab \le \|a\|^2 b^* b, \quad \text{for all } a, b \in A. \tag{1.2}$$

Every positive linear functional φ on A obeys a Cauchy-Schwarz inequality, i.e, one has

$$|\varphi(a^* b)|^2 \le \varphi(a^* a) \varphi(b^* b), \quad \text{for all } a, b \in A. \tag{1.3}$$

Prior to addressing hereditary subalgebras, we establish some notation associated to the GNS-construction. Per usual, *-homomorphism $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$ of an *involutive algebra* A will be called a *representation*. We refer to π as being *non-degenerate* provided that $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$. A vector ξ in $\mathcal{H}$ is *cyclic for* π if $\pi(A)\xi$ is dense in $\mathcal{H}$. Let ω be a state acting on a C*-algebra A and denote by $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ its associated GNS-triple. Then

$$\omega(\cdot) = \langle \pi_\omega(\cdot)\xi_\omega, \xi_\omega \rangle. \tag{1.4}$$

The vector ξ_ω is cyclic for π_ω. In particular, π_ω becomes faithful precisely whenever ω is faithful. The corresponding *universal representation* of A, meaning the maximal non-degenerate faithful representation of A, will be denoted by the pair $(\pi_u, \mathcal{H}_u)$.

For pure states additional salient features may be deduced. Recall that a representation π of A is *irreducible* if it has no nontrivial invariant subspaces, that is, any subspace $V \subseteq \mathcal{H}$ such that $\pi(a)V \subseteq V$ for each a in A must be a copy of $\mathbb{C}$. A result of Segel asserts that the GNS representation π_ω is irreducible if and only if ω is pure, see *proposition* 3.13.2 in [34].

1.2   Heredity, Approximate Units, Excision and Fullness

Properties regarding ideals and approximate units are taken into account here. We shall further discuss hereditary subalgebras, since they will appear constantly. Hereditary subalgebras play an important role for nuclear C*-algebras; these inherit nuclearity. The definition of an approximate unit varies slightly throughout the literature. As such settling some conventions must be done.

Definition. Let A be some C*-algebra.

- A *left approximate unit* of A is a net $(e_i)_{i \in J} \subseteq A$ of positive contractions such that $\lim_{i \in I} e_i a = a$. A *right approximate unit* of A is the obvious analogue and we further call $(e_i)_{i \in J}$ an approximate unit should it be both simultaneously.

- A is called *σ-unital* if it admits a countable approximate unit.

Every C*-algebra admits an approximate and every left (resp. right) closed algebraic ideal I inside some C*-algebra admits a right (resp. left) approximate unit. Ideals in C*-algebras determine C*-algebras heavily resembling their ambient algebra. The notion encapsulating this is heredity, which thus "enlarges the framework" of ideals.

Definition. Let A denote some C*-algebra.

- A C*-subalgebra B of A is *hereditary* if whenever $a \leq b$ for positive elements a in A and b in B, then a must be contained in B.

- Given a positive element b in B, the corresponding set $\mathrm{her}(b) := \overline{bAb}$ is referred to as *the hereditary C*-subalgebra generated by* b. The element b is said to be *strictly positive* if $\mathrm{her}(b) = A$.

The reader has been deceived slightly. A priori the hereditary algebra generated by some positive element does not appear hereditary, despite the subtle suggestions. We will justify the terminology including some correspondences to ideals shortly. Let us initially consider a typical example.

Example. Let A denote a C*-algebra containing a nonzero projection p. The unital C*-algebra pAp, commonly called *the corner of* p, constitutes a hereditary subalgebra in A. Indeed, if $b \leq pap$ for some elements a, b in A_+, then $p^\perp b p^\perp \leq p^\perp pap p^\perp$ wherein $p^\perp = 1_{A^+} - p$. The right-hand side must be zero since p is orthogonal to $p^\perp$, whereby $0 = \|p^\perp b p^\perp\| = \|b^{1/2} p^\perp\|^2$ holds. It follows that $p b^{1/2} = b^{1/2} p = b^{1/2}$ in A. Therefore one must have

$$b = p b^{1/2} b^{1/2} p = pbp \in pAp,$$

as required. Corners are encountered non-stop in the thesis, so do keep the example in mind.

The majority of used features for hereditary subalgebras is present beneath. Additionally, we shall rarely refer to these throughout the thesis and instead treat these as being standard results.

Proposition 1.2.1. *Suppose A denotes some C*-algebra.*

(i) *For every left closed ideal I in A, the set $I \cap I^*$ is a hereditary subalgebra in A.*

(ii) *The assignment $I \overset{\mu}{\mapsto} I \cap I^*$ defines a one-to-one correspondence from the set of ideals onto the set of hereditary subalgebras in A having $B \mapsto I_B$ as inverse, where*

$$I_B := \{a \in A : a^* a \in B\}.$$

In particular, every ideal corresponds to exactly one hereditary subalgebra.

(iii) *$B \subseteq A$ is a hereditary subalgebra if and only if $b a b_0 \in B$ is valid for all $a \in A$ and $b, b_0 \in B$.*

(iv) *For any positive element b, $\mathrm{her}(b)$ is the smallest hereditary subalgebra in A containing b.*

Proof. (i): It is evident that $I \cap I^*$ defines a C*-subalgebra in A. For heredity, suppose $0 \le a \le b$ occurs with $a \in A$ and $b \in I \cap I^*$. Choose a right approximate unit $(e_\alpha)_{\alpha \in J}$ of I. Due to

$$(1_{A+} - e_\alpha)a(1_{A+} - e_\alpha) \le (1_{A+} - e_\alpha)b(1_{A+} - e_\alpha)$$

being valid for each index α in J, one may deduce that

$$\|a^{1/2}(e_\alpha - 1_{A+})\|^2 = \|(e_\alpha - 1_{A+})a(e_\alpha - 1_{A+})\| \overset{(1.1)}{\le} \|b^{1/2}(e_\alpha - 1_{A+})\|^2.$$

The right-hand side tends to zero by hypothesis. Ergo the containment $a^{1/2} = a^{1/2} \lim_{\alpha \in J} e_\alpha \in I \cap I^*$ is valid, whereof a must likewise lie therein.

(ii): Let us justify that I_B in fact constitutes a left algebraic ideal, leaving the remaining properties to entertain the reader. Let a be an element in A and b be another in B. According to (1.2) one has $(ab)^*ab \le \|a\|^2 b^*b \in B$. By definition of I_B along with heredity of B, ab must belong to I_B. When a, b both belong to B one has

$$(a + b)^*(a + b) \le (a + b)^*(a + b)^* + (a - b)^*(a - b) = 2a^*a + 2b^*b.$$

Each term on the right-hand side belongs to B, hence $a + b$ must belong to I_B. To prove that $I \mapsto I \cap I^*$ is the inverse of $B \mapsto I_B$, we confine ourselves with one composition, i.e., $\mu\mu^{-1}$ is the identity. Let B be some hereditary subalgebra of A. The inclusion $B \subseteq I_B \cap I_B^*$ is clear. Upon decomposing into the positive and negative parts, verifying the reverse inclusion of the respective positive cones suffices. If b belongs to $(I_B \cap I_B^*)_+$, then b^2 must lie inside B by definition of I_B. This in turn forces b to belong to B_+ as required.

(iii): Based on the correspondence μ, any hereditary subalgebra $B \subseteq A$ is of the form $B = I \cap I^*$ for some closed left ideal $I \trianglelefteq A$. Thus any element $x = (ba)b_0$ with $a \in A$ and $b, b_0 \in B$ must be contained in I. The same holds for its adjoint x^* because $b \in I^*$ by assumption.

Conversely, let B be any C*-subalgebra inside A such that $bab_0 \in B$ whenever $a \in A$ and $b, b_0 \in B$ hold. Suppose $a \le b$ for positive elements $a \in A$ and $b \in B$. Fix an approximate unit $(e_\alpha)_{\alpha \in J}$ in B. In a manner resembling the proof of (i), one acquires

$$\|a^{1/2}(e_\alpha - 1_{A+})\|^2 \le \|b^{1/2}(e_\alpha - 1_{A+})\| \to 0.$$

Therefore $a = \lim_{\alpha \in J} e_\alpha a e_\alpha$ belongs to B as desired.

(iv): Part (iii) immediately reveals that $\mathrm{her}(a)$ must be hereditary. As such $a \in \mathrm{her}(a)$ remains to be justified. On the merits of (iii) once more, it suffices to express a as some norm limit of elements in $\mathrm{her}(a)$. For this, choose some approximate unit $(e_\alpha)_{\alpha \in J}$ therein. Then $a^2 = \lim_{\alpha \in J} ae_\alpha a \in \mathrm{her}(a)$, hence a must likewise lie therein. Voila. $\square$

The preceding proof is a testament to the strength of approximate units. However, one occasionally tackles commutator approximations, especially when approaching quasidiagonality. An empowered version of approximate units are therefore paramount to manufacture. This was achieved in [2] and [33], stated as the main results. We adopt the convention of writing $[a, b] := ba - ab$.

Definition. An approximate unit $(e_\alpha)_{\alpha \in J}$ contained in some C*-algebra A is *quasicentral* provided that it asymptotically commutes with elements in A, that is,

$$\lim_{\alpha \in J} \|[e_\alpha, a]\| = 0$$

for all a in A.

Proposition 1.2.2 (Arveson, Pedersen)**.** *A quasicentral approximate unit may be extracted from the convex hull of an existing approximate unit in some C*-algebra A. The approximate unit may further be chosen to be countable whenever A is separable.*

The next device to be derived is excision for pure states. The theorem is due to Akemann, Anderson and Pedersen in [1]. We present the notion and the aforementioned theorem modulo Kadison's transitivity theorem, which may be recovered in *theorem* 5.2.2 [31] for a rigorous proof.

Theorem 1.2.3 (Kadison's transitivity theorem)**.** *Let A be a non-trivial C^*-algebra admitting an irreducible representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$. If $\xi_1,\ldots,\xi_n$ form a linearly independent set of vectors in $\mathcal{H}$ and $\eta_1,\ldots,\eta_n$ are some additional vectors in $\mathcal{H}$, then there exists some element a in A such that $\pi(a)\xi_k = \eta_k$ for every $k = 1,\ldots,n$.*

Kadison's theorem is the saving grace when excising pure states. We present the proof in the unital scenario for quasi-completeness after introducing the notion of excision.

Definition. Let A be some C^*-algebra. A net $(e_\alpha)_{\alpha \in J}$ consisting of positive contractions is said to *excise* a state ω acting on A if

$$\lim_{\alpha \in J} \|e_\alpha^{1/2} a e_\alpha^{1/2} - \omega(a)e_\alpha\| = \lim_{\alpha \in J} \|e_\alpha a e_\alpha - \omega(a)e_\alpha^2\| = 0$$

and $\omega(e_\alpha) = 1$ for each index α and for every a in A.

Theorem 1.2.4 (Akemann-Anderson-Pederson)**.** *All pure states on a C^*-algebra may be excised.*

Proof. Suppose ω denotes a pure state on some C^*-algebra A. We restrict ourselves to the unital case for simplicity. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ be its associated irreducible GNS-triple. The set

$$\mathcal{L}_\omega = \{a \in A : \omega(a^*a) = 0\}$$

is known to constitute a closed left-algebraic ideal. We initially verify that one has $\ker \omega = \mathcal{L}_\omega + \mathcal{L}_\omega^*$, so suppose $a + b^*$ belongs to $\mathcal{L}_\omega + \mathcal{L}_\omega^*$. Using (1.3) repeatedly one obtains

$$|\omega((a + b^*)1_A)|^2 \leq \omega(a^*a) + \omega(bb^*) + \omega(aa^*)^{1/2}\omega(bb^*)^{1/2} + \omega(bb^*)^{1/2}\omega(a^*a)^{1/2} = 0,$$

granting the inclusion $\mathcal{L}_\omega + \mathcal{L}_\omega^* \subseteq \ker \omega$. For the reverse inclusion, let a be some element in the kernel of ω. Due to (1.4), the vectors $\pi_\omega(a)\xi_\omega$ and ξ_ω are orthogonal. Letting $\eta_1 = \xi_\omega$ and $\eta_2 = 0$, Kadison's transitivity theorem guarantees the existence of some element b in B for which $\pi_\omega(b)\xi_\omega = \xi_\omega$ together with $\pi_\omega(ba)\xi_\omega = 0$ hold. Thus (1.4) combined with ξ_ω being cyclic imply that $\omega((ba)^*ba) = 0$, so ba must belong to $\mathcal{L}_\omega$. A similar observation yields the containment $(1_A - b)a \in \mathcal{L}_\omega^*$. One acquires

$$a = ba + (1_A - b)a \in \mathcal{L}_\omega + \mathcal{L}_\omega^*.$$

The identity $\ker \omega = \mathcal{L}_\omega + \mathcal{L}_\omega^*$ follows. To produce the excising net, let $(e_\alpha)_{\alpha \in J}$ be any approximate unit of the hereditary subalgebra $\mathcal{L}_\omega \cap \mathcal{L}_\omega^*$. Put $u_\alpha := 1_A - e_\alpha$. Then $(u_\alpha)_{\alpha \in J}$ consists of positive contractions such that $\lim_{\alpha \in J} au_\alpha = 0$ for each element a in $\mathcal{L}_\omega$. Furthermore, the element $a - \omega(a)$ belongs to $\ker \omega$, so it must be of the form $v + w^*$ for some v, w in $\mathcal{L}_\omega$ according to the previous identity. Therefore one may deduce that

$$\lim_{\alpha \in J} \|u_\alpha a u_\alpha - \omega(a)u_\alpha^2\| = \lim_{\alpha \in J} \|u_\alpha(a - \omega(a))u_\alpha\|$$
$$= \lim_{\alpha \in J} \|u_\alpha v u_\alpha + u_\alpha w^* u_\alpha\|$$
$$\leq \lim_{\alpha \in J} \|v u_\alpha\| + \lim_{\alpha \in J} \|u_\alpha w^*\| = 0$$

Finally, since e_α belongs to $\mathcal{L}_\omega + \mathcal{L}_\omega^* = \ker \omega$, one has $\omega(u_\alpha) = 1$. This complete the proof. $\square$

We close the section by discussing fullness and simplicity. Simplicity is among the profound assumptions granting classification results for C^*-algebras and will appear often. Afterwards, we connect the notions by comparing strengths. Towards the end of the section a significant lemma will be proved for future purposes.

Definition. Suppose A and B are C*-algebras. Let a be any element in A. Define *the algebraic ideal generated by a* to be the algebraic two-sided ideal

$$I_{\mathrm{alg}}(a) := \left\{ \sum_{k=1}^{n} x_k a y_k : x_k, y_k \in A, \ n \in \mathbb{N} \right\}.$$

- A is said to be *simple* if it only admits trivial ideals.

- A is said to be *algebraically simple* if it only admits trivial algebraic ideals.

- A nonzero element $a \geq 0$ belonging to A is *full* provided that $I_{\mathrm{alg}}(a)$ is norm-dense in A.

- In the presence of units, a unital $*$-homomorphism $\gamma \colon A \longrightarrow B$ is *full* provided that $\gamma(a)$ is full in B for each nonzero a belonging to A.

Evidently, simplicity entails that every element must be full and any unital $*$-homomorphism having a simple C*-algebra as codomain must be full. For positive elements in the unital setting, fullness poses strict algebraic structure. Phenomena of such sort may strike the reader as being undesirable. However, it becomes a key-ingredient during the proof of the Tikuisis-White-Winter theorem.

Lemma 1.2.5. *Let A be some unital C*-algebra. If a denotes a positive full element in A, then there exist elements $b_1, \dots b_n$ in A such that*

$$1_A = \sum_{k=1}^{n} b_k a b_k^*.$$

Proof. The proof will be executed in three steps. At first, via density of $I_{\mathrm{alg}}(a)$ one may approximate the unit 1_A in terms of elements in $I_{\mathrm{alg}}(a)$. In particular, one may extract an element w in $I_{\mathrm{alg}}(a)$ fulfilling $1_A \approx_1 w$. Hence w must be invertible. Being an algebraic ideal, $I_{\mathrm{alg}}(a)$ must contain the element $w^{-1}w = 1_A$. One may thereof find some elements $x_1, \dots, x_m, y_1, \dots, y_m$ in A satisfying

$$1_A = \sum_{k=1}^{m} x_k a y_k.$$

Fix momentarily some elements x, y in A. Due to $0 \leq (x^* - y)^* a (x^* - y) = xax^* + y^*ay - xay - y^*ax^*$ one may deduce that $xay \leq xax^* + y^*ay$. Applying this observation to each pair of elements x_k and y_k from before, with $k \leq m$ being some positive integer, one may infer that

$$1_A \leq \sum_{k=1}^{m} (x_k a x_k^* + y_k^* a y_k). \tag{1.5}$$

Now, if $1_A \leq v$ for some element v in A, then $1_A = rvr^*$ for some additional element r contained in A. Indeed $v - 1_A \geq 0$ forces invertibility of v, hence one simply selects $r = v^{-1/2}$. Invoking this observation to the right-hand side of (1.5) yields such an element r. Let $n = 2m + 1$, then stipulate that $b_1 = rx_1, \dots, b_m = rx_m$ and $b_{m+1} = ry_1^*, \dots, b_{2m+1} = ry_m^*$ to acquire

$$1_A \overset{(1.5)}{=} \sum_{k=1}^{m} rx_k a (rx_k)^* + \sum_{k=1}^{m} ry_k^* a (ry_k^*)^* = \sum_{k=1}^{n} b_k a b_k^*.$$

This proves the claim. $\qquad\square$

Remark. The lemma provides an "if and only if" statement in the presence of a unit. Certainly, assume that A is a unital C*-algebra, containing some nonzero positive element a, admitting some elements $b_1, \dots, b_n$ for which we have $1_A = b_1 a b_1^* + \dots + b_n a b_n^*$. If so, the unit must belong to $I_{\mathrm{alg}}(a)$, whereby $\overline{I_{\mathrm{alg}}(a)} = A$.

1.3   Remarks on Tensors and Limits

Among the wealth of functorial constructions used during the thesis, the most unavoidable ones are arguably the tensor products and inductive limits. The purpose of the current section will be to establish notations and conventions attached, albeit we shall investigate traces on minimal tensor products in detail. The section will furthermore have additional emphasis on UHF-algebras.

Recalling Tensor Products

Theoretical background equivalent to the material gathered in section 3.1–3.5 of [9] is presumed well-known. However, certain statements leaning on states will be exploited during the progress. Due to the necessity of such, we tacitly include various results attached. Let in the following A, B, C and D be C*-algebras. The involutive algebraic tensor product of A with B is denoted by $A \odot B$, whereas the minimal and maximal tensor product closures are denoted by $A \otimes B$ and $A \otimes_{\max} B$, respectively. Let $\varphi \colon A \longrightarrow C$ and $\psi \colon B \longrightarrow D$ be bounded linear maps.

- The unique induced map from $A \odot B$ into $C \odot D$ defined on elementary tensors through the assignment $a \otimes b \mapsto \varphi(a) \otimes \psi(b)$ will be symbolically represented by $\varphi \otimes \psi$.

- For $C = D$, the unique induced map from $A \odot B$ into D defined on elementary tensors through the assignment $a \otimes b \mapsto \varphi(a)\psi(b)$ will be symbolically represented by $\varphi \times \psi$.

- For $C = D = \mathbb{C}$, the unique induced linear functional on $A \odot B$ defined on elementary tensors through the assignment $a \otimes b \mapsto \varphi(a)\psi(b)$ will be symbolically represented by $\varphi \odot \psi$.

If each map φ, ψ is positive or defines a $*$-homomorphism, then the induced maps evidently enjoy the respective properties.

Lemma 1.3.1. *Suppose A, B are unital C*-algebras admitting states ω_0, ω respectively. Suppose $\|\cdot\|_\alpha$ represents either the minimal or maximal norm on $A \odot B$. Under these premises, the induced positive linear functional $\omega \odot \omega_0$ extends to a state on $A \otimes_\alpha B$.*

Proof. According to the Hahn-Banach extension theorem, the sole obstruction is discontinuity. Hence it suffices to verify boundedness of $\omega_0 \odot \omega$. We adopt a Krein-Milman convexity type argument. Let initially ω, ω_0 be pure. Extract excising nets $(e_i)_{i \in I_0}$ and $(p_i)_{i \in I}$ for both pure states. Let $a \otimes b$ be an elementary tensor in $A \odot B$. Letting $J := I_0 \times I$ one may from

$$\|(e_i \otimes p_i)(a \otimes b)(e_i \otimes p_i) - \omega_0(a)\omega(b)(e_i \otimes p_i)^2\| \leq \|(e_i a e_i - \omega_0(a)e_i^2) \otimes p_i b p_i\|$$
$$+ \|\omega_0(a)e_i^2 \otimes (p_i b p_i - \omega(b)p_i^2)\| \to 0$$

conclude that $(e_i \otimes p_i)_{i \in J}$ excises $\omega_0 \odot \omega$. Ergo for each element x in $A \odot B$ one acquires

$$|(\omega_0 \odot \omega)(x)| = \lim_{j \in J} \|(\omega_0 \odot \omega)(x)(e_j \otimes p_j)^2\| = \lim_{j \in J} \|(e_j \otimes p_j)x(e_j \otimes p_j)\| \leq \|x\|.$$

It follows that ω_0, ω must be contractions if they are pure states, so that these extend to $A \otimes_\alpha B$. Suppose now ω_0 and ω are general states. According to the Krein-Milman theorem, for every tolerance $\varepsilon > 0$ one may find $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ in $\mathbb{R}^+$ subject to $\alpha_1 + \ldots + \alpha_n = \beta_1 + \ldots + \beta_n = 1$ along with pure states $\varphi_1, \ldots, \varphi_n$ on A, and $\psi_1, \ldots, \psi_m$ on B such that

$$\sum_{k=1}^n \alpha_k \varphi_k \approx_{\varepsilon/2} \omega_0, \quad \text{respectively,} \quad \sum_{k=1}^m \beta_k \psi_k \approx_{\varepsilon/2} \omega. \tag{1.6}$$

From the pure case one acquires

$$(\omega \odot \omega_0)(x) \approx_\varepsilon \sum_{k=1}^n \sum_{\ell=1}^m \alpha_k \beta_\ell (\varphi_k \odot \psi_\ell)(x) \leq \sum_{k=1}^n \sum_{\ell=1}^m \alpha_k \beta_\ell \|x\| = \|x\|.$$

for each x in $A \odot B$. Upon $\varepsilon > 0$ being arbitrary, the sought conclusion holds. $\square$

We will invoke Kirchberg's slice lemma to address faithfulness of induced traces on tensor products, in view of the preceding lemma. We avoid diving into the proof.

Corollary 1.3.2. *For unital C*-algebras A, B admitting faithful traces τ_A, τ_B, respectively, the uniquely determined induced trace $\tau_A \otimes \tau_B$ extending $\tau_A \odot \tau_B$ remains faithful.*

Proof. The tracial property of $\tau_A \otimes \tau_B$ stems from continuity and the state condition from lemma 1.3.1. Concerning faithfulness, write $\tau = \tau_A \otimes \tau_B$ and consider the ideal $\mathcal{L}_\tau = \{a \in A : \tau(a^*a) = 0\}$. Seeking a contradiction, suppose τ cannot be faithful . Being an ideal in $A \otimes B$, it must be hereditary, whereby Kirchberg's slice lemma, see *lemma* 4.1.9 in [35], provides some element z fulfilling $z^*z = a \otimes b \in A \odot B$ together with $zz^* \in \mathcal{L}_\tau$. Ergo one acquires $0 = \tau(z^*z) = \tau_A(a) \otimes \tau_B(b) > 0$, a contradiction. $\square$

A Remark on Inductive Limit Algebras

Inductive limits emerge occasionally, especially classes arising as such that are subject to classification theory, the C*-algebras in play are AF - and UHF algebras. These classes are crucial, especially the latter. The main objective of the current section is to eliminate potential notational conflict in the future, leaving details to a combination of [29] and [30]. The central types of inductive limits will be countable ones, hence we restricts ourselves hereto.

Definition. Let $\mathscr{A}$ denote some category. An *inductive sequence* in $\mathscr{A}$ is a sequence $(A_n, \pi_n)_{n \geq 1}$ consisting of pairs in which A_n denotes an object inside $\mathscr{A}$ and $\pi_n \colon A_n \longrightarrow A_{n+1}$ denotes a morphism therein. The maps π_n are called *the connecting morphisms*.

An *inductive limit* of the inductive sequence is a pairing $(A, \{\pi_n^\infty\}_{n \geq 1})$, where A is some object in $\mathscr{A}$ and each $\pi_n^\infty \colon A_n \longrightarrow A$ is a morphism making the diagram

$$A_n \xrightarrow{\ \ \pi_n\ \ } A_{n+1}$$
$$\pi_n^\infty \searrow \quad \swarrow \pi_{n+1}^\infty$$
$$A$$

commute for every positive integer n. Moreover, A must fulfill the following universal property: For any additional inductive limit $(B, \{\varrho_n^\infty\}_{n \geq 1})$ attached to the common inductive sequence, there exists a unique morphism $\gamma \colon A \longrightarrow B$ making the diagram

$$A \xrightarrow{\ \ \gamma\ \ } B$$
$$\pi_n^\infty \searrow \quad \nearrow \varrho_n^\infty$$
$$A_n$$

commute for every positive integer n.

Notice that upon invoking the universal property twice, an inductive limit is uniquely determined up to isomorphism in $\mathscr{A}$. Thus one may speak of "the inductive limit" if existence has been guaranteed. We denote the limit by $\varinjlim(A_n, \pi_n)$ whenever it exists. Proofs may be uncovered in chapter 6 of [29].

Proposition 1.3.3. *Let $(A_n, \pi_n)_{n \geq 1}$ denote an inductive sequence comprised of C*-algebras having $*$-homomorphisms as morphisms. Then the inductive sequence has a limit $(A, \{\pi_n^\infty\}_{n \geq 1})$ such that*

$$\bigcup_{n=1}^\infty \pi_n^\infty(A_n)$$

becomes norm-dense in A, that is, $\overline{\bigcup_{n=1}^\infty \pi_n^\infty(A_n)}$ serves as a model for $\varinjlim(A_n, \pi_n)$.

A highlighted application of inductive limits revolves around infinite tensor algebras of unital C*-algebras. Infinite tensor algebras yield fruitful characterizations of UHF-algebras. If the reader is unfamiliar with the concept of infinite tensor algebras, the technical details may be found in [30].

Definition. Suppose $A_1, A_2, \ldots$ are unital C*-algebras. The *spatial infinite tensor product* is defined to be the inductive limit

$$\bigotimes_{n=1}^{\infty} A_n := \varinjlim \left(\bigotimes_{k=1}^{n} A_k, \iota_k \right).$$

Here ι_k denotes the unital $*$-monomorphism given by $a \mapsto a \otimes 1_{A_{k+1}}$.

We turn our gaze towards concrete examples arising hereby, namely AF - and UHF algebras. We will afterwards depict a few properties associated to UHF-algebras specifically.

Definition. An inductive limit attached to an inductive sequence $(F_n, \pi_n)_{n \geq 1}$ consisting of finite dimensional C*-algebras F_n and $*$-homomorphisms $\pi_n \colon F_n \longrightarrow F_{n+1}$ is referred to as being *approximately finite dimensional*, abbreviated AF.

The subclass of AF-algebras whose members are inductive limits of sequences $(\mathbb{M}_{n(k)}, \pi_k)_{k \geq 1}$ in which $\pi_k \colon \mathbb{M}_{n(k)} \longrightarrow \mathbb{M}_{n(k+1)}$ is a unital $*$-homomorphism and $(n(1), n(2), \ldots)$ is some sequence of natural numbers is the class of *uniformly hyperfinite algebras*, abbreviated UHF.

For UHF-algebras keeping track of the positive integers $n(1), n(2), \ldots$ is the sole necessary ingredient required to unfold the building blocks. Indeed, suppose $(\mathbb{M}_{n(k)}, \pi_k)_{k \geq 1}$ denotes the inductive sequence of some UHF-algebra A. Since the $*$-homomorphisms π_k are unital, the integer $n(k)$ must divide $n(k+1)$ for each k. To realize this, fix some k in $\mathbb{N}$ and let e_j be the j'th diagonal unit matrix in $\mathbb{M}_{n(k)}$. Then $\mathrm{Tr}_{k+1}(\pi_k(e_j))$ must be an integer such that

$$n(k+1) = \mathrm{Tr}_{n(k+1)}(\pi_k(e_1)) + \ldots + \mathrm{Tr}_{n(k+1)}(\pi_k(e_{n(k)})) = n(k)\mathrm{Tr}_{n(k+1)}(\pi_k(e_1))$$

In particular, the morphism π_k may be regarded as the $*$-monomorphism $a \mapsto \mathrm{diag}(a, a, \ldots, a)$ with $d(k)$-copies occurring and $n(k)d(k) = n(k+1)$ being fulfilled. In this regard, using the isomorphism $\mathbb{M}_n \otimes \mathbb{M}_m \cong \mathbb{M}_{nm}$ one may with the convention $d(1) := n(1)$ rewrite A into

$$A = \bigotimes_{n=1}^{\infty} \mathbb{M}_{d(k)}.$$

The sequence of positive integers $(n(1), n(2), \ldots)$ induces a *supernatural number N*. For the formal definition, the reader is urged to consult section 1.3 in [30]. For those who find supernaturals quaint, we shall consistently regard N as being a formal prime factorization

$$N = \prod_{k=1}^{\infty} p_k^{\alpha_k},$$

where α_k is some positive integer or $\alpha_k = \infty$. The UHF-algebra attached to N is called the UHF-algebra of type N, denoted by $\mathbb{M}_N$. Notice that if N' is the supernatural number having the integers $\beta_1, \beta_2, \ldots$, including the possibility of ∞, as its exponents and one declares that NN' is the supernatural number having $\alpha_k + \beta_k$ as its k'th exponent, then

$$\mathbb{M}_N \otimes \mathbb{M}_{N'} \cong \mathbb{M}_{NN'}. \tag{1.7}$$

We primarily adopt the tensorial point of view for UHF-algebras in the thesis.

1.4 Completely Positive Morphisms and Nuclearity

Hitherto $*$-homomorphisms have been the morphisms in play, however, some weaker ones are preferred; completely positive maps. Completely positive maps are the heart of approximation properties for C*-algebras. The plethora of fundamental facts regarding these morphisms are assumed familiar. We shall in addition employ the empowered version known as order zero maps, so in order to convey sensible notation and overview, the core structural aspects have been assembled.

Definition. A bounded linear map $\psi\colon A \longrightarrow B$ between C*-algebras is *completely positive* if its n'th amplification map $\psi_n\colon \mathrm{M}_n(A) \longrightarrow \mathrm{M}_n(B)$ is positive for every positive integer n. The amplification is the induced bounded linear map given by

$$\psi_n([a_{ij}]) = [\psi(a_{ij})].$$

Under the identification $\mathrm{M}_n(A) \cong \mathbb{M}_n \otimes A$, the amplification ψ_n transforms into the tensor map $\psi \otimes \mathrm{id}_{\mathbb{M}_n}$. We shall pursue both points of view throughout the thesis.

We liberally abbreviate completely positive c.p, contractive completely positive c.p.c and unital completely positive as u.c.p. throughout. The crown-jewels of completely positive maps are Arveson's extension theorem and the Stinespring dilation theorem. The latter asserts that completely positive maps are twisted $*$-homomorphisms, whereas the former resolves the extension problem.

Theorem 1.4.1 (Stinespring)**.** *Let A be a C*-algebra and $\psi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ be completely positive. There exists a triple $(\sigma, w, \mathcal{K})$ consisting of a representation $\sigma\colon A \longrightarrow \mathrm{B}(\mathcal{K})$ and a linear contractive operator $w\colon \mathrm{B}(\mathcal{H}) \longrightarrow \mathrm{B}(\mathcal{K})$ such that*

$$\psi(\cdot) = w^*\sigma(\cdot)w.$$

The triple is referred to as the Stinespring dilation of ψ. It follows that $\|\psi\| = \|\psi(1_A)\|$.

Theorem 1.4.2 (Arveson's extension theorem)**.** $\mathrm{B}(\mathcal{H})$ *is an injective object in the category of C*-algebras having contractive completely positive maps as morphisms. The statement remains valid in the category of unital C*-algebras with unital completely positive maps as morphisms.*

The information contained in Stinespring's dilation theorem is of greater magnitude than one might expect. Determining multiplicativity of completely positive maps is completely enclosed as follows. Suppose $\psi\colon A \longrightarrow E$ is some completely positive map between C*-algebras. Let $(\sigma, w, \mathcal{K})$ be its Stinespring dilation, regarding A as being faithfully represented on some Hilbert space $\mathcal{H}$. For each a in A one has

$$\psi(a^*a) - \psi(a)^*\psi(a) = w^*\sigma(a)^*(1_\mathcal{K} - ww^*)\sigma(a)w \geq 0, \tag{1.8}$$

because $w \leq 1_\mathcal{H}$. An intriguing consequence hereof concerns the impact stemming from whenever equality is reached. For this, one defines *the multiplicative domain* of ψ to be

$$\mathrm{Mult}(\psi) = \{a \in A : \psi(a^*a) = \psi(a)^*\psi(a),\ \psi(aa^*) = \psi(a)\psi(a)^*\}.$$

The name of course arises from ψ restricting to a multiplicative map on $\mathrm{Mult}(\psi)$. To see this, maintain the notation prior to deducing (1.8). If $\psi(a^*a) = \psi(a)^*\psi(a)$, then the calculation in (1.8) in conjunction with the C*-identity entails that $(1_\mathcal{K} - w^*w)^{1/2}\sigma(a)w = 0$. Thus

$$\psi(ba) - \psi(b)\psi(a) = w^*\sigma(b)(1_\mathcal{K} - ww^*)^{1/2}\big[(1_\mathcal{K} - ww^*)^{1/2}\sigma(a)w\big] = 0, \tag{1.9}$$

$$\psi(ab) - \psi(a)\psi(b) = \big[w^*\sigma(a)(1_\mathcal{K} - ww^*)^{1/2}\big](1_\mathcal{K} - ww^*)^{1/2}\sigma(b)w = 0, \tag{1.10}$$

whenever a belongs to the multiplicative domain of ψ and b belongs to A.

An additional important and more restrictive kin of completely positive maps are conditional expectations. These tend to emerge, hence a few remarks are supplied. A relatively deep theorem due to Tomiyama will be used, whose statement we include without proof. The statement and proof may be found as *theorem* 1.5.10 in [9].

Definition. A linear map $E\colon A \longrightarrow B$ between C*-algebras, with $B \subseteq A$ being an inclusion of C*-algebras, is called a *projection* provided that E restricts to the identity on B. Moreover, if E defines a contractive completely positive projection obeying the rule $E(bab_0) = bE(a)b_0$ for each a in A and every b, b_0 in B, then E is called a *conditional expectation* onto B.

Theorem 1.4.3 (Tomiyama)**.** *Let $B \subseteq A$ be an inclusion of C*-algebras and let $E\colon A \longrightarrow B$ be a projection. Then E is a conditional expectation if and only if E is contractive completely positive.*

Completely positive maps realize salient C*-algebraic features in abstract[1] approximation-lingo. The benefits of such characterizations are aplenty, partly due to approximation properties being notoriously easier to deduce. Nuclearity is among the most prominent approximation-natured conditions to impose. To adequately present the full picture some terminology will be presented.

Definition. Let $\gamma\colon A \longrightarrow B$ be a completely positive map between C*-algebras.

- The map γ is said to be *finite dimensionally factorable* if there are contractive completely positive maps $\varphi\colon A \longrightarrow \mathbb{M}_n$ together with $\psi\colon \mathbb{M}_n \longrightarrow B$ such that $\gamma = \psi \circ \varphi$.

- The map γ is said to be *nuclear* if there exists a net $(\gamma_\alpha)_{\alpha \in J}$ consisting of finite dimensionally factorable maps from A into B such that $\|\gamma_\alpha(\cdot) - \gamma(\cdot)\| \to 0$.

- The map γ is said to have the *local completely positive approximation property* if there, for each finite subset $F \subseteq A$ and tolerance $\varepsilon > 0$, exists a finite dimensionally factorable map ψ such that $\gamma(a) \approx_\varepsilon \psi(a)$ for all a in F.

The reader is assumed to be aware of the equivalence between the latter two properties. The completely positive approximation property is now non-standard terminology due to the extensive work of Choi-Effros (Kirchberg gave another proof). Indeed, the property is merely nuclearity translated into approximation language. For proofs, we refer to *theorem 3.8.7* in [9].

Theorem 1.4.4 (Choi-Effros, Kirchberg)**.** *For any C*-algebra A the following are equivalent.*

- *The identity on A is nuclear.*

- *The identity on A has the local completely positive approximation property.*

- *There exists a unique C*-norm on $A \odot E$, for any additional C*-algebra E.*

The exactness counterpart is presented, leaving section 3.9 in [9] as a reference.

Theorem 1.4.5 (Kirchberg)**.** *For every C*-algebra A, the following are equivalent.*

- *The functor $E \mapsto A \otimes E$ is exact.*

- *There exists a faithful representation of A having the completely positive approximation property.*

Having established these notions, we may define nuclearity and exactness. Nuclearity in particular has dominated the C*-algebraic scene for decades. Several frequently occurring C*-algebras are nuclear including crossed products by amenable actions, reduced group C*-algebras associated to amenable groups, AF-algebras, the compact operators, commutative C*-algebras, the Cuntz-algebras $\mathcal{O}_n$ and more.

Definition. Suppose A denotes any C*-algebra. We call A *nuclear* should it fulfill either of the equivalent conditions occurring in theorem 1.4.4, and we call A *exact* if it fulfills either of the equivalent conditions occurring in theorem 1.4.5.

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[1] "Abstract" here meaning without reference to B($\mathcal{H}$). Quasidiagonality is an example, to be revealed later.

Nuclearity entails exactness whereas the converse is far from being true. For instance, the free group on n-generators gives rise to the exact but non-nuclear C*-algebra $C_r^*(\mathbb{F}_n)$. A distinguishing aspect between exactness and nuclearity concerns subalgebras. Indeed, exactness passes to subalgebras while the analogue statement for nuclearity fails. The class of nuclear C*-algebras luckily remains stable under passing to hereditary subalgebras. In particular, ideals will inherit nuclearity and the class of nuclear C*-algebras proves itself to be a well-behaving class. We justify this next.

Proposition 1.4.6. *Suppose A denotes some C*-algebras and let $A_1 \subseteq A_2 \subseteq \ldots$ be an increasing chain of C*-algebras.*

  (i) *If A_n is nuclear for every n, then the inductive limit $\overline{\bigcup_{n=1}^{\infty} A_n}$ is nuclear.*

 (ii) *If B is a hereditary subalgebra in A with A being nuclear, then B must be nuclear.*

(iii) *If A is an extension of nuclear C*-algebras in the category of C*-algebras having (not necessarily unital) $*$-homomorphisms as morphisms, then A must be nuclear.*

Proof. To spare time, the level of rigor has been reduced significantly. The main ideas are therefore exhibited, leaving the reader to delve into the gory ε-details.

(i): Let E denote the inductive limit of $A_1 \subseteq A_2 \ldots$ with inclusions along the way. Fix some finite subset $F \subseteq E$ and tolerance $\varepsilon > 0$. Finiteness of F guarantees the existence of some positive integer k together with an element e inside A_k such that $e \approx_\varepsilon a$ for all a in F. Since A_k is nuclear, there exists some factorization

$$A_k \xrightarrow{\psi_0} \mathbb{M}_n \xrightarrow{\varphi} A_k \subseteq A$$

of id_{A_k} by completely positive maps such that $(\varphi \circ \psi_0)(x) \approx_\varepsilon x$ whenever $x \in A_k$. According to Arveson's extension theorem, ψ_0 extends to a completely positive map $\psi \colon A \longrightarrow \mathbb{M}_n$. The completely positive map $\varphi \circ \psi$ has the sought properties, for given any $a \in A_k$ one has $(\varphi \circ \psi)(a) \approx_{3\varepsilon} a$.

(ii): Let $(e_j)_{j \in J_0}$ be an approximate unit of B. Define a completely positive map $\gamma_j \colon A \longrightarrow B$ via the compression $a \mapsto e_j a e_j$. Notice that γ_j has the prescribed codomain as B is hereditary. Let $(\psi_j)_{j \in J_1}$ be the net consisting of finite dimensional factorable maps converging point-norm wise to id_A. Then by collecting J_0, J_1 into the product directed set $J := J_0 \times J_1$, the compositions

$$B \xrightarrow{\iota} A \xrightarrow{\psi_j} A \xrightarrow{\gamma_j} B,$$

where ι denotes the inclusion map, provides a finite dimensional factorable map φ_j for each index j inside J. Due to $\|\gamma_j(b) - b\| = \|e_j b e_j - b\| \to 0$, it follows that $\iota \gamma_j \psi_j(b) \to b$.

(iii): This permanence property requires far more meticulous care. The strategy becomes more clear if we present the setup. Suppose A is an extension of an ideal I via some C*-algebra B. Let $\pi \colon A \longrightarrow B$ be the corresponding $*$-epimorphism. Since nuclearity entails exactness, the sequence

$$0 \longrightarrow I \otimes E \xrightarrow{\iota \otimes \mathrm{id}} A \otimes E \xrightarrow{\pi \otimes \mathrm{id}} B \otimes E \longrightarrow 0 \tag{1.11}$$

becomes short-exact for every C*-algebra E. We shall verify that $A \odot E$ admits a unique C*-norm, which amounts to verifying that $\|\cdot\| \geq \|\cdot\|_{\max}{}^2$. Since one has $\|\cdot\| \leq \|\cdot\|_{\max}$ in conjunction with $\|\cdot\|_{\min}$-continuity, the identity map on $A \odot E$ extends uniquely to a $*$-epimorphism $\sigma \colon A \otimes_{\max} E \longrightarrow A \otimes E$. Our task is to ensure injectivity of σ. To achieve this we arrange a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I \otimes E & \xrightarrow{\iota \otimes \mathrm{id}} & A \otimes E & \xrightarrow{\pi \otimes \mathrm{id}} & B \otimes E & \longrightarrow & 0 \\
& & \Big\uparrow{\scriptstyle \mathrm{id}} & & \Big\uparrow{\scriptstyle \sigma} & \nearrow{\scriptstyle \pi'} & \Big\downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & I \otimes E & \xrightarrow{\iota'} & A \otimes_{\max} E & \xrightarrow{\varrho} & \frac{A \otimes_{\max} E}{\mathrm{im}\, \iota'} & \longrightarrow & 0
\end{array}
$$

---

[2]The claim stems from minimality of the $\|\cdot\|_{\min}$-norm combined with maximality of $\|\cdot\|_{\max}$. The former is a non trivial result due to Takesaki, theorem 3.4.8 in [9]

Diagram chasing on the right-hand side entails that any element x in the kernel of σ must belong to the kernel of the quotient map ϱ, hence satisfies $x = \iota'(y)$ for some element y in $I \otimes E$ by exactness. Commutativity of the left-hand square yields $(\iota \otimes \mathrm{id})(y) = 0$, whereby exactness ensures that $y = 0$, granting $x = \iota'(y) = 0$. Altogether, it suffices to produce the $*$-homomorphisms π', θ and ι' along with the alleged commutativity.

The inclusion $I \odot E \subseteq I \otimes E$ induces a $*$-monomorphism $I \odot E \hookrightarrow A \otimes E$ by composing with $\iota \otimes \mathrm{id}$. Composing with the canonical inclusion $A \otimes E \hookrightarrow A \otimes_{\max} E$ gives a unique $*$-homomorphism $\iota'' \colon I \odot E \longrightarrow A \otimes_{\max} E$ such that $\iota''(a \otimes e) = \iota(a) \otimes e$ for every elementary tensor $a \otimes e$ in $I \odot E$. We wish to extend its domain to $I \otimes E$. Consider the function

$$\| \cdot \|_1 \colon I \odot E \longrightarrow \mathbb{R}^+; \quad x \mapsto \max\{\|\iota''(x)\|_{\max}, \|x\|\}.$$

Upon ι'' being a $*$-homomorphism one easily verifies that it defines a C*-norm. Nuclearity of I forces $\| \cdot \|_1$ to coincide with $\| \cdot \|_{\min}$. One hereby infers that ι'' must be $\| \cdot \|_{\min}$-contractive. Thus it extends uniquely to some $*$-homomorphism $\iota' \colon I \otimes E \longrightarrow A \otimes_{\max} E$ fulfilling $\sigma \iota'(a \otimes e) = \sigma(\iota(a) \otimes e) = \iota(a) \otimes e$ on every elementary tensor $a \otimes e$ in $I \otimes E$.

Constructing the morphism π' is done in an analogues manner, so details have been reduced to a bare minimum. Let π'' be the $*$-homomorphism $A \odot E \longrightarrow B \otimes E$ induced by $\pi \otimes \mathrm{id}$ and the inclusion $A \odot E \hookrightarrow A \otimes E$. Consider the C*-norm on $A \odot E$ defined as

$$\| \cdot \|_2 \colon A \odot E \longrightarrow \mathbb{R}^+; \quad x \mapsto \max\{\|\pi''(x)\|, \|x\|_{\max}\}.$$

Nuclearity of B implies $\| \cdot \|_{\max}$-continuity of π'' through an argument running parallel to the establishment of $\| \cdot \|_{\min}$-continuity of ι''. As such the morphism π'' extends to some $*$-homomorphism $\pi' \colon A \otimes_{\max} E \longrightarrow B \otimes E$ such that $\pi'(a \otimes e) = \pi(a) \otimes e$ for $a \in A$ and $e \in E$, uniquely. To construct the morphism θ some additional care must be taken. Define at first a bilinear map

$$\Lambda \colon B \times E \longrightarrow \frac{A \otimes_{\max} E}{\operatorname{im} \iota'} =: D; \quad (\pi(a), e) \mapsto \varrho(a \otimes e),$$

with ϱ being the quotient map in the diagram. The map is independent on the choice of lift, for if $\pi(a - a_0) = 0$ then $a - a_0$ must belong to $\operatorname{im} \iota \subseteq \operatorname{im} \iota'$ via exactness. The universal property of tensor products induces a unique $*$-homomorphism $\theta_0 \colon B \odot E \longrightarrow D$ for which $\theta_0(\pi(a) \otimes e) = \varrho(a \otimes e)$ for each elementary tensor $\pi(a) \otimes e$ inside $B \odot E$. Employing the previous trick on the C*-norm

$$\| \cdot \|_3 \colon B \odot E \longrightarrow \mathbb{R}^+; \quad x \mapsto \max\{\|\theta_0(x)\|, \|x\|\}$$

allows one to deduce $\| \cdot \|_{\min}$-continuity of θ_0, thus granting a unique $*$-homomorphism extension $\theta \colon B \otimes E \longrightarrow D$ of θ_0. Finally, for every element $a \otimes e$ in $A \odot E$ one has $\theta(\pi'(a \otimes e)) = \varrho(a \otimes e)$ upon which the required commutativity holds by invoking uniqueness of the involved maps. $\square$

Separable nuclear C*-algebra bring solutions to the lifting problem in the category of C*-algebras with completely positive maps as morphisms. The theorem was established by Choi-Effros in [10], although the reader is encouraged to skim section 5.1 in [30] for an alternative proof.

Theorem 1.4.7 (Choi-Effros' lifting theorem)**.** *Suppose A, E denote some C*-algebras such that E is separable. Assume in addition that A is unital and let I be an ideal in A. Under these premises, every nuclear completely positive map $\psi \colon E \longrightarrow A/I$ admits a completely positive lift $\varphi \colon E \longrightarrow A$, meaning the diagram*

commutes. Here $\varrho \colon A \longrightarrow A/I$ is the quotient map. In the event of E being unital and ψ being unital, the lift φ may be chosen to be unital. In particular, every completely positive (resp. unital completely positive) map from a nuclear C*-algebra into a unital quotient admits a completely positive (resp. unital completely positive) lift.

Order Zero Maps

One frequently strives to enhance the utility of completely positive maps in the sense that one desires to have these further resemble ∗-homomorphisms. The resulting morphisms in question are called *order zero maps* and were meticulously studied by Winter and Zacharias, based on the work of Wolff. Order zero maps are thoroughly introduced in [49] and section 5.3 of [30]; the reader is assumed to be fully acquainted with both. The notion giving birth to order zero maps is orthogonality.

Definition. Suppose A and B are two C∗-algebras.

- Two elements $a, b \in A$ are called *orthogonal* should $ab = ba = a^*b = ab^* = 0$ and we write $a \perp b$ to symbolize orthogonality of these elements. We refer to two subsets $\Omega, \Omega_0 \subseteq A$ as being *orthogonal* if $a \perp b$ for all $a \in \Omega$ and $b \in \Omega_0$, symbolically writing $\Omega \perp \Omega_0$ to denote this.

- A completely positive map $\varphi \colon A \longrightarrow B$ is *of order zero* should it preserve orthogonality, meaning $a \perp b$ in A implies $\varphi(a) \perp \varphi(b)$ in B for all $a, b \in A$. The corresponding collection of order zero maps $\varphi \colon A \longrightarrow B$ is symbolically denoted by $\mathcal{O}(A, B)$.

Given an order zero map $\varphi \colon A \longrightarrow B$, we denote by B_φ the C∗-algebra generated by its image. The main structure theorem due to Winter-Zacharias reveals the crux of order zero maps, namely a characterization resembling Stinespring dilations. Note that given a ∗-homomorphism $\pi \colon A \longrightarrow B$ and positive element b in B commuting with the image of π, the "translated" map $a \mapsto b\pi(a)$ defines an order zero map. Essentially, the aforementioned theorem states that all order zero maps arise in this manner. The entity $\mathcal{M}(A)$ attached to any C∗-algebra is the *multiplier algebra* of A[3]. The second paramount characterization of order zero maps is their correspondence with ∗-homomorphisms from cones, to be discussed afterwards.

Theorem 1.4.8 (Winter-Zacharias)**.** *Suppose $\varphi \colon A \longrightarrow B$ denotes an order zero map between C∗-algebras. If so, there exists a triple $(e_\varphi, \pi_\varphi, B_\varphi)$ consisting of a ∗-homomorphism $\pi_\varphi \colon A \longrightarrow \mathcal{M}(B_\varphi)$ together with an element $0 \leq e_\varphi$ in B_φ. Additionally, e_φ commutes with the image of φ and fulfills $\|e_\varphi\| = \|\varphi\|$. The triple witnesses φ by the formula*

$$\varphi(\cdot) = e_\varphi \pi_\varphi(\cdot) = \pi_\varphi(\cdot) e_\varphi.$$

Lastly, the positive element may be chosen as $e_\varphi = \varphi(1_A)$ in the event of A being unital.

Terminology. The associated triple $(e_\varphi, \pi_\varphi, B_\varphi)$ of an order zero φ is referred to as *the order zero triple* of φ and the corresponding formula witnessing φ will be referred to as *the order zero relation*, in spite of the chosen terminology being non-canonical. There is a precedence, however, to address the ∗-homomorphism π_φ as the *supporting morphism* of φ.

Proposition 1.4.9. *Let A, B be C∗-algebras. If so, there exists a one-to-one correspondence between contractive order zero maps $\varphi \colon A \longrightarrow B$ and ∗-homomorphisms $\pi \colon C_0(0,1] \otimes A \longrightarrow B$. Letting id denote the generating element of $C_0(0,1]$ and $\gamma \colon A \hookrightarrow C_0(0,1] \otimes A$ be the linear isometry $a \mapsto \mathrm{id} \otimes a$, the correspondence may be captured in terms of the commutative diagrams*

$$
\begin{array}{ccc}
A & \xrightarrow{\gamma} & C_0(0,1] \otimes A \\
& \searrow{\varphi} & \downarrow{\varrho_\varphi} \\
& & B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\gamma} & C_0(0,1] \otimes A \\
& \searrow{\varphi_\varrho} & \downarrow{\varrho} \\
& & B
\end{array}
$$

Here ϱ_φ is the ∗-homomorphism, induced by some contractive order zero map $\varphi \colon A \longrightarrow B$, defined by the assignment $\mathrm{id} \otimes a \mapsto \varphi(a)$, whereas φ_ϱ denotes the contractive order zero map, induced by some ∗-homomorphism $\varrho \colon C_0(0,1] \otimes A \longrightarrow B$, defined by the assignment $\mathrm{id} \otimes a \mapsto \varrho(a)$.

---

[3]The appendix contains additional and detailed information regarding $\mathcal{M}(A)$, although its existence is considered established in the thesis.

Proof. We briefly record important observations of the proof. The prescribed mappings are obviously mutual inverses should they exist. To achieve existence, suppose $\varphi \colon A \longrightarrow B$ denotes a contractive order zero map. Let $(e_\varphi, \pi_\varphi, B_\varphi)$ be the order zero triple attached to φ. Define a $*$-homomorphism $\varrho \colon C_0(0,1] \longrightarrow B$ via the assignment id $\mapsto e_\varphi$. The element e_φ commutes with the image of π_φ, thereby ensuring that ϱ, π_φ have commuting images. By nuclearity of the abelian C*-algebra $C_0(0,1]$, the associated tensor map $\varrho \times \pi_\varphi \colon C(0,1] \otimes A \longrightarrow B$ exists and yields the designated morphism ϱ_φ due to the order zero relation implying

$$\varrho_\varphi(\mathrm{id} \otimes a) = \varrho(\mathrm{id})\pi_\varphi(a) = e_\varphi \pi_\varphi(a) = \varphi(a).$$

Conversely, every $*$-homomorphism $\varrho \colon C_0(0,1] \otimes A \longrightarrow B$ induces an order zero map $\varphi_\varrho \colon A \longrightarrow B$ in terms of the assignment $a \mapsto (\varrho \circ \gamma)(a)$. Since γ is obviously orthogonality preserving, the map φ_ϱ must be contractive and of order zero. This finalizes the proof. $\qquad\square$

Remark. Let $\mathcal{O}_c(A, B)$ be the subspace of $\mathcal{O}(A, B)$ comprised of contractive morphisms. The correspondence may be expressed as an isomorphism of sets $\mathcal{O}_c(A, B) \cong \mathrm{Hom}(C_0(0,1] \otimes A, B)$ for any pair A, B of C*-algebras. In contrast to ordinary completely positive maps, in the sense of Stinespring dilation, the benefit of the correspondence revolves around $*$-homomorphisms on cones being relatively easy to tackle and manipulate. Loosely speaking, we may upgrade our order zero maps to $*$-homomorphism without paying too much. Moreover, passing issues to cones does not alter traces heavily, which becomes pivotal for the main theorem, as well shall witness.

For future purposes, we verify some standard observations for order zero maps before proceeding. Despite these observations being straightforward consequences of the correspondence theorem alongside the structure theorem, they develop some intuition behind order zero maps — in the unital case, they are very close to being multiplicative.

Corollary 1.4.10. *Suppose $\varphi \colon A \longrightarrow B$ is some contractive completely positive map between C*-algebras with A being unital. Under these premises, the following hold.*

(i) φ is of order zero if and only if one has $\varphi(ab)\varphi(1_A) = \varphi(a)\varphi(b)$ for all a, b in A.

(ii) If $\varphi \colon A \longrightarrow B$ is of order zero, then φ defines a $*$-homomorphism if and only if $\varphi(1_A)$ is a projection inside B.

Proof. (i): Let $\varphi \colon A \longrightarrow B$ be a contractive order zero map. Select some a, b in A. The induced $*$-homomorphism ϱ_φ for which the above diagram becomes commutative satisfies,

$$\varphi(ab)\varphi(1_A) = \varrho_\varphi(\gamma(ab)\gamma(1_A)) = \varrho_\varphi((\mathrm{id} \otimes a)(\mathrm{id} \otimes b)) = \varphi(a)\varphi(b).$$

The converse of (i) is trivial due to the left-hand side of the assumed relation being zero whenever $a \perp b$. Notice that an entirely analogue computation reveals that $\varphi(1_A)\varphi(ab) = \varphi(a)\varphi(b)$, hence $\varphi(1_A)$ must necessarily commute with the image of φ.

(ii): The "only if" part is obvious whereas the "if" part may be verified as follows. According to the structure theorem of contractive order zero maps, there exists a $*$-homomorphism $\pi_\varphi \colon A \longrightarrow B$ commuting with $e_\varphi = \varphi(1_A)$ and fulfilling $\pi_\varphi(\cdot)\varphi(1_A) = \varphi(\cdot)$. One then deduces that

$$\varphi(ab) = \pi_\varphi(ab)e_\varphi^2 = \pi_\varphi(a)e_\varphi\pi_\varphi(b)e_\varphi = \varphi(a)\varphi(b)$$

for all a, b in A, completing the proof. $\qquad\square$

The relation in (i) is commonly referred to as the *"order zero identity"*.

1.5   On Finiteness of Projections, Stability and K-Theory

K-theory in its various disguises is perhaps the leading device to classify C*-algebras. We shall invoke several properties attached, including classification results of UHF-algebras. Since K-theory captures projections on matrix algebras over the C*-algebra in question, some remarks regarding projections of different "sizes" and stabilizations are added. Let us at first discuss finiteness of projections, a notion related to quasidiagonality.

Definition. Let A be some C*-algebra containing a nonzero projection p.

- The projection p is *infinite* if p is equivalent to some proper subprojection, meaning there exists some projection q in A fulfilling $p \sim q < p$. The C*-algebra A is *infinite* should it contain a nonzero infinite projection.

- The projection p is called *finite* provided that it cannot be infinite. A C*-algebra is *finite* if every projection is finite.

- The projection p is *properly infinite* if there are orthogonal projections p_1, p_2 satisfying $p_1 + p_2 \leq p$ and $p \sim p_1 \sim p_2$. The C*-algebra A is *properly infinite* should an existing unit be so.

Lemma 1.5.1. *Suppose A denotes a unital C*-algebra. Then A is finite if and only if 1_A is finite. Furthermore, these conditions are equivalent to every isometry being a unitary.*

Proof. Omitted and may found in [29] as *lemma* 5.1.2. $\qquad\square$

Some remarks are in order. In the literature, the definition of finiteness varies. Some demand finiteness of a C*-algebra to be finiteness of the unit, passing to the unitization in the non-unital case. The one selected here is more potent. Therefore we shall refer to finiteness of the unit in the presence of one as *unitally finiteness*. The prime notion stemming from finiteness, for the purposes of the thesis, is stable finiteness.

Definition. A C*-algebra E is *stably finite* if $\mathrm{M}_n(E)$ is unitally finite for each n in $\mathbb{N}$.

The choice of word "stable" may be understood in the following context. A commonly occurring entity in the C*-algebraic setting is the stabilization. To forego future notational inconveniences, fix some positive integer n and C*-algebra A. The assignment $\mu\colon A^n \longrightarrow \mathrm{M}_n(A)$ defined by declaring

$$a_1 \oplus \ldots \oplus a_n \mapsto \mathrm{diag}(a_1, \ldots, a_n)$$

is a $*$-monomorphism, unital in the presence of one on A. Ergo the diagonal matrices in $\mathrm{M}_n(A)$ may be identified with A^n, which we liberally exploit throughout. Under this identification, every finite collection $\psi_1, \ldots, \psi_n$ of morphisms $\psi_k\colon A \longrightarrow B$ induce a morphism $\psi_1 \oplus \ldots \oplus \psi_n\colon A \longrightarrow \mathrm{M}_n(B)$ of the same type by $a \mapsto \psi_1(a) \oplus \ldots \oplus \psi_n(a)$. We abbreviate $\psi^n = \psi_1 \oplus \ldots \oplus \psi_n$ if $\psi_1 = \psi_2 = \ldots = \psi_n$. Consider now the sequence

$$A \xrightarrow{\ d_1\ } \mathrm{M}_2(A) \xrightarrow{\ d_2\ } \mathrm{M}_3(A) \xrightarrow{\ d_3\ } \ldots$$

wherein $d_n\colon \mathrm{M}_n(A) \longrightarrow \mathrm{M}_{n+1}(A)$ is the non-unital $*$-monomorphism sending an element to the upper-left corner of the zero matrix in $\mathrm{M}_{n+1}(A)$. If one regards $\mathrm{M}_n(A)$ as being contained within $\mathrm{M}_{n+1}(A)$ via the embedding d_n, then we refer to

$$\mathrm{M}_\infty(A) := \varinjlim(\mathrm{M}_n(A), d_n)$$

as the *stable matrix algebra of A*. The norm-closure, meaning the inductive limit associated to the aforementioned inductive sequence, is called the *stabilization of A*. It may be proven that $A \otimes \mathbb{K}$ provides a model of the stabilization. We avoid distinguishing between these two pictures and will exploit both viewpoints on several occasions. Here is the point:

Lemma 1.5.2. *Let A be some C^*-algebra. If so, A is stably finite if and only if $A \otimes \mathbb{K}$ is finite.*

Only the "only if " part demands justification. The proof relies on projections in inductive limits being norm-approximated via projections in the finite stages, i.e., on $M_n(A)$ which is assumed to be finite. We deter from dwelling further into the details.

Examples of stably finite C^*-algebras are not uncommon. For instance, every unital C^*-algebra A admitting a faithful trace τ_A must be stably finite: The induced trace $\tau_{A,n} := \tau_A \otimes \tau_n$ on $M_n(A)$ remains faithful due to corollary 1.3.2, so every isometry s in $M_n(A)$ satisfies $ss^* = 1_{A_n}$; otherwise

$$0 < \tau_{A,n}(1_{A_n} - ss^*) = \tau_{A,n}(s^*s) - \tau_{A,n}(ss^*) = 0.$$

The stabilization often emerges to "weaken" certain invariants into more frequent ones. An instance would be stable isomorphism; A is *stably isomorphic* to B whenever $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Stable isomorphism may seem misplaced, however, patience combined with the theorem of L. Brown found beneath reveals the mileage gathered. The proof of L. Brown's theorem resides in [6].

Theorem 1.5.3 (L.Brown). *Suppose A denotes some separable C^*-algebra. If E is any hereditary subalgebra in A whose elements are full, then A must be stably isomorphic to E.*

Having addressed the stabilization, we proceed to K-theory in brevity. For the K_1-group we are inclined to address homotopy of unitaries, which bears independent significance.

Definition. Suppose A denotes a unital C^*-algebra. Two unitaries u, v in A are *homotopic* if one may find a continuous path $(u_t)_{t \in [0,1]}$ comprised of unitaries from A, i.e the assignment $t \mapsto u_t$ is norm-continuous and each u_t is a unitary in A, such that $u_0 = u$ and $u_1 = v$. We symbolically write $u \sim_h v$ to represent this instance.

Let $G(\cdot)$ be the Gröthendieck construction, meaning the functor from the category of abelian semigroups to abelian groups. The stable matrix algebra contains two subsets, namely the ones comprised of projections and unitaries;

$$\mathcal{P}_\infty(A) := \bigcup_{n=1}^\infty \mathrm{Proj}_n(A), \quad \text{respectively,} \quad \mathcal{U}_\infty(A) := \bigcup_{n=1}^\infty \mathcal{U}_n(A).$$

One endows both these constructions with the following composition. Suppose p belongs to $M_n(A)$ for some integer n while q belongs to $M_k(A)$ for some integer k. One defines an associative composition $\oplus$ on $\mathcal{M}_\infty(A)$ via the assignment $(p, q) \mapsto p \oplus q$ with $M_{n+k}(A)$ being the target. The composition clearly restricts to compositions on $\mathcal{P}_\infty(A), \mathcal{U}_\infty(A)$. Consider the equivalence relations $\sim_0, \sim_1$ on $\mathcal{P}_\infty(A), \mathcal{U}_\infty(A)$, respectively, defined by stipulating that

$$p \sim_0 q \quad \overset{\mathrm{def}}{\Longleftrightarrow} \quad \exists v \in M_{k,n}(A) : \ p = v^*v, \ q = vv^*;$$

$$u \sim_1 v \quad \overset{\mathrm{def}}{\Longleftrightarrow} \quad \exists m \in \mathbb{N} : \ u \oplus 1_{m-n} \sim_h v \oplus 1_{m-k} \text{ relative to } \mathcal{U}_m(A) \ .$$

Herein $p \in M_n(A)$ and $q \in M_k(A)$, whereas $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_k(A)$. One then defines abelian groups, whenever A is unital, by

$$K_0(A) := G(\mathcal{P}_\infty(A)/ \sim_0) \quad \text{together with} \quad K_1(A) := \mathcal{U}_\infty(A)/ \sim_1 \ .$$

There are two canonical maps $[\cdot]_0 \colon \mathcal{P}_\infty(A) \longrightarrow K_0(A)$ and $[\cdot]_1 \colon \mathcal{U}_\infty(A) \longrightarrow K_1(A)$, namely $p \mapsto [p]_0$ and $u \mapsto [u]_1$, respectively. The canonical maps obviously restrict to additive maps on $\mathrm{Proj}(A)$ and $\mathcal{U}(A)$. Functoriality arises as a consequences hereby, i.e., any $*$-homomorphism $\pi \colon A \longrightarrow B$ induces an abelian group homomorphism $[\pi]_n \colon K_n(A) \longrightarrow K_n(B)$, with $n = 0, 1$. The definitions in the non-unital case are within reasonable proximity of the unital ones:

$$K_0(A) := \ker \left(K_0(A^+) \overset{K_0(q_A)}{\longrightarrow} K_0(\mathbb{C}) \right) \quad \text{and} \quad K_1(A) := \mathcal{U}_\infty(A^+)/ \sim_1 \ .$$

Due to projections of inductive limits being norm-approximated by projections, one may define $K_0(A)$ in terms of the stablization instead, replacing $A \otimes \mathbb{K}$ with $\mathcal{P}_\infty(A)$ throughout.

Definition. Let A and B be C*-algebras.

- Let $\pi, \varrho \colon A \longrightarrow B$ be $*$-homomorphisms of C*-algebras. We refer to these as being *homotopic* to one another, symbolically written $\pi \sim_h \varrho$, if there exists a continuous path $(\pi_t)_{t \in [0,1]}$, in the sense that $t \mapsto \pi_t$ is point-norm continuous, consisting of $*$-homomorphisms from A into B such that $\pi_0 = \pi$ and $\pi_1 = \varrho$ hold.

- A is said to be *homotopically equivalent to* B if there exist $*$-homomorphisms $\pi \colon A \longrightarrow B$ and $\varrho \colon B \longrightarrow A$ such that $\pi \circ \varrho \sim_h \mathrm{id}_B$ together with $\varrho \circ \pi \sim_h \mathrm{id}_A$ are valid.

The K-theory groups are subjects to a myriad of functorial properties and stability properties. We exhibit the most fundamental ones, albeit one ought to have mentioned Bott-periodicity and the existence of a six-term exact sequence.

Proposition 1.5.4. *The assignments $A \mapsto \mathrm{K}_n(A)$ define functors from the category C*-algebras into the category of abelian groups for $n = 0, 1$. Furthermore, the following properties are fulfilled.*

- $\mathrm{K}_n(\cdot)$ *is a stable functor, meaning $\mathrm{K}_n(A \otimes \mathbb{K}) \cong \mathrm{K}_n(A)$.*

- $\mathrm{K}_n(\cdot)$ *is finitely additive, meaning $\mathrm{K}_n(A \oplus B) \cong \mathrm{K}_n(A) \oplus \mathrm{K}_n(B)$.*

- $\mathrm{K}_n(\cdot)$ *is a continuous functor, meaning $\mathrm{K}_n(\varinjlim A_k) = \varinjlim \mathrm{K}_n(A_k)$.*

- $\mathrm{K}_n(\cdot)$ *is homotopy invariant, meaning $\pi \sim_h \varrho$ entails $[\pi]_n = [\varrho]_n$ for two $*$-homomorphisms π, ϱ.*

The final, and perhaps most peculiar, functorial property attached to K-theory revolves around preservation of exact sequences. Suppose

$$0 \longrightarrow A \overset{\iota}{\longrightarrow} B \overset{\pi}{\longrightarrow} C \longrightarrow 0 \tag{1.12}$$

represents a short-exact sequence of C*-algebras. Applying $\mathrm{K}_n(\cdot)$ yields an exact sequence

$$\mathrm{K}_n(A) \overset{[\iota]_n}{\longrightarrow} \mathrm{K}_n(B) \overset{[\pi]_n}{\longrightarrow} \mathrm{K}_n(C).$$

However, if the sequence (1.12) splits, then the induced sequence below becomes split short-exact.

$$0 \longrightarrow \mathrm{K}_n(A) \overset{[\iota]_n}{\longrightarrow} \mathrm{K}_n(B) \overset{[\pi]_n}{\longrightarrow} \mathrm{K}_n(C) \longrightarrow 0.$$

One of the profound results based on K-theory is the complete classification of UHF-algebras.

Theorem 1.5.5 (Elliott)**.** *Let A, B be two UHF-algebras. Then A becomes isomorphic to B if and only if there exists an isomorphism $\varphi \colon \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(B)$ such that $\varphi([1_A]_0) = [1_B]_0$.*

K-theory might fail to sustain sufficient data to capture the structure of the underlying C*-algebra[4]. One approach to remedy the hindrance would be to consider ordered K-theory. Recall that an *ordered abelian group* is some pair (G, G^+) consisting of an abelian group G together with an additively closed subset G^+ containing 0 such that $G^+ - G^+ = G$ and $G^+ \cap -G^+ = \{0\}$. For K_0 one declares that

$$\mathrm{K}_0(A)^+ = \{[p]_0 : p \in \mathcal{P}_\infty(A)\}$$

and refers[5] to the pair $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ as *the ordered K-theory*. One then defines an *ordered morphism* of K-groups to be an abelian group homomorphism $\varphi \colon \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(B)$ subject to the additional condition that $\varphi(\mathrm{K}_0(A)^+) \subseteq \mathrm{K}_0(B^+)$.

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[4]K_0 and K_1 without order are for instance unable to retain the structure of AF-algebras. In fact, for AF-algebras one leans on the ordered K_0-group. See [29] or [35] for an in depth survey.

[5]Caution! It need not always constitute and ordered abelian group. A sufficient condition would be stably finiteness and unitality, see *proposition* 5.1.5 in [29].

Chapter 2

Setting the Stage

Tikuisis, White and Winter construct morphisms that team-up to witness the designated quasidi-agonality of traces. Conjuring these maps is no easy task. The deus ex machina in building these is Connes' celebrated uniqueness theorem of injective separable hyperfinite II_1 factors. An additional powerful tool we must have at our disposal is ultraproducts. These permit one to characterize quasidiagonality in terms of a single algebra in the nuclear separable setting. Furthermore, ultraproducts are pivotal in acquiring the morphisms. The chapter pursues various indispensable results to provide these maps. Alas, some deep theorems are left without proof.

2.1 Induced von Neumann Algebras

von Neumann algebras have a crucial presence in the thesis. In fact, we appeal to very deep results concerning these and occasionally lean on von Neumann algebraic analogues to motivate certain notions in the purely C*-algebraic framework. The section develops the von Neumann algebraic theory that tends to be used. Afterwards, the setup permitting us to invoke Connes' uniqueness theorem will be established thoroughly.

Definition. Let $\mathcal{H}$ be any Hilbert space.

- The locally convex Hausdorff topology on $B(\mathcal{H})$ induced via the seminorms $a \mapsto \|a\xi\|$ for each ξ in $\mathcal{H}$ is called the *strong operator topology*, abbreviated sot.

- The locally convex Hausdorff topology on $B(\mathcal{H})$ induced via the seminorms $a \mapsto |\langle a\xi, \eta \rangle|$ for each ξ, η in $\mathcal{H}$ is called the *weak operator topology*, abbreviated wot.

- The locally convex Hausdorff topology on $B(\mathcal{H})$ induced via the seminorms $a \mapsto \left(\sum_{k=1}^{n} \|a\xi_n\|^2 \right)^{1/2}$ for each $(\xi_n)_{n \geq 1}$ in $\ell^2(\mathcal{H})$ is called the *σ-strong operator topology*, abbreviated σ-sot.

- The locally convex Hausdorff topology on $B(\mathcal{H})$ induced via the seminorms $a \mapsto \sum_{k=1}^{n} |\langle a\xi_n, \eta_n \rangle|$ for each pair of elements $(\xi_n)_{n \geq 1}, (\eta_n)_{n \geq 1}$ in $\ell^2(\mathcal{H})$ such that $\sum_{k=1}^{n} \|\xi_n\| \cdot \|\eta_n\|$ becomes finite is called the *σ-weak operator topology*, abbreviated σ-wot.

A σ-wot continuous linear functional acting on a σ-wot closed involutive subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$ is referred to as being *normal*. The vector space consisting of all normal functional on $\mathcal{M}$, symbolically represented by $\mathcal{M}_*$, is called *the predual of $\mathcal{M}$*.

The four topologies add powerful structure on subalgebras in $B(\mathcal{H})$. The study of algebras closed in these are von Neumann algebras. Prior to addressing their formal definition in detail, we address Kaplansky's density theorem; a remarkably useful density result.

Theorem 2.1.1 (Kaplansky's density theorem). *The closed unit ball A_1 of any non-degenerate involutive subalgebra $A \subseteq B(\mathcal{H})$ is τ-dense in $(\overline{A}^\tau)_1$, where τ denotes either of the above topologies.*

The core of von Neumann algebras is arguably von Neumann's bicommutant theorem, relating an algebraic characterization to involutive algebras closed under either of the previous four locally convex topologies on $\mathrm{B}(\mathcal{H})$. For the record, for an involutive algbera $A \subseteq \mathrm{B}(\mathcal{H})$ we denote by A' its commutant, meaning

$$A' := \{e \in \mathrm{B}(\mathcal{H}) : ae = ea \text{ for all } a \in A\}.$$

Theorem 2.1.2 (von Neumann's bicommutant theorem)**.** *For a non-degenerate involutive subalgebra $\mathscr{M} \subseteq \mathrm{B}(\mathcal{H})$ containing $1_{\mathcal{H}}$ as the identity the following are equivalent.*

- $\mathscr{M}'' = \mathscr{M}$ *with* $\mathscr{M}'' = (\mathscr{M}')'$.

- $\mathscr{M}$ *is closed in either the sot, wot, σ-sot or σ-wot topology.*

Definition. An involutive non-degenerate subalgebra $\mathscr{M} \subseteq \mathrm{B}(\mathcal{H})$ is called a *von Neumann algebra* should it fulfill one, hence all, of the equivalent conditions in theorem 2.1.2. Furthermore,

- $\mathscr{M}$ is *finite* if it admits a faithful normal trace;

- $\mathscr{M}$ is of *type* II_1 if $\mathscr{M}$ is infinite dimensional and finite;

- $\mathscr{M}$ is called a *factor* if $\mathcal{Z}(\mathscr{M}) := \{a \in \mathscr{M} : ab = ba \text{ for all } b \in \mathscr{M}\}$ is $*$-isomorphic to $\mathbb{C}$;

- $\mathscr{M}$ is called *hyperfinite* if there exists an increasing chain $F_1 \subseteq F_2 \subseteq \ldots$ consisting of finite dimensional C*-algebras such that $\bigcup_{n=1}^{\infty} F_n$ is strong-operator dense in $\mathscr{M}$.

The notion of a type II_1 here is non-standard, although equivalent to the original one. Since we avoid discussing the type-decomposition theorem of Murray and von Neumann, the current one has been adopted. We will construct (the) hyperfinite II_1 factors rigorously and discuss uniqueness of such. The framework one appeals to revolves around investigating the induced von Neumann algebra $\pi_{\tau}(A)''$ attached to any pairing (A, τ) consisting of a unital C*-algera and trace τ thereon, which paves the path from C*-algebras to von Neumann algebras. As such general considerations are taken into account, starting with realizing the inherited type of $\pi_{\tau}(A)''$.

Lemma 2.1.3. *Let M be an involutive subalgebra of $\mathrm{B}(\mathcal{H})$ and call a vector ξ in $\mathcal{H}$,,tracial for M'' if the vector state $a \mapsto \langle a\xi, \xi \rangle$ restricts to a trace on M. Then $a\xi = 0$ implies $a = 0$ for every a in A, whenever ξ is a cyclic tracial vector for M .*

Proof. Suppose $a\xi = 0$ for some element a in M. For an additional pair of elements b, c in M one has $\langle ab\xi, c\xi \rangle = \langle bc^*a\xi, \xi \rangle = 0$ due to ξ being tracial. Since ξ is cyclic, the norm-closure of $M\xi$ coincides with $\mathcal{H}$, so the observation remains valid when replacing $b\xi$ and $c\xi$ with general vectors in $\mathcal{H}$, i.e., $\langle a\eta, \eta_0 \rangle = 0$ must be true for all η, η_0 in $\mathcal{H}$. It follows that $a = 0$ as claimed. $\square$

Proposition 2.1.4. *Suppose A denotes an infinite dimensional C*-algebra admitting a trace τ and let $(\pi_{\tau}, \mathcal{H}_{\tau}, \xi_{\tau})$ be the attached GNS-triple. If so, $\mathcal{N} := \pi_{\tau}(A)''$ becomes a type II_1 von Neumann algebra having the weak-operator continuous extension $\tau_{\mathcal{N}}$ of the trace defined on $\tau_{\tau}(A)$ by*

$$\pi_{\tau}(a) \mapsto \langle \pi_{\tau}(a)\xi_{\tau}, \xi_{\tau} \rangle, \quad a \in A,$$

as normal faithful trace.

Proof. According to von Neumann's double commutant theorem, $\pi_{\tau}(A)$ is weak-operator dense in $\mathcal{N}$. Keeping this in mind, we define a trace $\tau_0 : \pi_{\tau}(A) \longrightarrow \mathbb{C}$ by $\pi_{\tau}(a) \mapsto \langle \pi_{\tau}(a)\xi_{\tau}, \xi_{\tau} \rangle$. The tracial property of τ_0 follows immediately from the trace property on τ in conjunction with (1.4). Since τ_0 is clearly weak-operator continuous, it extends to a normal trace $\tau_{\mathcal{N}}$ on $\mathcal{N}$. It remains to be proven that $\tau_{\mathcal{N}}$ is faithful to ensure that $\mathcal{N}$ must be of type II_1. The above lemma comes to our aid. Observe that ξ_{τ} must be tracial for $\pi_{\tau}(A)$ based on (1.4), and therefore relative to $\mathcal{N}$ by normality. If $\pi_{\tau}(a)$ belongs to $\mathcal{N}$ and fulfills $\tau_{\mathcal{N}}(\pi_{\tau}(a)^*\pi_{\tau}(a)) = 0$, then $\|\pi_{\tau}(a)\xi_{\tau}\| = 0$. It follows that $\pi_{\tau}(a) = 0$ by lemma 2.1.3, hence $\tau_{\mathcal{N}}$ if faithful by normality. Ergo, $\tau_{\mathcal{N}}$ must be a normal faithful trace acting on $\mathcal{N}$, completing the proof. $\square$

The von Neumann algebra $\mathscr{N} = \pi_\tau(A)''$ contains valuable informations in regards to A; A lies within a von Neumann algebra in this manner, in the faithful case at least. A peculiar subtlety is that the induced normal trace on $\mathscr{N}$ automatically becomes faithful despite τ not being faithful. Having established the type of $\mathscr{N}$, one ought to proceed further into answering whether $\mathscr{N}$ may become a factor. To accomplish this, we lean on a theorem due to Sakai, see *theorem* 7.3.6 in [24].

Theorem 2.1.5 (Sakai). *Suppose φ, ψ are positive linear functionals acting on a von Neumann algebra $\mathscr{M}$ such that $\varphi \leq \psi$. Then there exists a positive contraction e in $\mathscr{M}$ such that $\varphi(\cdot) = \psi(e \cdot e)$.*

Proposition 2.1.6. *Suppose A denotes a C^*-algebra admitting a trace τ. Abbreviate $\mathscr{N} = \pi_\tau(A)''$. There exists an order-preserving isomorphism of sets:*

$$\Delta \colon \big\{ a \in \mathcal{Z}(\mathscr{N}) : 0 \leq a \leq 1_\mathscr{N} \big\} \longrightarrow \big\{ \varphi \in A^* : \varphi \text{ is tracial and } 0 \leq \varphi \leq \tau \big\}.$$

The correspondence is explicitly given via the assignment $a \mapsto \tau_a$, where $\tau_a(\cdot) = \langle \pi_\tau(\cdot) a \xi_\tau, \xi_\tau \rangle$.

Proof. We start with bringing meaning to Δ. Given an arbitrary element a inside $\mathcal{Z}(\mathscr{N})$ one obtains $\sigma(a\pi_\tau(b)) \subseteq \sigma(a)\sigma(\pi_\tau(b)) \subseteq \mathbb{R}^+$ for every b in A, whereupon τ_a becomes a positive functional. The trace property of τ_a may be verified thus: Let $\tau_\mathscr{N}$ denote the faithful normal trace on $\mathscr{N}$, so that $\tau_a(\cdot) = \tau_\mathscr{N}(\pi_\tau(\cdot)a)$. Due to a commuting with the image of π_τ and $\tau_\mathscr{N}$ being tracial, τ_a clearly becomes a trace on A. The mapping Δ therefore has the designated codomain. Injectivity may be shown as follows. The involutive algebra $\pi_\tau(A)$ is weak-operator dense inside $\mathscr{N}$. Hence, under the assumption $\tau_a = \tau_b$ for positive contractions a, b in $\mathcal{Z}(\mathscr{N})$, the weak-operator extensions of τ_a, τ_b to $\mathscr{N}$ agree as well. As such,

$$\tau_\mathscr{N}((a-b)^*(a-b)) = \tau_\mathscr{N}(a^*a - b^*a - a^*b + b^*b)$$
$$= \tau_a(a^*) - \tau_a(b^*) - \tau_b(a^*) + \tau_b(b^*) = 0.$$

Since $\tau_\mathscr{N}$ is faithful, one has $a = b$, proving Δ to be an injection.

For surjectivity, suppose φ denotes a positive tracial functional dominated by τ. To employ Sakai's theorem, we must translate φ in terms of π_τ, ξ_τ. Define a $*$-linear map $\psi_0 \colon \pi_\tau(A)\xi_\tau \longrightarrow \mathbb{C}$ by the assignment $\pi_\tau(a)\xi_\tau \mapsto \varphi(a)$. The map ψ_0 becomes meaningful on the merits of lemma 2.1.3. Plainly, ψ_0 is a bounded positive functional, hence ψ_0 extends by density to a bounded positive functional ψ on $\mathcal{H}_\tau$. Invoking Riez's representation theorem, there exists some vector η in $\mathcal{H}_\tau$ fulfilling

$$\varphi(a) = \psi(\pi_\tau(a)\xi_\tau) = \langle \pi_\tau(a)\xi_\tau, \eta \rangle \tag{2.1}$$

Sakai's theorem now enters the scene. Let $\omega_\varphi \colon \mathscr{N} \longrightarrow \mathbb{C}$ be the positive functional given by the assignment $a \mapsto \langle a\xi_\tau, \eta \rangle$. Based on φ being tracial, the relation (2.1) alongside wot-density provides the trace property for ω_φ. Moreover, $\omega_\varphi \leq \tau_\mathscr{N}$ on the weak-operator dense subalgebra $\pi_\tau(A)$ because $\varphi \leq \tau$ by hypothesis, hence over the entirety of $\mathscr{N}$. Sakai's theorem applies to produce a positive contraction e in $\mathscr{N}$ obeying the rule

$$\omega_\varphi(\cdot) = \tau_\mathscr{N}(e^{1/2} \cdot e^{1/2}) = \tau_\mathscr{N}(e \cdot). \tag{2.2}$$

The element e must be central, for given any b in $\mathscr{N}$ one must have

$$0 \leq \tau_\mathscr{N}((eb - be)^*(eb - be)) \overset{(2.2)}{=} \omega_\varphi(bb^*e) - \omega_\varphi(beb^*) - \omega_\varphi(b^*eb) + \omega_\varphi(b^*be) = 0.$$

Faithfulness of $\tau_\mathscr{N}$ in turn implies that $eb - be = 0$. Furthermore, (2.1)-(2.2) combined with e being central yields surjectivity due to

$$\varphi(a) \overset{(2.1)}{=} \omega_\varphi(\pi_\tau(a)) \overset{(2.2)}{=} \big\langle \pi_\tau(a)e\xi_\tau, \xi_\tau \big\rangle = \tau_e(a).$$

This finalizes the proof. $\square$

Using the preceding proposition, the condition of $\mathcal{N}$ being a factor appears: τ has to be extremal. There are several benefits to such considerations. Situations frequently reduce to this case via a Krein-Milman theorem argument, a technique we shall employ in the future.

Corollary 2.1.7. *Let A be an infinite-dimensional unital C^*-algebra admitting a trace τ. If so, the type II_1 von Neumann algebra $\mathcal{N} = \pi_\tau(A)''$ is a factor if and only if τ is extremal.*

Proof. Let us rephrase the assertion into a more convenient language. On the merits of (1.4), the image $\tau_a = \Delta(a)$ of an element a inside $\mathcal{Z}(\mathcal{N})$ satisfies $\tau_a(\cdot) = a\tau(\cdot)$ whenever $\mathcal{N}$ is a factor. Invoking proposition 2.1.6, one may reformulate the statement thus: τ is extremal if and only if for each positive tracial bounded functional ψ on A fulfilling $\psi \leq \tau$, there exists some real number $0 \leq x \leq 1$ such that $\psi = x\tau$ holds.

Suppose at first τ is extremal. Let ψ be any positive tracial functional on A dominated by τ. Without loss of generality, assume in addition that ψ is non-zero and differs from τ (there is nothing to prove otherwise). Define strictly positive real numbers $t_1 = \psi(1_A)$ and $t_2 = \tau(1_A) - \psi(1_A)$. These numbers evidently satisfy $t_1 + t_2 = 1$ and

$$\tau = t_1(t_1^{-1}\psi) + t_2(t_2^{-1}(\tau - \psi)).$$

By our hypothesis imposed on τ, $t_1^{-1}\psi = t_2^{-1}(\tau - \psi)$ must hold, which rearranges into $t_1\tau = \psi$.

To prove the converse, suppose every positive tracial functional ψ on A dominated by τ gives rise to some real number $0 \leq x \leq 1$ such that $\psi = x\tau$. Suppose one has some convex combination $\tau = (1-t)\tau_0 + t\tau_1$ of τ. Then $(1-t)\tau_0$ and $t\tau_1$ are positive tracial functionals dominated by τ, implying the sought conclusion. This proves the claim. $\qquad\square$

Our long endeavor of von Neumann algebras induced from GNS representations of traces culminates into the prototype von Neumann algebra: The hyperfinite II_1 factor $\mathcal{R}$. We afterwards proceed to discussing injectivity and the uniqueness theorem of Connes.

Proposition 2.1.8. *There exists a hyperfinite II_1 factor $\mathcal{R}$ acting on a separable Hilbert space.*

Proof. All UHF-algebras are simple and monotracial; see for instance *proposition* 1.4.3 in [30]. In particular, the CAR-algebra $\mathbb{M}_{2^\infty}$ admits a unique faithful trace τ_0. Faithfulness stems from the left-ideal $\mathcal{L}_\tau$ being a bona-fide ideal via the trace property, hence simplicity forces this to be the zero-ideal. Upon τ_0 being faithful and $\mathbb{M}_{2^\infty}$ separable, the GNS-triple associated to τ_0 consists of a separable Hilbert space $\mathcal{H}_{\tau_0}$ and faithful representation π_{τ_0}. The prime candidate will be

$$\mathcal{R} := \pi_{\tau_0}(\mathbb{M}_{2^\infty})''.$$

This determines a hyperfinite von Neumann algebra, which must be of type II_1 according to proposition 2.1.4. Certainly, regarding the CAR-algebra as a UHF-algebra, it may be identified with the norm-closure of $\bigcup_{n=1}^\infty \mathbb{M}_{2^n}$. From uniqueness of τ_0, it must be extremal (the extreme points of traces are weak*-dense in the trace simplex), whereby $\mathcal{R}$ must be a factor based on corollary 2.1.7 $\qquad\square$

2.2 Injectivity of von Neumann Algebras

The device lacking from obtaining a uniqueness result of $\mathcal{R}$ is injectivity of von Neumann algebras. The subsection seeks to examine the notion and explore how nuclearity of a C^*-algebras translates into injectivity of the ambient von Neumann algebra $\pi_\tau(A)''$ attached to a trace τ. Deep and beautiful results of Connes are exploited here, to our dismay without proofs.

Definition. A von Neumann algebra $\mathscr{N} \subseteq \mathrm{B}(\mathcal{H})$ is *injective* if, for every unital completely positive map $\psi_0 \colon \mathscr{M}_0 \longrightarrow \mathscr{N}$ and unital inclusion $\iota \colon \mathscr{M}_0 \hookrightarrow \mathscr{M}$ of von Neumann algebras, there exists a unital completely positive map $\psi \colon \mathscr{M} \longrightarrow \mathscr{N}$ making the diagram below commute.

$$
\begin{array}{ccc}
 & \mathscr{M} & \\
{\scriptstyle\iota}\Big\uparrow & \searrow{\scriptstyle\psi} & \\
\mathscr{M}_0 & \xrightarrow[\psi_0]{} & \mathscr{N}
\end{array}
$$

In other words, $\mathscr{N}$ as an injective object in the category of von Neumann algebras having unital completely positive maps as morphisms.

The von Neumann-algebraic notion of injectivity may be described completely in terms of conditional expectations, a characterization distinguishing von Neumann algebras from C*-algebras.

Proposition 2.2.1. *A von Neumann algebra $\mathscr{N}$ acting on a Hilbert space $\mathcal{H}$ is injective if and only if there exists a conditional expectation $E \colon \mathrm{B}(\mathcal{H}) \longrightarrow \mathscr{N}$.*

Proof. Suppose at first $E \colon \mathrm{B}(\mathcal{H}) \longrightarrow \mathscr{N}$ is some conditional expectation. Let $\iota \colon \mathscr{M}_0 \hookrightarrow \mathscr{M}$ be a unital inclusion of von Neumann algebras and let $\psi_0 \colon \mathscr{M}_0 \longrightarrow \mathscr{N}$ be unital completely positive. Regarding ψ_0 as a map attaining values in $\mathrm{B}(\mathcal{H})$, one may invoke Arveson's extension theorem to produce a unital completely positive extension $\psi \colon \mathscr{M} \longrightarrow \mathrm{B}(\mathcal{H})$ of ψ_0. The sole thing that has transpired is obtaining the diagram

$$
\begin{array}{ccccc}
 & \mathscr{M} & & & \\
{\scriptstyle\iota}\Big\uparrow & \searrow{\scriptstyle\psi} & & & \\
\mathscr{M}_0 & \xrightarrow[\psi_0]{} & \mathrm{B}(\mathcal{H}) & \xrightarrow[E]{} & \mathscr{N}
\end{array}
$$

Gazing at the diagram, it is evident that composing ψ with the conditional expectation E yields a unital completely positive map extending ψ_0 as a map into $\mathscr{N}$.

Conversely, if $\mathscr{N}$ is injective, then the identity map thereon extends to a unital completely positive map $E \colon \mathrm{B}(\mathcal{H}) \longrightarrow \mathscr{N}$. The map E clearly defines a projection, so according to Tomiyama's theorem, see theorem 1.4.3, it must a conditional expectation. $\qquad\square$

The grand scheme behind bringing injective von Neumann algebras in the thesis is based on Connes' staggering work; his celebrated uniqueness theorem. The proof of the statement is beyond the scope of the thesis, so we settle with referring to [12].

Theorem 2.2.2 (Connes)**.** *Every injective type* II_1 *factor $\mathscr{M}$ acting on a separable Hilbert space admits a normal $*$-isomorphism $\mathscr{M} \cong \mathscr{R}$.*

We will, however, apply the theorem in the C*-algebraic scene. Doing so requires additional work and vast machinery that translates nuclearity into von Neumann lingo.

Lemma 2.2.3. *Let $\mathscr{M}$ be a von Neumann algebra acting on $\mathcal{H}$. Suppose I denotes some σ-wot closed ideal in $\mathscr{M}$. Under these premises, one may find a projection e in $\mathcal{Z}(\mathscr{M})$ such that $I = e\mathscr{M}$.*

Proof. Evidently, I must be norm-closed as well. It therefore determines an ideal in the C*-algebraic sense and thereby admits an approximate unit $(e_\alpha)_{\alpha \in J}$. Extract a quasicentral approximate unit from the existing approximate unit. Denote this by $(e_\alpha)_{\alpha \in J}$ again. Due to the net being an increasing net consisting of positive contractions, it admits a strong operator limit[1] e. The limit belongs to I

---

[1] This is a basic fact regarding von Neumann algebras, for instance *theorem* 17.1 in [51].

since I is σ-wot closed[2]. On the merits of multiplication to the left (resp. right) by a fixed bounded operator being sot-continuous and the norm topology being stronger than the sot-topology, one acquires

$$ae = \text{sot-}\lim_{\alpha \in J}(ae_\alpha) = \text{sot-}\lim_{\alpha \in J}(e_\alpha a) = ea$$

for all $a \in \mathscr{M}$ and similarly $ae = a$ together with $e^2 = e$ hold for each $a \in I$. Altogether, e constitutes a central projection such that $e\mathscr{M} = I$, proving the claim. $\qquad\square$

The machinery required is the enveloping von Neumann algebra, which happens to pop up throughout the thesis implicitly. Its existence along with universal property are assumed to be known. For a thorough proof, the reader is urged to consult section 2 of the third chapter in [41] with an emphasis on *theorem* 2.4 therein. Recall that every C*-algebra A admits a universal faithful non-degenerate representation $\pi_u \colon A \longrightarrow \mathrm{B}(\mathcal{H}_u)$. The associated von Neumann algebra $\pi_u(A)''$ is called *the enveloping von Neumann algebra.*

Theorem 2.2.4. *For every C*-algebra A, the universal representation $\pi_u \colon A \longrightarrow \mathrm{B}(\mathcal{H}_u)$ extends to an isometric surjective linear weak*-to-σ-wot continuous homeomorphism, still denoted by π_u for simplicity, from A^{**} onto $\pi_u(A)''$.*

The universal representation enjoys the following universal property: For every non-degenerate representation $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$ there exists a unique normal $*$-epimorphism $\varrho \colon \pi_u(A)'' \longrightarrow \pi(A)''$ making the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \pi_u\ } & \pi_u(A)'' \\
& \searrow{\scriptstyle \pi} & \ \ \downarrow{\scriptstyle \varrho} \\
& & \pi(A)''
\end{array}
$$

commutative. The point behind invoking these powerful tools is the following consequence. We intentionally delay the proof of A being nuclear forcing semidiscreteness of its double dual, instead apply it at first to spur some motivation behind the implementation, starting with a minor lemma.

Lemma 2.2.5. *Suppose $\mathscr{M}, \mathscr{N}$ denote two von Neumann algebras and let a normal $*$-epimorphism $\pi \colon \mathscr{M} \longrightarrow \mathscr{N}$ be given. Then $\mathscr{M}$ is injective if and only if both $\mathscr{N}$ and $\ker \pi$ are injective.*

Proof. Notice first that $\ker \pi$ is indeed σ-weakly closed ideal in $\mathscr{M}$ by normality of π. Invoking lemma 2.2.3 one may produce a central projection e in $\ker \pi$ such that $\ker \pi = e\mathscr{M}$. Due to $e \perp e^\perp$, one can form the decomposition $\mathscr{M} = \ker \pi \oplus e^\perp \mathscr{M}$. The restriction of π onto the second summand becomes a normal $*$-isomorphism onto $\mathscr{N}$, whereupon $\mathscr{M} \cong \ker \pi \oplus \mathscr{N}$ becomes valid. Now, finite direct sums of injective von Neumann algebras are injective. Indeed, if $\ker \pi$ and $\mathscr{N}$ are injective, then they admit unital conditional expectations $E_0 \colon \mathrm{B}(\mathcal{H}) \longrightarrow \ker \pi$ together with $E_1 \colon \mathrm{B}(\mathcal{H}) \longrightarrow \mathscr{N}$, and the map $E \colon \mathrm{B}(\mathcal{H}) \longrightarrow \ker \pi \oplus \mathscr{N}$ defined by the assignment $a \mapsto (E_0(a), E_1(a))$ is clearly a unital conditional expectation as required in proposition 2.2.1. $\qquad\square$

Corollary 2.2.6. *Let A be a nuclear C*-algebra admitting a non-degenerate separable representation $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$. Then the von Neumann algebra $\pi(A)''$ is injective. In particular, the von Neumann algebra $\pi_\tau(A)''$ associated to a nuclear unital separable C*-algebra admitting an extremal trace τ must be isomorphic to $\mathscr{R}$.*

Proof. According to corollary 2.2.11, the von Neumann algebra $\pi_u(A)''$ must injective. By the universal property of π_u, there exists a unique normal $*$-epimorphism $\varrho \colon \pi_u(A)'' \longrightarrow \pi(A)''$. The preceding proposition provides injectivity $\pi(A)''$. The remaining statement follows from Connes' uniqueness theorem in conjunction with corollary 2.1.7. $\qquad\square$

---

[2]The σ and ordinary topologies agree on bounded sets, hence on the projection lattice. Moreover, the weak and strong topologies agree on convex sets such as I.

In the previous proof it was implicitly exploited that semidiscreteness of the enveloping von Neumann algebra may be deduced via nuclearity of the underlying C*-algebra. For quasi-completeness we derive this modulo certain technical results alongside equivalence of injectivity and semidiscreteness. In essence, we restrict ourselves to the main perspectives of the proof, commencing by reacquainting ourselves with semidiscreteness.

Definition. Suppose $\mathscr{M}$ denotes a von Neumann algebra and let A be some C*-algebra.

- A bounded linear map $\psi \colon A \longrightarrow \mathscr{M}$ is *weakly-nuclear* if there is a net $(\psi_\alpha)_{\alpha \in J}$ consisting of completely positive finite-dimensionally factorable maps such that $\psi_\alpha(\cdot) \to \psi(\cdot)$ in σ-wot wise.

- The von Neumann algebra $\mathscr{M}$ is called *semidiscrete* if $\mathrm{id}_{\mathscr{M}}$ is weakly-nuclear.

Remark I. The convergence $\psi_\alpha(\cdot) \to \psi(\cdot)$ in the σ-wot topological sense may be rephrased via normal functionals. For each element a in A one has $\psi_\alpha(a) \to \psi(a)$ σ-wot wise if and only if for each normal functional ω on $\mathscr{M}$ one has $(\omega \circ \psi_\alpha)(a) \to (\omega \circ \psi)(a)$. Furthermore, in the presence of units, unital completely positive maps may be arranged; see *proposition* 2.2.7 in [9].

Remark II. Another essential observation resembling nuclearity of C*-algebras concerns the collection of weakly nuclear maps: It is point σ-wot closed in the unital case. As such one merely has to approximate a given unital completely positive map $\psi \colon A \longrightarrow \mathscr{M}$ up to any prescribed tolerance, finite subset in $\mathscr{M}$ and finite subset $N \subseteq \mathscr{M}_*$; see *proposition* 3.8.2 in [9].

Deriving semidiscreteness of the double dual associated to some nuclear C*-algebra requires an involved theorem by Kirchberg. He succeeded in verifying that semidiscreteness may be checked solely in terms of the commutant and vice versa. Having established this, one merely needs to deduce injectivity of $\pi_u(A)'$ with π_u being the universal representation associated to some nuclear C*-algebra, for then $\pi_u(A)'$ becomes semidiscrete if and only if $\pi_u(A)'' \cong A^{**}$ is semidiscrete. We start by ensuring injectivity of $\pi_u(A)'$.

Proposition 2.2.7. *Suppose A is some nuclear C*-algebra admitting a non-degenerate representation $\pi \colon A \longrightarrow \mathrm{B}(\mathcal{H})$. It follows that $\pi(A)'$ must be injective.*

Proof. Let $\mathscr{M}_0 \subseteq \mathscr{M}$ be a unital inclusion of von Neumann algebras and let $\varphi \colon \mathscr{M}_0 \longrightarrow \pi(A)'$ be a unital completely positive map. Our task is to extend φ to a unital completely positive map defined on $\mathscr{M}$. Since φ and π have commuting ranges by hypothesis, there exists a unique unital completely positive map

$$\pi \times \varphi \colon A \otimes \mathscr{M}_0 = A \otimes_{\max} \mathscr{M}_0 \longrightarrow \mathrm{B}(\mathcal{H}); \qquad a \otimes e \mapsto \pi(a)\varphi(e).$$

The inclusion $\mathscr{M}_0 \subseteq \mathscr{M}$ induces, by functoriality of the minimal tensor product, a $*$-monomorphism $A \otimes \mathscr{M}_0 \hookrightarrow A \otimes \mathscr{M}$, whereby Arveson's extension theorem yields a unital completely positive extension ψ_0 of $\pi \times \varphi$ defined on $A \otimes \mathscr{M}$. Setting $\psi \colon \mathscr{M} \longrightarrow \mathrm{B}(\mathcal{H})$ to be $\psi(e) = \psi_0(1_A \otimes e)$ gives a unital completely positive map. We claim that ψ does the job. Since $\psi(e_0) = (\pi \times \varphi)(1_A \otimes e_0) = \varphi(e_0)$ whenever e_0 belongs to $\mathscr{M}_0$, the morphism ψ extends φ. It remains to be shown that it attains values in $\pi(A)'$. Therefore, fix an a in A and an e in $\mathscr{M}$. As ψ_0 maps $a \otimes 1_{\mathscr{M}}$ into $\pi(a)$ one may deduce that $A \otimes \mathbb{C}1_{\mathscr{M}} \subseteq \mathrm{Mult}(\psi_0)$. Thus (1.9), the fact that ψ_0 extends $\pi \times \varphi$ in conjunction with $\pi(A)$ commuting with φ guarantee that

$$\begin{aligned}
\pi(a)\psi(e) &= \pi(a)\psi_0(1_A \otimes e) \\
&= \psi_0(a \otimes 1_{\mathscr{M}})\psi_0(1_A \otimes e) \\
&= \psi_0(1_A \otimes e)\psi_0(a \otimes 1_{\mathscr{M}}) \\
&= \psi(e)\pi(a).
\end{aligned}$$

Continuity combined with linearity of ψ yields the containment $\psi(\mathscr{M}) \subseteq \pi(A)'$ as claimed. $\qquad\square$

In light of the preceding discussion, we only lack Kirchberg's theorem to reach our goal. It becomes immensely more pleasant to prove the theorem with some preliminary work. There are two main steps involved in the difficult implication of Kirchberg's theorem. Firstly, one attempts to rewrite a plausible candidate factoring the given unital completely positive map in terms of vector states, which is achieved through a Radon-Nikodym type statement. It shall be addressed now.

Proposition 2.2.8. *Let ω be some state acting on a unital C^*-algebra A. Let further $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ be its GNS-triple. If σ is some positive functional on A dominated by ω on A_+, then there exists a unique positive contractive operator e_σ in $\pi_\omega(A)'$ fulfilling*

$$\sigma(a) = \langle \pi_\omega(a) e_\sigma \xi_\omega, \xi_\omega \rangle$$

for each element a inside A.

Proof. Let $q_\omega \colon A \longrightarrow \mathcal{H}_\omega$ be the quotient map. Define a sesquilinear form $\langle \cdot, \cdot \rangle_\sigma$ on $q_\omega(A)$ by the assignment $(q_\omega(a), q_\omega(b)) \mapsto \sigma(b^*a)$ for all a, b in A. Applying (1.3) together with $\sigma \leq \omega$ easily entails that the sesquilinear form must be bounded by ω on the norm-dense subspace $q_\omega(A) \subseteq \mathcal{H}_\omega$. Denoting the extension to $\mathcal{H}_\omega$ by $\langle \cdot, \cdot \rangle_\sigma$ as well permits one to invoke Riez' representation theorem to find a unique positive contractive operator $e_\sigma \colon \mathcal{H}_\omega \longrightarrow \mathcal{H}_\omega$ subject to

$$\langle e_\sigma \xi, \eta \rangle = \langle \xi, \eta \rangle_\sigma \tag{2.3}$$

for all vectors ξ, η in $\mathcal{H}_\omega$. Recall that $\pi_\omega(a) q_\omega(b) = q_\omega(ab)$ for any pair of elements a, b contained in A. Computing grants

$$\langle \pi_\omega(a) e_\sigma \xi_\omega, \xi_\omega \rangle = \langle e_\sigma q_\omega(1_A), q_\omega(a^* 1_A) \rangle \overset{(2.3)}{=} \sigma(a).$$

Moreover,

$$\langle (e_\sigma \pi_\omega(a) - \pi_\omega(a) e_\sigma) \xi_\omega, \xi_\omega \rangle \overset{(2.3)}{=} \langle \pi_\omega(a) \xi_\omega, \xi_\omega \rangle_\sigma - \langle \pi_\omega(a) e_\sigma \xi_\omega, \xi_\omega \rangle = \sigma(1_A^* a) - \sigma(a) = 0$$

for every a in A. Thus $\pi_\omega(A)$ commutes with e_σ due to ξ_ω being cyclic, proving the claim. $\qquad\square$

In order to approximate the given unital completely positive map in Kirchberg's theorem in terms of vector states, one appeals to a lemma by Glimm. Glimm's lemma allows such approximations of states under suitable, and modest, assumptions. For the record, given vector ξ in some Hilbert space one defines the vector functional of ξ, denoted by ω_ξ, as $\omega_\xi(\cdot) = \langle \cdot \xi, \xi \rangle$.

Lemma 2.2.9 (Glimm)**.** *Suppose A denotes some separable C^*-algebra represented faithfully on the separable Hilbert space $\mathcal{H}$ such that A trivially intersects the compact operators thereon. Under these premises, any state ω admits an orthonormal set of vectors $(\xi_n)_{n \geq 1}$ in $\mathcal{H}$ satisfying $\omega_{\xi_n}(a) \to \omega(a)$ for every a belonging to A.*

Proof. Fix some finite subset $F \subseteq A$ together with some tolerance $\varepsilon > 0$. Upon rescaling throughout we may assume that F consists solely of contractions. Select a finite dimensional subspace $\mathcal{H}_0 \subseteq \mathcal{H}$. To arrive at the desired conclusion, we produce a unit vector ξ inside $\mathcal{H}_0^\perp$ satisfying $\omega_\xi(a) \approx_{h(\varepsilon)} \omega(a)$ for all a in F, where h is some function for which $h(\varepsilon) \to 0$ whenever $\varepsilon \to 0$.

According to the Krein-Milman theorem, there is some convex combination $\psi = \sum_{k=1}^n c_k \psi_k$ consisting of pure states such that $\omega(a) \approx_\varepsilon \psi(a)$ for every a in F. Excising each of the pure states ψ_k supplies positive contractions e_k in A for which

$$e_k a e_k \approx_\varepsilon \psi_k(a) e_k^2, \quad a \in F. \tag{2.4}$$

Let p_0 be the orthogonal projection onto $\mathcal{H}_0$. Since the compact operators form a two-sided ideal, expanding $r := p_0^\perp e_1 p_0^\perp - e_1$ reveals that r must be compact. The restriction q to A of the canonical

map onto $B(\mathcal{H})/K(\mathcal{H})$ must be a $*$-monomorphism, since $A \cap K(\mathcal{H}) = \{0\}$ by hypothesis. Ergo, we are allowed conclude that

$$\|q(p_0^\perp e_1 p_0^\perp)\| = \|q(e_1)\| = \|e_1\| = 1.$$

It follows that $\|p_0^\perp e_1 p_0^\perp\| \geq 1$, (otherwise $\|q(p_0^\perp e_1 p_0^\perp)\| < 1$, which opposes the above). Hence $p_0^\perp e_1 p_0^\perp$ must have unit length. We claim one may extract some unit vector ξ_1 in $\mathcal{H}_0$ satisfying $e_1 \xi_1 \approx_\varepsilon \xi_1$. This is based on the upcoming estimates. Fix some unspecified tolerance $0 < \delta < 1$. It will suffice to provide bounded functions $g, g' \colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that both converge uniformly to zero whenever their input does, $\|p_0^\perp \eta\| < g'(\delta)$, $g'(\delta)^2 < 1$ and such that the latter estimate in

$$
\begin{aligned}
\|e_1 p_0^\perp \eta - \eta\| &\leq \|p_0 e_1 p_0^\perp \eta\| + \|p_0^\perp e_1 p_0^\perp \eta - \eta\| \\
&\leq \|p_0^\perp \eta\| + \|p_0^\perp e_1 p_0^\perp \eta - \eta\| \\
&< g(\delta) + g'(\delta)
\end{aligned}
$$

holds for some unit vector η. Indeed having found such maps, the identity $\|p_0^\perp \eta\|^2 + \|p_0 \eta\|^2 = 1$ combined with declaring that $\xi_1 = p_0^\perp \eta / \|p_0^\perp \eta\|$ yields

$$\|e_1 \xi_1 - \xi_1\| = \|p_0^\perp \eta\|^{-1} \|e_1 p_0^\perp \eta - p_0^\perp \eta\| < \frac{g(\delta) + g'(\delta)}{(1 - g'(\delta)^2)^{1/2}} \to 0.$$

Due to $p_0^\perp e_1 p_0^\perp$ being a positive contraction, its spectrum contains 1. Being a self-adjoint operator acting on $\mathcal{H}$, either 1 must be an eigenvalue of $p_0^\perp e_1 p_0^\perp$ or we may determine some sequence $(\eta_n)_{n \geq 1}$ of vectors such that $p_0^\perp e_1 p_0^\perp \eta_n \to \eta_n$ in norm[3]. Regardless of the outcome, there exists some unit vector η such that $p_0^\perp e_1 p_0^\perp \eta$ is within $g'(\delta) := 1 - (1 - \delta)^{1/2}$ distance of η. Now,

$$1 - \|p_0^\perp e_1 p_0^\perp \eta\| = \|\eta\| - \|p_0^\perp e_1 p_0^\perp \eta\| \leq \|p_0^\perp e_1 p_0^\perp \eta - \eta\| < 1 - (1 - \delta)^{1/2}.$$

Rearranging this leaves us with

$$(1 - \delta)^{1/2} < \|p_0^\perp e_1 p_0^\perp \eta\| \leq \|e_1 p_0^\perp \eta\| \leq 1. \tag{2.5}$$

According to the Pythagorean identity, one may infer that $\|p_0^\perp e_1 p_0^\perp \eta\|^2 + \|p_0 e_1 p_0^\perp \eta\|^2 = \|e_1 p_0^\perp \eta\|^2$. Combining this with (2.5) will supply the required relations, namely

$$
\begin{aligned}
\|p_0 e_1 p_0^\perp \eta\| &< \delta^{1/2} =: g(\delta), \\
\|p_0 \eta\| &= \|p_0(\eta - p_0^\perp e_1 p_0^\perp)\| < 1 - (1 - \delta)^{1/2} = g'(\delta).
\end{aligned}
$$

Mimicking the argument with respect to the finite-dimensional subspace

$$\mathcal{H}_1 = \mathrm{span}\{\mathcal{H}_0, a\xi_1, a^* \xi_1 : a \in F\},$$

will supply some unit vector ξ_2 in $\mathcal{H}_1^\perp$ such that $e_2 \xi_2 \approx_\varepsilon \xi_2$. Reiterating the process yields pairwise orthogonal unit vectors $\xi_1, \ldots, \xi_n$ subject to such estimates. We claim that the vector $\xi = \sum_{k=1}^n c_k^{1/2} \xi_k$ works, so fix some a in F. Due to $\xi_k \approx_\varepsilon e_k \xi_k$, one has

$$|\omega_{\xi_k}(a) - \omega_{e_k \xi_k}(a)|^2 \leq \|e_k a e_k \xi_k - a\xi_k\| \leq \|e_k \xi_k - \xi_k\| \cdot \|a\| + \|e_k a e_k \xi_k - a e_k \xi_k\| < 2\varepsilon. \tag{2.6}$$

Moreover,

$$|\omega_{e_k \xi_k}(a) - \psi_k(a)\|e_k \xi_k\|^2| \leq (\|e_k a e_k - \psi_k(a)e_k^2\| \cdot \|\xi_k\|)^{1/2} \overset{(2.4)}{<} \varepsilon^{1/2}, \tag{2.7}$$

while the estimate $\|e_k \xi_k\|^2 \leq (\varepsilon + 1)^2$ ensures that (as $F \subseteq A_1$ was arranged)

$$\left| \sum_{k=1}^n c_k \psi_k(a)\|e_k \xi_k\|^2 - \psi(a) \right| \leq \sum_{k=1}^n c_k |\psi_k(a)(\|e_k \xi_k\|^2 - 1)| < \varepsilon^2 + 2\varepsilon. \tag{2.8}$$

---

[3]For normal operators, the residual part of the spectrum is empty.

Altogether, one has

$$\omega_\xi(a) = \sum_{k=1}^n c_k \omega_{\xi_k}(a) \overset{(2.6)}{\underset{(2\varepsilon)^{1/2}}{\approx}} \sum_{k=1}^n c_k \omega_{e_k \xi_k}(a) \overset{(2.7)}{\underset{\varepsilon^{1/2}}{\approx}} \sum_{k=1}^n c_k \psi_k(a) \|e_k \xi_k\|^2 \overset{(2.8)}{\underset{\varepsilon^2 + 2\varepsilon}{\approx}} \psi(a).$$

Since $\psi(a) \approx_\varepsilon \omega(a)$, the assertion follows. $\qquad\square$

Theorem 2.2.10 (Kirchberg)**.** *Let* $\gamma\colon A \longrightarrow \mathscr{M} \subseteq \mathrm{B}(\mathcal{H})$ *be a unital completely positive map between some unital separable C*-algebras and some von Neumann algebra* $\mathscr{M}$*. Under these premises, the map* γ *is weakly nuclear if and only if the product map* $\gamma \times \mathrm{id}_{\mathscr{M}'}\colon A \odot \mathscr{M}' \longrightarrow \mathrm{B}(\mathcal{H})$ *is* $\|\cdot\|_{\min}$*-continuous. In particular, a von Neumann algebra* $\mathscr{M}$ *is semidiscrete if and only if* $\mathscr{M}'$ *is.*

Proof. Suppose at first γ is weakly nuclear. Let weak-nuclearity be witnessed via the factorizations

$$\gamma_n\colon A \xrightarrow{\quad\varphi_n\quad} \mathbb{M}_{k(n)} \xrightarrow{\quad\psi_n\quad} \mathscr{M}.$$

Define hereby maps $\mu_n\colon A \otimes \mathscr{M}' \longrightarrow \mathrm{B}(\mathcal{H})$ as the compositions

$$\mu_n\colon A \otimes \mathscr{M}' \xrightarrow{\varphi_n \otimes \mathrm{id}} \mathbb{M}_{k(n)} \otimes \mathscr{M}' = \mathbb{M}_{k(n)} \otimes_{\max} \mathscr{M}' \xrightarrow{\psi_n \times_{\max} \mathrm{id}_{\mathscr{M}'}} \mathrm{B}(\mathcal{H}).$$

The obtained sequence $(\mu_n)_{n \geq 1}$ must be contained in the closed unit ball of bounded linear maps from $A \otimes \mathscr{M}'$ into $\mathrm{B}(\mathcal{H})$. This space of morphisms inherits a weak*-topology, interpreted as the point σ-weak topology, by page 5 in [9]. Alaoglu's theorem thus guarantees the existence of some point σ-weak cluster point μ. Upon replacing the sequence with a suitable subsequence, we may assume that $\mu_n \to \mu$ point σ-wot wise. Since the space of unital completely positive maps is point σ-wot closed (see remark II) the map μ remains unital completely positive. Moreover,

$$\mu(a \otimes e) = \sigma\text{-wot} \lim_{n \to \infty} (\psi_n \varphi_n(a)e) = \left(\sigma\text{-wot} \lim_{n \to \infty} \gamma_n(a)\right)e = \gamma(a)e$$

shows that μ must be a continuous extension of $\gamma \times \mathrm{id}_{\mathscr{M}'}$ as desired. The second equality is based on σ-wot continuity of multiplication to the left (resp. right) by bounded operators. This shows that the implication "$\Rightarrow$" is valid.

The converse will be deduced in two steps. The underlying idea will be easier to visualize once we settle a general observation. Suppose $\mathcal{K}$ denotes some Hilbert space, upon which A is faithfully represented, admitting an orthonormal set $\{\delta_1, \ldots, \delta_n\}$. Let p be the orthogonal projection onto the linear span of the orthonormal set. The unital completely positive map $\varphi\colon A \longrightarrow p\mathrm{B}(\mathcal{K})p$ defined via compression by p attains values in $p\mathrm{B}(\mathcal{K})p \cong \mathbb{M}_n$, hence its image points $\varphi(a)$ may be identified with matrices expressed in terms of the orthonormal set, i.e.,

$$[\varphi(a)_{ij}]_{ij} = [\langle a\delta_i, \delta_j \rangle]_{ij}.$$

Here we naturally regard A as being contained in $\mathrm{B}(\mathcal{K})$. Suppose hereafter that $\{b_1, \ldots, b_n\} \subseteq \mathscr{M}$ is some fixed finite set. Let $\{e_{ij}\}_{i,j=1}^n$ be the set of matrix units in $\mathbb{M}_n$ arising from the aforementioned orthonormal set. The bounded linear map $\psi\colon p\mathrm{B}(\mathcal{K})p \longrightarrow \mathscr{M}$ defined by declaring that

$$\psi(e_{ij}) = b_i^* b_j,$$

then extended accordingly, is another unital completely positive map by *example* 1.5.13 in [9]. The point is that the unital completely positive map $\psi \circ \varphi\colon A \longrightarrow \mathscr{M}$ satisfies

$$(\psi \circ \varphi)(a) = \psi\left(\sum_{i,j=1}^n \langle a\delta_i, \delta_j \rangle e_{ij}\right) = \sum_{i,j=1}^n \langle a\delta_i, \delta_j \rangle b_i^* b_j. \tag{2.9}$$

According to remark I-II it suffices to verify that γ belongs to the σ-wot closure of factorable maps. Select two finite subsets $F \subseteq A$, $N \subseteq \mathscr{M}_*$ and a tolerance $\varepsilon > 0$. For each normal functional ω in N we construct a pair (φ, ψ) of unital completely positive maps such that $(\omega \circ \gamma)(a) \approx_\varepsilon (\omega \circ \psi \circ \varphi)(a)$ for every a in F may be arranged .

The first step towards achieving this will be to access the Radon-Nikodym type statement. Finiteness of N permits us to define an additional normal functional ω_a on $\mathscr{M}$ by summing each element in N and taking the average afterwards. As such ω_a majorizes N, so that proposition 2.2.8 yields positive contractions e_ω in $\pi_{\omega_a}(\mathscr{M})'$ subject to

$$\omega(m) = \langle \pi_{\omega_a}(m) e_\omega \xi_{\omega_a}, \xi_{\omega_a} \rangle \tag{2.10}$$

for every m in $\mathscr{M}$ and ω in N. By our hypothesis imposed on $\gamma \times \mathrm{id}_{\mathscr{M}'}$, *lemma* 3.8.4 of [9] entails $\| \cdot \|_{\min}$-continuity of $\theta := (\pi_{\omega_a} \circ \gamma) \times \mathrm{id}_{\pi_{\omega_a}(\mathscr{M})'}$, whereby $\sigma \colon A \otimes \pi_{\omega_a}(\mathscr{M})' \longrightarrow \mathbb{C}$ given by

$$\sigma(a \otimes e) = \langle \theta(a \otimes e) \xi_{\omega_a}, \xi_{\omega_a} \rangle = \langle (\pi_{\omega_a} \circ \gamma)(a) e_\omega \xi_{\omega_a}, \xi_{\omega_a} \rangle$$

defines a state on $A \otimes \pi_{\omega_a}(\mathscr{M})'$. The second step revolves around applying Glimm's lemma. Represent A faithfully onto some separable Hilbert space $\mathcal{H}$ via ϱ. By separability, we may add countably infinitely many copies of ϱ, meaning construct the assignment $a \mapsto (\varrho(a), \varrho(a), \ldots)$, to obtain a new faithful representation with separable target, which we tacitly identify with $\mathrm{B}(\mathcal{H})$ using separability. However, the new representation cannot contain finite rank operators, hence $A \subseteq \mathrm{B}(\mathcal{H})$ cannot intersect $\mathbb{K}$ non-trivially. Invoking Glimm's lemma with respect to the induced faithful representation $A \otimes \pi_{\omega_a}(\mathscr{M})' \subseteq \mathrm{B}(\mathcal{H} \otimes \mathcal{H}_{\omega_a})$ thus becomes permissible. Choose thereby an orthonormal set $\{\delta_1, \ldots \delta_n\} \subseteq \mathcal{H}$ together with elements $\{m_1, \ldots m_n\} \subseteq \mathscr{M}$ such that

$$\sigma(a \otimes e) \approx_\varepsilon \left\langle (a \otimes e_\omega) \left(\sum_{k=1}^n \delta_k \otimes q_{\omega_a}(m_k) \right), \sum_{k=1}^n \delta_k \otimes q_{\omega_a}(m_k) \right\rangle. \tag{2.11}$$

whenever a belongs to F and ω lies in N. Define ψ and φ as the unital completely positive maps obeying (2.9) from before with respect to these finite collection of elements. Due to

$$\left\langle (a \otimes e_\omega) \left(\sum_{k=1}^n \delta_k \otimes q_{\omega_a}(m_k) \right), \sum_{k=1}^n \delta_k \otimes q_{\omega_a}(m_k) \right\rangle = \sum_{k,\ell=1}^n \langle a\delta_k, \delta_\ell \rangle \langle \pi_{\omega_a}(m_k^* m_\ell) e_\omega \xi_{\omega_a}, \xi_{\omega_a} \rangle$$

$$\overset{(2.10)}{=} \sum_{k,\ell=1}^n \langle a\delta_k, \delta_\ell \rangle \omega(m_k^* m_\ell)$$

being valid for every ω in N and a inside A, one acquires

$$(\omega \circ \gamma)(a) = \sigma(a \otimes e_\omega) \overset{(2.11)}{\approx_\varepsilon} \sum_{k,\ell=1}^n \langle a\delta_k, \delta_\ell \rangle \omega(m_k^* m_\ell) \overset{(2.9)}{=} (\omega \circ \psi \circ \varphi)(a)$$

for every element a in F and ω in N. This completes the proof of the initial statement. For the remaining assertion, one merely applies the first in conjunction with von Neumann's bicommutant theorem: $\mathrm{id}_{\mathscr{M}}$ is semidiscrete if and only if $\mathrm{id}_{\mathscr{M}} \times \mathrm{id}_{\mathscr{M}'} = \mathrm{id}_{\mathscr{M}''} \times \mathrm{id}_{\mathscr{M}'}$ is $\| \cdot \|_{\min}$-continuous, the latter occurring if and only if $\mathrm{id}_{\mathscr{M}'}$ is weakly nuclear. $\qquad\square$

Corollary 2.2.11. *The enveloping von Neumann algebra associated to any nuclear separable C^*-algebra A is injective. In particular, A^{**} must be injective.*

Proof. Suppose A denotes a separable nuclear C*-algebra and let π_u be its universal representation. Then $\pi_u(A)'$ must be injective according to proposition 2.2.7, hence semidiscrete by the equivalence of these notions; see *theorem* 9.3.3 in [9]. However, Kirchberg's theorem entails that $\pi_u(A)''$ must be injective thereby as well. The final claim stems from $A^{**} \cong \pi_u(A)''$ as von Neumann algebras. $\square$

2.3 Limit Algebras

Limits algebras have a tantalizing ability to transform approximation properties into exact ones. For our specific purposes, we appeal to more refined versions relying on metric ultraproducts in the C*-algebraic framework as opposed to traditional ones. The rationale behind invoking ultraproducts will be revealed momentarily. Familiarity with limit algebras is assumed, however, establishing notational aspects has been deemed advantageous. Ultraproduct C*-algebras will be presented in detail, although with references[4] replacing some proofs .

Let $(A_i)_{i \in I}$ be a family of C*-algebras indexed over some directed set I. The formal product $\prod_{i \in I} A_i$ consisting of tuples $(a_i)_{i \in I}$, in which a_i belongs to A_i for each index i in I, may be endowed with the unbounded supremum norm. We will often use the shorthand (a_i) for an element in the product algebra, whenever the underlying indexing set is understood. We define $\ell^\infty(A_i, I)$ as the set consisting of *bounded* elements inside $\prod_{i \in I} A_i$. The space $\ell^\infty(A_i, I)$ equipped with componentwise operations turns into a C*-algebra with the supremum norm. Furthermore, the subspace $c_0(A_i, I) \subseteq \ell^\infty(A_i, I)$ comprised of elements (a_i) satisfying $\lim \|(a_i)\| = 0$ becomes an ideal inside $\ell^\infty(A_i, I)$. Hence

$$\ell(A_i, I) := \frac{\ell^\infty(A_i, I)}{c_0(A_i, I)}$$

defines a C*-algebra. Being the quotient of C*-algebras, $\ell(A_i, I)$ inherits a canonical $*$-epimorphism $\varrho \colon \ell^\infty(A_i, I) \longrightarrow \ell(A_i, I)$. Let now $p_i \colon \ell^\infty(A_i, I) \longrightarrow A_i$ be the i'th projection mapping and let $\iota_i \colon A_i \hookrightarrow \ell^\infty(A_i, I)$ be the i'th inclusion map. Using these maps, elements in the C*-algebra $\ell^\infty(A_i, I)$ may be characterized via the commutative diagrams

$$
\begin{array}{ccc}
 & \ell^\infty(A_i, I) & \\
{\scriptstyle \iota_i} \nearrow & & \searrow {\scriptstyle p_i} \\
A_i & \xrightarrow{\quad \mathrm{id} \quad} & A_i
\end{array}
$$

In this regard, an element a in $\ell^\infty(A_i, I)$ is completely characterized in terms of the point-images $p_i(a)$. An additional feature concerns morphisms. Suppose $\pi_i \colon E \longrightarrow A_i$ denotes a c.p.c., u.c.p or $*$-homomorphisms for each $i \in I$ or C*-algebras and define *the induced map of* $(\pi_i)_{i \in I}$ as the uniquely determined morphism $\pi \colon E \longrightarrow \ell^\infty(A_i, I)$ of same type given by $e \mapsto (\pi_i(e))_{i \in I}$.

Consider the specific case in which $A_i = A$ for each index i in I. We simplify, with an unpleasant abuse of notation[5], in the countable case by abbreviating $\ell^\infty(A) = \ell^\infty(A, \mathbb{N})$ and $c_0(A) = c_0(A, \mathbb{N})$. The specialized case, often referred to as *the sequence or the limit algebra*, of $\ell(A, \mathbb{N})$ is typically represented as

$$A_\infty = \frac{\ell^\infty(A)}{c_0(A)}.$$

It is well-known that in A_∞ the unique C*-norm defined through the canonical quotient map $\varrho_A \colon \ell^\infty(A_i, I) \longrightarrow \ell(A_i, I)$, may be expressed via the formula

$$\|(a_1, a_2, \ldots)\| = \limsup_{n \to \infty} \|a_n\|. \tag{2.12}$$

To A_∞ one may associate the *diagonal* map $\delta_A \colon A \longrightarrow \ell^\infty(A)$ given by $a \mapsto (a, a, \ldots)$. The limit algebra A_∞, in spite of being a powerful tool for separable C*-algebras, lacks a degree of flexibility due to traces on A_∞ often being a nuisance to tackle. Fortunately, there is room for vast improvements using ultrafilters.

---

[4]In general, one ought to read the section 1.2 in [30], wherein consistent notation and conventions with the present ones are exhibited. If the reader has no quarrel reading another project of the author, please consult this.

[5]Unfortunately, this cannot always be compared to the ordinary ℓ^∞ and c_0 spaces despite the notational overlap! Rest assured, the original Banach spaces will not occur.

Definition. Suppose S denotes a nonempty set. A family $\mathcal{F}$ of subsets in S is said to

- *be nontrivial* if $\emptyset \notin \mathcal{F}$;

- *be upward directed* if $A \subseteq B$ and $A \in \mathcal{F}$ imply $B \in \mathcal{F}$;

- *have the finite intersection property* if $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;

- *be maximal* if any subset A in S must satisfy either $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$.

A nontrivial upward directed family $\mathcal{F}$ of subsets in S having the finite intersection property is called a *filter*. An *ultrafilter* is a maximal filter on S. A filter $\mathcal{F}$ is *free* should $\bigcap_{M \in \mathcal{F}} M = \emptyset$.

Let $S \subseteq X$ be an inclusion of nonempty sets. The family $\mathcal{F}(S)$ whose elements are subsets $A \subseteq X$ containing S forms an ultrafilter. For any ultrafilter ω on $\mathbb{N}$, ω is free if and only if $\omega \neq \mathcal{F}(\{n\})$ for every positive integer n. To verify this, proving the contrapositive statement is easier. Indeed, if $\omega = \mathcal{F}(\{n\})$ for some integer n, then $n \in \bigcap_{M \in \omega} M$ which opposes freeness. For the converse, suppose each member in ω contains the singleton $\{n\}$, meaning $\omega \subseteq \mathcal{F}(\{n\})$ holds. For the reverse inclusion, let $M \subseteq \mathbb{N}$ contain n. By maximality of ω either M or $\mathbb{N} \setminus M$ belongs to ω. The latter cannot occur, for otherwise $\{n\} \not\subseteq \mathbb{N} \setminus M \in \omega$. Altogether $\omega = \mathcal{F}(\{n\})$ as claimed.

The sought intriguing properties of ultrafilters are, amongst several; guaranteed existence and their ability to ensure convergence of sequences in $\mathbb{C}$. We collect these two facts into a single proposition and refer to section 1.2 in [30] and the appendix in [9] for those demanding a proof. In order to understand the statements properly, we must address convergence along filters.

Definition. Let X be a topological space and let $\mathcal{F}$ be some filter on a directed set I. A net $(x_i)_{i \in I}$ *converges to x along* $\mathcal{F}$ if, for each open neighbourhood U around x, the set $S_U := \{i \in I : x_i \in U\}$ belongs to $\mathcal{F}$. Equivalently, the net $(x_i)_{i \in I}$ converges to x along $\mathcal{F}$ if, for every open U neighborhood of x, there exists some member S in $\mathcal{F}$ such that $\mathcal{N}_S(x_i) := \{x_i \in X : i \in S\}$ lies in U. We write $x = \lim_{i \to \mathcal{F}} x_i$ to denote a limit point along $\mathcal{F}$.

Proposition 2.3.1. *Let X, Y be topological spaces and $S \neq \emptyset$ be a set admitting a filter $\mathcal{F}$.*

(i) *There exists an ultrafilter containing $\mathcal{F}$.*

(ii) *The limit of any net $(x_i)_{i \in I}$ in X along $\mathcal{F}$ is unique should X be Hausdorff.*

(iii) *Continuity preserves convergence along $\mathcal{F}$, meaning one has $f(x) = \lim_{i \to \mathcal{F}} f(x_i)$ whenever $f \colon X \longrightarrow Y$ is a continuous map and $(x_i)_{i \in I}$ converges to x along $\mathcal{F}$.*

Furthermore, under the additional hypothesis of $\mathcal{F}$ being an ultrafilter, one has:

(i) *If X is compact, then any net in X converges along $\mathcal{F}$. In particular, bounded sequences in $\mathbb{C}$ converge along any ultrafilter on $\mathbb{N}$.*

Proof. We shall prove the third property regarding continuity and refer to [30] for the remainder. Suppose $(x_i)_{i \in I}$ is some net in X converging along $\mathcal{F}$ to some point x. Choose an open neighbourhood V around $f(x)$. Continuity of f forces $U = f^{-1}(V)$ to be an open neighborhood around x. By definition, there exists some member S in $\mathcal{F}$ such that $\mathcal{N}_S(x_i) \subseteq U$. Due to $f(\mathcal{N}_S(x_i)) \subseteq V$ and $f(\mathcal{N}_S(x_i)) = \mathcal{N}_S(f(x_i))$ by definition, the assertion follows at once. $\qquad\square$

Our objective will be to improve the limit algebra A_∞ associated to any C*-algebra A by including ultrafilters, especially due to the fourth statement in the above proposition. We need some preparation to achieve this. Let ω be any filter on $\mathbb{N}$. For any sequence $(x_n)_{n \geq 1}$ in $\mathbb{R}$, we define the filter-analogues of limes supremum, respectively limes inferior, for ω as

$$\limsup_{\omega} x_n = \inf_{X \in \omega} \sup_{n \in X} x_n, \quad \text{respectively}, \quad \liminf_{\omega} x_n = \sup_{X \in \omega} \inf_{n \in X} x_n.$$

Consider the scenario wherein ω is free. By hypothesis, freeness of ω ensures that for any fixed positive integer n and every $k \leq n$ there must exist some $X_k \in \omega$ subject to $k \notin X_k$. The finite intersection $X_n = \bigcap_{k=1}^n X_k$ therefore belongs to $\{n+1, n+2 \ldots\}$ and defines a member of ω due to the finite intersection property. The real number $\sup_{n \leq k} x_k$ will thus exceed $\sup_{k \in X_n} x_k$. Applying the infimum we deduce that

$$\limsup_\omega x_n \leq \limsup_{n \to \infty} x_n. \tag{2.13}$$

Another remark concerns the Frechét filter. The Frechét filter $F_\mathbb{N}$ is the family consisting of cofinite subsets in $\mathbb{N}$. If $(x_n)_{n \geq 1}$ is a sequence in $\mathbb{C}$ converging in the ordinary topological sense to some x, then each neighbourhood U around x admits some N in $\mathbb{N}$ such that x_n belongs to U for all integers n exceeding N, so there are infinitely many such integers n, whereupon $S_U \in F_\mathbb{N}$.

On the other hand, the convergence $x_n \to x$ along the Frechét filter entails, for each open neighbourhood U around x, the existence of some infinite set $M \subseteq \mathbb{N}$ fulfilling $\mathcal{N}_M(x_n) \subseteq U$. Altogether, sequential convergence along the Frechét filter is equivalent ordinary sequential convergence. We may now venture into the world of ultrapowers.

Definition. Suppose $(A_n)_{n \geq 1}$ denotes a sequence of C*-algebras and let ω be a filter on $\mathbb{N}$. To each such a sequence, we define $c_\omega(A_n, \mathbb{N})$ as the ideal in $\ell^\infty(A_n, \mathbb{N})$ consisting of elements $(a_1, a_2, \ldots)$ subject to the relation $\lim_{n \to \omega} \|a_n\| = 0$. The *ultraproduct* associated to the sequence $(A_n)_{n \geq 1}$ with respect to ω is defined as the quotient

$$\prod_\omega A_n = \frac{\ell^\infty(A_n, \mathbb{N})}{c_\omega(A_n, \mathbb{N})}.$$

In the case where $A_n = A$ for all n in $\mathbb{N}$, the ideal $c_\omega(A, \mathbb{N})$ is abbreviated $c_\omega(A)$ and the resulting ultraproduct algebra

$$A_\omega = \frac{\ell^\infty(A)}{c_\omega(A)}$$

is referred to as *the ultrapower of A* with respect to the filter ω. The canonical quotient mapping from $\ell^\infty(A)$ onto A_ω, which determines the norm on A_ω, is denoted by ϱ_ω and we define the *diagonal embedding* $\delta_\omega \colon A \longrightarrow A_\omega$ given by the composition $\varrho_\omega \circ \delta_A \colon A \hookrightarrow \ell^\infty(A) \longrightarrow A_\omega$.

If skeptical eyes arise from the reader, you are completely on track. Let us maintain the notation exhibited in the definition. First of all, one ought to justify that $c_\omega(A_n, \mathbb{N})$ in fact defines an ideal. We give an argument in brevity. One easily deduces subadditivity of $\limsup_\omega$ in a manner resembling the case for ordinary sequential convergence. Moreover, if a sequence $(x_n)_{n \geq 1}$ converges to x along ω or if $(x_n)_{n \geq 1}$ is an element of $\ell^\infty(A)$, then one may verify that $\limsup_\omega x_n = \lim_\omega x_n$. The ideal structure of $c_\omega(A)$ follows here, since one for instance has

$$0 \leq \limsup_\omega \|a_n + b_n\| \leq \lim_\omega \|a_n\| + \lim_\omega \|b_n\| = 0$$

for elements $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ in $c_\omega(A)$. Here we implicitly exploited that $\limsup_\omega x_n \geq 0$ for a sequence $(x_n)_{n \geq 1}$ of positive reals, which clearly holds. Therefore, $c_\omega(A)$ is additively closed, whereas the remaining algebraic properties are verified in similar fashions. To prove that $c_\omega(A)$ is norm-closed let $(a_n^k)_{k \geq 1}$ be a sequence of elements in $c_\omega(A)$ converging to some point a_n. Suppose some $\varepsilon < 0$ is given and let $K \in \mathbb{N}$ be large enough to force $\sup_{n \in \mathbb{N}} \|a_n^k - a_n\| < \varepsilon$ for any k exceeding K. Subadditivity of $\limsup_\omega$ yields

$$\limsup_{n \to \omega} |\|a_n^k\| - \lim_\omega \|a_n\|| \leq \limsup_{n \to \omega} \|a_n - a_n^k\| \leq \sup_{n \in \mathbb{N}} \|a_n - a_n^k\| < \varepsilon.$$

By hypothesis, (a_n^k) belongs to $c_\omega(A)$ for each positive integer k, so the first term on the left-hand side vanishes as $k \to \infty$. Thus $\lim_\omega \|a_n\| = 0$ from which $c_\omega(A)$ becomes closed. Before proceeding further, we derive an analogue of (2.12) for ultrafilters.

Proposition 2.3.2. *Let ω be any filter on $\mathbb{N}$ and let A be any C^*-algebra. Suppose further that the tuple $a = (a_1, a_2, \ldots)$ is in $\ell^\infty(A)$. Then $\|\varrho_\omega(a)\|_{A_\omega} = \limsup_\omega \|a_n\|$ holds. Under the additional hypothesis that ω defines an ultrafilter on $\mathbb{N}$, one has $\|\varrho_\omega(a)\|_{A_\omega} = \lim_{n\to\omega} \|a_n\|$.*

Proof. Uniqueness of C^*-norms reduces the task into verifying that $\pi_\omega(a) \mapsto \limsup_\omega \|a_n\|$, where $a = (a_1, a_2, \ldots)$ lies in $\ell^\infty(A)$, defines a C^*-norm on A_ω. Denote the map by σ to ease the notation. The axioms of being a C^*-norm are clearly satisfied, perhaps save $\sigma(\pi_\omega(a)) = 0$ implying $a = 0$, provided that σ is well-defined. To prove well-definedness, suppose $\pi_\omega(a) = \pi_\omega(b)$ for $a = (a_n)$ and $b = (b_n)$ in $\ell^\infty(A)$. Subadditivity yields the inequality

$$\limsup_\omega \|\|a_n\| - \|b_n\|\| \le \limsup_\omega \|a_n - b_n\| = 0,$$

whereupon $\sigma(\pi_\omega(a)) = \sigma(\pi_\omega(b))$ follows immediately,. Hence σ must be a C^*-seminorm on A_ω. To verify that it in fact determines an actual C^*-norm, assume $\sigma(\pi_\omega(a)) = 0$ for some $a = (a_1, a_2, \ldots)$ inside $\ell^\infty(A)$. By definition of σ, for each prescribed tolerance $\varepsilon > 0$ we may find some X in ω subject to $\sup_{n \in X} \|a_n\| < \varepsilon$. This in turn implies that $\|a_n\| < \varepsilon$ for any n in X. As our choice of $\varepsilon > 0$ was arbitrary, we deduce that $\limsup_{n\to\omega} \|a_n\| = 0$ and ergo $a_n = 0$ for every positive integer n, so $\sigma = \|\cdot\|_{A_\omega}$ by uniqueness of C^*-norms.

For the second assertion, suppose ω is an ultrafilter on $\mathbb{N}$. Given an element $(a_1, a_2, \ldots)$ in $\ell^\infty(A)$, the associated sequence $(\|a_n\|)_{n\ge 1}$ of positive real number becomes bounded, so it must belong to some compact subset in $\mathbb{R}$. According to proposition 2.3.1, it therefore converges along ω. Since the limes superior and ordinary limes along ω agree whenever the limit along ω exists, one may infer that $\sigma(\pi_\omega(a)) = \lim_w \|a_n\|$ for any such element a, completing the proof. $\qquad\square$

As a final preparation prior to discussing testing-results and tracial ultrapowers, we discuss some relations between A_ω and A_∞ for any C^*-algebra A. Appealing to (2.13), one may conclude that ordinary convergence of real numbers entails convergence along any free ultrafilter ω on $\mathbb{N}$. The comparison provides us with an inclusion $c_0(A) \subseteq c_\omega(A)$, whereby we acquire a $*$-homomorphism $\varrho \colon A_\infty \longrightarrow A_\omega$ such that the associated diagram

$$
\begin{array}{ccc}
\ell^\infty(A) & & \\
{\scriptstyle \varrho_A}\Big\downarrow & \searrow^{\varrho_\omega} & \\
A_\infty & \xrightarrow[\varrho]{} & A_\omega
\end{array}
$$

commutes, namely $\varrho_A(a_1, a_2, \ldots) \mapsto \varrho_\omega(a_1, a_2, \ldots)$.

Remark. Suppose A, B are C^*-algebras. Let ω be a free ultrafilter on $\mathbb{N}$. If for each positive integer n there exists a contractive linear map $\pi_n \colon A \longrightarrow B$, the map $\pi \colon A_\omega \longrightarrow B_\omega$ given by

$$\pi(\varrho_\omega^A(a)) = \varrho_\omega^B(\pi_1(a_1), \pi_2(a_2), \ldots),$$

for every $a = (a_1, a_2, \ldots)$ in $\ell^\infty(A)$, becomes a bounded linear map. Well-definedness, meaning independence on the choice of lift for an element in A_ω, may effortlessly be derived from the preceding proposition. Evidently, the induced morphism π becomes a $*$-homomorphism, commonly referred to the *induced morphism* of $(\pi_n)_{n\ge 1}$, if π_n defines a $*$-homomorphism for all n in $\mathbb{N}$. Now, suppose each $\pi_n \colon A \longrightarrow B$ denotes a $*$-monomorphism. Since $*$-homomorphisms are monic precisely whenever they are isometric, one may apply proposition 2.3.2 once more to infer that

$$\|\pi(\varrho_\omega^A(a))\| = \lim_\omega \|\pi_n(a_n)\|_B = \lim_\omega \|a_n\|_A = \|\varrho_\omega^A(a)\|.$$

The induced map π associated to a sequence of $*$-monomorphisms therefore becomes isometric. The analogue statement for epimorphisms remains valid, although easier to verify.

Proposition 2.3.3. *Let A be a (unital) C^*-algebra and let ω be a filter on $\mathbb{N}$.*

(i) *Every projection p in A_ω lifts to a projection in $\ell^\infty(A)$.*

(ii) *Every unitary u in A_ω lifts to a unitary in $\ell^\infty(A)$.*

Proof. (i): Write $\pi_\omega(a) = p$ for some projection p in A_ω and self-adjoint element $a = (a_n)$ in $\ell^\infty(A)_+$. Applying proposition 2.3.2 we have $\limsup_\omega \|a_n^2 - a_n\| = 0$. For every positive integer n, define a member E_n of ω by letting

$$E_n = \left\{ k \in \mathbb{N} : \|a_k^2 - a_k\| < \frac{1}{n^2} \right\}.$$

A fundamental functional calculus result states that $\sigma(a_k) \subseteq [-\frac{1}{n}, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1 + \frac{1}{n}]$ for each positive integer n provided that $k \in E_n$, see for instance *lemma* 2.2.3 in [29] for a proof of the matter. The obtained family $\{E_n\}_{n=1}^\infty$ forms a descending chain of members in ω. We thus decompose

$$\mathbb{N} = (\mathbb{N} \setminus E_1) \cup \left(\bigcup_{n \in \mathbb{N}} E_n \setminus E_{n+1} \right) \cup \left(\bigcap_{n \in \mathbb{N}} E_n \right).$$

To ease the notational burden, we introduce the shorthand $I_n = [1 - \frac{1}{n}, 1 + \frac{1}{n}]$ for each n in $\mathbb{N}$. The associated indicator map $\chi_{I_n} \colon \mathbb{N} \longrightarrow \{0, 1\}$ becomes continuous on $\sigma(a_k)$ for each k in E_n due the previously established inclusion of $\sigma(a_k)$. The whole idea is of course that for indices in the intersection $\bigcap_n E_n$, the elements a_k turn into idempotents. Define accordingly for each positive integer k, elements

$$b_k = \begin{cases} a_k, & \text{if } k \in \bigcap_{n \in \mathbb{N}} E_n, \\ \chi_{I_n}(a_k), & \text{if } k \in E_n \setminus E_{n+1}. \end{cases}$$

On the merits of the functional calculus being a homomorphism, $\chi_{I_n}(a_k)$ becomes a projection for each $n \in \mathbb{N}$ and $k \in E_n$, whereas for k belonging to the intersection $\bigcap_{n \in \mathbb{N}} E_n$ the size of $\|a_k^2 - a_k\|$ can be made arbitrarily small. Hence $\|a_k^2 - a_k\| = 0$ holds whenever one has $k \in \bigcap_{n \in \mathbb{N}} E_n$. As such the sequence $b = (b_n)_{n \geq 1}$ is a projection in $\ell^\infty(A)$ with $\varrho_\omega(a) = \varrho_\omega(b) = p$. The latter relation stems from $\limsup_\omega \|a_n - b_n\|$ being zero by construction of b.

(ii): Assume that A admits a unit 1_A. The ultrapowers A_ω admits $\pi_\omega(1_A, 1_A, \ldots)$ as a unit. Suppose $a = (a_1, a_2, \ldots)$ denotes a lift of some unitary u in A_ω. Now, proposition 2.3.2 ensures that

$$\limsup_\omega \|a_n^* a_n - 1_A\| = \limsup_\omega \|a_n a_n^* - 1_A\| = 0. \tag{2.14}$$

Using (2.14), choose some X in ω such that both $\|a_n^* a_n - 1_A\| < 1$ and $\|a_n a_n^* - 1_A\| < 1$ become valid. Being within unit distance of the unit, $a_n^* a_n$ together with $a_n a_n^*$ are invertible for each n in X. The unitary polar decomposition for C^*-algebras thus supplies some unitary v_n in A such that $a_n = v_n |a_n|$ for all n in X. Keeping this in mind, set

$$b_n = \begin{cases} 1_A, & \text{if } n \notin X, \\ v_n, & \text{if } n \in X. \end{cases}$$

Plainly, $b = (b_1, b_2, \ldots)$ is a unitary in $\ell^\infty(A)$, so it remains to be proven that it lifts u. For each given $\varepsilon > 0$, choose some member X in ω subject to $\|a_n^* a_n - 1_A\| < \varepsilon \|a\|^{-1}$ for any $n \in X$. Here we invoke (2.14). From the polar decomposition we infer[6] that for $b_n = v_n$ one has

$$\|b_n - a_n\| = \|a_n(|a_n|^{-1} - 1_A)\| \leq \|a\| \cdot \|(a_n^* a_n)^{-1/2} - 1_A\| < \varepsilon.$$

As a consequence, one acquires $\limsup_\omega \|a_n - b_n\| = 0$, whence $\varrho_\omega(b) = u$, proving the claim. $\qquad\square$

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[6]Mischief occurs here. However, a standard continuous functional calculus trick reveals that $\|a_n^* a_n - 1_A\| < \varepsilon$ implies $\|(a_n^* a_n)^{-1/2} - 1_A\| < \varepsilon$.

Testing Results

During the previous section, several prestigious properties of ultraproducts were unveiled. The present section seeks to exhibit another crucial tool in the ultraproduct arsenal: Kirchberg's ε-test. It provides conditions from which one may produce elements of ultraproduct C*-algebras having unique properties attached to them. The conditions are captured in "testing functions". Statements of such types have substantial impacts and are frequently applied. We initially derive Kirchberg's test and afterwards discuss a selected portion of additional testing results. Throughout the entire section, we regard ω as being some fixed free ultrafilter on $\mathbb{N}$ for simplicitiy.

Lemma 2.3.4 (Kirchberg's ε-test)**.** *Let* $X_1, X_2, \ldots$ *be any sequence of sets. Suppose there exists a sequence* $(f_n^k)_{n \geq 1}$ *of functions* $f_n^k \colon X_n \longrightarrow \mathbb{R}^+$ *for each positive integer* k. *Define accordingly a new sequence of, possibly unbounded, functions* $f_\omega^k \colon \prod_n X_n \longrightarrow [0, \infty]$ *by*

$$f_\omega^k(x_1, x_2, \ldots) = \lim_{n \to \omega} f_n^k(x_n).$$

If, for every positive integer m *and every prescribed tolerance* $\varepsilon > 0$, *there exists a tuple* $(x_1, x_2, \ldots)$ *in* $\prod_n X_n$ *fulfilling* $f_\omega^k(x_1, x_2, \ldots) < \varepsilon$ *for each positive integer* $k \leq m$, *then there exists some element* $(y_1, y_2, \ldots)$ *belonging to* $\prod_n X_n$ *such that* $f_\omega^k(y_1, y_2, \ldots) = 0$ *for all* k *in* $\mathbb{N}$

Proof. Fix a positive integer m. For each n in $\mathbb{N}$ let $(X_{n,\ell})_{\ell \geq 0}$ be the sequence of nonempty sets constructed inductively by setting $X_{n,0} = X_n$ and

$$X_{n,\ell} = \left\{ x \in X_n : \max_{k \leq \ell} f_n^k(x) < \frac{1}{\ell} \right\}.$$

Set $\mu \colon \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$ to be the assignment $n \mapsto \max\{\ell : X_{n,\ell} \neq \emptyset, \ \ell \leq n\}$. By hypothesis, one may find some element x in $\prod_n X_n$ such that $\lim_{n \to \omega} f_n^k(x_n) = f_\omega^k(x) < \frac{1}{\ell}$ whenever one has $1 \leq k \leq m$. Thus the set

$$Y_m := \left\{ n \in \mathbb{N} : \max_{i \leq m} f_n^i(x_n) < \frac{1}{m} \right\}$$

belongs to the filter ω, for every m in $\mathbb{N}$. None of the sets Y_m can therefore be empty. It follows that $X_{n,m}$ cannot be empty whenever n lies inside Y_m. Hence $\min\{n, m\} \leq \mu(n) \leq n$ for all integers n in Y_m. Set now $Z_m = \{n \in \mathbb{N} : m \leq \mu(n)\}$ and $Y_m' = Y_m \setminus \{1, 2, \ldots, m-1\}$. One certainly has $m \leq \mu(n) \leq n$ if n belongs to Y_m' because of our previous estimate. As such the inclusion $Y_m' \subseteq Z_m$ must be valid. The ultrafilter ω is free, so it must contain any cofinite set[7], including Y_m'. Since ω is upward directed, the inclusion $Y_m' \subseteq Z_m$ hereby entails $Z_m \in \omega$. These observations yield

$$\lim_{n \to \omega} \frac{1}{\mu(n)} \leq \liminf_{n \to \omega} \frac{1}{\mu(n)} = \sup_{X \in \omega} \left(\inf_{n \in X} \frac{1}{\mu(n)} \right) \leq \inf_{k \in \mathbb{N}} \left(\sup_{n \in Z_k} \frac{1}{\mu(n)} \right) \leq \inf_{k \in \mathbb{N}} \frac{1}{k} = 0.$$

One may extract some y_n inside the nonempty set $X_{n,\mu(n)}$ for each positive integer n to produce a tuple $y = (y_1, y_2, \ldots)$ in $\prod_n X_n$ satisfying

$$0 \leq f_\omega^k(y) \leq \lim_{n \to \omega} \frac{1}{\mu(n)} \leq 0.$$

The element y satisfies the sought property, completing the proof. $\square$

We adopt the convention of calling the associated maps f_ω^k *testing functions*. Testing functions and results hereof will be useful when having to construct an order zero map on the ultraproduct of the universal UHF-algebra. We will dwell further into these assertions. However, for the sake of generality some considerations concerning induced maps on ultraproducts ought to be addressed.

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[7] ω if free if and only if it contains the Frechét filter.

Observation. Let $A, B_1, B_2, \ldots$ be some C*-algebras. Suppose $(\varphi_n)_{n \geq 1}$ denotes a uniformly bounded sequence of $*$-linear maps $\varphi_n \colon A_0 \longrightarrow B_\omega$, where A_0 is some norm-dense involutive $\mathbb{Q}[i]$-subalgebra of A. The sequence $(\varphi_n)_{n \geq 1}$ induces a bounded $*$-linear map $\varphi_0 \colon A_0 \longrightarrow \prod_\omega B_n$ via

$$\varphi(\cdot) = \varrho_\omega(\varphi_1(\cdot), \varphi_2(\cdot), \ldots). \tag{2.15}$$

Linearity combined with preservation of the involution follows from an easy application of proposition 2.3.1. Thus, it extends by continuity together with density to a bounded $*$-linear map from A into $\prod_\omega B_n$. Our objective will be to encapsulate criteria upon which properties, including complete positivity and order zero, are guaranteed for φ. During the proof, we implicitly exploit the following commutativity rule based on exactness of $\mathbb{M}_n$:

$$\left(\prod_\omega B_n \right) \otimes \mathbb{M}_n \cong \frac{\ell^\infty(B_n, \mathbb{N}) \otimes \mathbb{M}_n}{c_\omega(B_n, \mathbb{N}) \otimes \mathbb{M}_n} \cong \frac{\ell^\infty(\mathbb{M}_n \otimes B_n, \mathbb{N})}{c_\omega(\mathbb{M}_n \otimes B_n, \mathbb{N})} = \prod_\omega \mathrm{M}_n(B_n).$$

The middle isomorphism arises from the map sending an element $\sum_{i,j=1}^n (b_1^{ij}, b_2^{ij}, \ldots) \otimes e_{ij}$ into the element $(b_1, b_2, \ldots)$, where $b_k = \sum_{i,j=1}^n b_k^{ij} \otimes e_{ij}$.

Lemma 2.3.5. *Let A be some separable C*-algebra, $B_1, B_2, \ldots$ be some sequence of C*-algebras and A_0 be a countable norm-dense involutive $\mathbb{Q}[i]$-subalgebra of A. For each n, let L_n be the set of $*$-linear maps from A_0 into B_n. Let $L \subseteq \prod_n L_n$ consist of uniformly bounded tuples.*

Under these premises, there exists a sequence $(f_n^k)_{k \geq 1}$ consisting of functions $f_n^k \colon L_n \longrightarrow \mathbb{R}^+$ such that every tuple $(\varphi_n)_{n \geq 1}$ in L induces a contractive completely positive map $\varphi \colon A \longrightarrow \prod_\omega B_n$ if and only if $\lim_{n \to \omega} f_n^k(\varphi_n) = 0$ for each $k \in \mathbb{N}$.

Proof. The proof has been split into two parts. First of all, we construct testing functions that detect whether φ becomes contractive or not. To achieve this, fix throughout an enumeration $a_1, a_2, \ldots$ of A_0. Define accordingly testing functions $f_n^k \colon L_n \longrightarrow [0, \infty]$ by declaring

$$f_n^k(\varphi_n) = \max\{\|\varphi_n(a_k)\| - \|a_k\|, 0\}.$$

Based on the characterization of norms on ultraproducts[8] one may deduce that $\|\varphi(a_k)\| = \lim_\omega \|\varphi_n(a_k)\|$ for each positive integer k. Then φ_0 must be contractive if and only if one has $\lim_\omega f_n^k(\varphi_n) = 0$ for every such k. By continuity combined with density, the continuous extension φ of φ_0 onto A becomes contractive if and only if one has $\lim_\omega f_n^k(\varphi_n) = 0$ for all k in $\mathbb{N}$

This tackles the contractive property. Fix momentarily some positive integer n. Since A_0 is norm-dense in A, one effortlessly realizes that $\mathrm{M}_n(A_0)$ becomes an involutive norm-dense $\mathbb{Q}[i]$-subalgebra of $\mathrm{M}_n(A)$. Furthermore, the norm-closure of $\mathrm{M}_n(A_0)_+$ must coincide with $\mathrm{M}_n(A)_+$. Enumerating $\mathrm{M}_\ell(A_0)_+ = (b_{\ell,k})_{k \geq 1}$ for each positive integer ℓ, one may define testing functions $g_{\ell,n}^k \colon L_n \longrightarrow \mathbb{R}^+$, indexed in accordance with the functions f_n^k above, by

$$\varphi_n \mapsto \mathrm{dist}(\varphi_{n,\ell}(b_{\ell,k}), \mathrm{M}_\ell(B_n)_+).$$

Here $\varphi_{n,\ell}$ denotes the ℓ'th amplification of φ_n. Let $\varphi_\ell \colon \mathrm{M}_\ell(A) \longrightarrow \mathrm{M}_\ell(\prod_\omega B_n)$ be the ℓ'th amplification of the map φ induced by $(\varphi_n)_{n \geq 1}$ as in (2.15). Using this particular notation, φ_ℓ is positive for all integers ℓ if and only if φ is completely positive. Due to $\mathrm{M}_\ell(A_0)_+$ being norm-dense in $\mathrm{M}_\ell(A)$,

$$\varphi \text{ is completely positive,} \quad \text{if and only if,} \quad \psi_\ell := \varphi_{\ell \,|\mathrm{M}_\ell(A_0)} \text{ is positive for all } \ell \in \mathbb{N}. \tag{2.16}$$

The target of φ_ℓ is an isomorphic copy of $\prod_\omega \mathrm{M}_\ell(B_n)$. In terms of this identification, φ_ℓ becomes the bounded linear map induced from the sequence $(\varphi_{1,\ell}, \varphi_{2,\ell}, \ldots)$. That is, if γ_ℓ denotes the map induced from $(\varphi_{1,\ell}, \varphi_{2,\ell}, \ldots)$, then

$$\gamma_\ell \colon \mathrm{M}_\ell(A) \xrightarrow{\ \varphi_\ell\ } \mathrm{M}_\ell(\textstyle\prod_\omega B_n) \cong \prod_\omega \mathrm{M}_\ell(B_n).$$

Altogether, $\lim_{n \to \omega} g_{\ell,n}^k(\varphi_1, \varphi_2, \ldots) = 0$ for all indices k, ℓ if and only if γ_ℓ is positive on $\mathrm{M}_\ell(A_0)$ for all ℓ, which occurs if and only if ψ_ℓ. The assertion therefore follows from (2.16). $\quad\square$

---

[8]Although proposition 2.3.2 does not cover $\prod_\omega B_n$, the proof may be adjusted to include this generality.

Lemma 2.3.6. *Let A be some separable unital C^*-algebra, $B_1, B_2, \ldots$ be a sequence of C^*-algebras and A_0 be a countable norm-dense involutive $\mathbb{Q}[i]$-subalgebra of A. For each n, let L_n be the set of $*$-linear maps from A_0 into B_n. Let $L \subseteq \prod_n L_n$ consist of uniformly bounded tuples.*

Under these premises, there exists a sequence $(f_n^k)_{k \geq 1}$ consisting of functions $f_n^k \colon L_n \longrightarrow \mathbb{R}^+$ such that every tuple $(\varphi_n)_{n \geq 1}$ in L induces a contractive order zero map $\varphi \colon A \longrightarrow \prod_\omega B_n$ if and only if $\lim_{n \to \omega} f_n^k(\varphi_n) = 0$ for each $k \in \mathbb{N}$.

Proof. As lemma 2.3.5 provides testing functions for φ to be contractive completely positive, we are only required to test for orthogonality preservation. To this end, enumerate the closed unit ball of A_0 by the elements $a_1, a_2, \ldots$ in A_0. By density, one may infer that $(A_0)_1$ becomes norm-dense in the unit ball of A. For each pair of integers $k, \ell \in \mathbb{N}$, define functions $h_{k,n}^\ell \colon L_n \longrightarrow \mathbb{R}^+$ via

$$\varphi_n \mapsto \|\varphi_n(a_k a_\ell)\varphi(1_A) - \varphi_n(a_k)\varphi_n(a_\ell)\|.$$

Due to φ being order zero if and only if it satisfies the order zero identity, see corollary 1.4.10(i), the $*$-linear map $\varphi_0 \colon A_0 \longrightarrow \prod_\omega B_n$ induced by the sequence $(\varphi_n)_{n \geq 1}$ becomes an order zero map if and only if $\lim_{n \to \omega} h_{k,n}^\ell(\varphi_n) = 0$. According to the density of $(A_0)_1$ in $(A)_1$, the continuous extension $\varphi \colon A \longrightarrow \prod_\omega B_n$ must be of order zero if and only if $\lim_{n \to \omega} h_{k,n}^\ell(\varphi_n) = 0$. Voila. $\qquad \square$

2.4   Tracial Ultrapowers

Having defined ultrapowers of C*-algebras, we embark on a journey into the wonders of tracial ultrapowers. The current exposition might seem unrelated to the overall theme of ultrapowers. However, the reader is assured of a greater epiphany to surface in due time. The admirable aspect of trace-ideals revolves around the obtained quotients together with their close relationship to von Neumann algebraic features. Naturally, some basics are in order.

Definition. Suppose A denotes a C*-algebra admitting a trace τ. Let ω be a free ultrafilter on $\mathbb{N}$. We define seminorms on A, and the corresponding ultrapower A_ω, by

$$\| \cdot \|_{2,\tau} \colon A \longrightarrow \mathbb{R}^+; \quad \|a\|_{2,\tau} = \tau(a^*a),$$
$$\| \cdot \|_{\tau,\omega} \colon A_\omega \longrightarrow \mathbb{R}^+; \quad \|(a_n)\|_{\tau,\omega} = \lim_\omega \|a_n\|_{2,\tau}.$$

Notice that no mischief occurs for the latter seminorm; an element $a = (a_1, a_2, \ldots)$ in $\ell^\infty(A)$ must be norm-bounded by hypothesis, hence the sequence $(\|a_n\|)_{n \geq 1}$ belongs to some compact subset of $\mathbb{R}$ and attains a limit along ω according to proposition 2.3.1. The fact that $\| \cdot \|_{2,\tau}$ becomes a seminorm stems from the GNS-identity (1.4). Furthermore, in the event of τ being faithful, (1.4) entails that $\| \cdot \|_{2,\tau}$ on A turns into a norm.

Definition. Suppose ω denotes a free ultrafilter on $\mathbb{N}$. Given a C*-algebra A, we define

$$J_{\tau,\omega} := \left\{ \varrho_\omega(a_1, a_2, \ldots) \in A_\omega : \lim_\omega \|a_n\|_{2,\tau} = 0 \right\}.$$

We refer to the quotient $A_\tau^\omega = A_\omega / J_{\tau,\omega}$ as the *tracial ultrapower of A with respect to τ*.

Remark. When dealing with tracial ultrapowers, it is customary to omit referring to the quotient map $\varrho_\omega \colon \ell^\infty(A) \longrightarrow A_\omega$. Ergo we shall not distinguish between elements in either and simply write a generic in A_τ^ω as $\varrho_\tau^\omega(a_1, a_2, \ldots)$ for brevity, unless confusion may occur, where ϱ_τ^ω is the quotient map onto A_τ^ω. Secondly, the diagonal embedding $\delta_\omega \colon A \hookrightarrow A_\omega$ induces an embedding $\varrho_\tau^\omega \varrho_\omega \delta_A \colon A_\omega \longrightarrow A_\tau^\omega$, once again identifying A with images of constant sequences.

For the sake of comforting ourselves with these entities, we shall verify that $J_{\tau,\omega}$ in fact determines an ideal within A_ω, whereupon the tracial ultrapowers become meaningful (and hence are C*-algebras

themselves). Suppose a belongs to A and b belongs to $J_{\tau,\omega}$. The Cauchy-Schwarz inequality for positive functionals, i.e. (1.3), yields

$$
\begin{aligned}
\|ab\|_{2,\omega} &= \lim_\omega \|a_n b_n\|_{2,\tau} \\
&= \lim_\omega \tau\big(b_n^*(a_n^* a_n b_n)\big) \\
&\leq \lim_\omega \big(\tau(b_n^* b_n)\tau((a_n^* a_n b_n)^*(a_n^* a_n b_n))\big)^{1/2} \\
&= \lim_\omega \tau((a_n^* a_n b_n)^*(a_n^* a_n b_n))^{1/2} \cdot \|b\|_{2,\omega}^{1/2}.
\end{aligned}
$$

Due to the right-hand factor being zero, ab must belong to $J_{\tau,\omega}$. We implicitly exploited continuity of multiplication in $\mathbb{C}$ during the final equality in conjunction with continuity preserving ω-limits. The remaining properties an ideal are verified in resembling manners.

A remarkable aspect to be revealed is that $A_\tau^\omega, \mathscr{N}_\tau^\omega$ become identical von Neumann algebras, whose type we may track solely in terms of the underlying C*-algebra A. Further, in order to appeal to von Neumann algebraic trickery, we are poised to address the types tracial ultrapowers of von Neumann algebras attain, including finiteness. We will not establish the von Neumann algebraic structure of tracial ultrapowers and instead confine ourselves solely to taking types, finiteness and factors into account.

Observation. Let A be a C*-algebra and let ω be a free ultrafilter on $\mathbb{N}$. Suppose $(\varphi_n)_{n \geq 1}$ denotes a uniformly bounded sequence of linear functionals $\varphi \colon A \longrightarrow \mathbb{C}$. Define a map $\varphi_\omega \colon A_\omega \longrightarrow \mathbb{C}$ by the following formula:

$$
\varphi_\omega(\varrho_\omega(a_1, a_2, \ldots)) = \lim_\omega \varphi_n(a_n). \tag{2.17}
$$

Well-definedness of φ_ω stems from a computation resembling the ones already encountered alongside the limit $\lim_\omega \varphi(a_n)$ being meaningful thanks to proposition 2.3.1. The induced map φ_ω obviously must be positive, bounded, contractive and tracial provided each φ_n satisfies the respective properties. One may further induce a faithful trace $\tau \colon A_\omega \longrightarrow \mathbb{C}$ from A by

$$
[\varrho_\omega(a_1, a_2, \ldots)] \mapsto \lim_\omega \tau(a_n). \tag{2.18}
$$

Here the choice of lift $(a_1, a_2, \ldots)$ for both quotient maps becomes irrelevant, similar to previous cases. Notice that τ_ω automatically becomes faithful by the definition of $J_{\tau,\omega}$. In this sense, the same ideal measures the failure of the induced trace τ_ω on A_ω from being faithful. Mimicking the argument presented in the proof of proposition 2.3.2, one acquires the characterization

$$
\|\varrho_\tau^\omega \varrho_\omega(a_1, a_2, \ldots)\|_{A_\tau^\omega} = \lim_\omega \|a_n\|_{2,\tau}. \tag{2.19}
$$

of the norm on A_τ^ω, for any such pairing (A, τ).

Proposition 2.4.1. *The tracial ultrapower $\mathscr{M}_\tau^\omega$ associated to a von Neumann algebra $\mathscr{M}$ admitting a normal trace defines a finite von Neumann algebra*

Proof. The reader may consult the appendix in [9] on the matter. Finiteness, however, is immediate from the existence of a faithful trace, namely the one depicted in (2.18). $\qquad \square$

Lemma 2.4.2. *Let $\mathscr{M}$ be a von Neumann algebra admitting a faithful trace τ. Then $\mathscr{M}$ is a factor if and only if for each nonzero projection p in $\mathscr{M}$ with $\tau(p) \leq 1/2$ one has $p \preceq 1_{\mathscr{M}} - p$.*

Proof. Suppose at first $\mathscr{M}$ is a factor. For any non-zero projection $p \in \mathscr{M}$, one has either $p \preceq 1_{\mathscr{M}} - p$ or $1_{\mathscr{M}} - p \preceq p$ according to the comparability theorem, see *corollary* 25.5 in [51]. We have to discard the latter from being a possibility. Since $\tau(1_{\mathscr{M}} - p) \geq 1/2$ we may deduce that $\tau(p) \leq \tau(1_{\mathscr{M}} - p)$. If $\tau(p) < 1/2$, then $1_{\mathscr{M}} - p \preceq p$ cannot be valid. In the event of $\tau(p) = 1/2$ we have $\tau(p) = \tau(1_{\mathscr{M}} - p)$. We therefore must have $p \preceq 1_{\mathscr{M}} - p$.

For the converse suppose $\mathscr{M}$ is not a factor and choose a non-trivial central projection p in $\mathscr{M}$ hereby. Upon interchanging p with $1_{\mathscr{M}} - p$, we may assume that $\tau(p) \leq 1/2$. Suppose q denotes any projection such that $p \sim q$, say $p = v^*v$ together with $q = vv^*$ hold for some partial isometry v. Since p is central, the inequality (1.2) implies that $q = vv^*vv^* = vp^2v^* = pvv^*p \leq p$. Hence $q \npreceq 1_{\mathscr{M}} - p$ and thus $p \preceq 1_{\mathscr{M}} - p$ fails. This proves the claim. $\qquad\square$

Proposition 2.4.3. *Suppose $\mathscr{M}$ denotes a von Neumann algebra admitting a faithful trace τ and let ω be a free ultrafilter on $\mathbb{N}$.*

   (i) $\mathscr{M}_\tau^\omega$ is a factor provided that $\mathscr{M}$ is a factor.

   (ii) $\mathscr{M}_\tau^\omega$ is of type II_1 provided that $\mathscr{M}$ is a type II_1 factor.

In addition, $\mathscr{M}_\tau^\omega$ is always finite.

Proof. (i): Let p be any nonzero projection in $\mathscr{M}_\tau^\omega$ and lift it to a projection $(p_1, p_2, \ldots)$ in $\ell^\infty(\mathscr{M})$ using proposition 2.3.3. Substituting the projections p and p_n with their corresponding orthogonal complements if necessary we may safely assume that

$$F := \{n \in \mathbb{N} : \tau(p_n) \leq 1/2\} \in \omega.$$

According to lemma 2.4.2, one has $p_n \preceq 1_{\mathscr{M}} - p_n$ for each positive integer n, say $v_n^*v_n = p_n$ and $v_n v_n^* \leq 1_{\mathscr{M}} - p_n$ for partial isometries $(v_n)_{n \geq 1}$ within $\mathscr{M}$. Letting $u_n = v_n$ for each integer n in F and $u_n = 0$ otherwise yields a partial isometry $u = \varrho_\tau^\omega(v_1, v_2, \ldots)$ in $\mathscr{M}_\tau^\omega$. Since u satisfies

$$\|v^*v - p\| \overset{(2.19)}{=} \lim_\omega \|v_n^*v_n - p_n\|_{2,\tau} = \lim_\omega \tau(v_n^*v_n - v_n^*v_n p_n - p_n v_n^*v_n + p_n) = 0$$

one obtains $v^*v = p$ while $vv^* \preceq 1_{\mathscr{M}_\tau^\omega} - p$ stems from $*$-homomorphisms preserving the order $\preceq$ (hence so do $\varrho_\omega, \varrho_\tau^\omega$). The claim now immediately follows from lemma 2.4.2.

(ii): Based on (i) in conjunction with (2.18), the von Neumann algebra $\mathscr{M}_\tau^\omega$ becomes a finite factor, hence must be of type I_n (i.e. isomorphic to $\mathbb{M}_n$) for some positive integer n or type II_1. We assert the former must be false. Fix any positive integer k. According the halving lemma[9], there exists mutually orthogonal projections $p_1, p_2, \ldots, p_k$ in $\mathscr{M}$. By embedding these into $\mathscr{M}_\tau^\omega$ via the diagonal map δ_ω composed with the usual quotient map ϱ_τ^ω, they form a k-dimensional $\mathbb{C}$-linear space therein. Due to this being valid for each $k < n$, $\mathscr{M}_\tau^\omega$ cannot be of type I_n for any n in $\mathbb{N}$, so that (ii) follows whereas the latter assertion was established previously. $\qquad\square$

As promised, the great epiphany achieved and point in considering the strong-operator closure $\mathscr{N}$ of the GNS triple associated to a pair (A, τ) consisting of a unital C*-algebra and trace hereon: Its tracial ultrapower A_τ^ω becomes isomorphic to $\mathscr{N}_\tau^\omega$.

Proposition 2.4.4. *Suppose A denotes a unital C*-algebra admitting some trace τ and let further $\mathscr{N} = \pi_\tau(A)''$. Under these premises, there exists a $*$-epimorphism $\Lambda: A_\tau^\omega \longrightarrow \mathscr{N}_\tau^\omega$ making*

commute. In particular, $A_\omega^\tau \cong \mathscr{N}_\tau^\omega$ and A_τ^ω becomes a finite II_1 factor.

---

[9]Consult for instance *lemma* 4.9.2 in [13], noting that type II do not contain abelian projections.

Proof. Let $\tau_{\mathcal{N}}$ be the normal faithful trace on $\mathcal{N}$ induced by the trace τ. To ensure existence of the $*$-homomorphism Λ, consider the ideal $J_{\tau,\omega} \trianglelefteq A_{\omega}$. Fix some $a = (a_1, a_2, \ldots)$ in $\ell^{\infty}(A)$. If the element a belongs to $I := \ker \gamma\pi_{\infty}$, then (2.19) implies that $\lim_{\omega} \|\pi_{\tau}(a_n)\|_{2,\tau_{\mathcal{N}}} = 0$. Upon $\tau_{\mathcal{N}}$ extending τ, the point-image $\varrho_{\omega}^{A}(a)$ satisfies

$$\|\varrho_{\omega}^{A}(a)\|_{\tau,\omega} \stackrel{(2.19)}{=} \lim_{\omega} \|a_n\|_{2,\tau} \stackrel{(1.4)}{=} \lim_{\omega} \tau_{\mathcal{N}}(\pi_{\tau}(a_n^* a_n)) = \lim_{\omega} \|\pi_{\tau}(a_n)\|_{2,\tau_{\mathcal{N}}} \stackrel{(2.19)}{=} \|\gamma\pi_{\infty}(a)\|_{\mathcal{N}_{\tau_{\mathcal{N}}}^{\omega}}$$

We thus infer that $\varrho_{\omega}^{A}(I) = J_{\tau,\omega}$ must be valid. Define accordingly the map $\Lambda \colon A_{\tau}^{\omega} \longrightarrow \mathcal{N}_{\tau}^{\omega}$ through the, now meaningful, assignment $\sigma(a) \mapsto \gamma\pi_{\tau}^{\infty}(a)$. Commutativity of the daunting diagram is automatic. We claim that Λ is a $*$-epimorphism. Let $\gamma(z)$ be any positive element inside $\mathcal{N}_{\tau}^{\omega}$ lifting to some positive element $z = (z_1, z_2, \ldots)$ in $\ell^{\infty}(\mathcal{N})$. According to Kaplansky's density theorem, any closed bounded ball in $\pi_{\tau}(A)$ must be strong-operator dense in $\mathcal{N}$. This permits one to select, for each positive integer n, some element a_n in A fulfilling $\|a_n\| \leq \|z_n\|$ and

$$\|\pi_{\tau}(a_n) - z_n\|_{2,\tau_{\mathcal{N}}} = \|(\pi_{\tau}(a_n) - z_n)\xi_{\tau}\| \leq \frac{1}{n}. \tag{2.20}$$

The first equality is based on a straightforward computation leaning solely on (1.4) in conjunction with the definition of $\tau_{\mathcal{N}}$. It follows that

$$\|\gamma\pi_{\tau}^{\infty}(a_1, a_2, \ldots) - \gamma(z_1, z_2, \ldots)\| \stackrel{(2.19)}{=} \lim_{\omega} \|\pi_{\tau}(a_n) - z_n\|_{2,\tau_{\mathcal{N}}} \stackrel{(2.20)}{\leq} 0,$$

Surjetivity of σ follows at once. Since one has $\ker \Lambda = \sigma(I)$, then

$$A_{\tau}^{\omega} = \sigma(A_{\omega})/\sigma(I) = A_{\tau}^{\omega}/\ker \Lambda \cong \mathcal{N}_{\tau}^{\omega},$$

This completes the proof. $\qquad\square$

We close the section with an application of Kirchberg's ε-test due to Rørdam and Kirchberg in [26], *proposition* 4.6. The assertion concerns additional structure that $J_{\tau,\omega}$ contains. For those unacquainted with the notion: An ideal I inside a C*-algebra is referred to as a σ-*ideal* should there exist some positive contraction s in I fulfilling $s \in B' \cap I$ and $sb = b$ for each $b \in B \cap I$, where B is any separable C*-subalgebra in A.

Proposition 2.4.5 (Kirchberg-Rørdam). *Let A be a unital C*-algebra admitting a trace τ. If so, $J_{\tau,\omega}$ determines a σ-ideal within A_{ω}.*

Proof. The crux of the proof revolves around Kirchberg's ε test. Suppose B denotes any separable C*-subalgebra in A. Let $(a_1^k, a_2^k, \ldots)$ be any contraction in $\ell^{\infty}(A)$ such that the sequence of elements $a^k = \varrho_{\omega}(a_1^k, a_2^k, \ldots)$ becomes norm-dense in B_1. Due to B being separable, we only have to produce a positive contraction s in $J_{\tau,\omega} \cap B'$ subject to $sb = b$ for some strictly positive element b of $B \cap J_{\tau,\omega}$. Choose through separability a strictly positive element b in $B \cap J_{\tau,\omega}$, then lift it to a positive element $(b_1, b_2, \ldots)$. Now, $J_{\tau,\omega} \cap B$ admits an approximate unit $(e(1), e(2), \ldots)$ in $J_{\tau,\omega}$, quasicentral in B. Let $(e_1(k), e_2(k), \ldots)$ be a lift of $e(k)$ and set X_n to be the set of all positive contractions in A, for all $n \in \mathbb{N}$. Define $f_n^k \colon X_n \longrightarrow \mathbb{R}^+$ by

$$f_n^1(x) = \|(1-x)b_n\|, \quad f_n^2(x) = \|x\|_{2,\tau} \quad \text{and} \quad f_n^{k+2}(x) = \|a_n^k x - x a_n^k\|.$$

Due to $(e(1), e(2), \ldots)$ being an approximate unit and quasicentral in B, one has $f_{\omega}^1(e_1^k, e_2^k, \ldots) = 0$ together with $f_{\omega}^{k+2}(e_1^k, e_2^k, \ldots) = 0$. By hypothesis, $\lim_{n \to \omega} \tau((e_n(k)^2)_{n \geq 1}) = 0$, whereby

$$f_{\omega}^2(e_1^k, e_2^k, \ldots) = \lim_{n \to \omega} \|(e^k)\|_{2,\tau} = \lim_{n \to \omega} \tau((e_n^k))^2) = 0.$$

Invoking Kirchberg's ε-test, we find some sequence consisting of positive contractions $(s_n)_{n \geq 1}$ in A such that $f_{\omega}^k(s_1, s_2, \ldots) = 0$ for every $k \in \mathbb{N}$. The first two testing functions (f_{ω}^1 and f_{ω}^2) ensure that $s := \varrho_{\omega}(s_1, s_2, \ldots)$ defines a positive contraction belonging to $J_{\tau,\omega}$ with $sb = b$. The final testing functions f_{ω}^k for $k \geq 3$ guarantee that s commutes with $C^*(a^k) \subseteq (B)_1$, hence with all of B_1 by norm-density, and thereof all of B. This concludes the proof. $\qquad\square$

Chapter 3

Quasidiagonality in the Nuclear Separable Framework

We delve into the realm of quasidiagonal C*-algebras and their connections to Nuclear separable C*-algebras. The theorem of Tikuisis-White-Winter requires a tremendous amount of setup. The chapter is devoted solely to investigating the concept of quasidiagonality alongside alternative characterizations. Statements occurring in the first section are mainly classical, hence assumed familiar. For a rigorous treatment of the matter, please skim the second chapter in [30].

3.1 Quasidiagonal C*-algebras

Quasidiagonality is a rigid structure to impose, however , we do gain several strong properties including existence of a quasidiagonal trace in the unital case. The notion originates back to Paul Halmos in the seventies, who introduced the notion of "block-diagonality", the notion whereby quasidiagonality was spawned as an approximation version. One frequently desires approximation-shaped properties to be detected via completely positive maps. Quasidiagonality has such a characterization, an observation by Voiculescu's pioneering work on the subject. In the thesis, Voiculescu's abstract formulation is our primary focus. For completion the full package is brought forth.

Theorem 3.1.1 (Voiculescu). *Let A be some C*-algebra. The following are equivalent.*

- *There exists a faithful representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$ and a net $(p_\alpha)_{\alpha \in J}$ comprised of finite rank projections in $\mathrm{B}(\mathcal{H})$, quasicentral in $\pi(A)$, such that $p_\alpha \to 1_\mathcal{H}$ strong-operator wise.*

- *There exists a net $(\psi_\alpha)_{\alpha \in J}$ comprised of contractive completely positive maps $\psi_\alpha\colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ which is asymptotically multiplicative - and isometric, meaning for each a, b in A one has*

$$\|\psi_\alpha(ab) - \psi_\alpha(a)\psi_\alpha(b)\| \to 0, \quad \text{respectively,} \quad \|\psi_\alpha(a)\| \to \|a\|.$$

- *For each prescribed finite subset $F \subseteq A$ and tolerance $\varepsilon > 0$, there exists a contractive completely positive map $\psi\colon A \longrightarrow \mathbb{M}_n$ fulfilling the estimates below for all a, b belonging to F:*

$$\psi(ab) \approx_\varepsilon \psi(a)\psi(b) \quad \text{together with} \quad \|\psi(a)\| > \|a\| - \varepsilon.$$

Moreover, in the presence of a unit on A the completely positive maps may chosen to be unital and sequences replace nets whenever A is separable.

Definition. A C*-algebra satisfying either of the conditions in theorem 3.1.1 is *quasidiagonal*.

It is apparent that quasidiagonality must be an invariant of C*-algebras. More may be deduced in fact; Voiculescu observed in [46] that quasidiagonality is a homotopy-invariant (see section 2.3, in [30] for a detailed proof). Furthermore, the standard permanence properties are listed.

Proposition 3.1.2. *Suppose $(A_n)_{n \geq 1}$ denotes some sequence of C^*-algebras and let $I \subseteq \mathbb{N}$ be arbitrary. Under these premises, the following properties are fulfilled.*

(i) *Quasidiagonality passes to subalgebras.*

(ii) *If A_n and A_m are quasidiagonal, then $A_n \otimes A_m$ is quasidiagonal.*

(iii) *The C^*-algebra A is quasidiagonal if and only if A^+ is quasidiagonal.*

(iv) *If there exists a $*$-monomorphism $\pi_n \colon A_n \longrightarrow A_{n+1}$ for each positive integer n, then the inductive limit $\varinjlim(A_n, \pi_n)$ remains quasidiagonal.*

Some examples arising are finite-dimensional C*-algebras along with certain AF-algebras. Other lists of examples include algebras such as the compact operators, residually finite dimensional ones, irrational rotation algebras and commutative C*-algebras. Notice that property (iv) cannot be generalized to general inductive limits. More exotic examples include the Bernoulli crossed product associated to a countable elementary amenable group (*theorem* 4.3.6, [30]). The property (iii) often comes in handy due to unital C*-algebras being notoriously more pleasant to tackle. The property (i) should not be underestimated either; it differs from nuclearity since heredity is required there. One should also be warned about quasidiagonality not passing to quotients. To provide the full picture, we exhibit the two known obstructions towards quasidiagonality.

Proposition 3.1.3. *Every quasidiagonal C^*-algebras is unitally stably finite. The converse if false.*

Proof. Let A be any quasidiagonal C*-algebra. Passing to the unitization, we may assume A to be unital and we verify the claim in the separable setting, the general being proven similarly. Stable finiteness of A amounts to $M_n(A)$ being finite for each positive integer n in the unital sense. Since $\mathbb{M}_n \otimes A \cong M_n(A)$ is a $\|\cdot\|_{\min}$-tensor product of quasidiagonal C*-algebras, it must be quasidiagonal. As such it suffices to establish finiteness of A. To accomplish this, we show that every isometry must be a unitary. Let $(\psi_n)_{n \geq 1}$ be the sequence of asymptotically multiplicative - and isometric unital completely positive maps attaining values in $\mathbb{M}_{k(n)}$. Let u be any isometry in A, so that

$$\|\psi_n(u)^*\psi_n(u) - 1_{k(n)}\| = \|\psi_n(u)^*\psi_n(u) - \psi_n(u^*u)\| \to 0.$$

Choose some sufficiently large integer N such that $\psi_n(u)^*\psi_n(u) \approx_1 1_{k(n)}$ whenever n exceeds N. It follows that $\psi_n(u)^*\psi_n(u)$ must be invertible in $\mathbb{M}_{k(n)}$, hence $\psi_n(u)$ must be invertible therein as well, having a nonzero determinant. Ergo the above convergence yields

$$\lim_{n \to \infty} \|\psi_n(u)\psi_n(u)^* - 1_{k(n)}\| = \lim_{n \to \infty} \|\psi_n(u)(1_{k(n)} - \psi_n(u)^{-1}\psi_n(u^*)^{-1})\psi_n(u^*)\|$$
$$\leq \lim_{n \to \infty} \|1_{k(n)} - (\psi_n(u)^*\psi_n(u))^{-1}\| \cdot \|\psi_n(u)\|^2$$
$$\leq \|(\psi_n(u)^*\psi_n(u))^{-1}\| \cdot \|\psi_n(u)^*\psi_n(u) - 1_{k(n)}\|$$
$$\to 0.$$

whenever n exceeds N. As the unital completely positive maps are asymptotically isometric - and multiplicative, one may infer that

$$\|uu^* - 1_A\| \leq \lim_{n \to \infty} \|\psi_n(uu^*) - \psi_n(u)\psi_n(u)^*\| + \lim_{n \to \infty} \|\psi_n(u)\psi_n(u)^* - 1_{k(n)}\| = 0,$$

proving the first claim. However, $C_r^*(\mathbb{F}_2)$ is a counter example to the converse. $\square$

Proposition 3.1.4. *Every unital quasidiagonal C^*-algebras admits a trace.*

Proof. We sketch the proof; computations may be found in *proposition* 2.2.7 of [30]. Let $(\psi_\alpha)_{\alpha \in J}$ be the net implementing quasidiagonality on some C*-algebra A. Composing each map ψ_α with the corresponding unique normalized trace on its target algebra, one obtains a weak*-cluster point of traces by Alaoglu's theorem, which becomes a trace on A itself. $\square$

The preceding propositions, albeit relatively easy to prove, make quasidiagonality rather restrictive. Moreover, the latter result should indicate that traces serve a pivotal role for quasidiagonal C*-algebras. Non-examples emerge hereby, to wit:

Corollary 3.1.5. *C*-algebras having infinite projections cannot be quasidiagonal. In particular, unital properly infinite C*-algebras cannot be quasidiagonal*

Proof. An immediate consequence of quasidiagonality implying stable finiteness. An alternative proof of the latter is given as follows. Let A be a unital quasidiagonal properly infinite C*-algebra. Let τ be its trace inherited from quasidiagonality and suppose $p \sim q < p$ occurs for some nonzero projections p, q in A. Then $\tau(p) = \tau(q) < \tau(p)$, since traces cannot distinguish between Murray - von Neumann equivalent projections, a contradiction. □

A concrete non-quasidiagonal example would then be the Cuntz algebras $\mathcal{O}_n$. Regarding C*-algebras admitting traces, the notion of quasidiagonality may be modified to quasidiagonal traces. Quasidiagonal traces form the core property we pursue throughout the thesis. These were introduced in [8], arguably based on a mix between Voiculescu's abstract characterization of quasidiagonality and proposition 3.1.4 (the proof in fact supplies a quasidiagonal trace).

Definition. Let A be a C*-algebra admitting a trace. A trace τ acting on A is called *quasidiagonal* provided there, for each finite $F \subseteq A$ and tolerance $\varepsilon > 0$, exists some contractive completely positive map $\psi \colon A \longrightarrow \mathbb{M}_k$ such that

$$\psi(ab) \approx_\varepsilon \psi(a)\psi(b) \quad \text{and} \quad (\tau_k \circ \psi)(a) \approx_\varepsilon \tau(a) \tag{3.1}$$

for all a, b in F. Equivalently, τ is quasidiagonal if it admits a net $(\psi_\alpha)_{\alpha \in J}$ consisting of c.p.c maps $\psi_\alpha \colon A \longrightarrow \mathbb{M}_{n(\alpha)}$ fulfilling $\|\psi_\alpha(ab) - \psi_\alpha(a)\psi_\alpha(b)\| \to 0$ and $\tau_{n(\alpha)} \circ \psi_\alpha \to \tau$ in the weak*-sense for all a, b in A. Furthermore, sequences replace nets whenever A is separable.

Remarks.

- The notion of quasidiagonal traces stated in its current shape above differs from the one exhibited in [42]. Therein, the definition has been restricted to unital C*-algebras while demanding the existing completely positive maps to be unital in addition. The versions agree in the unital case.

- Another notable non-unital consideration revolves around the unitization A^+. Suppose A denotes a non-unital C*-algebra admitting a trace τ. The induced positive functional $\tau_+ \colon A^+ \longrightarrow \mathbb{C}$, i.e.,

$$\tau_+(a + \lambda 1_{A^+}) = \lambda + \tau(a) \tag{3.2}$$

is effortlessly seen to constitute a trace on A^+. It is easy to deduce that τ_+ becomes quasidiagonal (in the unital sense) if τ is; the induced linear maps implementing quasidiagonality of τ induce unital completely positive maps on A^+ detecting quasidiagonality of τ_+. Moreover, the induced trace τ_+ is faithful whenever τ is so. To verify this, let some positive element $x = a + \lambda 1_{A^+}$ in $A^+ \setminus A$ be given. Then x must be of the form

$$b^*b + b\overline{\mu} + b^*\mu + |\mu|^2 1_{A^+}$$

for some complex value μ and element b in A. The first three summands may collected into a self-adjoint element, whereof rescaling appropriately allows to assume that x attains the form $1_{A^+} - b$ for some self-adjoint element b inside A. Write $b = b_+ - b_-$ for positive elements b_+, b_-. Due to b_+ differing from the unit in A^+ (otherwise $b = 1_{A^+} - b_-$, whereby $x = b_- \in A$), one may choose some element z inside A such that $z(1_{A^+} - b_+)z^* \neq 0$. Hence $\tau(z(1_{A^+} - b_+)z^*) > 0$. Upon b_+ being a contraction, the relation $\tau(b_+) = 1$ cannot occur, for else

$$0 < \tau(z(1_{A^+} - b_+)z^* = \tau\big((1_{A^+} - b_+)^{1/2}z^*z(1_{A^+} - b_+)^{1/2}\big) \overset{(1.2)}{\leq} \|z^*z\|1_{A^+} \cdot \tau(1_{A^+} - b_+) = 0.$$

One thus obtains $\tau(b_+) < 1$ and ergo $\tau_+(x) \geq 1 - \tau(b_+) > 0$.

3.2 A Lifting Theorem and Tracially Large Order Zero Maps

In order to aptly present the general ideas, the author chose to convey some perspective in the matter. The motivation behind proceeding in this manner is simply to keep track of the underlying strategy. The proof of Tikuisis-White-Winter's theorem invokes a plethora of apparatus and tricks, the largest player entering being an algebra one must be awed by: The universal UHF-algebra $\mathcal{Q}$.

We commence the current section by examining $\mathcal{Q}$ in the quasidiagonal context. The „universality" mentioned is synonymous to „maximal" in the sense that it unitally contains isomorphic copies of every UHF-algebra $\mathbb{M}_N$, whence every matrix algebra $\mathbb{M}_n$. We primarily adopt the infinite tensor product picture of $\mathcal{Q}$, meaning

$$\mathcal{Q} = \bigotimes_{k=1}^{\infty} \mathbb{M}_{p_k^\infty}.$$

Here $\{p_1, p_2, \ldots\}$ is the set of all primes. For each positive integer n, the matrix algebra $\mathbb{M}_n$ unitally embeds into $\mathcal{Q}$ in a trace preserving manner. Indeed writing $\mathcal{Q} = \mathbb{M}_n \otimes U$ for some UHF-algebra U ensures this and the assignment $E_n \colon \mathcal{Q} \longrightarrow \mathbb{M}_n$ given by

$$E_n(a \otimes e) = a \otimes \tau_U(e), \tag{3.3}$$

where τ_U denotes the unique trace on U, is a unital conditional expectation onto $\mathbb{M}_n$. The conditional expectation E_n recovers the trace in the sense that $\tau_n \circ E_n = \tau_{\mathcal{Q}}$. *Throughout the entire chapter, let some free ultrafilter ω on $\mathbb{N}$ be chosen.* One, amongst several, motivation behind bringing $\mathcal{Q}$ forward is its ability to witness quasidiagonality of traces. We further lean on some absorption principles of $\mathcal{Q}$ and K-theoretic consequences. *Henceforth the induced trace on $\mathcal{Q}_\omega$ will be denoted by τ_ω.*

Proposition 3.2.1. *Suppose p, q are projections in $\mathcal{Q}$.*

   (i) *$\tau_{\mathcal{Q}}(p) = \tau_{\mathcal{Q}}(q)$ if and only if $p \sim q$, while $\tau_{\mathcal{Q}}(p) \leq \tau_{\mathcal{Q}}(q)$ if and only if $p \preceq q$.*

   (ii) *For each positive real number $0 \leq s \leq 1$ there exists a projection p in $\mathcal{Q}_\omega$ such that $\tau_\omega(p) = s$.*

Proof. (i): Recall that the isomorphism $\mathrm{K}_0(\mathcal{Q}) \cong \mathbb{Q}$ stems from the map $\tau_* \colon \mathrm{K}_0(\mathcal{Q}) \longrightarrow \mathbb{Q}$ induced by the unique trace $\tau_{\mathcal{Q}}$, i.e., for each generic element $[p]_0 - [q]_0$ in $\mathrm{K}_0(\mathcal{Q})$,

$$\tau_*([p]_0 - [q]_0) = \tau_{\mathcal{Q}}(p) - \tau_{\mathcal{Q}}(q).$$

Consequently, if $\tau_{\mathcal{Q}}(p) = \tau_{\mathcal{Q}}(q)$ for two projections p, q in $\mathcal{Q}$, then $[p]_0 = [q]_0$. Since equality in K_0 translates into stable equivalence, there exists some projection r in $\mathcal{P}_\infty(\mathcal{Q})$ such that $p \oplus r \sim_0 q \oplus r$. The cancellation property of $\mathcal{Q}$ yields $p \sim q$ relative to $\mathcal{Q}$. For the second statement, we similarly acquire $\tau_{\mathcal{Q}}(q) - \tau_{\mathcal{Q}}(e) = \tau_{\mathcal{Q}}(p)$ for a projection e in Q. Thus $p \sim q - e \leq q$, so that $p \preceq q$.

   (ii): Fix some s in $[0, 1]$ and select a sequence $(q_1, q_2, \ldots)$ of rational numbers in $[0, 1]$ converging to s. Using the isomorphism τ_* one may find nonzero projections $p_1, p_2, \ldots$ in $\mathcal{Q}$ fulfilling $\tau_{\mathcal{Q}}(p_k) = q_k$ for each $k \leq n$. The element $p := (p_1, p_2, \ldots)$ belongs to $\ell^\infty(\mathcal{Q})$ and $\tau_\omega(p) = s$. $\square$

Remark. UHF-algebras are $\mathcal{Z}$-stable in the sense that $\mathcal{Z} \otimes U \cong U$ for any UHF-algebra U, with $\mathcal{Z}$ being the Jiang-Su algebra, see *theorem* 5 in [22]. According to Ozawa's theorem, found in [32], the trace τ_ω must be unique due to uniqueness of $\tau_{\mathcal{Q}}$. In other words, $\mathcal{Q}_\omega$ must be monotracial.

Proposition 3.2.2. *Let n be any positive integer. The following hold.*

   (i) *For every nonzero projection p in $\mathrm{M}_n(\mathcal{Q})$, there exists a $*$-isomorphism $p\mathrm{M}_n(\mathcal{Q})p \cong \mathcal{Q}$.*

   (ii) *For every nonzero projection p in $M_n(\mathcal{Q}_\omega)$, there exists a $*$-isomorphism $pM_n(\mathcal{Q}_\omega)p \cong \mathcal{Q}_\omega$.*

   (iii) *For every nonzero projection p in $\mathcal{Q}_\omega$ satisfying $\tau_\omega(p) > 0$, there exists some k in $\mathbb{N}$ and $*$-monomorphism $\mathcal{Q}_\omega \hookrightarrow p\mathcal{Q}_\omega p \otimes \mathbb{M}_k$ subject to $pap \mapsto pap \otimes e_{11}$ for all a in $\mathcal{Q}$.*

Proof. (i): According to the classification theorem of UHF-algebras, it suffices to show that $\mathrm{M}_n(\mathcal{Q})$ and $p\mathrm{M}_n(\mathcal{Q})p$ admit isomorphic invariants, since hereditary subalgebras of UHF-algebras are automatically UHF-algebras themselves. In symbols, we must establish the existence of a unit-preserving group isomorphism

$$\mu\colon (\mathrm{K}_0(\mathcal{Q}), [1_{\mathcal{Q}}]_0) \xrightarrow{\cong} (\mathrm{K}_0(p\mathrm{M}_n(\mathcal{Q})p), [p]_0). \tag{3.4}$$

The K_0-group of $\mathcal{Q}$ is precisely $\mathbb{Q}$. Invoking theorem 1.5.3, one obtains some $*$-isomorphism of stabilizations $p\mathrm{M}_n(\mathcal{Q})p \otimes \mathbb{K} \cong \mathrm{M}_n(\mathcal{Q}) \otimes \mathbb{K}$, applicable upon UHF-algebras being simple (projections are then automatically full). Thus

$$\mathrm{K}_0(p\mathrm{M}_n(\mathcal{Q})p) \cong \mathrm{K}_0(p\mathrm{M}_n(\mathcal{Q})p \otimes \mathbb{K}) \cong \mathrm{K}_0(\mathrm{M}_n(\mathcal{Q}) \otimes \mathbb{K}) \cong \mathrm{K}_0(\mathcal{Q}) \cong \mathbb{Q}$$

Note that stability of K_0 was employed during the first and third identification. Since the zeroth K-group of both the corner $p\mathrm{M}_n(\mathcal{Q})p$ and $\mathrm{M}_n(\mathcal{Q})$ are copies of $\mathbb{Q}$, the unit $[q]_0$ may identified with some nonzero rational number x. Define $\mu\colon \mathbb{Q} \longrightarrow \mathbb{Q}$ by mapping $1 \mapsto x$ to acquire (3.4).

(ii): Let p be some nonzero projection in $\mathrm{M}_k(\mathcal{Q}_\omega)$. Using proposition 2.3.3 lift p to some projection $(p_1, p_2, \ldots)$ inside $\ell^\infty(\mathcal{Q})$. According to (i), there exists an isomorphism $\pi_n\colon \mathrm{M}_k(\mathcal{Q}) \longrightarrow p_n\mathrm{M}_k(\mathcal{Q})p_n$ for each n in $\mathbb{N}$. The induced map

$$\pi\colon \mathrm{M}_k(\mathcal{Q}_\omega) \cong \prod_\omega \mathrm{M}_k(\mathcal{Q}) \longrightarrow \prod_\omega (p_n\mathcal{Q}p_n \otimes \mathbb{M}_k) \cong \mathbb{M}_k(p\mathcal{Q}_\omega p)$$

does the job. The identifications stem from matrix algebras commuting with ultraproducts.

(iii): We will initially deduce a claim that solves the embedding issue in $\mathcal{Q}$, whereafter we pass the embedding to the corresponding ultrapowers.

Claim. For each nonzero projection p in $\mathcal{Q}$ for which $\tau_{\mathcal{Q}}(p)m \leq 1$ holds for some positive integer m, there exists a $*$-monomorphism $\pi_n\colon \mathcal{Q} \longrightarrow p\mathcal{Q}p \otimes \mathbb{M}_m$ such that $pap \mapsto pap \otimes e_{11}$ for each element a in $\mathcal{Q}$, where e_{11} denotes the $(1,1)$'th unit matrix in $\mathbb{M}_m$.

Proof of claim. The projection $1 - p$ is orthogonal to $p_{11} := p$ and $\tau_{\mathcal{Q}}(1 - p) \geq \tau_{\mathcal{Q}}(p)$. According to proposition 3.2.1(i) we may infer that $p_{11} \sim p_{22} \leq 1 - p_{11}$ for a projection p_{22}. Iterating the argument, we may choose a finite collection $\{p_{ii}\}_{i=1}^m$ of pairwise orthogonal projection such that $\sum_{i=1}^m p_{ii} = 1_{\mathcal{Q}}$, $p_{11} = p$ and $p_{ii} \sim p_{jj}$ for all indices. Let v_{ij} be any partial isometry for each pair of indices $i, j \leq m$ witnessing the equivalence $p_{ii} \sim p_{jj}$. Extend $\{p_{ii}\}_{i=1}^m$ to matrix units $\{u_{ij}\}_{i,j=1}^m$ via the formulas

$$v_j^* v_j = p_{11}, \quad v_j v_j^* = p_{jj}, \quad u_{1j} := v_j^* \quad \text{and} \quad u_{ij} = (u_{1i})^* u_{1j} = v_i v_j^*.$$

The verifications of this turning into a collection of matrix units are straightforward, although cumbersome. Defining $\pi\colon \mathcal{Q} \longrightarrow p\mathcal{Q}p \otimes \mathbb{M}_m$ by

$$\pi(a) = \sum_{i,j=1}^m u_{1i} a u_{j1} \otimes e_{ij}$$

yields a $*$-monomorphism fulfilling $\pi(pap) = pap \otimes e_{11}$, because

$$\begin{aligned}
\pi(pap) &= \sum_{i,j=1}^m u_{1i}(u_{11} a u_{11}) u_{j1} \otimes e_{ij} \\
&= \sum_{i=1}^m u_{1i} u_{11} a u_{11} \otimes e_{jj} \\
&= pap \otimes e_{11}
\end{aligned}$$

Injectivity of π may be verified through similar, albeit tedious, calculations. We omit these to stay on track and consider the claim proven.

Suppose now p denotes a projection in $\mathcal{Q}_\omega$ with $0 < z < \tau_\omega(p)$. Lift p some projection $(p_1, p_2, \ldots)$ in $\ell^\infty(\mathcal{Q})$ via proposition 2.3.3 such that $\tau_\mathcal{Q}(p_n) \leq 1/k$ for some positive integer k and every positive integer n. Apply the claim to each projection p_n to produce $*$-monomorphisms $\pi_n \colon \mathcal{Q} \hookrightarrow p_n \mathcal{Q} p_n \otimes \mathbb{M}_m$ such that $p_n a p_n \mapsto p_n a p_n \otimes e_{11}$. The induced $*$-monomorphism

$$\pi \colon \mathcal{Q}_\omega \longrightarrow \prod_\omega (p_n \mathcal{Q} p_n \otimes \mathbb{M}_k) \cong p \mathcal{Q}_\omega p \otimes \mathbb{M}_m$$

satisfies $\pi(pap) = \varrho_\omega((p_n a_n p_n \otimes e_{11})_{n \geq 1}) = pap \otimes e_{11}$ for any a in $\mathcal{Q}_\omega$ having $(a_1, a_2, \ldots)$ as lift. This verifies (ii) and completes the proof. $\qquad\square$

Using proposition 3.2.2(ii) together with the Choi-Effros lifting theorem, we may deduce the sought characterization of quasidiagonality of traces, encapsulated purely in terms of $\mathcal{Q}_\omega$. The statements somewhat please ones intuition behind ultrapowers and their utility — they conceptually translate approximation properties into exact ones. Indeed, since quasidiagonality may be phrased in terms of morphisms that "asymptotically" approaches a $*$-monomorphism, one should expect quasidiagonality to induce embeddings into $\mathcal{Q}_\omega$.

Proposition 3.2.3. *Suppose A denotes a unital separable nuclear C^*-algebra admitting a trace τ. Then the following are equivalent.*

(i) τ is quasidiagonal.

(ii) There exists a unital $*$-homomorphism $\pi \colon A \longrightarrow \mathcal{Q}_\omega$ such that $\tau_\omega \circ \pi = \tau$.

(iii) There exists some $t \in (0, 1]$ such that for every finite subset $F \subseteq A$ and tolerance $\varepsilon > 0$, one may find a completely positive map $\psi \colon A \longrightarrow \mathcal{Q}_\omega$ satisfying

$$\psi(ab) \approx_\varepsilon \psi(a)\psi(b) \quad and \quad (\tau_\omega \circ \psi)(a) \approx_\varepsilon t\tau(a) \tag{3.5}$$

whenever $a, b \in F$. In particular, A is quasidiagonal whenever τ is an existing faithful quasidiagonal trace acting on A.

Proof. (i) $\Rightarrow$ (ii): Suppose $(\psi_n)_{n \geq 1}$ is the sequence of unital completely positive $\psi_n \colon A \longrightarrow \mathbb{M}_{k(n)}$ detecting quasidiagonality of τ. Regard these maps as attaining values in $\mathcal{Q}$. Let now $\psi_\omega \colon A \longrightarrow \mathcal{Q}_\omega$ be the map induced via the sequence $(\psi_n)_{n \geq 1}$. Due to ordinary sequential convergence entailing convergence along any free ultrafilter, one may infer that φ_ω becomes a unital $*$-homomorphism via asymptotic multiplicativity. The final condition stems from

$$(\tau_\omega \circ \psi_\omega)(a) = \lim_\omega (\tau_{k(n)} \circ \psi_n)(a) = \tau(a)$$

being true. This verifies the implication.

(iii) $\Rightarrow$ (ii): The proof relies on an application of Kirchberg's ε test. Assume that such a constant t satisfying (iii) exists. By separability of A, choose some countable dense subset $\{a_1, a_2, \ldots\}$ in A and write $F_k = \{a_1, \ldots a_k\}$ for each positive integer k. In the language of Kirchberg's ε-test, let X_n denote the set of contractive completely maps $\psi \colon A \longrightarrow \mathcal{Q}$ for each n in $\mathbb{N}$. Define, for every k in $\mathbb{N}$, maps $f_n^k \colon X_n \longrightarrow \mathbb{R}^+$ by the expressions

$$f_n^1(\psi) = \|\tau_\omega \circ \psi - t\tau\| \quad and \quad f_n^k(\psi) = \max_{i,j \leq k} \|\psi(a_i a_j) - \psi(a_i)\psi(a_j)\|, \; k \geq 2.$$

Let a positive integer m together with some tolerance $\varepsilon > 0$ be chosen. By hypothesis, we may find a completely positive map $\psi \colon A \longrightarrow \mathcal{Q}_\omega$ such that (3.5) holds on F_k for all $k \leq m$. The Choi-Effros

lifting theorem provides a completely positive lift $\psi_\infty = (\psi_1, \psi_2, \ldots)$ of ψ for each n, meaning each map ψ_n constitutes a completely positive map from A into $\mathcal{Q}$. Furthermore,

$$
\begin{aligned}
f_\omega^k(\psi_1, \psi_2, \ldots) &= \lim_{n \to \omega} \left(\max_{i,j \leq k} \| \psi_n(a_i a_j) - \psi_n(a_i)\psi_n(a_j) \| \right) \\
&= \max_{i,j \leq k} \| \varrho_\omega[(\psi_n(a_i a_j) - \psi_n(a_i)\psi_n(a_j))_{n \geq 1}] \| \\
&= \max_{i,j \leq k} \| \psi(a_i a_j) - \psi(a_i)\psi(a_j) \| \overset{(3.5)}{<} \varepsilon
\end{aligned}
$$

and $f_\omega^1(\psi_1, \psi_2, \ldots) = 0$ are valid whenever $2 \leq k \leq m$. According to Kirchberg's ε-test, there exists some sequence $(\pi_1, \pi_2, \ldots)$ consisting of contractive completely positive maps $\pi_n \colon A \longrightarrow \mathcal{Q}$ fulfilling $f_\omega^k(\pi_1, \pi_2, \ldots) = 0$ for each k in $\mathbb{N}$. The map $\pi \colon A \longrightarrow \mathcal{Q}_\omega$ induced by the sequence $(\pi_1, \pi_2, \ldots)$ is a $*$-homomorphism such that $\tau_\omega \circ \pi = t\tau$ on the dense subspace $(a_1, a_2, \ldots)$, thereof on A.

The map π almost serves the purpose of the designated morphism in (ii). We argue that t may be neglected by passing to the corner arising from $\pi(1_A)$. The projection $p = \pi(1_A)$ must be nonzero as $\tau_\omega(p) = t > 0$. Upon proposition 3.2.2 granting $p\mathcal{Q}_\omega p \cong \mathcal{Q}_\omega$, we may regard π as being a unital $*$-homomorphism into $B := p\mathcal{Q}_\omega p$. Uniqueness of trace ensures that $\tau_B \circ \pi = \lambda(\tau_\omega \circ \pi)$ for some positive real value λ, where τ_B denotes the trace on B. We therefore have $1 = \tau_B(p) = \lambda \tau_\omega(p) = t\lambda$. Rearranging this implies $\lambda = 1/t$, whereupon $\tau_B \circ \pi = (\tau_\omega \circ \pi)t^{-1} = \tau$ holds, proving (ii).

(ii) $\Rightarrow$ (i): By hypothesis, there exists some unital $*$-homomorphism $\pi \colon A \longrightarrow \mathcal{Q}_\omega$ subject to $\tau_\omega \circ \pi = \tau$. Due to nuclearity, π admits a completely positive lift $\psi = (\psi_1, \psi_2, \ldots) \colon A \longrightarrow \ell^\infty(\mathcal{Q})$. Suppose $E_n \colon \mathcal{Q} \longrightarrow \mathbb{M}_n$ denotes the canonical conditional expectation on $\mathcal{Q}$ for each n. The composed map $E_n \circ \psi_n \colon A \longrightarrow \mathbb{M}_n$ remains completely positive. Letting $(\varphi_n)_{n \geq 1}$ be the sequence of completely positive maps arising hereby grants a sequence implementing quasidiagonality of τ by the following reasoning. Let $\iota_n \colon \mathbb{M}_n \hookrightarrow \mathcal{Q}$ be the $*$-monomorphism attached to (3.3). Then

$$
\begin{aligned}
\lim_\omega \| \varphi_n(ab) - \varphi_n(a)\varphi_n(b) \| &= \lim_\omega \| \iota[E_n\psi_n(ab) - E_n\psi_n(a)E_n\psi(b)] \| \\
&= \lim_\omega \| \psi_n(ab) - \psi_n(a)\psi_n(b) \| \\
&= \| \pi(ab) - \pi(a)\pi(b) \| = 0.
\end{aligned}
$$

Here the third equality is based on ψ lifting π. The verification of $\tau_n \circ \psi_n = \tau$ is accomplished in an analogues manner, exploiting that E_n in (3.3) recovers the trace, and has been omitted for brevity.

Concerning the remaining claim, notice that the unital $*$-homomorphism $\pi \colon A \longrightarrow \mathcal{Q}_\omega$ arising from quasidiagonality of τ must be faithful whenever τ is, for indeed $\tau_\omega \circ \pi = \tau$ demands this. Since any unital nuclear separable C*-algebra is quasidiagonal if and only if the existence of a unital $*$-monomorphism $A \hookrightarrow \mathcal{Q}_\omega$ is assured, see for instance *theorem* 5.1.6 in [30] for a proof, the assertion follows immediately. This finalizes the proof. $\qquad\square$

For the proof of the main theorem, we conjure two $*$-homomorphisms (π_0, π_1) on cones of A which recover the trace τ on A to some extend. In an effort to spur some overview, we take a step backwards. During chapter 2, the connections illustrated via the figure beneath have been accessed. Let some separable C*-algebra A admitting a trace τ be given.
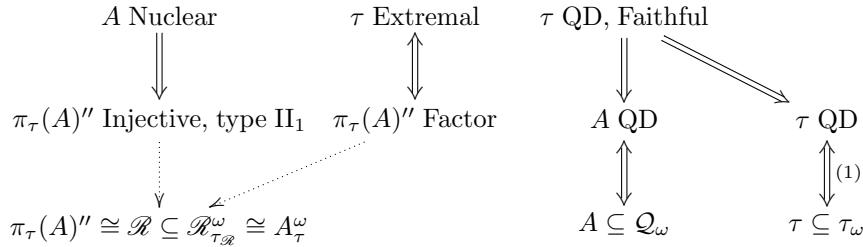
Here $\tau \subseteq \tau_\omega$ symbolically represents the existence of a $*$-homomorphism $\pi \colon A \hookrightarrow \mathcal{Q}_\omega$ such that $\tau_\omega \circ \pi = \tau$, while dashed arrows indicate that both combine into an implication. Up to technical

adjustments, we will provide quasidiagonality of our trace via (1). To achieve this, one constructs an order zero map $\psi\colon A \longrightarrow \mathcal{Q}_\omega$ that, in the language of [39], has "tracial large order", meaning $\tau_\omega \circ \psi$ recovers τ even when adding $\psi(1_A)^n$ to the input. Thus one remembers the trace τ.

The left-hand side of the display permits one to acquire ψ; using Krein-Milman's theorem one may pass to the extremal case, in which case the von Neumann algebra associated to the traces on $A, \mathcal{Q}$ both become copies of $\mathcal{R}$. Passing to the ultrapowers, we may employ Choi-Effros' lifting theorem to produce a unital completely positive map from A into $\mathcal{Q}_\omega$, which essentially speaking works once we twist it using the σ-ideal property of $J_{\tau,\omega}$, partly due to $\mathcal{R}$ being monotracial. This is the overall plan, so without further ado, let us dive into the gory details.

Proposition 3.2.4. *Let A be any separable unital nuclear C*-algebra admitting a trace τ. Under these premises, there exists an order zero map $\psi\colon A \longrightarrow \mathcal{Q}_\omega$ fulfilling*

$$\tau_\omega(\psi(a)\psi(1_A)^{n-1}) = \tau(a) \tag{3.6}$$

for every a in A and each positive integer n.

Proof. We initially consider the case in which τ is extremal. Afterwards, we extend to the general case using the Krein-Milman theorem to the weak*-compact convex space $T(A)$. Establishing the claim in the extremal case amounts to deriving *proposition* 3.2 in [39].

Claim. Suppose A, B denote separable infinite-dimensional nuclear C*-algebras admitting extremal traces τ_A, τ_B, respectively. Then there exists a contractive order zero map $\psi_0\colon A \longrightarrow B_\omega$ for which $1_{B_\omega} - \psi_0(1_A)$ belongs to the ideal $J_{\tau_B,\omega}$ and $\tau_{B_\omega} \circ \psi_0 = \tau_A$.

Proof of claim. Due to both traces being extremal, the von Neumann algebras $\mathcal{N}_A := \pi_{\tau_A}(A)''$ and $\mathcal{N}_B := \pi_{\tau_B}(B)''$ must be finite factors according to corollary 2.1.7, hence type II$_1$ factors by infinite-dimensionality. These von Neumann algebras are furthermore injective by corollary 2.2.6. Here $\pi_{\tau_A}\colon A \longrightarrow \mathrm{B}(\mathcal{H}_{\tau_A})$ naturally denotes the GNS-representation associated to τ_A and π_{τ_B} denotes the corresponding one for τ_B. As such they both become isomorphic copies of the injective type II$_1$ factor $\mathcal{R}$ according to Connes' uniqueness theorem. Let τ be the unique faithful trace on $\mathcal{R}$. Based on proposition 2.4.4, these identifications provide commutative diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\varrho_{\tau_A}} & \mathcal{R} \\
\scriptstyle \pi_A^\omega \downarrow & & \downarrow \scriptstyle \varrho_{\mathcal{R}}^\omega \\
A_{\tau_A}^\omega & \xrightarrow{\simeq} & \mathcal{R}_\tau^\omega
\end{array}
\qquad
\begin{array}{ccc}
B & \xrightarrow{\pi_{\tau_B}} & \mathcal{R} \\
\scriptstyle \varrho_B^\omega \downarrow & & \downarrow \scriptstyle \varrho_{\mathcal{R}}^\omega \\
B_{\tau_B}^\omega & \xrightarrow{\simeq} & \mathcal{R}_\tau^\omega
\end{array}
$$

such that $A_{\tau_A}^\omega \cong \mathcal{R}_\tau^\omega \cong B_{\tau_B}^\omega$. We may embed A into $B_{\tau_B}^\omega$ as follows. Let $\Lambda\colon A_{\tau_A}^\omega \longrightarrow B_{\tau_B}^\omega$ denote the acquired *-isomorphism. Due to A lying faithfully inside $A_{\tau_A}^\omega$ via the *-monomorphism δ_ω, composing with Λ yields another *-monomorphism $\Lambda_0\colon A \hookrightarrow B_{\tau_B}^\omega$. The Choi-Effros lifting theorem guarantees the existence of a unital completely positive lift $\psi_0\colon A \longrightarrow B_\omega$, meaning $\varrho_\omega^B \circ \psi_0 = \Lambda_0$. The scenario is displayed in the commutative diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\varrho_{\tau_A}^\omega \delta_\omega} & A_{\tau_A}^\omega & \xrightarrow{\Lambda} & B_{\tau_B}^\omega \\
 & \searrow_{\psi_0} & & & \uparrow \scriptstyle \varrho_\omega^B \\
 & & & \searrow & B_\omega
\end{array}
$$

It was previously deduced that $J_{\tau_B,\omega}$ defines a σ-ideal inside B_ω, see proposition 2.4.5. Invoking this property, we may relatively easily produce the required morphism via ψ_0. Choose some positive contraction e in $J_{\tau_B,\omega} \cap C'$ subject to $ec = c$ whenever c belongs to $C \cap J_{\tau_B,\omega}$ with C being the separable C*-subalgebra generated by $\psi_0(A)$. Define accordingly a bounded linear map $\psi\colon A \longrightarrow B_\omega$ by declaring that

$$\psi(\cdot) := (1_{B_\omega} - e)\psi_0(\cdot)(1_{B_\omega} - e).$$

The map ψ becomes a contractive completely positive map being the conjugation of such a map by positive contractions. The point of using e revolves around it commuting with the image of ψ_0; ψ_0 must be of order zero because if $ab = 0$ in A, then $\psi_0(a)\psi_0(b)$ belongs to $\ker \varrho_\omega^B = J_{\tau_B,\omega}$ by commutativity of the second diagram. Since e subsumes the role of a unit on $J_{\tau_B,\omega} \cap C$, the element $(1_{B_\omega} - e)\psi_0(a)\psi_0(b)$ must be zero. Due to e commuting with the image of ψ_0, it follows that

$$\psi(a)\psi(b) = (1_{B_\omega} - e)\psi_0(a)(1_{B_\omega} - e)^2\psi_0(b)(1_{B_\omega} - e)$$
$$= (1_{B_\omega} - e)^2\psi_0(a)\psi_0(b)(1_{B_\omega} - e)^2 = 0$$

Moreover, one may infer that

$$1_{B_\omega} - \psi_0(1_A) = 1_{B_\omega} - (1_{B_\omega} - e)^2 1_{B_\omega} = 2e - e^2 \in J_{\tau_B,\omega}.$$

As $\mathscr{R}$ admits a unique faithful trace, one obtains $\tau_{B_\omega} \circ \Lambda = \tau_{A_\omega}$, where τ_{A_ω} denotes the canonical trace on $A_{\tau_A}^\omega$ induced by τ_A. Thus, for each a belonging to A one has

$$\tau_{B_\omega}(\psi(a)) = \tau_{B_\omega}(\psi_0(a)) = \tau_{B_\omega}(\Lambda\delta_\omega(a)) = \tau_{A_\omega}(\delta_\omega(a)) = \tau(a).$$

This proves the claim.

Returning to our proof. Let $(\pi_\tau, \mathcal{H}_\tau, \xi_\tau)$ denote the GNS-triple associated to τ. If the trace τ is extremal and the corresponding von Neumann algebra $\mathscr{N} = \pi_\tau(A)''$ determines a II_1 factor, then the claim establishes an order zero map $\psi\colon A \longrightarrow \mathcal{Q}_\omega$ satisfying the conditions $\tau_\omega \circ \psi = \tau$ and $1_{\mathcal{Q}_\omega} - \psi_0(1_A) \in J_{\tau_{\mathcal{Q}},\omega}$. The first condition immediately entails the relation (3.6) for $n = 1$. When n exceeds 1, one applies the ordinary Cauchy-Schwarz inequality to the inner-product induced via the faithful trace $\tau_{\mathcal{Q}_\omega}$ to acquire the estimate

$$|\tau_{\mathcal{Q}_\omega}[x(1_{\mathcal{Q}_\omega} - \psi(1_A))]| \leq \tau_{\mathcal{Q}_\omega}(x^*x)^{1/2}\tau_{\mathcal{Q}_\omega}([1_{\mathcal{Q}_\omega} - \psi(1_A)]^2)^{1/2} = 0,$$

for every x belonging to $\mathcal{Q}_\omega$. Substituting $x = \psi(a)\psi(1_A)^{n-2}$ and exploiting that $\tau_\omega \circ \psi = \tau$ in the above provides (3.6) via a straightforward induction argument, which we omit for brevity.

On the other hand, if $\mathscr{N}$ is of type I_n for some n in $\mathbb{N}$, then I_n admits a unique faithful trace τ_n. Moreover, it embeds unitally into $\mathcal{Q}_\omega$ while preserving the trace[1], say via the $*$-monomorphism $\sigma\colon \mathscr{N} \hookrightarrow \mathcal{Q}_\omega$. Since τ is faithful, the representation π_τ unitally embeds A into $\mathscr{N}$ while making

$$\tau_{\mathscr{N}}(\pi_\tau(\cdot)) = \tau(\cdot)$$

hold. Composing these $*$-monomorphisms yields a unital $*$-monomorphism (hence an order zero map) $\psi\colon A \hookrightarrow \mathcal{Q}_\omega$ such that

$$\tau_{\mathcal{Q}_\omega}\psi = (\tau_{\mathcal{Q}_\omega}\sigma)\pi_\tau = \tau_{\mathscr{N}}\pi_\tau = \tau.$$

In conclusion, ψ satisfies (3.6) in the event of $\mathscr{N}$ being of type I_n. Due to $\mathscr{N}$ being a finite factor, it must be either of type I_n for some positive integer n or of type II_1. Regardless of the outcome, the asserted order zero map exists when τ is extremal.

For a general faithful trace τ, we combine a Krein-Milman convexity argument with Kirchberg's ε-test. To apply the ε-test, some setup needs to be settled. For every positive integer n, let X_n denote the collection of $*$-linear maps from A into $\mathcal{Q}$. According to lemma 2.3.6, there exists a countable collection of maps $f_n^k\colon X_n \longrightarrow \mathbb{R}^+$ such that a tuple $(\psi_1, \psi_2, \ldots)$ in $\prod_{n\in\mathbb{N}} X_n$ induces a contractive order zero map $\psi_\omega\colon A \longrightarrow \mathcal{Q}_\omega$ if and only if the associated testing functions $f_\omega^k\colon \prod_{n\in\mathbb{N}} X_n \longrightarrow [0, \infty]$ fulfill $f_\omega^k(\psi_1, \psi_2, \ldots) = 0$ for all positive integers k.

The principle behind invoking Kirchberg's test is merely to add testing functions that encode the property (3.6) as follows. By separability, choose some countable norm-dense subset $\{a_1, a_2, \ldots\}$ of A and define functions $g_n^{k,\ell}\colon X_n \longrightarrow [0, \infty]$, indexed over the natural numbers again, by

$$g_n^{k,\ell}(\psi_n) = \left|\tau_{\mathcal{Q}}(\psi_n(a_k)\psi_n(1_A)^{(\ell-1)}) - \tau(a_k)\right|.$$

---

[1]Elaboration upon, one appeals to the embedding that induces the conditional expectations E_n of (3.3).

By density, an element $(\psi_1, \psi_2, \ldots)$ in $\prod_{n \in \mathbb{N}} X_n$ induces a contractive order zero map $\psi \colon A \longrightarrow \mathcal{Q}_\omega$, extended from the dense subset, such that (3.6) holds for ψ if and only if

$$f_\omega^k(\psi_1, \psi_2, \ldots) = g_\omega^{k,\ell}(\psi_1, \psi_2, \ldots) = 0 \quad \text{for all } k, \ell \in \mathbb{N}. \tag{3.7}$$

Now the Krein-Milman theorem arrives to our aid. Fix some m in $\mathbb{N}$. Suppose some tolerance $\varepsilon > 0$ is given. By convexity and compactness of $T(A)$, we may find positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with $\sum_{k=1}^m \lambda_k = 1$ in conjunction with extremal traces $\tau_1, \tau_2, \ldots, \tau_m$ acting on A that together satisfy

$$\left| \sum_{k=1}^m \lambda_k \tau_k(a_i) - \tau(a_i) \right| < \varepsilon \tag{3.8}$$

for every positive integer $i \le k$. Choose a finite collection $p_1, \ldots, p_m$ consisting of pairwise orthogonal projections with $\tau_\omega(p_k) = \lambda_k$ for each $k \le m$. Due to proposition 3.2.2, the corners $B_k = p_k \mathcal{Q}_\omega p_k$ are isomorphic to $\mathcal{Q}_\omega$. Orthogonality is of course arranged to provide the order zero condition. From the extremal cases, there are contractive order zero maps $\psi_k \colon A \longrightarrow p_k \mathcal{Q}_\omega p_k$ satisfying

$$\tau_{B_k}(\psi_k(a)\psi_k(1_A)^{\ell-1}) = \tau_k(a) \tag{3.9}$$

for all positive integers ℓ and any element a in A. Since the projections are orthogonal, we merely define $\psi \colon A \longrightarrow \mathcal{Q}_\omega$ by the assignment $a \mapsto \sum_{k=1}^m \psi_k(a)$ to obtain a contractive completely positive map; it is the pointwise sum of contractive completely positive maps. Suppose $a \perp b$ inside A. Due to orthogonality of the projections, one may deduce that

$$\psi(a)\psi(b) = \sum_{i,j=1}^m \psi_i(a)\psi_j(b) = \sum_{i=1}^m \psi_i(a)\psi_i(b) = 0.$$

The remaining conditions for orthogonality preservation are verified in resembling manners. The latter equality stems from ψ_i being of order zero itself. It follows that ψ must be a contractive order zero map. Furthermore, for each positive integer ℓ one has

$$\tau_{\mathcal{Q}_\omega}(\psi(a)\psi(1_A)^{\ell-1}) = \sum_{k=1}^m \lambda_k \tau_{B_k}(\psi_k(a)\psi_k(1_A)^{\ell-1}) \overset{(3.9)}{=} \sum_{k=1}^m \lambda_k \tau_k(a)$$

for every a in A. Combining this particular identity with (3.8) yields $g_\omega^{k,\ell}(\psi_1, \psi_2, \ldots) < \varepsilon$ for every pair of integers $k, \ell \in \mathbb{N}$ not exceeding m. As $\varepsilon > 0$ was arbitrarily chosen, Kirchberg's ε-test ensures the existence of an element $(\psi_1, \psi_2, \ldots)$ in the product $\prod_{n \in \mathbb{N}} X_n$ fulfilling (3.7). On the merits of the remarks attached to (3.7), the induced map $\psi \colon A \longrightarrow \mathcal{Q}_\omega$ determines a contractive order zero map satisfying the property (3.6), completing the proof. $\square$

3.3 Comparison Theory

Having established an *order zero* map recovering the trace τ through τ_ω, our next step will be pass the property of the corresponding $*$-homomorphism on $C_0(0,1] \otimes A$ in an attempt to include multiplicativity. To fully encode multiplicativity one forms another morphism on $C_0[0,1) \otimes A$. Achieving this will be done by assembling unitaries witnessing the second $*$-homomorphism through the other. Allowing such unitaries to exist relies on strict comparison. The purpose of the section will be to establish the necessary theory to introduce strict comparison and verify the property for $\mathcal{Q}$.

The Cuntz Semigroup

Strict comparison relies on the Cuntz semigroup and dimension functions, concepts which are imperative to introduce with some preliminary setting established beforehand; ordered abelian semigroups.

Definition. Let S be an abelian semigroup. We call S *partially ordered* if it admits a partial order $\leq$ compatible with the additive structure on S. The aforementioned compatibility amounts to $s + t \leq s_0 + t_0$ whenever $s \leq s_0$ together with $t \leq t_0$ hold inside S.

An alternative point of view is regarding partially ordered semigroups as pairs (S, S^+) with $S^+ \subseteq S$ being some additively closed subset containing 0, whereon one stipulates that $s \geq 0$ if and only if $s \in S^+$. Thus one declares that $s \leq t$ if and only if $t - s \geq 0$ to define a partial order on S granting the structure of a partially ordered semigroup. Furthermore:

- An element u in S is called an *order unit* provided there for each s in S exists some positive integer n fulfilling $s \leq n \cdot u$.

- A partially ordered abelian semigroup S is said to be *almost unperforated* if, for each pairs $k, k_0 \in \mathbb{N}$ and $s, t \in S$, the conditions $k_0 < k$ and $k \cdot s \leq k_0 \cdot t$ imply $s \leq t$.

- An additive map $\varphi \colon S \longrightarrow T$ between partially ordered abelian semigroups preserving the order, meaning $s \leq s_0$ within S entails $\varphi(s) \leq \varphi(s_0)$ in T, is called an *ordered morphism*. An ordered morphism $\varphi \colon S \longrightarrow (\mathbb{R}, +)$ is commonly referred to as a *state*. We denote the collection of states on a partially ordered abelian semigroup S by S^*.

The theory of partially ordered abelian semigroups is wealthy and particularly rewarding in K-theory. For the thesis, we shall study another intriguing example, called the Cuntz semigroup. The Cuntz semigroup partly attempts to generalize the comparison theory of Murray - von Neumann. Since projections may be in few numbers for general C*-algebras, one considers positive elements instead.

Definition. Let A be any C*-algebra. For two elements a, b in A, we call a *Cuntz subequivalent* to b should the existence of a sequence $(v_n)_{n \geq 1}$ in A fulfilling $v_n b v_n^* \to a$ be guaranteed. We symbolically write $a \lesssim b$ to denote this and write $a \sim_c b$ whenever both $a \lesssim b$ and $b \lesssim a$ are valid, in which case the elements are said to be *Cuntz equivalent* to one another.

Through some effort, it can shown that $\sim_c$ must be an equivalence relation, so that the following notion becomes meaningful.

Definition. The *Cuntz semigroup* associated to a C*-algebra A is the monoid

$$W(A) = \mathrm{M}_\infty(A)_+ / \sim_c$$

equipped with the composition induced from $\oplus$. We denote an equivalence class with respect to $\sim_c$ by $\langle a \rangle$, so that $\langle a \rangle + \langle b \rangle := \langle a \oplus b \rangle$ becomes the composition. We endow $W(A)$ with the ordering

$$\langle a \rangle \leq \langle b \rangle \quad \overset{\mathrm{def}}{\Longleftrightarrow} \quad a \lesssim b.$$

A few remarks linking the notions are in order. First of all, for a C*-algebra A, the Cuntz semigroup $W(A)$ evidently becomes a semigroup. Due to the enlarged matrix $a \oplus b$ associated to elements in $a, b \in \mathrm{M}_\infty(A)_+$ conjugating to $b \oplus a$ via the permutation matrix having the value 1 in the off-blockdiagonal parts and zeroes elsewhere (and vice versa), one has $a \oplus b \sim_c b \oplus a$. Thus, $\langle a \oplus b \rangle = \langle b \oplus a \rangle$ upon which $W(A)$ becomes a partially ordered abelian semigroup.

Furthermore, if A admits a unit, the corresponding class $\langle 1_A \rangle$ defines an order unit in $W(A)$. We present an outline of the proof. In the event of a denoting an element in $\mathrm{M}_n(A)_+$, one certainly has $a^{1/2} 1_{\mathrm{M}_n(A)} a^{1/2} = a$, hence $a \lesssim 1_{\mathrm{M}_n(A)}$. The unit of A embeds into the algebra $\mathrm{M}_n(A)$ via the corner

map $d_{n-1} \circ d_{n-2} \circ \ldots \circ d_1$. Conjugating 1_A by suitable permutation matrices afterwards ensures that $1_A \otimes e_{kk} \lesssim 1_A$ with e_{kk} denoting the (k,k)'th unit matrix in $\mathbb{M}_n$ as usual. Ergo,

$$a \lesssim 1_{\mathrm{M}_n(A)} = \sum_{k=1}^{n} 1_A \otimes e_{kk} \lesssim n \cdot 1_A.$$

We conclude that $\langle 1_A \rangle$ becomes an order unit for $W(A)$. For future purposes, it is convenient to convey some calculus for $W(A)$ and the ordering $\leq$ thereon. For each $\varepsilon > 0$ and positive element a in $\mathrm{M}_\infty(A)$ we define a function $f_\varepsilon \colon \sigma(a) \longrightarrow \mathbb{C}$ by

$$f_\varepsilon(t) = \begin{cases} 0, & \text{if } t \leq \varepsilon, \\ \frac{t-\varepsilon}{\varepsilon}, & \text{if } \varepsilon \leq t \leq 2\varepsilon, \\ 1, & \text{if } 2\varepsilon \leq t. \end{cases}$$

Moreover, we let $(a - \varepsilon)_+$ to be the functional calculus of a applied to the map $t \mapsto \max\{0, t - \varepsilon\}$. Adopting this notation, one may deduce the following rules:

- Given $\varepsilon, \delta > 0$ and positive element a one has $((a - \varepsilon)_+ - \delta)_+ = (a - (\varepsilon + \delta))_+$.

- For every $\varepsilon > 0$ and positive elements a one has $f_\varepsilon(a) \sim_c (a - \varepsilon)_+$.

- $f_\varepsilon(a) \to a$ as $\varepsilon \to 0$ for any positive element a.

- If supp $f \subseteq$ supp g occurs in $C_0(\mathbb{R}^+)_+$, then $f \lesssim g$ in $C_0(\mathbb{R}^+)$. Here supp f is a placeholder for the *open* support of the function f.

Prior to truly diving into the theory, some characterizations and intuition is supplied. We avoid proving every implication in the following fundamental proposition, the full proof of which may be recovered in [36]. However, we establish a frequently applied lemma.

Lemma 3.3.1. *Let a, b be positive elements in some unital C^*-algebra A and set $\delta = \|a - b\|$. Then $f_\varepsilon(a) \leq r^* b r$ and $(a - \varepsilon)_+ \leq e b e^*$ for some r, e in A whenever $\delta < \varepsilon$, for any $\varepsilon > 0$.*

Proof. Due to $a - b$ being positive, one has $a - b \leq \|a - b\| 1_A$. Rearranging this grants $a - \delta \leq b$, whereby some straightforward functional calculus grants

$$(\varepsilon - \delta) f_\varepsilon(a) \leq f_\varepsilon(a)^{1/2}(a - \delta) f_\varepsilon(a)^{1/2} \leq f_\varepsilon(a)^{1/2} b f_\varepsilon(a)^{1/2}.$$

Letting $r = (\varepsilon - \delta)^{-1/2} f_\varepsilon(a)^{1/2}$ provides $f_\varepsilon(a) \leq r^* b r$. The proof of $(a - \varepsilon)_+ \leq e b e^*$ is proven in a manner entirely identical to the previous part. $\square$

The following establishes the most frequently exploited characterizations of the relation $\lesssim$.

Proposition 3.3.2. *Suppose A denotes a unital C^*-algebra containing two positive elements a, b. Under these premises, the following conditions are equivalent.*

(i) $a \lesssim b$.

(ii) For every $\varepsilon > 0$, there exists some x in A such that $f_\varepsilon(a) \leq x b x^*$.

(iii) For every $\varepsilon > 0$, there exists some x in A such that $(a - \varepsilon)_+ \leq x b x^*$.

(iv) For every $\varepsilon > 0$, there exists some $\delta > 0$ and x in A such that $f_\varepsilon(a) = x f_\delta(b) x^*$.

(v) For every $\varepsilon > 0$, there exists some $\delta > 0$ and x in A such that $(a - \varepsilon)_+ = x(b - \delta)_+ x^*$.

Proof. We only concern ourselves with (i)$\Rightarrow$(ii) and (iv)$\Rightarrow$(ii), leaving the remaining implications to *proposition* 2.4 found in [36].

(i) $\Rightarrow$ (ii): Let $a \lesssim b$ be implemented by the sequence $(v_n)_{n \geq 1}$. Choose an integer n large enough to force $v_n b v_n^* \approx_{\varepsilon/2} a$, so that by letting $x_n := v_n b^{1/2}$ and $y_n := b^{1/2} v_n^*$ one has

$$\|x_n y_n y_n^* x_n^* - a^2\| = \|v_n b v_n^* v_n b v_n^* - a^2\| \leq \|v_n b v_n^* a - a^2\| + \|v_n b v_n^* a - v_n b v_n^* v_n b v_n^*\| < \varepsilon.$$

According to lemma 3.3.1 it follows that

$$f_\varepsilon(a^2) \leq r x_n y_n y_n^* x_n^* r^* \overset{(1.2)}{\leq} \|y_n\|^2 r x_n x_n^* r^*$$

for some element r inside A. Rearranging the above yields $f_\varepsilon(a^2) \leq z b z^*$, where one lets $z = \|y_n\| r x_n$. Due to the support of the function $t \mapsto f(t)$ equaling the support of $t \mapsto f(t)^2$ for any positive element f in $C_0(\mathbb{R}^+)$, one may deduce that $f_\varepsilon(a) \leq z_0 b z_0^*$ for some z_0 in A.

(iv)$\Rightarrow$(ii): Suppose $f_\varepsilon(a) = x f_\delta(b) x^*$ holds for some element x and $\varepsilon, \delta > 0$. Let $y = g(b)$ with g being the function $t \mapsto 1$. Since $f_\delta(b) \leq b = g(b) b g(b)^*$, one may infer that $f_\varepsilon(a) \leq xyb(xy)^*$. $\square$

The Cuntz equivalence may be regarded as a replacement for the ordinary comparison theory of von Neumann algebras in the C*-algebraic realm in the sense that $p \lesssim q$ for projections occurs if and only if $p = vv^*$ and $v^*v \leq q$ for some partial isometry v (see *proposition* 2.1 in [36]).

Dimension Functions and Strict Comparison

Understanding strict comparison calls for an understanding of dimension functions. We investigate these from a strict comparison point of view, although not in the greatest generality. After addressing the notion, we immediately derive some minor properties.

Definition. Suppose A denotes a unital C*-algebra. A state φ acting on $W(A)$ is called a *dimension function* provided that $\varphi(\langle 1_A \rangle) = 1$. A dimension function φ is *lower-semicontinuous* if

$$\varphi(\langle a \rangle) \leq \liminf_{n \to \infty} \varphi(\langle a_n \rangle)$$

whenever $a_n \to a$ occurs in norm on A. The set of dimension functions on A is denoted by $\mathcal{D}(A)$, while the subset consisting of lower-semicontinuous ones is written as $\mathcal{LD}(A)$. Additionally, we define a state $\overline{d}_\varphi$ via the formula

$$\overline{d}_\varphi(\langle a \rangle) = \lim_{\varepsilon \to 0} \varphi(\langle f_\varepsilon(a) \rangle). \tag{3.10}$$

We shall frequently refer to $\overline{d}_\varphi$ as the *dimension function induced by φ*.

Lemma 3.3.3. *Let A denote a unital C*-algebra A admitting a dimension function φ. Then $\overline{d}_\varphi$ exists, determines a member of $\mathcal{LD}(A)$ and obeys the rules found beneath.*

(i) One has $\overline{d}_\varphi(\cdot) \leq \varphi(\cdot)$.

(ii) One has $\overline{d}_\varphi(\langle f_\varepsilon(a) \rangle) \leq \varphi(\langle a \rangle)$ for every class $\langle a \rangle$ in $W(A)$.

Proof. For existence of $\overline{d}_\varphi$, it suffices to ensure that $\overline{d}_\varphi(\langle a \rangle) \leq \overline{d}_\varphi(\langle b \rangle)$ whenever $a \lesssim b$. To achieve this let some tolerance $\varepsilon > 0$ be given and suppose $a \lesssim b$ relative to A. Invoking proposition 3.3.2 twice, there exists some $\delta > 0$ for which $f_\varepsilon(a) \lesssim f_\delta(b)$. Order preservation of states implies

$$\varphi(\langle f_\varepsilon(a) \rangle) \leq \varphi(\langle f_\delta(b) \rangle) \leq \overline{d}_\varphi(\langle b \rangle).$$

The net $(f_\varepsilon(a))_{\varepsilon > 0}$ tends to a as $\varepsilon \to 0$. Therefore, upon taking the supremum on both sides above, we deduce that $\overline{d}_\varphi(\langle a \rangle) \leq \overline{d}_\varphi(\langle b \rangle)$ as was needed.

If $a_n \to a$ occurs in A and some $\varepsilon > 0$ is given, then lemma 3.3.1 yields the relation $f_{\varepsilon/2}(a_n) \lesssim a_n$ for some sufficiently large positive integer n. We may thereof find a $\delta_n > 0$ such that $f_\varepsilon(a) \lesssim f_{\delta_n}(a_n)$. Combining these observations guarantees that

$$\varphi(\langle f_\varepsilon(a)\rangle) \leq \varphi(\langle f_{\delta_n}(a_n)\rangle) \leq \overline{d}_\varphi(\langle a_n\rangle).$$

The dimension function $\overline{d}_\varphi$ is thus lower-semicontinuous, i.e, a member of $\mathcal{LD}(A)$.

(i)-(ii): For any tolerance $\varepsilon > 0$ one has $f_\varepsilon(a) \lesssim a$, hence $\varphi(\langle f_\varepsilon(a)\rangle) \leq \varphi(\langle a\rangle)$ holds for any element a belonging to A. Letting $\varepsilon \to 0$ reveals that φ dominates the induced dimension function $\overline{d}_\varphi$. The property (ii) trivially holds, completing the proof. $\square$

The interesting point of view, for the thesis at least, concerns the tracial case. Let A be a unital exact C*-algebra admitting a trace τ. The general theory generalizes to quasi-traces. However, due to the incredibly work of Haagerup in [21], quasitraces on exact C*-algebras are traces, hence the additional exactness assumption. Define $d_\tau \colon W(A) \longrightarrow \mathbb{R}^+$ by

$$d_\tau(\langle a\rangle) = \lim_{n\to\infty} (\tau \otimes \mathrm{Tr}_k)\big(a^{1/n}\big).$$

Here a denotes a representative of $\langle a\rangle$ in $\mathrm{M}_k(A)^+$. The map d_τ exists due to the following observation. For a positive contraction a in $\mathrm{M}_k(A)$, the sequence $\{(\tau \otimes \mathrm{Tr}_k)(a^{1/n})\}_{n\geq 1}$ becomes increasing and the limit exists, being bounded by k. Hence the sequence converges in norm. For a general element a in $\mathrm{M}_k(A)$, one exploits the contractive case on

$$\lim_{n\to\infty} (\tau \otimes \mathrm{Tr}_k)\big(a^{1/n}\big) = \lim_{n\to\infty} (\tau \otimes \mathrm{Tr}_k)\big((a\|a\|^{-1})^{1/n}\|a\|^{1/n}\big)$$
$$= \lim_{n\to\infty} (\tau \otimes \mathrm{Tr}_k)\big((a\|a\|^{-1})^{1/n}\big)$$

Altogether, the map d_τ exists. The map d_τ a priori appears unrelated to the topic at hand, so we characterize it in familiar terms. Consider the corresponding map $d_{\tau,\varepsilon} \colon W(A) \longrightarrow \mathbb{R}^+$ given by

$$\langle a\rangle \mapsto \lim_{\varepsilon\to 0} (\tau \otimes \mathrm{Tr}_k)(f_\varepsilon(a))$$

with a being some representative of $\langle a\rangle$ in $\mathrm{M}_k(A)^+$. We assert that it equals d_τ. For this let a belong to $\mathrm{M}_k(A)_+$. The C*-algebra B generated by a and 1_A becomes abelian, so $B \cong C(\Omega)$ for some compact Hausdorff topological space Ω. It therefore suffices to verify that d_τ and $d_{\tau,\varepsilon}$ agree hereon. However, the trace $\tau \otimes \mathrm{Tr}_k$ restricted to B corresponds to the functional $\gamma \colon B \longrightarrow \mathbb{R}^+$ defined as

$$\gamma(f) = \int_{[0,\|a\|]} f \, d\mu,$$

with μ being some regular Borel measure. The aforementioned correspondence stems from the Riesz-Markov-Kakutani representation theorem. In the respective scenarios, when applying the restricted functional on $a^{1/n}$ and $f_\varepsilon(a)$, one merely has to verify that the integrals

$$\lim_{n\to\infty} \int_{[0,\|a\|]} t^{1/n} \, d\mu(t) \quad \text{respectively,} \quad \lim_{\varepsilon\to 0} \int_{[0,\|a\|]} f_\varepsilon(t) \, d\mu(t)$$

agree. Dominated convergence entails that both integrals equal $\mu((0,\|a\|])$, since $f_\varepsilon(t) \to 1$ as $\varepsilon \to 0$ whereas $t^{1/n} \to 1$ whenever $n \to \infty$, for $t \neq 0$. As such no distinction between $d_{\tau,\varepsilon}$ and d_ε occurs on B, so d_τ indeed attains the form in (3.10). Here is the point:

Theorem 3.3.4 (Handelman-Goodearl, Haagerup)**.** *The assignment $\tau \mapsto d_\tau$ on an exact unital C*-algebra defines a bijection from $T(A)$ onto $\mathcal{LD}(A)$. In particular, the dimension function d_τ is the unique lower-semicontinuous dimension function on an exact unital monotracial C*-algebra.*

Proof. The correspondence was proven in *theorem* II.2.2 in [5]. $\square$

Lower-semicontinuous dimension functions are the core of strict comparison. We exhibit the notion and proceed afterwards towards a theorem due to Rørdam in [36], regarding strict comparison for tensor products with UHF-algebras.

Definition. Let A be a C*-algebra admitting a trace.

- A is said to have *strict comparison of positive elements*, provided that given any pair $a, b \in W(A)$ fulfilling $\varphi(a) < \varphi(b)$ for all lower-semicontinuous dimension functions φ one has $a \lesssim b$.

- A is said to have *strict comparison of positive elements with respect to bounded traces* if given any $k \in \mathbb{N}$ and $a, b \in \mathrm{M}_k(A)^+$ fulfilling $d_\tau(a) < d_\tau(b)$ for any trace τ on A one has $a \lesssim b$.

We lean on $\mathcal{Q}$ having strict comparison with respect to its unique bounded trace. This may be recaptured via Rørdam's theorem of [36] whose proof we avoid to remain on track. Rørdam's theorem reaches UHF-algebras, which we account for after stating it.

Theorem 3.3.5 (Rørdam)**.** *Suppose E denotes a simple unital C*-algebra, let B be a UHF-algebra and set $A = E \otimes B$. Under these premises, A has strict comparison of positive elements.*

Proof. See *theorem* 5.2 in [36] for a proof. □

Corollary 3.3.6. *UHF-algebras have strict comparison with respect to their unique trace.*

Proof. Let A be any UHF-algebra. Upon UHF-algebras being simple, unital and monotracial, Rørdam's theorem applies to $A \cong \mathbb{C} \otimes A$. Therefore A has strict comparison of positive elements with respect to their unique trace. According to the Handlemann-Goodearl-Haagerup theorem, the sole lower-semicontinuous dimension function of $W(A)$ is precisely d_τ, meaning A in this case attains strict comparison of positive elements with respect to its unique bounded trace as claimed. □

We close the section by passing strict comparison to ultrapowers and deriving a lemma for future use, thereby establishing strict comparison for $\mathcal{Q}_\omega$, leaving us with the tools to adequately construct our tracially larges *-homomorphisms from suitable order zero maps. To bypass confusion, for any ultraproduct $A = \prod_\omega A_n$ let

$$T_\omega(A) := \{\varrho_\omega(a_1, a_2, \dots) \mapsto \lim_\omega \tau(a_n) : \tau \in T(A)\} \subseteq T\left(\prod_\omega A_n\right)$$

and denote the weak*-closure of $T_\omega(A)$ by $\mathcal{L}T_\omega(A)$.

Proposition 3.3.7. *The following hold.*

(i) *Let $(A_n)_{n \geq 1}$ be a sequence of unital C*-algebras having strict comparison of positive elements with respect to bounded traces. Then $A = \prod_\omega A_n$ has strict comparison of positive elements with respect to traces in $\mathcal{L}T_\omega(A)$.*

(ii) *Ultrapowers of $\mathcal{Q}$ have strict comparison of positive elements with respects to its unique trace.*

Proof. (i): Suppose $a, b \in \mathbb{M}_k \otimes A$ are positive contractions subject to $d_\tau(a) < d_\tau(b)$ for any trace τ in $\mathcal{L}T_\omega(A)$. Fix some tolerance $\varepsilon > 0$ and fix a monotonically decreasing sequence $(\delta_n)_{n \geq 1}$ of strictly positive real numbers such that $\delta_n \to 0$ in $\mathbb{R}^+$. Define a family of point-evaluation maps $\psi_{\delta_n}, \psi_\varepsilon \colon \mathcal{L}T_\omega(A) \longrightarrow \mathbb{C}$ by declaring that

$$\psi_{\delta_n}(\rho) = \rho(f_{\delta_n}(b)) \quad \text{and} \quad \psi_\varepsilon(\rho) = \rho(f_\varepsilon(a)).$$

The corresponding sequence $(\psi_{\delta_n} - \psi_\varepsilon)_{n \geq 1}$ belongs to the weak*-compact space $\mathcal{L}T_\omega(A) \subseteq (A^*)_1$ according to Alaoglu's theorem. Due to $(\delta_n)_{n \geq 1}$ being monotonically decreasing, the acquired sequence $(\psi_{\delta_n} - \psi_\varepsilon)_{n \geq 1}$ must be a monotonically increasing sequence on a compact Hausdorff space converging

pointwise to $d_{(\cdot)}(b) - \psi_\varepsilon(\cdot)$, and Dini's theorem thereby ensures uniform convergence. Upon the limit being strictly positive there exists some positive integer m such that $\tau(f_{\delta_m}(b)) > \tau(f_\varepsilon(a))$ for every limit trace τ on A. Choose positive contractive lifts $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of a and b, respectively. One checks that

$$\lim_{n\to\omega}\left(\inf_{\tau\in T(A_n)}\tau(f_{\delta_m}(b_n) - f_\varepsilon(a_n))\right) = \inf_{\tau\in\mathcal{L}T_\omega(A)}\tau(f_{\delta_m}(b_n) - f_\varepsilon(a_n)) > 0.$$

The above forces the set

$$I = \{n \in \mathbb{N} : \tau(f_{\delta_m}(b_n)) < \tau(f_\varepsilon(a_n)) \text{ for all } \tau \in T(A_n)\}$$

to be a member of ω. As $(a_n - \varepsilon)_+ \lesssim f_\varepsilon(a_n)$ for each n in $\mathbb{N}$, one has $d_\tau((a_n - \varepsilon)_+) \leq d_\tau(f_\varepsilon(a_n))$ for all integers n in I. Using proposition 3.3.2, strict comparison with respect to bounded traces in A_n for each n supplies us with positive elements $(v_n)_{n\geq 1}$ in $\mathbb{M}_k \otimes A_n$ such that

$$v_n^* f_{\delta_m} v_n \approx_{1/n} (a_n - \varepsilon)_+, \quad n \in I.$$

Letting $w_n = f_{\delta_m}(b_n)^{1/2}v_n$ whenever n belongs to I and otherwise $w_n = 1_{A_n}$ defines a bounded sequence $(w_1, w_2, \ldots)$, whose image w under the quotient map attached to A satisfies

$$\|w^*w - (a - \varepsilon)_+\| = \limsup_\omega \|w_n^* w_n - (a_n - \varepsilon)_+\| = 0.$$

Formulated differently, $w^*w = (a - \varepsilon)_+$ and one then verifies that $f_{\delta_m/2}(b)w = w$. Combining these two observations yields $(a - \varepsilon)_+ \lesssim b$ for each tolerance $\varepsilon > 0$, ensuring $a \lesssim b$ as desired.

(ii): Immediate from uniqueness of trace on $\mathcal{Q}_\omega$ as was argued for in the mark on page 49. This relies heavily on the use of the Jiang-Su algebra and UHF-algebras absorbing it. $\square$

3.4 Conjuring Tracial Lebesgue Cones

The central purpose of the section seeks to employ the theory presented prior to current moment. The underlying strategy is essentially to build two *-homomorphisms (π_0, π_1) over cones with values in $\mathcal{Q}_\omega$, whose scalar parts are recovered in a third one θ on $C([0,1])$ while recovering τ_ω. The tracially large order zero map constructed during the first section of the chapter enables us to conjure *-homomorphisms on the cones, having the ability to detect the trace remain intact. Here strict comparison provides the second map as a unitary flip of the first.

First of foremost, the versatile tool arising from strict comparison in the stable rank one scenario. Recall that two *-homomorphisms $\pi, \varrho\colon A \longrightarrow B$ with B unital are *approximately unitarily equivalent* if there are unitaries $(u_n)_{n\geq 1}$ in B such that $u_n\pi(\cdot)u_n^* \to \varrho(\cdot)$ in norm. Approximate unitary equivalence forms an equivalence relation $\sim_u$. To aptly state the theorem, we must take the complete Cuntz semigroup into account, albeit we shall not dwell in the theory thereof the slightest. The *complete Cuntz semigroup* attached to some C*-algebra A is the partially ordered semigroup $\mathrm{Cu}(A) := (A \otimes \mathbb{K})/\sim_c$ endowed with a structure resembling the one for $W(A)$.

Theorem 3.4.1 (Ciuperca-Elliott). *Suppose A denotes some C*-algebra of stable rank one. Under this hypothesis, two *-homomorphisms $\pi, \varrho\colon C_0(0,1] \longrightarrow A$ become approximately unitarily equivalent if their induced morphisms*

$$\pi_{cu}, \varrho_{cu}\colon \mathrm{Cu}(C_0(0,1] \otimes \mathbb{K}) \longrightarrow \mathrm{Cu}(A \otimes \mathbb{K}); \quad \pi_{cu}(\langle a\rangle) = \langle\pi(a)\rangle,$$

and likewise for ϱ, agree. As a special case, two elements a, b in A become approximately unitarily equivalent provided that $f(a) \sim_c f(b)$ for every positive nonzero element f in $C_0(0,1]$.

Proof. The validity of the initial statement was achieved in [11] in the disguise of *theorem* 4.1. For the second assertion, apply the first part to the pair of *-homomorphisms $\pi, \varrho\colon C_0(0,1] \longrightarrow A$ defined on the generating element $\mathrm{id}_{(0,1]}$ in $C_0(0,1]$ by $\pi(\mathrm{id}_{(0,1]}) = a$ together with $\varrho(\mathrm{id}_{(0,1]}) = b$. $\square$

Lemma 3.4.2. *Suppose A denotes a unital C^*-algebra of stable rank one and with strict comparison of positive elements with respect to bounded traces.*

(i) *Let a, b be two positive contractions in A such that $\tau(f(a)) = \tau(f(b)) > 0$ for every trace τ on A and every nonzero positive element f in $C_0(0, 1]$. It follows that a and b are approximately unitarily equivalent.*

(ii) *Any pair of contractions a, b in $\mathcal{Q}_\omega$ satisfying $\tau_\omega(f(a)) = \tau_\omega(f(b)) > 0$ for each nonzero positive element f in $C(0, 1]$ are unitarily equivalent.*

Proof. (ii): The second assertion regarding $\mathcal{Q}_\omega$ basically stems from the first due to $\mathcal{Q}$ having strict comparison of positive elements with respect to its unique trace and having stable rank one; see corollary 3.3.6 along with appendix B. These observations imply approximate unitary equivalence of positive contractions a, b belonging to $\mathcal{Q}_\omega$, which we may upgrade into a bona fide unitary equivalence in the following manner.

Suppose a, b are approximately unitarily equivalent elements in $\mathcal{Q}_\omega$. Choose lifts $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ in $\ell^\infty(\mathcal{Q})$ of a and b, respectively. In the language of Kirchberg's ε-test, set $X_n = \mathcal{U}(\mathcal{Q})$ for every positive integer n. For each additional positive integer k, define functions $f_n^k \colon X_n \longrightarrow \mathbb{R}^+$ via the formula

$$f_n^k(w) = \|w a_n w^* - b_n\|.$$

Suppose an integer $m \geq 1$ together with tolerance $\varepsilon > 0$ is given. By hypothesis, there exists some unitary u in $\mathcal{Q}_\omega$ such that $uau^* \approx_\varepsilon b$. Lift the unitary u to some unitary $(u_1, u_2, \ldots)$ inside $\ell^\infty(\mathcal{Q})$ using proposition 2.3.3. Then we have

$$f_w^k(u) = \limsup_{n \to \omega} \|u_n a_n u_n^* - b_n\| = \|uau^* - b\| < \varepsilon.$$

Kirchberg's ε-test thus provides an element $w = (w_1, w_2, \ldots)$ in $\prod_{n=1}^\infty X_n$ such that $f_\omega^k(w) = 0$ for all k in $\mathbb{N}$. Since each w_n defines a unitary, $\varrho_\omega(w)$ becomes a unitary in $\mathcal{Q}_\omega$ witnessing the unitary equivalence of a, b by construction. This proves (ii) modulo (i).

(i): Let $a, b \geq 0$ be contractions in A such that $\tau(f(a)) = \tau(f(b)) > 0$ for an arbitrary trace τ on A and nonzero positive f in $C_0(0, 1]$. According to theorem 3.4.1, verifying that $f(a) \sim_c f(b)$ suffices. Let therefore $\varepsilon > 0$ be given. Consider the nonempty set

$$U = \{0 < t \leq 1 : 0 < f(t) < \varepsilon\}.$$

Continuity of f makes U open, whereupon one may determine some unit vector g in $C_0(U)_+$. For general orthogonal positive elements x, y in A one clearly has $C^*(x + y) = C^*(x) + C^*(y)$. The identification in turn justifies the computation

$$
\begin{aligned}
d_\tau(x + y) &= \lim_{\varepsilon \to 0} \tau(f_\varepsilon(x + y)) \\
&= \lim_{\varepsilon \to 0} \tau(f_\varepsilon(x)) + \lim_{\varepsilon \to 0} \tau(f_\varepsilon(y)) \\
&= d_\tau(x) + d_\tau(y)
\end{aligned}
\tag{3.11}
$$

Due to a being a contraction, one has $0 < \tau(g(a)) \leq d_\tau(g(a))$. Since $(f(a) - \varepsilon)_+$ is orthogonal to $g(a)$ (g belongs to $C_0(U)$) and $(f(a) - \varepsilon)_+ + g(a) \lesssim f(a)$ by construction, the ordinary calculus rules of the dimension function d_τ, including (3.11), ensure that

$$
\begin{aligned}
d_\tau((f(a) - \varepsilon)_+) &< d_\tau[(f(a) - \varepsilon)_+ + g(a)] \\
&= d_\tau((f(a) - \varepsilon)_+) + d_\tau(g(a)) \\
&\leq d_\tau(f(a)) \\
&= d_\tau(f(b)).
\end{aligned}
$$

Upon τ being arbitrary, strict comparison of A forces $(f(a) - \varepsilon)_+ \lesssim f(b)$ for all $\varepsilon > 0$. We then deduce that $f(a) \lesssim f(b)$ via proposition 3.3.2 and the general rule $((a - \varepsilon)_+ - \delta)_+ = (a - (\varepsilon + \delta))_+$, completing the proof in view of our initial remark. □

With the unitary equivalence condition arranged, we may assemble the whole machinery upon which the theorem of Tikuisis-White-Winter is founded. Due to certain properties emerging repeatedly throughout, it seems prudent to introduce general terminology.

Definition. Let A and B be unital C*-algebras. Let $I \subseteq [0,1]$ be an interval and $\theta \colon C([0,1]) \longrightarrow B$ be a unital $*$-homomorphism. A $*$-homomorphism $\pi \colon C_0(I) \otimes A \longrightarrow B$ is *compatible with* θ if

$$\pi_{|C_0(I) \otimes \mathbb{C}1_A} = \theta_{|C_0(I) \otimes \mathbb{C}1_A}.$$

An element f in $C_0(I)$ may naturally be regarded as an element in $C_0(I) \otimes A$ via the $*$-monomorphism $f \mapsto f \otimes 1_A$. Without distinguishing between f and its copy in the cone, the restriction of θ onto $C_0(I) \otimes \mathbb{C}1_A$ thus becomes meaningful. Notice that compatibility permits us to conclude that

$$\theta(t)\pi(s) = \pi(ts) = \pi(st) = \pi(s)\theta(t) \tag{3.12}$$

for all t in $C([0,1])$ and s in $C_0(I) \otimes A$.

The sought duo (π_0, π_1) of maps will remember the original trace with some modifications. The "error" occurring arises from traces on the commutative C*-algebraic parts attached to the cones. We examine such traces now prior to constructing the maps.

Definition. The *Lebesgue trace* is the trace $\tau_{\mathcal{L}} \colon C([0,1]) \longrightarrow \mathbb{C}$ defined by

$$\tau_{\mathcal{L}} = \int_{[0,1]} f \, dm,$$

with m denoting the Lebesgue measure on $\mathbb{R}$. For any unital C*-algebra A, a positive contraction z therein with spectrum $[0,1]$ is said to have *Lebesgue spectral measure* with respect to a trace τ acting on A, provided that $\tau(f(z)) = \tau_{\mathcal{L}}(f)$ holds for every f in $C([0,1])$.

Notice that in the presence of a unit, a positive contraction a has Lebesgue spectral measure if and only if $1_A - a$ does. We are finally in position to build our $*$-homomorphisms — remembering that we ought to exploit the correspondence between $\mathcal{O}_c(A, \mathcal{Q}_\omega)$ and $\mathrm{Hom}(C_0(0,1] \otimes A, \mathcal{Q}_\omega)$, then recover the trace via proposition 3.2.4. To lower the proof length, we isolate some preliminary tricks of $\mathcal{Q}$. We need an old theorem due to Tomiyama in the nuclear case and which was improved to the exact scenario by Kirchberg.

Theorem 3.4.3 (Kirchberg, Tomiyama). *Let A be an exact C*-algebra. For any additional C*-algebra B, every ideal I in $A \otimes B$ satisfies*

$$\begin{aligned} I &= \overline{\mathrm{span}}\{I_A \odot I_B : I_A \trianglelefteq A, \ I_B \trianglelefteq B \text{ such that } I_A \odot I_B \subseteq I\} \\ &= \overline{\mathrm{span}}\{x \otimes y : x \otimes y \in I\}. \end{aligned}$$

Proof. Omitted. See *corollary* 9.4.6 in [9] for a proof. $\square$

The preceding theorem permit us let to $\mathcal{Q}_\omega$ become the target algebra in contrast to the initial codomain $\mathcal{Q} \otimes \mathcal{Q}_\omega$ that our maps (π_0, π_1) will have. We prove the precise property required in larger generality (type p^∞-UHF algebra will then work; we throw an additional observation required into the mix. Recall that A is *self-absorbing* if $A \otimes A \cong A$.

Lemma 3.4.4. *Suppose A denotes a unital selfabsorbing simple nuclear C*-algebra.*

(i) *There exists a unital $*$-monomorphism $A \otimes A_\omega \hookrightarrow A_\omega$.*

(ii) *There exists a positive contraction in $\mathcal{Q}$ having spectrum $[0,1]$ and having Lebesgue spectral measure with respect to the unique trace on $\mathcal{Q}$.*

Proof. (i): Consider the map $\Lambda\colon A \longrightarrow \ell^\infty(A \otimes A)$ given by the assignment $a \mapsto (a \otimes 1_A, a \otimes 1_A, \ldots)$. It defines a unital $*$-monomorphism, inducing a unital $*$-monomorphism $\Lambda_\omega\colon A \longrightarrow (A \otimes A)_\omega \cong A_\omega$ by composing with the canonical quotient map $\varrho_\omega^\otimes\colon \ell^\infty(A \otimes A) \longrightarrow (A \otimes A)_\omega$, because

$$\|\varrho_\omega^\otimes(a \otimes 1_A, a \otimes 1_A, \ldots)\| = \limsup_{n \to \omega} \|a \otimes 1_A\| = \|a\|.$$

Keeping this in mind, consider hereafter the $*$-monomorphism $\Lambda^0\colon A \longrightarrow A \otimes A$ given by $a \mapsto 1_A \otimes a$. The new $*$-homomorphism $\Lambda_\omega^0\colon A_\omega \longrightarrow (A \otimes A)_\omega \cong A_\omega$ is a unital $*$-monomorphism whose image commutes with $\Lambda_\omega(A)$. Universality of $\| \cdot \|_{\max}$ supplies us with a unique unital $*$-homomorphism $\Lambda_\omega \times \Lambda_w^0\colon A \otimes_{\max} A_\omega \longrightarrow A_\omega$ fulfilling

$$(\Lambda_\omega \times \Lambda_\omega^0)(a \otimes b) = \Lambda_\omega(a)\Lambda_\omega^0(b).$$

We assert that it must be an injection. Due to A being nuclear, $\Lambda_\omega \times \Lambda_\omega^0$ defines a map on $A \otimes A_\omega$. According to theorem 3.4.3 any ideal I in $A \otimes A_\omega$ is the norm-closed linear span of elementary tensors $x \otimes y$ with $x \in I_A$ and $y \in I_{A_\omega}$ for some ideals $I_A \trianglelefteq A$ and $I_{A_\omega} \trianglelefteq A_\omega$. In particular, this must be valid for the kernel of $\Lambda_\omega \times \Lambda_\omega^0$, having $I_A = A$ by simplicity of A. However, for any elementary tensor $1_A \otimes a$ belonging to $\ker(\Lambda_\omega \times \Lambda_\omega^0)$ one has

$$0 = (\Lambda_\omega \times \Lambda_\omega^0)(1_A \otimes a) = \Lambda_\omega(1_A)\Lambda_\omega^0(a) = \Lambda_\omega^0(a).$$

This observation forces $\ker \Lambda_\omega \times \Lambda_\omega^0$ to be isomorphic to $\{0\}$.

(ii): Consider the following sequence $(b_n)_{n \geq 1}$ of quadratic matrices;

$$\begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 3/4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/8 & 0 & 0 & \cdots \\ 0 & 0 & 1/4 & 0 & \cdots \\ 0 & 0 & 0 & 3/8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \cdots \quad .$$

That is, b_n is the diagonal matrix in $\mathbb{M}_{2^n}$ attaining the value $\frac{k-1}{2^n}$ on the k'th diagonal entry for $k = 0, 1, \ldots 2^n$. Let $(\mathbb{M}_{2^n}, \varphi_n)_{n \geq 1}$ be the inductive sequence attached to $\mathbb{M}_{2^\infty}$. Permuting with suitable permutation matrices gives unitaries $u_1, u_2, \ldots$ permitting us to force

$$\varphi_n(b_n) \approx_{2^{-(n+1)}} u_n b_{n+1} u_n^*.$$

Letting $a_n = u_n b_n u_n^*$ for each positive integer n thus defines a Cauchy-sequence $(a_n)_{n \geq 1}$ whose limit in $\mathbb{M}_{2^\infty}$ we denote by a. Conjugation by unitaries does not alter spectra, hence $\sigma(a_n) \subseteq \sigma(a_{n+1})$ for each n in $\mathbb{N}$ and each stage $\sigma(a_n)$ must be contained in $\sigma(a)$. Ergo,

$$\mathbb{Z}[1/2] \cap [0,1] = \bigcup_{n=1}^\infty \sigma(a_n) \subseteq \sigma(a).$$

Taking closure yields $\sigma(a) = [0,1]$, since $0 \leq a \leq 1_\mathcal{Q}$. Choose hereafter an arbitrary continuous function $f\colon [0,1] \longrightarrow \mathbb{C}$. Due to uniqueness of traces, one has

$$\tau_\mathcal{Q}(f(a_n)) = \tau_{2^n}(f(a_n)) = \frac{1}{2^n} \sum_{k=0}^{2^n} f\left(\frac{k-1}{2^n}\right).$$

Regarding the right-hand side as a Riemannian-sum, the limit for $n \to \infty$ becomes the associated Riemann-integral of f on the closed interval $[0,1]$ through continuity. The integral coincides with the Lebesgue integral on $[0,1]$ by continuity of f once more. It follows that

$$\tau_\mathcal{Q}(f(a)) = \lim_{n \to \infty} \tau_\mathcal{Q}(f(a_n)) = \int_{[0,1]} f \, dm,$$

whereof a attains Lebesgue spectral measure with respect to $\tau_\mathcal{Q}$, for $\tau_\mathcal{Q}(f(a_n)) \to \tau_\mathcal{Q}(f(a))$ by continuity of the involved maps. $\square$

Proposition 3.4.5. *Suppose A denotes a unital separable nuclear C^*-algebra admitting a trace τ. Then are there $*$-homomorphisms*

$$\theta\colon C([0,1]) \longrightarrow \mathcal{Q}_\omega, \quad \pi_0\colon C_0(0,1] \otimes A \longrightarrow \mathcal{Q}_\omega \quad and \quad \pi_1\colon C_0[0,1) \otimes A \longrightarrow \mathcal{Q}_\omega$$

with θ unital. Furthermore, π_0, π_1 are both compatible with θ and

$$\tau_\omega \circ \pi_0 = \tau_{\mathcal{L}} \otimes \tau = \tau_\omega \circ \pi_1.$$

Proof. By proposition 3.2.4 there exists a contractive order zero map $\psi\colon A \longrightarrow \mathcal{Q}_\omega$ such that

$$\tau_\omega\big(\psi(a)\psi(1_A)^{n-1}\big) = \tau(a) \tag{3.13}$$

for each $a \in A$ and $n \in \mathbb{N}$. Extract a sequence of positive contractions in $\mathcal{Q}$ whose norm-limit a_0 has spectrum $[0,1]$, remains a positive contraction and has Lebesgue spectral measure with respect to the unique trace $\tau_{\mathcal{Q}}$, justifiable by lemma 3.4.4(ii). Define accordingly $\Lambda\colon A \longrightarrow \mathcal{Q} \otimes \mathcal{Q}_\omega$ as

$$\Lambda(\cdot) = a_0 \otimes \psi(\cdot).$$

Plainly, Λ is of order zero, since ψ is. Let $\pi_0\colon C(0,1] \otimes A \longrightarrow \mathcal{Q} \otimes \mathcal{Q}_\omega$ be the $*$-homomorphism corresponding to Λ, meaning the continuous linear extension of the map defined on generating elementary tensors by $\mathrm{id}_{(0,1]} \otimes a \mapsto \Lambda(a)$, see proposition 1.4.9. Now, lemma 3.4.4(i) enables to us to regard π_0 as a unital $*$-homomorphism attaining values in $\mathcal{Q}_\omega$.

Recall that the minimal unitization of $C_0(0,1]$ is $*$-isomorphic to the unital C^*-algebra consisting of continuous functions on its one-point compactification, $[0,1]$. Letting $\theta\colon C([0,1]) \longrightarrow \mathcal{Q}_\omega$ be the map induced hereon via the restriction of π_0 onto $C_0(0,1] \cong C_0(0,1] \otimes \mathbb{C}1_A$ certainly yields a $*$-homomorphism such that π_0 becomes compatible with θ. One thereafter observes that

$$\begin{aligned}
\pi_0\big(\mathrm{id}_{(0,1]}^n \otimes a\big) &= \pi_0\big((\mathrm{id}_{(0,1]} \otimes a)(\mathrm{id}_{(0,1]} \otimes 1_A)^{n-1}\big) \\
&= \Lambda(a)\Lambda(1_A)^{n-1} \\
&= a_0^n \otimes \psi(a)\psi(1_A)^{n-1}
\end{aligned}$$

must be valid for every $a \in A$ and each $n \in \mathbb{N}$. The above computation is precisely why we arranged the empowered version (3.6) instead of settling with $\tau = \tau_\omega \circ \psi$. Applying $\tau_{\mathcal{Q}} \otimes \tau_\omega$, which corresponds to τ_ω due to the unital embedding $\mathcal{Q} \otimes \mathcal{Q}_\omega \hookrightarrow \mathcal{Q}_\omega$ and uniqueness of trace on $\mathcal{Q}$, gives

$$\begin{aligned}
\big((\tau_{\mathcal{Q}} \otimes \tau_\omega) \circ \pi_0\big)\big(\mathrm{id}_{(0,1]}^n \otimes a\big) &= \tau_{\mathcal{Q}}(a_0^n)\tau_\omega\big(\psi(a)\psi(1_A)^{n-1}\big) \\
&\overset{(3.19)}{=} \tau_{\mathcal{Q}}(a_0^n)\tau(a) \\
&= (\tau_{\mathcal{L}} \otimes \tau)\big(\mathrm{id}_{(0,1]}^n \otimes a\big).
\end{aligned}$$

The final equality is based on a_0 having Lebesgue spectral measure with respect to $\tau_{\mathcal{Q}}$. As elementary tensors of the form $\mathrm{id}_{(0,1]}^n \otimes a$ canonically span the $\mathbb{C}$-algebraic involutive tensor product $C(0,1] \odot A$, continuity of the acting linear maps ensures that $\tau_\omega \circ \pi_0 = \tau_{\mathcal{L}} \otimes \tau$.

We craft the remaining $*$-homomorphism π_1 as a unitary conjugation of π_0. Consider at first the positive contraction $x = \theta(\mathrm{id}_{[0,1]})$. Compatibility of π_0 guarantees that $x = \pi_0(\mathrm{id}_{[0,1]} \otimes 1_A)$. Let h belong to $C([0,1])$. Using some functional calculus leaning on 1_A being a projection, one acquires $h(x) = \pi_0(h(\mathrm{id}_{[0,1]}) \otimes 1_A) = \pi_0(h \otimes 1_A)$. Then $\tau_\omega \circ \pi_0 = \tau_{\mathcal{L}} \otimes \tau$ implies that

$$\tau_\omega(h(x)) = (\tau_\omega \circ \pi_0)(h \otimes 1_A) = \tau_{\mathcal{L}}(h).$$

Hence $1_{\mathcal{Q}_\omega} - x$ has Lebesgue spectral measure and we are thus permitted to invoke lemma 3.4.2(ii), thereby granting us some unitary u in $\mathcal{Q}_\omega$ fulfilling the relation $u\pi_0(\mathrm{id}_{[0,1]})u^* = 1_{\mathcal{Q}_\omega} - \pi_0(\mathrm{id}_{[0,1]})$. Suppose hereafter that $\sigma\colon C_0[0,1) \otimes A \longrightarrow C_0(0,1] \otimes A$ denotes the flip-map, meaning if f_0 is the map $t \mapsto f(1-t)$ associated to any f in $C_0(0,1]$, then

$$\sigma(f \otimes a) = f_0 \otimes a, \quad a \in A, \ f \in C_0(0,1].$$

Notice that $\sigma(1_{[0,1]} - \mathrm{id}_{[0,1]}) = \mathrm{id}_{[0,1]}$. Keeping this in mind, let π_1 be the composition

$$\pi_1 \colon C_0[0,1) \otimes A \xrightarrow{\ \sigma\ } C_0(0,1] \otimes A \xrightarrow{\ \pi_0\ } \mathcal{Q}_\omega \xrightarrow{\ \mathrm{Ad}_u\ } \mathcal{Q}_\omega.$$

Being composed by $*$-homomorphisms, π_1 must be one as well. Compatibility with θ in conjunction with the recovery of trace remain to be verified. For the former, it suffices to achieve equality on the element $1_{[0,1]} - \mathrm{id}_{[0,1]}$, for it generates $C_0[0,1)$. However, compatibility then stems from the established compatibility of π_0 with θ and unitality of θ;

$$\pi_1\big(1_{[0,1]} - \mathrm{id}_{[0,1]}\big) = u\pi_0\big(\mathrm{id}_{[0,1]}\big)u^* = \theta\big(1_{[0,1]} - \mathrm{id}_{[0,1]}\big).$$

As the Lebesgue integral is flip-invariant, one may easily deduce that $(\tau_{\mathcal{L}} \otimes \tau) \circ \sigma = \tau_{\mathcal{L}} \otimes \tau$, so

$$\tau_\omega \circ \pi_1 = \tau_\omega \circ (\mathrm{Ad}_u \circ \pi_0 \circ \sigma) = (\tau_\omega \circ \pi_0) \circ \sigma = (\tau_{\mathcal{L}} \otimes \tau) \circ \sigma = \tau_{\mathcal{L}} \otimes \tau.$$

This finalizes the proof. $\qquad\square$

The tracial large cone $*$-homomorphisms constructed during the preceding chapter are the prime candidates to build completely positive maps witnessing quasidiagonality of a given faithful trace. The idea may vaguely be depicted in the following way. Suppose τ denotes a faithful trace acting on a nuclear separable C*-algebra A. To the associated triple

$$\theta \colon C([0,1]) \longrightarrow \mathcal{Q}_\omega, \quad \pi_0 \colon C_0(0,1] \otimes A \longrightarrow \mathcal{Q}_\omega, \quad \text{and} \quad \pi_1 \colon C[0,1) \otimes A \longrightarrow \mathcal{Q}_\omega,$$

let Λ_0 be the restriction of π_0 to $C_0(0,1) \otimes A$ and let Λ_1 be the corresponding restriction for π_1. Imagine one had access to some unitary u in $\mathcal{Q}_\omega$ fulfilling $\Lambda_1 = \mathrm{Ad}_u \circ \Lambda_0$. Then one may build a $*$-homomorphism $\pi \colon A \longrightarrow \mathrm{M}_2(\mathcal{Q}_\omega)$ subject to

$$(\mathrm{tr}_2 \otimes \tau_\omega) \circ \pi = \frac{\tau}{2}. \tag{3.14}$$

The formula for π may be expressed by the formula

$$\pi(a) = \begin{bmatrix} \Lambda_1(h_0 \otimes a) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & u^* \end{bmatrix} \theta_2(R^*) \begin{bmatrix} \Lambda_1(h_1 \otimes a) & 0 \\ 0 & 0 \end{bmatrix} \theta_2(R) \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix} + \begin{bmatrix} \Lambda_0(h_2 \otimes a) & 0 \\ 0 & 0 \end{bmatrix}$$

Here h_0 may be chosen as any positive continuous map on $[0,1]$ attaining the value 1 at $t = 0$, linear on $(0, 1/4)$ and 0 elsewhere, h_2 denotes the map reflecting h_0 at $t = 1/2$ and h_1 may be any positive continuous piecewise linear map such that the set $\{h_0, h_1, h_2\}$ comprises a partition of unity. Lastly, R denotes a continuous rotation in $\mathrm{M}_2(C([0,1]))$ such that

$$R|_{0,1/4} = \begin{bmatrix} 1_{C([0,1])} & 0 \\ 0 & 1_{C([0,1])} \end{bmatrix} \quad \text{together with} \quad R|_{[3/4,1]} = \begin{bmatrix} 0 & 1_{C([0,1])} \\ 1_{C([0,1])} & 0 \end{bmatrix}$$

Verifying multiplicativity leans on the unitary equivalence $\Lambda_1 \sim \Lambda_0$ while (3.20) stems from

$$((\mathrm{tr}_2 \otimes \tau_\omega) \circ \pi)(a) = \frac{1}{2}\big(\tau_\omega \pi_1(h_0 \otimes a) + \tau_\omega \pi_1(h_1 \otimes a) + \tau_\omega \pi_0(h_2 \otimes a)\big)$$
$$= \frac{\tau(a)}{2}\big(\tau_{\mathcal{L}}(h_0 + h_1 + h_2)\big)$$
$$= \frac{\tau(a)}{2}$$

One then considers the associated unital $*$-homomorphism $\pi_{qd} \colon A \longrightarrow \pi(1_A)\mathrm{M}_2(\mathcal{Q}_\omega)\pi(1_A)$. The target algebra is in fact $*$-isomorphic to $\mathcal{Q}_\omega$ thanks to proposition 3.2.2 ensuring that $\mathrm{M}_2(\mathcal{Q}_\omega) \cong \mathcal{Q}_\omega$ and $\pi(1_A)$ being a non-trivial projection. Furthermore, uniqueness of trace on $\mathcal{Q}_\omega$ enables one to conclude that (3.14) remains valid under the identification, hence quasidiagonality stems from proposition 3.2.3. However, for a separable nuclear C*-algbera A, manufacturing such a unitary u is futile. Even searching for a unitary detecting approximate unitary equivalence of Λ_1, Λ_0 will be difficult. We must fix this issue to some reasonable extend.

Chapter 4

The Stable Uniqueness Theorem

This chapter will attempt to unravel how Tikuisis, White and Winter bypass the issue of establishing the aforementioned unitary equivalence using KK-theory. The principle behind tacitly invoking KK-theoretic aspect revolves around "stable uniqueness" results due to Dadarlat and Eilers in [18], whose work we are inclined to address. The chapter starts with a brief recap on Hilbert C*-modules alongside some fundamental observations based on Kasparov's work.

4.1   Hilbert C*-modules and KK-Theory

The theory established by Dadarlat and Eilers seeks to partially provide an answer to the following question: For a pair of ∗-homomorphisms defining the same class in KK-theory, can we arrange to some extend approximate unitary equivalence of these? Their work relies heavily on ideas due to Lin and his work on the same matter, from which they achieve approximate unitary equivalence provided that one passes to larger matrices with some ∗-homomorphism added.

   To derive the stability theorem, one requires both standard pictures of KK-theory. We therefore introduce these without proving neither the group axioms nor the properties such as homotopy invariance. For a far more detailed survey of the Cuntz picture, the reader may consult sections 3.1-3.2 in [30] or Cuntz' original paper [15].

Definition. Suppose A denotes some C*-algebra. A *Hilbert C*-module over A* is a $\mathbb{C}$-vector space E admitting an A-module structure and an *A-valued inner product*, meaning a map $\langle \cdot, \cdot \rangle \colon E \times E \longrightarrow A$ subject to the following axioms:

- $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for all $\xi, \eta \in E$;

- $\langle \cdot, \cdot \rangle$ is $\mathbb{C}$-linear in the second variable;

- $\langle \xi, a\eta \rangle = a\langle \xi, \eta \rangle$ for all $\xi, \eta \in E$ and $a \in A$;

- $\langle \xi, \xi \rangle \geq 0$ with equality occurring if and only if $\xi = 0$, for every $\xi \in E$;

- E endowed with the norm $\xi \mapsto \|\langle \xi, \xi \rangle\|_A^{1/2}$ defines a complete normed space.

It is apparent that Hilbert spaces define Hilbert C*-modules over $\mathbb{C}$. As such the notion generalizes Hilbert spaces and one obtains an analogue of the Cauchy-Schwarz inequality: If E denotes a Hilbert C*-modules over A containing elements ξ and η, one has for all a in A that

$$\|\langle \xi, \eta \rangle\|_A \leq \|\xi\| \cdot \|\eta\| \quad \text{and} \quad \|a\xi\| \leq \|\xi\| \cdot \|a\|_A. \tag{4.1}$$

For a proof, the reader is urged to consult section 3.3 in [30]. Hilbert C*-modules form a vital ingredient in the construction of the KK-functor and we shall often be working with a prototypical one. Before exhibiting the example, some additional terminology will be fruitful.

Definition. Suppose E, E_0 denote Hilbert C*-module over A. A linear map $t\colon E \longrightarrow E_0$ is called *adjointable* if there exists some linear map $s\colon E_0 \longrightarrow E$ fulfilling

$$\langle s\xi, \eta \rangle_{E_0} = \langle \xi, t\eta \rangle_E$$

for all $\xi \in E$ and $\eta \in E_0$. The map t is unique, hence commonly denoted by s^*. We write $\mathcal{L}_A(E, E_0)$ to represent the collection of all adjointable maps $t\colon E \longrightarrow E_0$, abbreviating $\mathcal{L}_A(E) = \mathcal{L}_A(E, E)$. In the latter scenario, one acquires a unital C*-algebra having $1_E = \mathrm{Id}_E$ as the unique unit, composition of adjointable operators as multiplication and with the "operator" norm

$$\|s\|_{\mathcal{L}_A(E)} := \sup_{\xi \in E_1} \|s\xi\|.$$

For verifications of the claim, we refer to [28] and/or [30]. A *unitary* u in $\mathcal{L}_A(E, E_0)$ is an adjointable map such that $uu^* = 1_{E_0}$ and $u^*u = 1_E$. Completely parallel to ordinary Hilbert spaces, E and E_0 are called *isomorphic* should there exist a unitary between them, any such being identified with one another upon unitaries preserving the A-valued inner-product.[1]

Suppose A denotes any C*-algebra. Then A admits a Hilbert C*-module structure over itself having $\langle a, b \rangle = a^*b$ as its associated A-valued inner product. In accordance with this particular example, the unital C*-algebra $\mathcal{L}(A) := \mathcal{L}_A(A)$ represents a generalization of $\mathrm{B}(\mathcal{H})$. Keeping the comparison in mind, let E be some Hilbert C*-module over B. Define, for any pair of elements ξ, η in E, the "rank one operator" $\omega_{\xi, \eta}$ as

$$\omega_{\xi, \eta}(\mu) = \xi \langle \eta, \mu \rangle.$$

Mimicking rank-one operators of $\mathrm{B}(\mathcal{H})$, we denote by $\mathcal{F}_B(E)$ the vector space having the operators $\omega_{\xi, \eta}$ form a basis and denote its norm-closure in $\mathcal{L}_B(E)$ as $\mathcal{K}_B(E)$. One may check that $\mathcal{K}_B(E)$ constitutes an ideal in $\mathcal{L}_B(E)$. We refer to elements of the latter as *compact operators* and those in former the *finite rank operators*. The proposition below was proven by Kasparov in [25] and bridges the concept of adjointables and multiplier algebras.

Proposition 4.1.1 (Kasparov). *One has $\mathcal{L}_B(E) \cong \mathcal{M}(\mathcal{K}_B(E))$ for every Hilbert B-module E.*

Proof. We adopt the multiplier picture of $\mathcal{M}(\mathcal{K}_B(E))$ throughout the proof. Define a linear map $\Delta\colon \mathcal{L}_B(E) \longrightarrow \mathcal{M}(\mathcal{K}_B(E))$ by sending an adjointable operator s to the pair $\Delta(s) := (s_0, s_1)$ consisting of the adjointable operators fulfilling

$$s_0(\omega_{\xi, \eta}) = \omega_{s\xi, \eta}, \quad \text{respectively,} \quad s_1(\omega_{\xi, \eta}) = \omega_{\xi, s^*\eta}$$

for all ξ, η in E. Due to (4.1) yielding $\|s_k(a)\| \leq \|s_k\| \cdot \|a\|$ for every finite rank operator a acting on E and $k = 0, 1$, the map Δ extends to the compact operators through density. We refer to the extension by Δ as well, hopefully without causing confusion.

   The assignment is easily seen to be injective, for $x_0(\omega_{\xi, \eta}) = 0$ when $\xi, \eta \in E$ forces $\omega_{x\xi, x\xi} = 0$ for each ξ in E and $x = 0$ thereof. The tricky part concerns surjectivity. Suppose (x, y) denotes a multiplier of $\mathcal{K}_B(E)$. Let p be the operator on E defined by the action

$$p(\xi) = \lim_{\varepsilon \to 0} x(\omega_{\xi, \xi})\xi(\|\xi\|^2 + \varepsilon)^{-1}.$$

The limit exists according to a standard application of (4.1) in conjunction with boundedness of the multipliers. We assert that the linear map $p^*\colon E \longrightarrow E$ given by

$$p^*(\xi) = \lim_{\varepsilon \to 0} y(\omega_{\xi, \xi})^*\xi(\|\xi\|^2 + \varepsilon)^{-1}$$

---

[1]In general, notions such as projections, self-adjoint elements etc. have analogues for modules, properties being verified in the same fashion as Hilbert spaces, almost verbatim. For orthogonal decompositions, see 15.3.9 in [47].

must be the adjoint of p. Due to $\omega_{\eta,\eta}(\eta)(\|\eta\|^2 + \varepsilon)^{-1}$ tending to η as $\varepsilon \to 0$, it follows that

$$
\begin{aligned}
\langle \eta, p^*(\xi) \rangle &= \lim_{\varepsilon \to 0} \left\langle \omega_{\eta,\eta}(\eta)(\|\eta\|^2 + \varepsilon)^{-1}, y(\omega_{\xi,\xi})^*\xi(\|\xi\|^2 + \varepsilon)^{-1} \right\rangle \\
&= \lim_{\varepsilon \to 0} \left\langle y(\omega_{\xi,\xi})\omega_{\eta,\eta}(\eta)(\|\eta\|^2 + \varepsilon)^{-1}, \xi(\|\xi\|^2 + \varepsilon)^{-1} \right\rangle \\
&= \lim_{\varepsilon \to 0} \left\langle \omega_{\xi,\xi} \cdot x(\omega_{\eta,\eta}(\eta))(\|\eta\|^2 + \varepsilon)^{-1}, \xi(\|\xi\|^2 + \varepsilon)^{-1} \right\rangle \\
&= \lim_{\varepsilon \to 0} \left\langle x(\omega_{\eta,\eta}(\eta))(\|\eta\|^2 + \varepsilon)^{-1}, \omega_{\xi,\xi}(\|\xi\|^2 + \varepsilon)^{-1} \right\rangle = \langle p(\eta), \xi \rangle.
\end{aligned}
$$

must be valid for any ξ, η inside E. The third equality is based on the pairing (x, y) being a multiplier. The operator p thus becomes adjointable. Setting $s(a) = x(\omega_{\xi,\eta}(a)) - \omega_{x\xi,\eta}(a)$ for each a in E, followed by a slightly unpleasant computation entails that $\omega_{e,e_0}(s(a)) = 0$ for any pair $e, e_0 \in E$. The obtained special case $e = e_0 = s(a)$ therefore forces $s(a) = 0$, whereby one may deduce that $p_0(\omega_{\xi,\eta}) = \omega_{x\xi,\eta}$ for all ξ, η belonging to E with $\Delta(p) = (p_0, p_1)$. A completely analogues argument will reveal that $p_1(\omega_{\xi,\eta}) = \omega_{\xi,y^*\eta}$, hence $\Delta(p) = (x, y)$ as desired. $\qquad\square$

Due to the two pictures of KK-theory having, a priori, rather distinct points of view in regards to their elements, we insist on moderately understanding how they interact with one another. This demands a connection between the C*-algebraic formulation of the adjointables, starting with the interpretation of A as the compact adjointables.

Proposition 4.1.2. *For every C*-algebra A one has $\mathcal{K}(A) \cong A$. Hence $\mathcal{L}(A) \cong \mathcal{A}$.*

Proof. Consider the $*$-homomorphism $\pi \colon A \longrightarrow \mathcal{L}(A)$ given by $\pi(a)b = ab$ for all a, b in A. For each nonzero a in A, observe that $\|a\| = \|\pi(a^*)a/\|a\|\| \leq \|\pi(a)\|$. This in turn reveals that π is a $*$-monomorphism. Define hereafter another map $\varphi_0 \colon \mathcal{F}(A) \longrightarrow \pi(A)$ by the assignment $\omega_{a,b} \mapsto \pi(ab^*)$. Due to π being an isometry, φ_0 extends to an isometry $\varphi \colon \mathcal{K}(A) \longrightarrow \pi(A)$. The image of φ equals the closure of the linear space spanned by elements of the form $\pi(ab^*)$. Let $\{e_\alpha\}_{\alpha \in I}$ be some approximate unit of A. Then any element a in A fulfills $\pi(a) = \lim_{\alpha \in I} \pi(ae_\alpha)$ with the latter belonging to $\varphi(\mathcal{K}(A))$ as φ has closed image. We conclude that $\mathcal{K}(A) \cong \pi(A) \cong A$. The final assertion is immediate from the preceding proposition. $\qquad\square$

The aforementioned prototypical Hilbert C*-module we shall consider will be addressed now. *Let $\mathcal{H}$ be some separable infinite dimensional Hilbert space throughout the entire remainder of the chapter.* Every C*-algebra B acts on the $\mathbb{C}$-algebraic tensor product $\mathcal{H}_B = \mathcal{H} \otimes B$ by multiplication on the B-coordinate while leaving $\mathcal{H}$ unaltered. Equipping the B-module $\mathcal{H}_B$ with the B-valued inner product $\langle \cdot, \cdot \rangle \colon \mathcal{H}_B \times \mathcal{H}_B \longrightarrow B$ defined by declaring that

$$
\langle \xi \otimes b, \xi_0 \otimes b_0 \rangle = \langle \xi, \xi_0 \rangle_{\mathcal{H}} \cdot b^* b_0
$$

forms a Hilbert C*-module structure of B. The point of $\mathcal{H}_B$ is its associated C*-algebra $\mathcal{L}_B(\mathcal{H}_B)$; an acquaintance in cognito.

Proposition 4.1.3. *Let B be some C*-algebra. Then $\mathcal{K}_B(\mathcal{H}_B) \cong \mathbb{K} \otimes \mathcal{K}(B)$ as C*-algebras. In particular, one has $\mathcal{L}_B(\mathcal{H}_B) \cong \mathcal{M}(B \otimes \mathbb{K})$.*

Proof. The identification stems from the following observation. Let $\xi, \eta, \mu \in \mathcal{H}$ and $a, b, c \in B$ be arbitrary elements. Based on

$$
\omega_{\xi \otimes a, \eta \otimes b}(\mu \otimes c) = (\omega_{\xi,\eta} \otimes \omega_{a,b})(\mu \otimes c)
$$

we may conclude that the canonical isometry $\beta \colon \mathcal{L}_{\mathbb{C}}(\mathcal{H}) \otimes \mathcal{L}(B) \longrightarrow \mathcal{L}_B(\mathcal{H}_B)$ defined in terms of the bilinear map $(s, t) \mapsto s \times t$ with $(s \times t)(\xi \otimes b) := s\xi \otimes tb$ (see the appendix for an elaboration), becomes a $*$-epimorphism via density of the finite rank operators. Indeed, the above gives

$$
\beta(\omega_{\xi,\eta} \otimes \omega_{a,b}) = \omega_{\xi \otimes a, \eta \otimes b}
$$

for any $\xi, \eta \in \mathcal{H}$ and $a, b \in B$, granting surjectivity immediately. The remaining statement stems from $\mathcal{K}(B) \cong B$ yielding $\mathcal{L}_B(\mathcal{H}_B) \cong \mathcal{M}(B \otimes \mathbb{K})$ thereof. $\qquad\square$

Prior to venturing further towards our KK-theoretic notions, we present the heart of $\mathcal{H}_B$ in the shape of Kasparov's Stabilization theorem. We omit proving it, in spite of exploiting it regularly. For the record, a Hilbert C*-module E over B is called *countably generated* should the existence of a countable set $\{e_n\}_{n \geq 1}$ of elements in E such that

$$\mathcal{D}(\{e_n\}_{n \geq 1}, B) := \left\{ \sum_{k=1}^{m} e_k b_k : b_k \in B, \ m \in \mathbb{N} \right\}$$

is norm-dense in E be guaranteed. A reoccurring example is $\mathcal{H}_B$ whenever B is σ-unital. Additionally, given Hilbert C*-modules E and F over a common C*-algebra A, the direct sum $E \oplus F$ as $\mathbb{C}$-vector spaces admits an A-module structure in the ordinary manner for A-modules. We endow the module with the map $\langle \cdot, \cdot \rangle \colon (E \oplus F) \times (E \oplus F) \longrightarrow \mathbb{C}$ given by

$$\langle (a, b), (a_0, b_0) \rangle = \langle a, a_0 \rangle_E + \langle b, b_0 \rangle_F$$

as an A-valued inner product.

Theorem 4.1.4 (Kasparov's stabilization theorem). *Suppose E denotes any countable generated Hilbert B-module with B being σ-unital. Under these premises, one has $E \oplus \mathcal{H}_B \cong \mathcal{H}_B$.*

Proof. See *theorem* 2 of [25] for a proof. $\qquad \square$

KK-theory adopts several points of view. For our primary concern, the most prominent ones revolve around representations of Hilbert C*-modules. We discuss the concept attached now, including some topological aspects that facilitate Skandalis' modified version of KK-theory. Afterwards, we tacitly derive special cases of representation producing elements of KK-theory.

Remark. Throughout the remaining chapter, B will denote a fixed σ-unital C*-algebra and E will denote a countably generated Hilbert C*-module over B, unless specified otherwise.

Definition. Let A, B be arbitrary C*-algebras and let E, F be Hilbert B-modules

- A $*$-homomorphism $\pi \colon A \longrightarrow \mathcal{L}_B(E)$ will be referred to as *a representation of A*.

- Given two representations $\pi \colon A \longrightarrow \mathcal{L}_B(E)$ and $\varrho \colon A \longrightarrow \mathcal{L}_B(F)$, we define

$$\pi \oplus \varrho \colon A \longrightarrow \mathcal{L}_B(E) \oplus \mathcal{L}_B(F) \cong \mathcal{L}_B(E \oplus F); \quad a \mapsto (\pi(a), \varrho(a)).$$

Here the latter identification is merely the assignment mapping a pair (a, b) into the adjointable operator $T_{a,b} \colon E \oplus F \longrightarrow E \oplus F$ fulfilling $T_{a,b}(\xi, \eta) = (a\xi, b\xi)$ for all $\xi \in E$ and $\eta \in F$.

The multiplier algebra associated to any C*-algebra may be endowed with a natural locally convex Hausdorff topology. Due to convergence becoming absolutely pivotal once we introduce homotopies of representations, we are poised to address topological aspects. Suppose therefore that A is some C*-algebra. Define for each a in A seminorms $\| \cdot \|_{ra}, \| \cdot \|_{\ell a} \colon \mathcal{M}(A) \longrightarrow \mathbb{R}^+$ by the formulas

$$\|x\|_{ra} = \|xa\|_A, \quad \text{respectively}, \quad \|x\|_{\ell a} = \|ax\|_A.$$

The locally convex Hausdorff topology generated by the set $\{\| \cdot \|_{ra}, \| \cdot \|_{\ell a} : a \in A\}$ is commonly referred to as *the strict topology*. In this topology, convergence of a net $(x_i)_{i \in I}$ occurs if and only if

$$\|xa - x_i a\|_A \to 0 \quad \text{together with} \quad \|ax - ax_i\|_A \to 0 \tag{4.2}$$

hold for all a belonging to A. Notice further that upon identifying $\mathcal{L}_B(E)$ with $\mathcal{M}(\mathcal{K}_B(E))$, one may equip the former with the strict topology.

The strict topology, multiplier algebras and adjointables permit us to properly investigate KK-theory in two equivalent, although different, pictures. Since each perspective has strengths over its equivalent counterpart, we shall lean on the usage of each variance throughout the entire chapter. As such we exhibit both rigorously.

Definition (Cuntz' picture). Suppose A denotes any C*-algebra.

- A *quasihomomorphism* from A into B is a pair (π, ϱ) consisting of two not necessarily unital representations $\pi, \varrho \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ for which the difference $\pi(a) - \varrho(a)$ belongs to $B \otimes \mathbb{K}$ for all a in A. We denote by $\mathbb{E}_c(A, B)$ the collection of quasihomomorphisms from A into B.

- A *quasihomotopy* connecting two quasihomomorphisms (π, ϱ) and (φ, ψ) is a family of quasiho-momorphisms $(\pi_t, \varrho_t)_{t \in [0,1]}$ from A into B, indexed over $[0, 1]$, fulfilling

$$(\pi, \varrho) = (\pi_1, \varrho_1) \quad \text{together with} \quad (\varphi, \psi) = (\pi_0, \varrho_0).$$

Additionally, we demand that $t \mapsto \pi_t(a)$ and $t \mapsto \varrho_t(a)$ are strictly continuous maps for each a in A whereas $t \mapsto \pi_t(a) - \varrho_t(a)$ is required to be norm continuous for every such a. The obtained equivalence relation is symbolically denoted by $\sim_h$.

- The space $\mathbb{E}_c(A, B)$ may be equipped with an associative composition $\oplus$ given by

$$(\pi, \varrho) \oplus (\pi_0, \varrho_0) := (\pi \oplus \pi_0, \varrho \oplus \varrho_0)$$

for any two pairs of quasihomomorphisms.

Definition (Kasparov's picture). Let A denote some C*-algebra and let E, F be Hilbert B-modules.

- A triple (π, ϱ, u) consisting of two representations $\pi \colon A \longrightarrow \mathcal{L}_B(E)$, $\varrho \colon A \longrightarrow \mathcal{L}_B(F)$ and an adjointable operator $u \colon E \longrightarrow F$ satisfying

$$u\pi(a) - \varrho(a)u \in \mathcal{K}_B(E, F),$$

$$\pi(a)(u^*u - 1_E) \in \mathcal{K}_B(E),$$

$$\varrho(a)(uu^* - 1_F) \in \mathcal{K}_B(F),$$

for all a belonging to A, is called a KK-*cycle* from A into B. We define $\mathbb{E}(A, B)$ to be the space of all KK-*cycles* from A into B, and endow it with the following associative composition:

$$(\pi, \varrho, u) \oplus (\pi_0, \varrho_0, u_0) := (\pi \oplus \pi_0, \varrho \oplus \varrho_0, u \oplus u_0).$$

A cycle (π, ϱ, u) is *degenerate* should the attached containments found above amount to the expressions on the left-hand side being identically zero. The space of degenerate cycles from A into B is denoted by $\mathbb{D}(A, B)$.

- An *operatorial homotopy* between KK-*cycles* (π, ϱ, u) and (φ, ψ, v) is a collection $(\pi_t, \varrho_t, u_t)_{t \in [0,1]}$ of cycles, indexed over $[0, 1]$, such that $t \mapsto u_t$ is norm continuous, $t \mapsto \pi_t(a)$ and $t \mapsto \varrho_t(a)$ are strictly continuous for each a in A, while fulfilling

$$(\pi_1, \varrho_1, u_1) = (\pi, \varrho, u) \quad \text{together with} \quad (\pi_0, \varrho_0, u_0) = (\varphi, \psi, v).$$

In this scenario, we call (π, ϱ, u) *operatorially homotopic* to (φ, ψ, v), symbolically represented via $(\pi, \varrho, u) \sim_{\mathrm{oh}} (\varphi, \psi, v)$. This is easily seen to be an equivalence relation.

- Two KK-*cycles*, say (π, ϱ, u) and (φ, ψ, v), are referred to as being *equivalent* if there exists a degenerate cycle (γ, γ_0, w) for which

$$(\pi, \varrho, u) \oplus (\gamma, \gamma_0, w) \sim_{\mathrm{oh}} (\varphi, \psi, v) \oplus (\gamma, \gamma_0, w).$$

We write $(\pi, \varrho, u) \sim_d (\pi, \varrho, v)$ to represent this occurrence.

Notice that the right-hand side of the compositions comprise a representation from A into

$$\mathcal{M}(B \otimes \mathbb{K}) \oplus \mathcal{M}(B \otimes \mathbb{K}) \cong \mathcal{M}((\mathbb{K} \oplus \mathbb{K}) \otimes B) \cong \mathcal{M}(B \otimes \mathbb{K}).$$

The first identification stems from the permanence property (iv) of proposition A.0.4 while the latter amounts to an exploitation of separability. For Kasparov's KK-theoretic picture, one employs proposition 4.1.3. It is customary to regard $(\varphi \oplus \varphi_0)(a)$ as the 2×2-diagonal matrix, acting on $\mathcal{L}_B(E) \oplus \mathcal{L}_B(E)$, attaining the value $\varphi(a)$ in the upper-left corner with $\varphi_0(a)$ in the lower-right corner. The general isomorphism $\mathrm{M}_n(\mathcal{L}_A(E)) \cong \mathcal{L}_A(E^n)$ of C*-algebras allows such interpretations. The isomorphism may be deduced in a fashion resembling the analogues statement for $\mathrm{B}(\mathcal{H})$.

Definition. Suppose A denotes a C*-algebra and let B be some σ-unital C*-algebra. We hereto define the KK-*groups* in the Kasparov - and Cuntz picture by

$$\mathrm{KK}_c(A, B) = \mathbb{E}_c(A, B) / \sim_h, \quad \text{respectively,} \quad \mathrm{KK}(A, B) = \mathbb{E}(A, B) / \sim_d,$$

endowed with the induced associative commutative compositions

$$[\varphi, \psi, u] + [\pi, \varrho, v] := [(\varphi, \psi, u) \oplus (\pi, \varrho, v)] \quad \text{and} \quad [\varphi, \psi] + [\pi, \varrho] := [(\varphi, \psi) \oplus (\pi, \varrho)]$$

in the respective order.

The type of elements in KK-theory we shall primarily encounter arise from existing *-homomorphisms. Suppose $\pi \colon A \longrightarrow B$ denotes some *-homomorphism. Selecting any finite rank projection p acting on $\mathcal{H}$, one may embed B into $B \otimes \mathbb{K}$ via the *-monomorphism given by $b \mapsto p \otimes b$. As such π may be regarded as a representation into $\mathcal{M}(B \otimes \mathbb{K}) \cong \mathcal{L}_B(\mathcal{H}_B)$. Therefore $(\pi, 0)$ becomes a quasihomomorphism whereas $(\pi, 0, 0)$ defines a degenerate cycle, hence

$$[\pi]_c := [\pi, 0] \quad \text{together with} \quad [\pi] := [\pi, 0, 0] \tag{4.3}$$

define elements in $\mathrm{KK}_c(A, B)$ and $\mathrm{KK}(A, B)$, respectively, referred to as *induced morphisms*.

Remark. One may be fooled to believe that the group axioms are automatic. However, it ought to be emphasized that the compositions on $\mathbb{E}(A, B)$ and $\mathbb{E}_c(A, B)$ are non-commutative. Moreover, they lack inverses. In the quotients, everything fortunately pans out neatly. Due to details becoming important momentarily, it seems prudent to elaborate moderately.

Consider any degenerate KK-*cycle* of the form $(\pi, \varrho, 1)$. Then $[\pi, \varrho, 1] = 0$ holds in $\mathrm{KK}(A, B)$. Similarly, the quasihomomorphism (π, π) associated to a representation $\pi \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ induces the zero class in $\mathrm{KK}_c(A, B)$, being homotopic to $(0, 0)$. Given a cycle $[\pi, \varrho, u]$, one acquires the equality $-[\pi, \varrho, u] = [-\pi, -\varrho, -u]$. This may be verified by considering the operator homotopy

$$\left(\begin{bmatrix} \pi & 0 \\ 0 & -\pi \end{bmatrix}, \begin{bmatrix} \varrho & 0 \\ 0 & -\varrho \end{bmatrix}, \begin{bmatrix} u \cos t & \sin t \\ \sin t & -u \cos t \end{bmatrix} \right),$$

which connects $[\pi, \varrho, u] + [-\pi, -\varrho, -u]$ at $t = 0$ to

$$\left(\begin{bmatrix} \pi & 0 \\ 0 & -\pi \end{bmatrix}, \begin{bmatrix} \varrho & 0 \\ 0 & -\varrho \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

at $t = \pi/2$. Since the latter expression has a degenerate cycle as representative, the claim follows. In an analogues manner, one may verify that $-[\pi, \varrho] = [\varrho, \pi]$, because the sum equals the induced class of $(\pi \oplus \varrho, \varrho \oplus \pi)$ which may be continuously rotated into $(\pi \oplus \varrho, \pi \oplus \varrho)$.

As a final note, the abelian groups $\mathrm{KK}(A, B)$ and $\mathrm{KK}_c(A, B)$ are isomorphic via the mapping induced from $(\pi, \varrho) \mapsto (\pi, \varrho, 1)$. We forego the proof completely for brevity.

As opposed to ordinary K-theory, KK-theory engages morphisms instead of projections or unitaries of some given C*-algebra. In K-theory, equality translates into *stable equivalence*, meaning Murray

von-Neumann equivalence up to adding a a projection: $[p]_0 = [q]_0$ relative to some C*-algebra A if and only if $p \oplus r \sim q \oplus r$ for some projection r in $\mathrm{M}_k(A)$. One may ponder and pose the question whether a resembling feature occurs in KK-theory. Our objective will be to (partially) supply an affirmative answer. First some terminology.

Definition. Let A be some C*-algebra and let E, F be Hilbert B-modules.

- Two representations $\pi \colon A \longrightarrow \mathcal{L}_B(E)$ and $\varrho \colon A \longrightarrow \mathcal{L}_B(F)$ are said to be *approximately unitarily equivalent* if there exists a sequence $(u_n)_{n \geq 1}$ of unitaries in $\mathcal{L}_B(F, E)$ such that

$$\lim_{n \to \infty} \|\pi(a) - u_n \varrho(a) u_n^*\| = 0 \quad \text{together with} \quad \pi(a) - u_m \varrho(a) u_m^* \in \mathcal{K}_B(E)$$

  for every a in A and each positive integer m. We write $\pi \sim_{a.u} \varrho$ to symbolically represent this.

- Two representations $\pi, \varrho \colon A \longrightarrow \mathcal{L}_B(E)$ are said to be *properly approximately unitarily equivalent* if there exists a family $(u_t)_{t \in \mathbb{R}^+}$ consisting of unitaries in $\mathcal{K}_B(E)^+$ such that

$$\lim_{t \to \infty} \|\pi(a) - u_t \varrho(a) u_t^*\| = 0 \quad \text{together with} \quad \pi(a) - u_t \varrho(a) u_t^* \in \mathcal{K}_B(E)$$

  for all a in A and every positive real number t. Moreover, the corresponding assignment $t \mapsto u_t$ is required to be norm-continuous. We write $\pi \approx_{p.u} \varrho$ to symbolically represent this.

Approximately unitarily equivalence is commonly very restrictive to demand under the assumption that $[\pi] = [\varrho]$, or in the Cuntz picture $[\pi]_c = [\varrho]_c$, so we will instead attempt to detect proper asymptotic unitary equivalence modulo adding a suitable representation. First of all, the easy implication revealing that proper unitary equivalence removes KK-theoretic obstructions. It further serves the purpose of justifying that the equivalence relation $\approx_{\mathrm{p.u}}$ has potential.

Proposition 4.1.5. *Let $\pi, \varrho \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ be properly unitarily equivalent representations of some C*-algebra A. Then (π, ϱ) becomes a quasihomomorphism with $[\pi, \varrho] = 0$ in $\mathrm{KK}(A, B)$.*

Proof. Suppose $(u_t)_{t \in \mathbb{R}^+}$ represents the continuous path of unitaries witnessing proper unitary equivalence of π and ϱ. Due to $(B \otimes \mathbb{K})^+$ containing $B \otimes \mathbb{K}$ as an ideal, we may deduce that $\pi(a) - \varrho(a) \in B \otimes \mathbb{K}$ since each u_t belongs to $\mathcal{K}(\mathcal{H}_B)^+ \cong (B \otimes \mathbb{K})^+$. To verify that $[\pi, \varrho, 1]$ must be zero, assume for the time being that $[\pi, \varrho, 1] = [\pi, u_1 \pi(\cdot) u_1^*, 1]$ must be valid. Evidently, $(\pi, u_1 \pi(\cdot) u_1^*, 1)$ and $(\pi, \pi, 1)$ are unitarily equivalent, i.e., the coordinates of the cycles are unitarily equivalent. As unitary equivalent cycles induce the same KK-classes, we have

$$[\pi, \varrho, 1] = [\pi, u_1 \pi(\cdot) u_1^*, 1] = [\pi, \pi, 1] = 0.$$

Ergo the proof reduces to establishing $[\pi, \varrho] = [\pi, u_1 \pi u_1^*]$. To accomplish this, put

$$\mu_t(a) = \pi(a) \quad \text{and} \quad \sigma_t(a) = \begin{cases} u_{1/t} \pi(a) u_{1/t}^*, & \text{if } t > 0, \\ \varrho(a), & \text{if } t = 0, \end{cases}$$

for each real number $0 \leq t \leq 1$. By our hypothesis imposed on the unitaries $(u_t)_{t \in \mathbb{R}^+}$ in conjunction with (π, ϱ) determining a quasihomomorphism, the family $\{(\mu_t, \sigma_t) : t \in \mathbb{R}^+\}$ defines a quasihomotopy transforming (π, ϱ) into $(\pi, u_1 \pi(\cdot) u_1^*)$. This proves the claim. $\square$

As described previously, our objective will be to develop a converse statement, at least in the nuclear setting; such statements are commonly referred to as *uniqueness* statements of morphisms.

4.2   Strict Nuclearity

The observation in proposition 4.1.5 greatly enhances our ability to detect the lack of KK-theoretic distinction between $*$-homomorphisms. Deriving a converse statement requires substantial effort and in fact ordinary KK-theory a priori seems inadequate to tackle the issue. Therefore, we add a cherry on top: Strict nuclearity. The crux of restricting to a particular subgroup stems from the ability to enable nuclearity, the idea originating from Skandalis in [40], whose work we record. Throughout the entire section, we shall regard B as being a fixed σ-unital C$*$-algebra.

Definition. Let A be any C$*$-algebra.

- A completely positive map $\psi \colon A \longrightarrow \mathcal{L}_B(E)$ is *strictly nuclear* if there exists a net $(\psi_\alpha)_{\alpha \in J}$ consisting of finite-dimensionally factorable maps such that $\psi_\alpha(\cdot) \to \psi(\cdot)$ strictly.

- We define the *nuclear KK-theory group* of A and B as follows. Suppose $\mathbb{E}_{\mathrm{nuc}}(A,B)$ denotes the subset in $\mathbb{E}_c(A,B)$ consisting of strictly nuclear quasihomomorphisms and let $\sim_{\mathrm{nuc}}$ be the equivalence stemming from restricting homotopy of quasihomomorphisms to strictly nuclear quasihomomorphisms. Define accordingly
$$\mathrm{KK}_{\mathrm{nuc}}(A,B) = \mathbb{E}_{\mathrm{nuc}(A,B)} / \sim_{\mathrm{nuc}} .$$

- Parallel to the previous definition, the nuclear version of Kasparov's picture may be defined in the following manner. Let $\mathbb{E}_{\mathrm{nuc}}(A,B)$ represent the collection of all KK-cycles[2] whose morphisms are strictly nuclear and let $\mathbb{D}_{\mathrm{nuc}}(A,B)$ be the subcollection of strictly nuclear degenerate cycles. If $\sim_{\mathrm{n.d}}$ denotes the equivalence relation obtained from $\sim_d$ by restricting to $\mathbb{D}_{\mathrm{nuc}}(A,B)$, then
$$\mathrm{KK}_{\mathrm{nuc}}(A,B) = \mathbb{E}_{\mathrm{nuc}}(A,B) / \sim_{\mathrm{n.d}}$$

  defines the nuclear version of Kasparov's KK-group.

Remark. If one mimics the proof concerning the statement that nuclearity of completely positive maps defined on nuclear C$*$-algebras is automatic, one may deduce the same statement for strict nuclearity — certainly, the norm-topology is stronger than the strict topology.

   Observe that $\pi \oplus \varrho$ remains strictly nuclear whenever each coordinate is strictly nuclear. As such $\mathrm{KK}_{\mathrm{nuc}}(A,B)$ determines a normal subgroup within $\mathrm{KK}(A,B)$, hence induces a canonical map $\theta \colon \mathrm{KK}_{\mathrm{nuc}}(A,B) \longrightarrow \mathrm{KK}(A,B)$.

The initial step towards establishing uniqueness of induced elements in KK-theory and lack of KK-theoretic obstruction between them will be to translate the current distinction of representatives in the same class into unitary equivalence. For any representation $\pi \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$, let
$$D_\pi := \{ b \in \mathcal{M}(B \otimes \mathbb{K}) : [b, \pi(A)] \subseteq B \otimes \mathbb{K} \}.$$

Lemma 4.2.1. *Let A be some unital separable C$*$-algebra, let $\pi \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ be a unital representation and suppose w_0, w_1 belong to D_π. If $[\pi, \pi, w_0] = [\pi, \pi, w_1]$ in $\mathrm{KK}(A,B)$, then there exists a unital representation $\gamma \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ such that*
$$(\pi \oplus \gamma, \pi \oplus \gamma, w_0 \oplus 1) \sim_{\mathrm{oh}} (\pi \oplus \gamma, \pi \oplus \gamma, w_1 \oplus 1).$$

Under the hypothesis that π is strictly nuclear with $[\pi, \pi, w_1] = [\pi, \pi, w_2]$ in $\mathrm{KK}_{\mathrm{nuc}}(A,B)$ the representation γ may be chosen to be strictly nuclear.

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[2]Some abuse of notation is being extended here. However, it ought to emphasize on no distinction occurring for neither Kasparov's nor Cuntz' nuclear picture. We shall employ both versions and will remark when either enters.

Proof. Due to $[\pi, \pi, w_0] - [\pi, \pi, w_1] = 0$ in $\mathrm{KK}(A, B)$, the representatives (π, π, w_0) and (π, π, w_1) must differ by a degenerate cycle (γ_0, γ_1, u) up to operatorial homotopy, where $\gamma_i \colon A \longrightarrow \mathcal{L}_B(E_i)$ for $i = 0, 1$ is the representation and $u \colon E_0 \longrightarrow E_1$ is an adjointable operator, meaning

$$(\pi \oplus \gamma_0, \pi \oplus \gamma_1, w_0 \oplus u) \sim_{\mathrm{oh}} (\pi \oplus \gamma_0, \pi \oplus \gamma_1, w_1 \oplus u). \tag{4.4}$$

Suppose $(u_t)_{t \in [0,1]}$ denotes the continuous path of operators implementing the transformation of $w_0 \oplus u$ into $w_1 \oplus u$. We will tacitly reduce the proof into the unital scenario with u being a unitary. To achieve this, look at the projection $p_i = \gamma_i(1_A)$ in $\mathcal{L}_B(E_i)$. Upon decomposing E_i into the direct sum $p_i E_i \oplus p_i^\perp E_i$ then restrcting γ_i to the first summand, we may assume that γ_i becomes unital. Having arranged this, one may have the relation (4.4) withstand by replacing u with $p_1 u p_0$ and each u_t with $(1 \oplus p_1) u_t (1 \oplus p_0)$. The new cycle $(\gamma_0, \gamma_1, p_1 u p_0)$ remains degenerate, $(1 \oplus p_1) u_0 (1 \oplus p_0) = w_0 \oplus p_1 u p_0$ and $(1 \oplus p_1) u_1 (1 \oplus p_0) = w_1 \oplus p_1 u p_0$, thereby constituting an operatorial homotopy between the designated substitutions of E_i, u, u_t and γ_i. For instance, degeneracy of (γ_0, γ_1, u) forces $u p_0 = p_1 u$, so that degeneracy once more implies that

$$\gamma_0(a)(p_0 - (p_1 u p_0)^*(p_1 u p_0)) = \gamma_0(a)(p_0 - u^* p_1 u) p_0 = \gamma_0(a)(1_{E_0} - u^* u) p_0 = 0.$$

The remaining degeneracy-conditions are verified in resembling fashions. Furthermore, degeneracy of the new cycle easily reveals that $p_1 u p_0$ becomes a unitary. Having established (4.4) in the unital case with u being a unitary, notice that the family of triples $(\pi \oplus \gamma_0, \pi \oplus \gamma_0, (1 \oplus u^*) u_t)$, continuously indexed over $[0, 1]$, defines an operatorial homotopy

$$(\pi \oplus \gamma_0, \pi \oplus \gamma_0, w_0 \oplus 1) \sim_{\mathrm{oh}} (\pi \oplus \gamma_0, \pi \oplus \gamma_0, w_1 \oplus 1). \tag{4.5}$$

If one picks an element t in $[0, 1]$ and some a in A, then

$$(\pi \oplus \gamma_0)(a)(1_{\mathcal{H}_B \oplus E_0} - u_t^*(1_{\mathcal{H}_B} \oplus u)(1_{\mathcal{H}_B} \oplus u^*) u_t) = (\pi(a) \oplus \gamma_0(a))(1_{\mathcal{H}_B \oplus E_0} - u_t^* u_t)$$

yields the first condition of a cycle, since the elements $(u_t)_{t \in [0,1]}$ implementing the operatorial homotopy (4.4) entail that the right-hand side above must be compact for each t in $[0, 1]$. The remaining axioms of a cycle are verified similarly. To finish the proof, we need some trickery allowing the choice $E_i = \mathcal{H}_B$ for each index $i = 0, 1$. Let $\sigma \colon A \longrightarrow \mathcal{L}_B(\mathcal{H}_B)$ be any unital representation. Due to (4.5), the degenerate cycle $(\sigma, \sigma, 1_{\mathcal{H}_B})$ evidently satisfies

$$(\pi \oplus \gamma_0 \oplus \sigma, \pi \oplus \gamma_0 \oplus \sigma, w_0 \oplus 1_{E_0} \oplus 1_{\mathcal{H}_B}) \sim_{\mathrm{oh}} (\pi \oplus \gamma_0 \oplus \sigma, \pi \oplus \gamma_0 \oplus \sigma, w_1 \oplus 1_{E_0} \oplus 1_{\mathcal{H}_B}). \tag{4.6}$$

Invoking Kasparov's stabilization theorem, we may extract a unitary $v \colon E_0 \oplus \mathcal{H}_B \longrightarrow \mathcal{H}_B$ witnessing the isomorphism $E_0 \oplus \mathcal{H}_B \cong \mathcal{H}_B$ of Hilbert B-modules. Defining $\gamma = \mathrm{Ad}_v \circ (\gamma_0 \oplus \sigma)$ gives a unital representation $\gamma \colon A \longrightarrow \mathcal{L}_B(\mathcal{H}_B)$ such that $(\pi \oplus \gamma, \pi \oplus \gamma, w_i \oplus 1)$ becomes a cycle for each index $i = 0, 1$. On the merits of (4.6), the cycle fulfills

$$(\pi \oplus \gamma, \pi \oplus \gamma, w_0 \oplus 1) = (\pi \oplus \mathrm{Ad}_v(\gamma_0 \oplus \sigma), \pi \oplus \mathrm{Ad}_v(\gamma_0 \oplus \sigma), w_0 \oplus \mathrm{Ad}_v(1_{E_0} \oplus 1_{\mathcal{H}_B}))$$
$$\sim_{\mathrm{oh}} (\pi \oplus \mathrm{Ad}_v(\gamma_0 \oplus \sigma), \pi \oplus \mathrm{Ad}_v(\gamma_0 \oplus \sigma), w_1 \oplus \mathrm{Ad}_v(1_{E_0} \oplus 1_{\mathcal{H}_B}))$$
$$= (\pi \oplus \gamma, \pi \oplus \gamma, w_1 \oplus 1).$$

This tackles the assertion in the non-nuclear case. Repeating the proof and noting that the degenerate cycle (γ_0, γ_1, u) may be chosen to be strictly nuclear whenever π is with $[\pi, \pi, w_1] = [\pi, \pi, w_2]$ in $\mathrm{KK}_{\mathrm{nuc}}(A, B)$, the newly assigned map $\gamma = \mathrm{Ad}_v \circ (\gamma_0 \oplus \sigma)$ becomes strictly nuclear provided that both summands γ_0 and σ are strictly nuclear, being the conjugation of a strictly nuclear map. Since γ_0 could be chosen to be strictly nuclear, one needs to justify that a strictly nuclear representation $\sigma \colon A \longrightarrow \mathcal{L}_B(\mathcal{H}_B)$ exists. We refer to *proposition* 2.18 in [18] for this. $\qquad\square$

For convenience, we add a lifting lemma of unitaries.

Lemma 4.2.2. *Suppose A denotes a unital C^*-algebra containing a non-trivial ideal I. Let*

$$0 \longrightarrow I \overset{\iota}{\longrightarrow} A \overset{q}{\longrightarrow} B \longrightarrow 0$$

be a short-exact sequence of C^*-algebras. If so, every unitary w in A fulfilling $q(w) \sim_h 1_B$ relative to $\mathcal{U}(B)$ provides a unitary w_0 in I^+ homotopic to w relative to $\mathcal{U}(A)$.

Proof. Suppose the relation $q(w) \sim_h 1_B$ is detected via some continuous path $(u_t)_{t\in[0,1]}$ of unitaries in B. Compactness of the unit interval and continuity of the map $t \mapsto u_t$ provides a positive integer n making the family $1_B = u_0, u_1, \ldots, u_n = q(w)$ of unitaries in B satisfy $u_{k+1} \approx_1 u_k$ for each $k \le n-1$. We inductively construct unitaries $v_0, v_1, \ldots, v_n$ in A such that v_j lifts u_j while $v_j \approx_1 v_{j+1}$ for every $1 \le j \le n-1$. Indeed, since unitaries within a distance of 2 from one another are homotopic, these unitaries extend to a homotopy $v_0 \sim_h v_n$ relative to $\mathcal{U}(A)$ in which $q(v_0) = 1_B$ (hence v_0 lies in I^+ by exactness). It therefore suffices to establish the existence of v_{n-1}, upon setting $v_n := w$ and repeating the process.

Write $z_n = u_{n-1}u_n^*$ for each n. Since $u_{n-1} \approx_1 u_n$, the normal element $z_n - 1_B = u_{n-1}u_n^* - 1_B$ defines a strict contraction, so its spectral radius cannot exceed 1. Hence -1 cannot belong to the spectrum of z_n. We must thus have $\exp(i\theta) \notin \sigma(z_n)$ for some real value θ. The acquired continuous map $f\colon (\theta, \theta + 2\pi) \longrightarrow \mathbb{R}$ given by $\exp(it) \mapsto t$ satisfies $\lambda = \exp(if(\lambda))$ for each λ in the spectrum of z_n. Letting $b_n = f(z_n)$ yields a self-adjoint element in B subject to $z_n = \exp(ib_n)$. Lift b_n via q to some self-adjoint a_n in A with $\|a_n\| = \|b_n\|$. Set $v_{n-1} = \exp(ia_n)v_n$ and observe that

$$q(v_{n-1}) = \exp(iq(a_n))q(v_n) = \exp(ib_n)u_n = u_{n-1}.$$

Furthermore, we have

$$\|v_n - v_{n-1}\| \le \|1 - \exp(ia_n)\| = \|1 - \exp(ib_n)\| \le \|u_n - u_{n-1}\| < 1.$$

Thus we have constructed v_{n-1}, proving the claim in view of our previous remarks. $\qquad\square$

Our next step towards a characterization of equality in KK-theory begins with introducing a generalization of the Calkin algebra, *the corona algebras*. Let A be a non-unital C^*-algebra and symbolically write $\mathcal{Q}_*(A) := \mathcal{M}(A)/A$. For our purposes A will serve as $B \otimes \mathbb{K}$ in which case β will represent the canonical $*$-epimorphism $\beta\colon \mathcal{M}(B \otimes \mathbb{K}) \longrightarrow \mathcal{Q}_*(B \otimes \mathbb{K})$.

A peculiar feature hereof is the following short-exact sequence. Consider the image of β when restricting to D_π. An element $\beta(x)$ herein is zero whenever x defines an element in $B \otimes \mathbb{K}$ whose commutator $[x, \pi(A)]$ with $\pi(A)$ remains in $B \otimes \mathbb{K}$. Setting for each subset $M \subseteq \mathcal{Q}_*(B \otimes \mathbb{K})$,

$$M^c := \{y \in \mathcal{Q}_*(B \otimes \mathbb{K}) : [y, M] \subseteq B \otimes \mathbb{K}\},$$

we acquire a short-exact sequence

$$0 \longrightarrow B \otimes \mathbb{K} \overset{j}{\longrightarrow} D_\pi \overset{\beta}{\longrightarrow} (\beta\pi(A))^c \longrightarrow 0 \tag{4.7}$$

of C^*-algebras with $*$-homomorphisms as morphisms. We unveil the use of (4.7) momentarily. As a final intermezzo, we mimic the ℓ^2-construction associated to a C^*-algebra for Hilbert modules. Let E be some Hilbert B-module and write $E^\infty = E \oplus E \oplus \ldots$ as a $\mathbb{C}$-vector space. Define accordingly a Hilbert B-module in terms of the following structural data:

$$E^\infty := \left\{ (\xi_n)_{n\ge 1} \in E^\infty : \sum_{n=1}^\infty \|\xi_n\|_E^2 < \infty \right\}; \quad \langle (\xi_1, \xi_2, \ldots), (\eta_1, \eta_2, \ldots) \rangle = \sum_{n=1}^\infty \langle \xi_n, \eta_n \rangle_E.$$

The ordinary techniques proving completeness of $\ell^2(\mathbb{N})$ may be adapted to verify completeness of E^∞. Here the algebraic operations are the obvious/usual ones for B-modules. Given a (unital) representation $\pi\colon A \longrightarrow \mathcal{L}_B(E)$ there exists an induced (unital) representation $\pi_\infty\colon A \longrightarrow \mathcal{L}_B(E^\infty)$ defined by the assignment $\pi_\infty(a)(\xi_1, \xi_2, \ldots) = (\pi(a)\xi_1, \pi(a)\xi_2, \ldots)$ for $(\xi_n)_{n\ge 1}$ in E^∞.

Theorem 4.2.3 (Dadarlat-Eilers). *Suppose A and B are separable C^*-algebras. Let (π, ϱ) be any quasihomomorphism from A into B. Then the following statements are equivalent.*

(i) $[\pi, \varrho] = 0$ *in* $\mathrm{KK}(A, B)$.

(ii) There exists a representation $\sigma \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ such that $\pi \oplus \sigma \approx_{p.u} \varrho \oplus \sigma$.

Additionally, the representation σ may be chosen to be unital, should A admit one, whenever π, ϱ are unital. Lastly, if π and ϱ are strictly nuclear, then the following are equivalent.

(i) $[\pi, \varrho] = 0$ *in* $\mathrm{KK}_{\mathrm{nuc}}(A, B)$.

(ii) There exists a strictly nuclear representation $\sigma \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ such that $\pi \oplus \sigma \approx_{p.u} \varrho \oplus \sigma$.

Proof. The implication "(ii) $\Leftarrow$ (i)" in each case is immediate from proposition 4.1.5. We will commence by reducing the whole situation to the unital case. Suppose (i) is equivalent to (ii) under the hypothesis that the maps π, ϱ are unital. Upon the unitization A^+ inducing unital representations $\pi_+, \varrho_+ \colon A^+ \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$, each being strictly nuclear whenever π, ϱ are so, in conjunction with $[\pi_+, \varrho_+] = 0$ remaining valid in $\mathrm{KK}(A, B)$, we may appeal to the unital scenario to construct σ_+. The sought representation σ may be chosen as the restriction of σ_+ to the ideal $A \trianglelefteq A^+$. Thus, it suffices to verify the implication (i) $\Rightarrow$ (ii) with π, ϱ being unital.

Step 1. We shall manufacture σ by invoking lemma 4.2.1, then adjust the placement of the unitaries occurring therefrom. The map $\gamma_0 := \pi_\infty \oplus \varrho_\infty$ must be a unital representation from A into $\mathcal{L}_B((\mathcal{H}_B)^\infty \oplus (\mathcal{H}_B)^\infty)$, whose codomain may be identified with $\mathcal{L}_B(\mathcal{H}_B)$ due to Kasparov's stabilization theorem. Exploiting the same argument enables us to deduce that $\pi \oplus \gamma_0$ and $\varrho \oplus \gamma_0$ are unital representations of A into $\mathcal{M}(B \otimes \mathbb{K})$. Kasparov's stabilization theorem ensures the existence of a unitary u_0 in $\mathcal{L}_B(\mathcal{H}_B)$ conjugating $\pi \oplus \gamma_0$ onto $\varrho \oplus \gamma_0$, yielding $(\pi \oplus \gamma_0, \pi \oplus \gamma_0, u_0) \sim_{\mathrm{oh}} (\pi \oplus \gamma_0, \varrho \oplus \gamma_0, 1)$. We therefore have

$$[\pi \oplus \gamma_0, \pi \oplus \gamma_0, u_0] = [\pi \oplus \gamma_0, \varrho \oplus \gamma_0, 1] = [\pi, \varrho, 1] = 0 = [\pi \oplus \gamma_0, \pi \oplus \gamma_0, 1].$$

The second equality is based on the cycle $(\gamma_0, \gamma_0, 1)$ being degenerate. Invoking lemma 4.2.1 grants us a unital representation $\gamma \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ such that

$$(\pi \oplus \gamma_0 \oplus \gamma, \pi \oplus \gamma_0 \oplus \gamma, u_0 \oplus 1) \sim_{\mathrm{oh}} (\pi \oplus \gamma_0 \oplus \gamma, \pi \oplus \gamma \oplus \gamma_0, 1 \oplus 1). \tag{4.8}$$

Observe that the strictly nuclear version of lemma 4.2.1 permits (4.8) to withstand with γ being strictly nuclear. Stipulate that $\Lambda_\pi := \pi \oplus \gamma_0 \oplus \gamma$, $\Lambda_\varrho := \varrho \oplus \gamma_0 \oplus \gamma$ together with $u := u_0 \oplus 1$. Then $\Lambda_\varrho = u \Lambda_\pi(\cdot) u^*$, and we may infer that $(\Lambda_\pi, \Lambda_\pi, u) \sim_{\mathrm{oh}} (\Lambda_\pi, \Lambda_\pi, 1)$. We take $\sigma = \gamma \oplus \gamma_0$.

Step 2. We shall later on apply a result concerning asymptotically inner automorphisms. However, doing so requires the unitary u to be homotopic to another unitary in $(B \otimes \mathbb{K})^+$ relative to $\mathcal{U}(D_{\Lambda_\pi})$, which we access now. Through to the operatorial homotopy $(\Lambda_\pi, \Lambda_\pi, u) \sim_{\mathrm{oh}} (\Lambda_\pi, \Lambda_\pi, 1)$ established during the preceding step, we may find a norm continuous path of unitaries $(w_s)_{s \in [0,1]}$ in $\mathcal{M}(B \otimes \mathbb{K})$ subject to $w_0 = u$, $w_1 = 1$ and the conditions

$$[\Lambda_\pi(a), w_s], \ \Lambda_\pi(a)(w_s w_s^* - 1), \ \Lambda_\pi(a)(w_s^* w_s - 1) \in B \otimes \mathbb{K}. \tag{4.9}$$

for each s in $[0,1]$ and a in A. Due to the former containment of (4.9) combined with unitality of β, $\beta(w_s)$ defines a unitary in $(\beta \Lambda_\pi(A))^c$ for every such s in $[0,1]$. Additionally, the acquired norm continuous path of unitaries $(\beta(w_s))_{s \in [0,1]}$ joints $\beta(u)$ to the unit of $\mathcal{Q}_*(B \otimes \mathbb{K})$. For instance, the latter in conjunction with the former containment of (4.9) ensure that

$$1 - \beta(w_s)^* \beta(w_s) = \beta(\Lambda_\pi(1_A) - w_s^* w_s) = 0,$$

with the remaining conditions being shown similarly. Due to lemma 4.2.2 applied to the short exact sequence (4.7), we are guaranteed the existence of a unitary v in $(B \otimes \mathbb{K})^+$ homotopic to u, relative to $\mathcal{U}(D_{\Lambda_\pi})$ as desired.

Step 3. Let E be the C*-algebra $\Lambda_\pi(A) + B \otimes \mathbb{K}$. We will now arrange a norm-continuous path of unitaries $(z_t)_{t \in \mathbb{R}^+}$ such that

$$\lim_{t \to \infty} \|z_t \Lambda_\pi(a) z_t^* - \Lambda_\varrho(a)\| = 0 \tag{4.10}$$

for every a inside A. To accomplish this, we equip the group of automorphisms on any C*-algebra with the uniform topology, i.e., the point-norm topology. In this regard, Ad_u becomes homotopic to Ad_v relative to $\mathrm{Aut}(E)$ by the second step. The homotopy of unitaries entails that $\alpha := \mathrm{Ad}_{v^*u}$ becomes homotopic to the id_E relative to $\mathrm{Aut}(E)$, for $u \sim_h v$ ensures that $v^*u \sim_h 1$. Let $(\alpha_s)_{s \in [0,1]}$ be the uniformly continuous path of unitaries joining α to the identity on E. Consider the C*-subalgebra $D(n, j_1, \ldots, j_n)$ generated by the set

$$(\alpha_{s_1}^{j_1} \circ \alpha_{s_2}^{j_2} \circ \cdots \circ \alpha_{s_n}^{j_n})(\Lambda_\pi(A)), \quad n \in \mathbb{N}, \ j_k \in \mathbb{Z}, \ s_k \in [0,1] \cap \mathbb{Q}.$$

Let D be the C*-subalgebra generated by all the subalgebras $D(n, j_1, \ldots, j_n)$. Then D must be unital and separable, since Λ_π is unital while A is assumed to be separable. Due to D being invariant under $(\alpha_t)_{t \in [0,1]}$, meaning $\alpha_t(D) = D$ for each $t \in [0,1]$, the automorphism α must be homotopic to id_D via automorphisms therein. As such, *proposition* 2.14 in [16] provides a continuous path of unitaries $(z_t)_{t \in \mathbb{R}^+} \subseteq vD \subseteq E$ subject to $z_0 = v$ and

$$\lim_{t \to \infty} \|z_t d z_t^* - u d u^*\| = 0 \tag{$*$}$$

for each element d in D. Thus the relation $\Lambda_\pi(\cdot) = u\Lambda_\varrho(\cdot)u^*$ implies that

$$\lim_{t \to \infty} \|z_t \Lambda_\pi(a) z_t^* - \Lambda_\varrho(a)\| = \lim_{t \to \infty} \|z_t \Lambda_\pi(a) z_t^* - u\Lambda_\pi(a)u^*\| \overset{(*)}{=} 0$$

for every a belonging to A, granting (4.10).

Final Step. Recall that $\sigma = \gamma_0 \oplus \gamma$. We now craft our unitaries detecting proper asymptotic unitary equivalence of $\pi \oplus \sigma$ and $\varrho \oplus \sigma$. Since $\Lambda_\pi = \pi \oplus \sigma$ together with $\Lambda_\varrho = \varrho \oplus \sigma$ hold, the task boils down to determining a continuous path of unitaries $(u_t)_{t \in \mathbb{R}^+}$ in $(B \otimes \mathbb{K})^+$ fulfilling

$$\lim_{t \to \infty} \|u_t \Lambda_\pi(a) u_t^* - \Lambda_\varrho(a)\|.$$

By construction of E, we may determine some $x_t \in A$ and $y_t \in B \otimes \mathbb{K}$ subject to $z_t = \Lambda_\pi(x_t) + y_t$ for any t inside $\mathbb{R}^+$. Let $\beta \colon D_{\Lambda_\pi} \longrightarrow \beta(\pi(A))^c$ be the *-epimorphism of (4.7). The *-homomorphism $\beta\Lambda_\pi$ is unital by construction, and we may without loss of generality assume that $\beta\Lambda_\pi$ is injective, hence must be an isometry. Let some a in A be fixed. As (π, ϱ) is a quasihomomorphism, the difference $\pi(\cdot) - \varrho(\cdot)$ targets the compact adjointables, whence $\beta\pi = \beta\varrho$. Since $z_t z_t^* = 1$ and $\Lambda_\pi(1_A) = 1$, the family $(x_t)_{t \in \mathbb{R}^+}$ must necessarily be a continuous path of unitaries satisfying

$$\lim_{t \to \infty} \|x_t a x_t^* - a\| = \lim_{t \to \infty} \|q(\Lambda_\pi(x_t)\Lambda_\pi(a)\Lambda_\pi(x_t)^* - \Lambda_\varrho(a))\| \leq \lim_{t \to \infty} \|z_t \Lambda_\pi(a) z_t^* - \Lambda_\varrho(a)\| \overset{(4.10)}{=} 0.$$

The first equality stems from the isometric property of $q\Lambda_\pi$. Setting $u_t := z_t \Lambda_\pi(x_t)^* = 1 + y_t \Lambda_\pi(x_t)^*$ for each $t \in \mathbb{R}^+$ yields the desired unitaries, since

$$\|u_t \Lambda_\pi(a) u_t^* - \Lambda_\varrho(a)\| \leq \|z_t \Lambda_\pi(a) z_t^* - \Lambda_\varrho(a)\| + \|z_t \Lambda_\pi(x_t^* a x_t - a) z_t^*\|$$
$$\leq \|z_t \Lambda_\pi(a) z_t^* - \Lambda_\varrho(a)\| + \|x_t^* a x_t - a\|$$

tends to zero as $t \to \infty$, completing the proof in view of our initial remark and our occasional mentioning of how strict nuclearity may be arranged throughout the proof. $\square$

The implication (i)$\Rightarrow$(ii) is within reasonable proximity of uniqueness result. Uniqueness here should be read as some approximation natured version of $[\pi] = [\varrho]$ in KK-theory implying $\mathrm{Ad}_u \circ \pi = \varrho$ for some suitable unitary u. The weakened version we shall endorse concerns achieving approximate unitary equivalence upon stabilizing with a nuclearly absorbing representation. Before deriving this, we bring precision to the statement.

Definition. Let A be a (unital) C*-algebra and let E, F be Hilbert B-modules.

- A non-unital representation $\pi\colon A \longrightarrow \mathcal{L}_B(E)$ is called *nuclearly absorbing* if $\pi \oplus \varrho \sim_{\mathrm{a.u}} \pi$ for every strictly nuclear representation $\varrho\colon A \longrightarrow \mathcal{L}_B(F)$.

- A unital representation $\pi\colon A \longrightarrow \mathcal{L}_B(E)$ is called *unitally nuclearly absorbing* if $\pi \oplus \varrho \sim_{\mathrm{a.u}} \pi$ for every strictly nuclear unital representation $\varrho\colon A \longrightarrow \mathcal{L}_B(F)$.

In order to circumvent the necessity of unitality, we deduce some properties of nuclear absorption.

Proposition 4.2.4. *Suppose A denotes a separable C*-algebra.*

(i) *If $\pi\colon A \longrightarrow \mathcal{L}_B(E)$ is non-unital nuclearly absorbing, the induced map $\pi_+\colon A^+ \longrightarrow \mathcal{L}_B(E)$ given by $a + \lambda 1_{A^+} \mapsto \pi(a) + \lambda 1_E$ becomes unitally nuclearly absorbing.*

(ii) *With A unital, every unitally nuclearly absorbing representation $\pi\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ induces a non-unital nuclearly absorbing representation $\pi_s\colon A \longrightarrow \mathcal{M}(\mathcal{K}(\mathcal{H} \oplus \mathcal{H}) \otimes B)$ given by the sum $\pi_s = 0 \oplus \pi$, where the first summand acts on $\mathcal{H}_B$.*

Proof. (i): Let $\sigma\colon A^+ \longrightarrow \mathcal{L}_B(F)$ be a unital strictly nuclear representation. The restriction σ_0 of σ onto A remains strictly nuclear, whereupon $\pi \oplus \sigma_0 \sim_{\mathrm{a.u}} \pi$ holds by hypothesis. Let $(u_n)_{n \geq 1}$ be the sequence of unitaries from $E \oplus F$ into E witnessing the equivalence. Then

$$u_n(\pi_+ \oplus \sigma)(a + \lambda 1_{A^+})u_n^* = u_n(\pi(a) \oplus \sigma_0(a) + \lambda(1_{E \oplus F}))u_n^* \to \pi(a) + \lambda 1_E = \pi_+(a + \lambda 1_{A^+}).$$

(ii): Let $\sigma\colon A \longrightarrow \mathcal{L}_B(F)$ be some strictly nuclear representation. Write $p = \sigma(1_A)$ to obtain a projection acting on E and decompose accordingly $E = pE \oplus p^\perp E$. The representation σ thus attains the form $\sigma = p\sigma(\cdot)p \oplus 0$ having the second summand act on $p^\perp E$, so that $\sigma_p := p\sigma(\cdot)p$ defines a unital strictly nuclear representation $\sigma_p\colon A \longrightarrow \mathcal{L}_B(pE)$, being the compression of a strictly nuclear map by a projection. Due to π being unitally nuclearly absorbing, one has $\sigma_p \oplus \pi \sim_{\mathrm{a.u}} \pi$. Invoking Kasparov's stabilization theorem, $p^\perp E \oplus \mathcal{H}_B \cong \mathcal{H}_B$ follows and hence

$$\sigma \oplus \pi_s = (\sigma_p \oplus 0_{p^\perp E}) \oplus (0_{\mathcal{H}_B} \oplus \pi) \sim_{\mathrm{a.u}} \sigma_p \oplus \pi \oplus 0_{\mathcal{H}_B} \sim_{\mathrm{a.u}} 0_{\mathcal{H}_B} \oplus \pi = \pi_s.$$

This proves the claim. $\square$

Lemma 4.2.5. *Suppose $\pi, \varrho, \gamma, \sigma\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ are unital representations fulfilling the equivalences $\pi \oplus \gamma \approx_{p.u} \varrho \oplus \gamma$ and $\gamma \sim_{a.u} \sigma$. Then there exists a sequence $(u_n)_{n \geq 1}$ comprised of unitaries belonging to $(B \otimes \mathbb{K})^+$ subject to*

$$\|u_n(\pi \oplus \sigma)(a)u_n^* - (\varrho \oplus \sigma)(a)\| \to 0$$

for every element a in A.

Proof. On one hand, there is a continuous path of unitaries $(w_t)_{t \in \mathbb{R}^+} \subseteq (\mathcal{K}(\mathcal{H}^2) \otimes B)^+$ such that

$$\lim_{t \to \infty} \|w_t(\pi(a) \oplus \gamma(a))w_t^* - \varrho(a) \oplus \gamma(a)\| = 0$$

for every $a \in A$. On the other hand, we may find unitaries $(v_n)_{n \geq 1} \subseteq (\mathbb{K} \otimes B)^+$ fulfilling

$$\lim_{n \to \infty} \|\gamma(a) - v_n \sigma(a)v_n^*\| = 0$$

for every a inside A. Setting $u_n = (1 \oplus v_n)w_n(1 \oplus v_n)^*$ defines a unitary in $(\mathcal{K}(\mathcal{H}^2) \otimes B)^+$ such that

$$
\begin{aligned}
&\|u_n(\pi(a) \oplus \sigma(a))u_n^* - \varrho(a) \oplus \sigma(a)\| \\
&\leq \|w_n(\pi(a) \oplus v_n\sigma(a)v_n^*)w_n^* - \varrho(a) \oplus v_n\sigma(a)v_n^*\| \\
&= \|w_n(\pi(a) \oplus v_n\sigma(a)v_n^*)w_n^* \pm \varrho(a) \oplus \gamma(a) - \varrho(a) \oplus v_n\sigma(a)v_n^* \pm w_n(\pi(a) \oplus \gamma(a))w_n^*\| \\
&\leq \|\gamma(a) - v_n\sigma(a)v_n^*\| + \|v_n\sigma(a)v_n^* - \gamma(a)\| + \|w_n(\pi(a) \oplus \gamma(a))w_n^* - \varrho(a) \oplus \gamma(a)\| \\
&= 2\|\gamma(a) - v_n\sigma(a)v_n^*\| + \|w_n(\pi(a) \oplus \gamma(a))w_n^* - \varrho(a) \oplus \gamma(a)\| \to 0.
\end{aligned}
$$

This verifies the claim. $\square$

Proposition 4.2.6 (Dadarlat-Eilers)**.** *Let A be some separable C^*-algebra, B be some σ-unital C^*-algebra and let further $\pi, \varrho\colon A \longrightarrow B \otimes \mathbb{K}$ be two nuclear $*$-homomorphisms inducing the class in $\mathrm{KK}_{nuc}(A, B)$. Let $\gamma\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ be any nuclearly absorbing representation. Under these premises, there exists, for each finite $F \subseteq A$ and $\varepsilon > 0$, a unitary $u \in \mathrm{M}_3(B \otimes \mathbb{K})^+$ fulfilling*

$$\|u(\pi(a) \oplus 0 \oplus \gamma(a))u^* - (\varrho(a) \oplus 0 \oplus \gamma(a))\| < \varepsilon$$

for every element a belonging to F.

Proof. Choose some finite subset $F \subseteq A$ and tolerance $\varepsilon > 0$. By our hypothesis imposed on the $*$-homomorphisms π, ϱ, they form a strictly nuclear quasihomomorphism (π, ϱ) subject to

$$[\pi, \varrho] = [\pi, 0] + [0, \varrho] = [\pi, 0] - [\varrho, 0] = 0$$

in $\mathrm{KK}_{\mathrm{nuc}}(A, B)$. Invoking theorem 4.2.3, we may deduce that $\pi \oplus \sigma \approx_{\mathrm{p.u}} \varrho \oplus \sigma$ for some strictly nuclear representation $\sigma\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$. Since the induced representation $\gamma_s = 0 \oplus \gamma$ defines a non-unital nuclear absorbing representation regardless of whether γ is unital or not, see proposition 4.2.4, we may infer that $\sigma \oplus \gamma_s \sim_{\mathrm{a.u}} \gamma_s$. This in turn yields

$$\pi \oplus \sigma \oplus 0 \oplus \gamma \approx_{\mathrm{p.u}} \varrho \oplus \sigma \oplus 0 \oplus \gamma \quad \text{and} \quad \sigma \oplus 0 \oplus \gamma \sim_{\mathrm{a.u}} 0 \oplus \gamma.$$

The sought unitaries are hereby obtained by applying the preceding lemma to the representations $\pi, \varrho, \sigma \oplus \gamma$ and γ_s in the respective order of the lemma, completing the proof. $\qquad\square$

4.3   Adding Quasidiagonality

We draw near our designated stable uniqueness. However, we should locate a unitary in some matrix algebra of B as opposed to its stabilization. The added representation γ along the lower-right diagonal entry there needs to be controlled with greater efficiency. To adjust the result into a more manageable uniqueness result, Dadarlat and Eilers throw quasidiagonality into the mix.

Definition. A representation $\gamma\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ for any pair of C^*-algebras A, B is *quasidiagonal* if there exists an approximate unit $(e_n)_{n \geq 1}$, quasicentral in $\gamma(A)$, of finite-rank projections in $B \otimes \mathbb{K}$.

Remark. Regarding $B \otimes \mathbb{K}$ as the inductive limit of the sequence $(\mathrm{M}_n(B), d_n)_{n \geq 1}$, we may assume each projection e_n lies in $\mathrm{M}_{r_n}(B)$ for some integers $(r_n)_{n \geq 1}$. Using this convention, the corresponding sequence $(\gamma_n)_{n \geq 1}$ consisting of completely positive maps $\gamma_n\colon A \longrightarrow \mathrm{M}_{r_n}(B)$ given by $a \mapsto e_n \gamma(a) e_n$ is referred to as *the quasidiagonalization* of γ.

Theorem 4.3.1 (Dadarlat-Eilers)**.** *Suppose A and B denote unital C^*-algebras with A being separable. Let $\gamma\colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ be a quasidiagonal unitally nuclearly absorbing representation, quasidiagonality being implemented via the projections $(e_n)_{n \geq 1}$. Let further $(\gamma_n)_{n \geq 1}$ be the quasidiagonalization of γ with targets $\mathrm{M}_{r_n}(B)$. Suppose $\pi, \varrho\colon A \longrightarrow B$ are two nuclear $*$-homomorphisms inducing the same class in $\mathrm{KK}_{nuc}(A, B)$ such that $\pi(1_A)$ is unitarily equivalent to $\varrho(1_A)$, and assume in addition one may arrange that*

$$e_n \pi(a) e_n = \pi(a) \quad \text{and} \quad e_n \varrho(a) e_n = \varrho(a) \tag{4.11}$$

for every a inside A. Under these premises, there exists, for every finite subset $F \subseteq A$ and tolerance $\varepsilon > 0$, a positive integer n together with a unitary u in $\mathrm{M}_{r_n+1}(B)$ fulfilling

$$\|u(\pi(a) \oplus \gamma_n(a))u^* - (\varrho(a) \oplus \gamma_n(a))\| < \varepsilon$$

whenever a lies in F.

Proof. Certain estimates occurring during the proof are omitted for brevity. However, we will flag any scenario wherein quasidiagonality enters the scene for emphasis. To ease the notational burden, we add the abbreviations $\pi_\gamma(\cdot) = (\pi \oplus 0 \oplus \gamma)(\cdot)$ and $\varrho_\gamma(\cdot) = (\varrho \oplus 0 \oplus \gamma)(\cdot)$. Upon replacing the map ϱ with the nuclear $*$-homomorphism $v\varrho(\cdot)v^*$, where v denotes an existing unitary implementing the unitary equivalence of $\varrho(1_A)$ and $\pi(1_A)$, we may assume that $\pi(1_A) = \varrho(1_A)$ throughout the proof. Apply proposition 4.2.6 to obtain a unitary v in $(\mathrm{M}_3(\mathbb{K}) \otimes B)^+$ satisfying

$$v\pi_\gamma(a)v^* \approx_\varepsilon \varrho_\gamma(a), \quad a \in F. \tag{4.12}$$

Define a projection by $p_n = e_n \oplus e_n \oplus e_n$ for each positive integer n. Due to $(e_n)_{n \geq 1}$ being quasicentral on the image of γ, we have $\|[v, p_n]\| \to 0$ by (4.11). Our following step will be to perturb $p_n v p_n$ towards a unitary. Indeed setting $x_n = p_n v p_n$ for any integer n, we acquire

$$\|x_n^* x_n - p_n\| \leq \|(p_n v^* - v^* p_n)p_n v p_n\| + \|v^* p_n(v p_n - p_n v)\| + \|v^* p_n v - p_n\|$$
$$\leq \|[p_n, v^*]\| + 2\|[v, p_n]\| \to 0.$$

Note that the quasicentral property of $(e_n)_{n \geq 1}$ arising from quasidiagonality is crucial here. Therefore $x_n x_n^*$ becomes invertible in the unital corner algebra induced by p_n, hence it admits a unitary polar decomposition $x_n x_n^* = z|x_n^* x_n|$ therein. The unitary $z := x_n x_n^* |x_n^* x_n|^{-1}$ in $\mathrm{M}_{3r_n}(B)$ will be within $0 < \delta < 1^3$ distance of x_n. For simplicity, write

$$\pi_{\gamma_n}(a) = \pi(a) \oplus 0 \oplus \gamma_n(a) \quad \text{and} \quad \varrho_{\gamma_n}(a) := \varrho(a) \oplus 0 \oplus \gamma_n(a)$$

for every a in A. Then (4.11) entails that $\pi_{\gamma_n}(a) = p_n \pi_{\gamma_n}(a) p_n$ and $\varrho_{\gamma_n}(a) = p_n \varrho_{\gamma_n}(a) p_n$. Combining this particular observation with the bound $x_n \approx_\delta z$ alongside (4.12), some triangle inequality trickery will grant an estimate

$$z\pi_{\gamma_n}(a)z^* \approx_{h(\delta)} \varrho_{\gamma_n}(a) \tag{4.13}$$

for some bounded function h satisfying $h(\delta) \to 0$ should $\delta \to 0$. To finalize the proof, consider the projection $e = \pi(1_A) \oplus \gamma_n(1_A) = \varrho(1_A) \oplus \gamma_n(1_A)$, which evidently satisfies $e\pi_{\gamma_n}(\cdot)e = \pi_{\gamma_n}(\cdot)$ with a similar relation being valid for ϱ_{γ_n}. Repeating the perturbation procedure for the unitary z, we may assume without loss of generality that $zez^* = e$. Having arranged this, $w := eze$ becomes a partial isometry belonging to $\mathrm{M}_{r_n+1}(B)$ such that $w^*w = ww^* = e$, whereby the element

$$u := w + 1_{r_n+1} - e$$

must be a unitary in $\mathrm{M}_{r_n+1}(B)$. Let $a \in F$ be arbitrarily chosen. Letting δ be small enough to force $h(\delta) < \varepsilon$ and heavily exploiting that $e\pi_\gamma(a)e = \pi_\gamma(a)$ will give

$$\|u(\pi_{\gamma_n}(a))u^* - \varrho_{\gamma_n}(a)\| = \|w\pi_{\gamma_n}(a))w^* - \varrho_{\gamma_n}(a)\|$$
$$\leq \|eze\pi_{\gamma_n}(a)ez^*e - e\varrho_{\gamma_n}(a)e\|$$
$$\leq \|z\pi_{\gamma_n}(a)z^* - \varrho_{\gamma_n}(a)\| < \varepsilon,$$

wherein the latter bound stems from (4.13), completing the proof. $\square$

The Dadarlat-Eilers stable uniqueness theorem almost ends the section. In what follows, we seek to encapsulate the strategy whereby we enable it by posing the question: Why does even γ exist? The remainder of the section carries an exposition of building such maps.

Fullness will play an essential role momentarily and the crucial consideration is the induced "diagonal map" that captures γ in the shape of a representation on A. Suppose $\gamma \colon A \longrightarrow B$ defines a

---

[3]Specifying δ is unnecessary. We will only need to to be sufficiently small to force $\|x_n^* x_n - p_n\| < 1$ for invertibility and making the following estimates strictly smaller than ε.

unital full $*$-homomorphism. Let $\{e_{ij}\}_{i,j \geq 1}$ denote the ordinary matrix units in $\mathbb{M}_n$ of any dimension n. We define an associated "diagonal map" $d_\gamma \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ induced by γ via the formula

$$d_\gamma(a) = \sum_{k=1}^{\infty} \gamma(a) \otimes e_{kk}.$$

For completeness, we justify its existence and quasidiagonality: For an increasing sequence of rank n projections $(p_n)_{n \geq 1}$ converging strictly to $1_\mathcal{H}$, consider the sequence $e_n = p_n \otimes 1_B$ of projections in $B \otimes \mathbb{K}$. The existence of the projections p_n may be justified by noting that the strict topology on $\mathrm{B}(\mathcal{H}) = \mathcal{M}(\mathbb{K})$ coincides with the σ-strong topology, which on bounded subsets such as $\mathrm{Proj}(\mathcal{H})$ agrees with the strong-operator topology. We claim that the acquired sequence $(\gamma_n)_{n \geq 1}$ of maps $\gamma_n \colon A \longrightarrow \mathrm{M}_n(B)$ defined by

$$\gamma_n(a) := \sum_{k=1}^{n} \gamma(a) \otimes e_{kk} \quad \big(= \gamma^n(a) \big)$$

subsumes the role of a quasidiagonalitzation of γ. Certainly, the sequence $(\gamma_n)_{n \geq 1}$ converges in the point-strict topology (easily checked), hence d_γ becomes meaningful. Since each e_n is of rank n, the corner algebra $e_n \mathcal{M}(B \otimes \mathbb{K}) e_n$ embeds into $\mathbb{M}_n$ for each positive integer n. Ergo,

$$\gamma_n(a) = \sum_{k=1}^{n} \gamma(a) \otimes e_{kk} = e_n d_\gamma(a) e_n$$

must be valid for any a in A, proving the claim. The map d_ι is unital by strict continuity of left - and right multiplication. Having established quasidiagonality, we only lack nuclear absorption. We will lean on an alternative characterization found as *proposition* 2.19 in [18]. For the proof $\mathrm{CP}(A, B)$ denotes all c.p maps $\psi \colon A \longrightarrow B$.

Proposition 4.3.2. *Let A be a unital separable C^*-algebra, let B be some σ-unital C^*-algebra and let $\pi \colon A \longrightarrow \mathcal{L}_B(E)$ be a unital representation. Then the following are equivalent.*

(i) *π is unitally nuclearly absorbing.*

(ii) *For every unital completely positive map $\psi \colon A \longrightarrow \mathbb{M}_n \subseteq \mathcal{L}_B(B^n)$ there exists some norm-bounded sequence $(w_n)_{n \geq 1}$ in $\mathcal{K}_B(B^n, E)$ such that*

$$\lim_{n \to \infty} \|\psi(a) - w_n^* \pi(a) w_n\| \to 0 \quad \text{together with} \quad \lim_{n \to \infty} \|w_n^* e\| = 0$$

hold for each a in A and every e in $\mathcal{K}_B(E)$.

Theorem 4.3.3 (Dadarlat-Eilers, Lin)**.** *Suppose A denotes a unital separable C^*-algebra and let $\gamma \colon A \hookrightarrow B$ be unital and full. Then d_γ becomes unitally nuclearly absorbing.*

Proof. Let $\psi \colon A \longrightarrow \mathbb{M}_n \hookrightarrow \mathrm{M}_n(B) = \mathcal{L}_B(B^n)$ be a unital completely positive map. We extract nuclear absorption through the characterization of proposition 4.3.2. We reduce the task into the scenario in which ψ is a state, that is, whenever $n = 1$. Firstly, recall the one-to-one correspondence $\Delta \colon \mathrm{CP}(A, \mathrm{M}_n(B)) \longrightarrow \mathrm{CP}(\mathrm{M}_n(A), B)$ defined by.

$$\Delta \bigg(\sum_{i,j=1}^{n} e_{ij} \otimes \varphi_{ij}(\cdot) \bigg) \bigg(\sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} \bigg) := \sum_{i,j=1}^{n} \varphi_{ij}(a_{ij})$$

where $\{e_{ij}\}_{i,j}$ denotes the canonical unit matrices in $\mathbb{M}_n$, $a = [a_{ij}]$ in $\mathrm{M}_n(A)$ and $\varphi_{ij} \colon A \longrightarrow B$ is the $(i,j)'th$-coordinate map of some completely positive map $\varphi \colon A \longrightarrow \mathrm{M}_n(B)$.

This reduces to the case $n = 1$, because the amplification γ_n remains full, whereupon replacing γ with γ_n provides the general case from the $n = 1$ situation. Having accomplished this, the idea revolves around invoking excision of states, then supplying a Krein-Milman argument after settling the extremal case. By hypothesis, the map γ is faithful (fullness implies injectivity), hence we identify $\gamma(A)$ with A throughout. Suppose that $F \subseteq A$ is any finite subset and let some tolerance $\varepsilon > 0$ be given. Our objective will be to find some isometry v in $\mathcal{L}_B(B, B^m)$ for some m in $\mathbb{N}$ fulfilling

$$\psi(a) \approx_\varepsilon v^*(a \otimes 1_m)v \tag{4.14}$$

for every a inside F, having ψ be some state acting on A. Suppose initially that ψ is pure. Appealing to excision for pure states, we may find some $x \geq 0$ in A such that

$$\psi(a)x^2 \approx_\varepsilon xax. \tag{4.15}$$

Define next, for each $z \in (0, 1)$, continuous functions $f_z, g_z \colon [0, 1] \longrightarrow [0, 1]$ by

$$f_z(t) = \begin{cases} 0, & \text{if } t = 0 \\ \text{affine}, & \text{if } 0 \leq t \leq z \ , \\ 1, & \text{if } t \geq z \end{cases} \qquad g_z(t) = \begin{cases} 0, & \text{if } t \leq z \\ 1, & \text{if } 2^{-1}(z+1) \leq t \\ \text{affine}, & \text{if } z \leq t \leq 2^{-1}(z+1) \end{cases} .$$

Some continuous functional calculus easily entails the following properties.

- For a sufficiently large $z_0 \in (0, 1)$, one may assume that $x = f_{z_0}(x)$ due to $f_z(t) \to \mathrm{id}_{[0,1]}$ as $z \to 1$.

- The element $f_{z_0}(x) \geq 0$ still satisfies the estimate in (4.15).

- Setting $y = g_{z_0}(x)$ one acquires $\|x\| = \|y\| = 1$ and $xy = y = yx$.

Since π is full, one may appeal to lemma 1.2.5 to produce elements $b_1, \ldots, b_m \in B$ such that one has $b_1^* y^2 b_1 + b_2^* y^2 b_2 \ldots + b_m^* y^2 b_m = 1_B$. Here comes the trick: Letting v denote the column matrix $[yb_1, \ldots, yb_m]^t$ produces an element in $\mathcal{L}_B(B, B^n)$ satisfying $v^*v = 1_B$, meaning an isometry therein. Furthermore, the property $xy = y$ ensures that

$$(x \otimes 1_m)v = [xyb_1, \ldots, xyb_m]^t = [yb_1, \ldots yb_m]^t = v.$$

This in turn implies that $v^*(x \otimes 1_m) = v^*$. Therefore,

$$\|\psi(a)1_B - v^*(a \otimes 1_m)v\| = \|\psi(a)v^*(x^2 \otimes 1_m)v - v^*(x \otimes 1_m)(a \otimes 1_m)(x \otimes 1_m)v\|$$

$$\leq \|\psi(a)x^2 \otimes 1_m - xax \otimes 1_m\| \overset{(4.15)}{<} \varepsilon.$$

Therefore (4.14) has been established for pure states. For a general state ψ, determine positive real numbers $\alpha_1, \ldots, \alpha_k$ summing to 1 and pure states $\psi_1, \ldots, \psi_k$ on A such that

$$\left\| \psi(a) - \sum_{i=1}^{k} \alpha_i \psi_i(a) \right\| < \varepsilon/2, \quad a \in F. \tag{4.16}$$

Choose for each pure state an isometry v_i satisfying $v_i v_i^* \leq (0, \ldots, 0, 1_m \otimes 1_B, 0 \ldots)$ and (4.14) with respect to the tolerance $\varepsilon/2$, with the non-trivial entry occurring in the i'th stage. The isometry condition alongside $\alpha_1 + \ldots + \alpha_k = 1$ guarantee that $v := \sum_{i=1}^{k} \alpha_i^{1/2} v_i$ becomes an isometry. Finally, (4.14) in conjunction with (4.16) entail that

$$\|\psi(a) - v^*(a \otimes 1_{mk})v\| = \left\| \psi(a) - \sum_{i=1}^{k} \alpha_i(v_i^*(a \otimes 1_m)v_i) \right\|$$

$$\leq \left\| \psi(a) - \sum_{i=1}^{k} \alpha_i \psi_i(a) \right\| + \sum_{i=1}^{k} \alpha_i \|\psi_i(a) - v_i^*(a \otimes 1_m)v_i\| < \varepsilon$$

for every a in F, completing the proof. $\qquad\square$

4.4   Applying the UCT

To enable the stable uniqueness theorem, one must arrange a myriad of assumptions to activate it. The troublesome aspect arising hereby stems from K-theory not preserving general products, which causes a certain obstacle when dealing with ultrapowers. An additional hindrance concerns the control of the integer n occurring in Dardalat-Eilers' theorem; we need a n. The final section of the chapter attempts to unravel the machinery permitting these changes.

Let a sequence $(B_n)_{n \geq 1}$ consisting of C*-algebras be given. Denote the product algebra $\ell^\infty(B_k, \mathbb{N})$ by B for notational convenience. Let now $p_n \colon B \longrightarrow B_n$ be the n'th projection *-homomorphism. From functoriality, there are induced group homomorphisms

$$\mathrm{K}_i(p_n) \colon \mathrm{K}_i(B) \longrightarrow \mathrm{K}_i(B_n) \quad \text{and} \quad p_*^i \colon \mathrm{K}_i(B) \longrightarrow \prod_{n \in \mathbb{N}} \mathrm{K}_i(B_n)$$

for each $i = 1, 2$. The latter morphism is the product map induced by the homomorphisms $\mathrm{K}_i(p_n)$. The unfortunate situation which may occur is that p_*^i might fail to be an isomorphism. To encompass product - and limit algebras, including ultrapowers, Dadarlat and Eilers investigate the total K-theory. We introduce the notion, albeit without much further elaboration.

Definition. Suppose A denotes some C*-algebra. We define the i'th *K-theory of A with coefficients in* $\mathbb{Z}/n\mathbb{Z}$ as the abelian group

$$\mathrm{K}_i(A; \mathbb{Z}/n\mathbb{Z}) := \mathrm{K}_i(A \otimes B),$$

where B is any C*-algebra fulfilling $\mathrm{K}_i(B) \cong \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{K}_{i+1}(B) \cong \{0\}$[4] for each i in $\mathbb{N}$. The *total K-theory* of A is then the abelian group

$$\underline{\mathrm{K}}(A) = \prod_{n \in \mathbb{N}} \mathrm{K}_n(A; \mathbb{Z}/n\mathbb{Z}).$$

Here the product is interpreted as the direct product.

Maintain the previous notation with p_*^n being the group homomorphism induced by the projections $\mathrm{K}_n(p_k \otimes \mathrm{id}) \colon \mathrm{K}_n(B \otimes Z_n) \longrightarrow \mathrm{K}_n(B_k \otimes Z_n)$ with Z_n being any C*-algebra having $\mathrm{K}_n(Z_n) \cong \mathbb{Z}/n\mathbb{Z}$ and $\mathrm{K}_{n+1}(Z_n) \cong \{0\}$ as K-theoretic data. Consider the acquired homomorphism

$$\sigma \colon \underline{\mathrm{K}}(B) = \prod_{n \in \mathbb{N}} \mathrm{K}_n(B; \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\Pi_n \, p_*^n} \prod_{n \in \mathbb{N}} \left(\prod_{k \in \mathbb{N}} \mathrm{K}_n(B_k; \mathbb{Z}/n\mathbb{Z}) \right) = \prod_{k \in \mathbb{N}} \underline{\mathrm{K}}(B_k). \tag{4.17}$$

Group homomorphism between total K-theory are in general ,,flawed". One instead considers group homomorphisms preserving *Bockstein operations*, whose set we denote by $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(C))$ for any pair of C*-algebras A, B. However, to avoid deterring completely from the overall aim, we avoid the formal definition. Hopefully, the expert will not have qualms hereto.

Regardless, (4.17) produces morphisms between sets of morphisms as follows. Due to σ attaining values in the product occurring on the right-hand side of (4.17), it may uniquely be presented on the form $\sigma = (\sigma_1, \sigma_2, \ldots)$. Let σ^* denote the mapping assigning a morphism $\varphi \colon \underline{\mathrm{K}}(A) \longrightarrow \underline{\mathrm{K}}(B)$ to the morphism $(\varphi_k^\sigma)_{k \geq 1}$, having φ_k^σ be the composition

$$\varphi_k^\sigma \colon \underline{\mathrm{K}}(A) \xrightarrow{\varphi} \underline{\mathrm{K}}(B) \xrightarrow{\sigma_k} \underline{\mathrm{K}}(B_k).$$

Then one acquires the morphism

$$\sigma_* \colon \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)) \longrightarrow \prod_{k \in \mathbb{N}} \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B_k)). \tag{4.18}$$

---

[4]The choice is irrelevant for us. So one could select $\mathrm{K}_n(A; \mathbb{Z}_n) = \mathrm{K}_n(A \otimes \mathcal{O}_{n+1})$ as Cuntz in [14] computes the zeroth K-group of $\mathcal{O}_{n+1}$, namely $\mathrm{K}_0(\mathcal{O}_{n+1}) \cong \mathbb{Z}/n\mathbb{Z}$.

The first result we invoke tackles injectivity of (4.18). In order to have induced elements of limit algebras of KK-theory kept at bay, Dadarlat and Eilers impose constraints to these sequences, such as $(B_n)_{n \geq 1}$, to prevent unpredictable behavior. We address C*-algebras of this kin.

Definition. A C*-algebra A is an *admissible target of finite type* if A is unital, of real rank zero and fulfills the following K-theoretic properties.

· The canonical map $[\cdot]_1 \colon \mathcal{U}(A) \longrightarrow \mathrm{K}_1(A)$ is surjective.

· For any p in $\mathrm{K}_0(A)$, one has $[1_B]_0 + p \geq 0$ whenever $np \geq 0$ for some n in $\mathbb{N}$.

· For every k in $\mathbb{N}$ and projections p, q in $\mathrm{M}_k(A)$, $[p]_0 = [q]_0$ implies that $p \oplus 1_B \sim q \oplus 1_B$.

· For every p in $\mathrm{K}_0(A)$ and every n in $\mathbb{N}$, there exists some q in $\mathrm{K}_0(A)$ such that $-[1_B]_0 \leq q \leq [1_B]_0$ together with $p - q \in n\mathrm{K}_0(A)$ become valid.

The prime example, at least for us, is $\mathcal{Q}$. Indeed $\mathrm{K}_1(\mathbb{M}_n) = \{0\}$, hence continuity of the functor $\mathrm{K}_1(\cdot)$ yields $\mathrm{K}_1(A) = \{0\}$ for every UHF-algebra A, granting the first condition. The second one follows immediately for UHF-algebras due to their K_0-group being unperforated. Since $\mathbb{M}_n$ has the cancellation property, so does any finite dimensional C*-algebra being the finite direct sum of such. Cancellation passes to inductive limits, whereupon $\mathcal{Q}$ obtains cancellation and this clearly implies the third condition above. For the remaining one, notice that $n\mathrm{K}_0(\mathcal{Q}) = n\mathbb{Q} = \mathbb{Q}$, so that one may select $q = 0$. Real rank zero of $\mathcal{Q}_\omega$ stems from proposition B.0.7.

We proceed to stating the main theorem attached to the UCT. The theorem was proven by Dardalat and Eilers in [18], whose proof we omit. One ought to notice that their proof merely tackles limit algebras as opposed to ultraproducts. An inspection of their proof will permit one to adjust various passages to include ultraproduct algebras, verbatim. Therefore, our version deviates slightly from the original. The statement demands some setup

Theorem 4.4.1 (Kasparov)**.** *Let A, B, C be some C*-algebras with A separable. Then there exists an associative $\mathbb{Z}$-bilinear map "$\cdot$": $\mathrm{KK}(A, B) \times \mathrm{KK}(B, C) \longrightarrow \mathrm{KK}(A, C)$, written multiplicatively. The map is called the Kasparov product and has the following properties.*

(i) $[\pi] \cdot [\varrho] = [\pi \circ \varrho]$ for all $*$-homomorphisms $\pi \colon A \longrightarrow B$ and $\varrho \colon B \longrightarrow C$.

(ii) The abelian group $\mathrm{KK}(A, A)$ admits a unital ring-structure using the Kasparov product as multiplication while having $[\mathrm{id}_A]$ act as the unique multiplicative unit.

(iii) The Kasparov product restricts to $\mathbb{Z}$-bilinear maps

$$\text{``} \cdot \text{''}\colon \mathrm{KK}(A, B) \times \mathrm{KK}_{\mathrm{nuc}}(B, C) \longrightarrow \mathrm{KK}_{\mathrm{nuc}}(A, C);$$

$$\text{``} \cdot \text{''}\colon \mathrm{KK}_{\mathrm{nuc}}(A, B) \times \mathrm{KK}(B, C) \longrightarrow \mathrm{KK}_{\mathrm{nuc}}(A, C),$$

fulfilling the functorial property (i).

Remark. The third condition in fact contains additional information, each of which are straightforward although important. Suppose $\theta \colon \mathrm{KK}_{\mathrm{nuc}}(A, B) \longrightarrow \mathrm{KK}(A, B)$ denotes the canonical map for C*-algebras A, B with A being separable. Then the products of (iii) are compatible with θ, i.e.,

$$\theta(x \cdot y) = \theta(x) \cdot y \quad \text{and} \quad \theta(y' \cdot x') = y' \cdot \theta(x') \tag{4.19}$$

whenever $x \in \mathrm{KK}_{\mathrm{nuc}}(A, B)$, $y \in \mathrm{KK}(B, C)$ and $x' \in \mathrm{KK}_{\mathrm{nuc}}(B, C)$, $y' \in \mathrm{KK}(A, B)$, with C being an additional C*-algebra.

The condition (iii) together with (4.19) will be useful in the future. Our current main objective will be to activate the UCT-condition. We are therefore inclined to address the correspondences of ordinary K-theory with the bivariant KK-theory.

Proposition 4.4.2. *Suppose A denotes a separable C^*-algebra. Then there are isomorphisms*

$$\mathrm{KK}(\mathbb{C}, A) \cong \mathrm{K}_0(A) \quad and \quad \mathrm{KK}(C_0(\mathbb{R}), A) \cong \mathrm{K}_1(A)$$

of abelian groups. In particular, the Kasparov product induces group homomorphisms

$$\kappa_0 \colon \mathrm{KK}(A, B) \longrightarrow \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)); \quad \kappa_0([\pi, \varrho])([p]_0) = [p]_0 \cdot [\pi, \varrho],$$
$$\kappa_1 \colon \mathrm{KK}(A, B) \longrightarrow \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B)); \quad \kappa_1([\pi, \varrho])([u]_1) = [u]_1 \cdot [\pi, \varrho].$$

Here we adopt the Cuntz-picture of KK-theory.

Definition. A separable C^*-algebra A satisfies the *UCT-condition* should the sequence

$$0 \longrightarrow \mathrm{Ext}(\mathrm{K}_i(A), \mathrm{K}_{i+1}(B)) \longrightarrow \mathrm{KK}(A, B) \xrightarrow{\kappa_0 \oplus \kappa_1} \mathrm{Hom}(\mathrm{K}_i(A), \mathrm{K}_i(B)) \longrightarrow 0$$

be split short-exact for each σ-unital C^*-algebra B and any index $i = 0, 1$.

The UCT stands for „the universal coefficient theorem", connecting it to the original theorem of homological algebra. The impact of the UCT-condition reaches products of K-theory, i.e., the structure of the natural maps such as (4.17) and (4.18). One of the obstacles one would encounter during the proof of achieving quasidiagonality arises when attempting to apply the stable uniqueness theorem to $\mathcal{Q}_\omega$; a priori we cannot control induced by $*$-homomorphisms on ultraproducts in KK-theory. We now resolve the issue using admissible targets of finite type. In [17], it has been established that the UCT-condition entails surjectivity of the induced group homomorphism

$$\Lambda_A^B \colon \mathrm{KK}(A, B) \longrightarrow \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)) \tag{4.20}$$

associated to any separable C^*-algebra A and any σ-unital C^*-algebra B.

Proposition 4.4.3 (Dardalat-Eilers)**.** *Suppose $(B_n)_{n \geq 1}$ denotes a sequence consisting of admissible target algebras of finite type and write $B = \ell^\infty(B_n, \mathbb{N})$. Let A be some C^*-algebra.*

  (i)   *The C^*-algebras B and $\prod_\omega B_n$ are admissible targets of finite type.*

  (ii) *The map $\sigma \colon \underline{\mathrm{K}}(B) \longrightarrow \prod_{n \in \mathbb{N}} \underline{\mathrm{K}}(B_n)$ in (4.17) is injective*

  (iii) *The map $\sigma_* \colon \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)) \longrightarrow \prod_{n \in \mathbb{N}} \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B_n))$ in (4.18) is injective*

  (iv) *The induced map $\Lambda_A^E \colon \mathrm{KK}(A, E) \longrightarrow \mathrm{Hom}_\Lambda\big(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(E)\big)$, in which $E = \prod_\omega B_n$ or $E = B$, in (4.20) is an isomorphism if A fulfills the UCT-condition,*

An additional feature that the UCT resolves is the distinction between Skandalis' nuclear KK-theory and Kasparov's KK-theory; there are none. However, proving this requires a deep result due to Rosenberg and Schochet. The statement reformulates the UCT-condition in terms of invertibility in KK-theory. The proof is beyond the scope of the thesis. Alas, we merely state it. As a minor comfort we may fortunately explain the salient feature attached.

Definition. Suppose A, B are C^*-algebras with A separable. Let $\theta \colon \mathrm{KK}_{\mathrm{nuc}}(A, B) \longrightarrow \mathrm{KK}(A, B)$ be the canonical map.

• An element x in $\mathrm{KK}(A, B)$ is *invertible* provided the existence of an element y in $\mathrm{KK}(B, A)$ fulfilling $x \cdot y = 1_A$ and $y \cdot x = 1_B$ is guaranteed.

• Two C^*-algebras A, B are *KK-equivalent* if $\mathrm{KK}(A, B)$ contains an invertible element.

• We refer to A as being *KK-nuclear* if there exists some u in $\mathrm{KK}_{\mathrm{nuc}}(A, A)$ such that $\theta(u) = 1_A$.

Theorem 4.4.4 (Schochet-Rosenberg)**.** *A separable C^*-algebra E satisfies the UCT-condition if and only if E is KK-equivalent to some commutative C^*-algebra. Furthermore, the class of separable C^*-algebras satisfying the UCT-condition, denoted by $\mathcal{N}$, satisfies the following properties.*

(i) $\mathcal{N}$ is closed under taking inductive limits.

(ii) $A \in \mathcal{N}$ and $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$ entails that $B \in \mathcal{N}$.

(iii) $\mathcal{N}$ is closed under taking extensions by members of $\mathcal{N}$.

(iv) For actions α, β on A, both $A \rtimes_\alpha \mathbb{Z}$ and $A \rtimes_\beta \mathbb{R}$ belong to $\mathcal{N}$ whenever A does.

Proposition 4.4.5. *Let A be a separable C^*-algebra. Then the following are equivalent.*

(i) A is KK-nuclear.

(ii) For each separable C^*-algebra B, the canonical map $\theta \colon \mathrm{KK}_{\mathrm{nuc}}(A, B) \longrightarrow \mathrm{KK}(A, B)$ is an isomorphism of abelian groups.

(iii) For each separable C^*-algebra B, the canonical map $\theta \colon \mathrm{KK}_{\mathrm{nuc}}(B, A) \longrightarrow \mathrm{KK}(B, A)$ is an isomorphism of abelian groups.

Moreover, A is KK-nuclear whenever A is KK-equivalent to a nuclear C^*-algebra.

Proof. Obviously (ii) and (iii) both imply (i), hence we are only required to justify the implication (i)⇒(ii) by reproducing the argument symmetrically for (i)⇒(iii). Let u be an element in $\mathrm{KK}_{\mathrm{nuc}}(A, A)$ such that $\theta(u) = 1_A$. The map $\iota \colon \mathrm{KK}(A, B) \longrightarrow \mathrm{KK}_{\mathrm{nuc}}(A, B)$ defined by the assignment $x \mapsto u \cdot x$ must according to (4.19) obey the rule

$$\theta(\iota(x)) = \theta(u \cdot x) = \theta(u) \cdot x = 1_A \cdot x = x$$

for every x belonging to $\mathrm{KK}(A, B)$. Ergo θ determines an isomorphism. To verify the final statement, let u be some invertible element in $\mathrm{KK}(A, B)$ with B being K-nuclear. Using the recently established surjectivity, lift u via θ to some element x in $\mathrm{KK}_{\mathrm{nuc}}(A, B)$. Then $\theta(x \cdot u^{-1}) = u \cdot u^{-1} = 1_A$. $\qquad\square$

The sought consequence is that the UCT-condition detects KK-nuclearity. Collecting the plethora of observations with an added emphasis on sujrectivity of (4.18), we are fully prepared to modify the stable uniqueness theorem. The initial step will be to control product morphisms.

Proposition 4.4.6. *Let A be a member of $\mathcal{N}$, B_n be an admissible target of finite type algebra for $n \in \mathbb{N}$. Set furthermore $B = \ell^\infty(B_n, \mathbb{N})$. Suppose there are $*$-homomorphisms $\pi_n, \varrho_n \colon A \longrightarrow B_n$ such that they induce the same morphism*

$$\underline{\mathrm{K}}(\pi_n) = \underline{\mathrm{K}}(\varrho_n) \colon \underline{\mathrm{K}}(A) \longrightarrow \underline{\mathrm{K}}(B_n).$$

Then the induced $*$-homomorphisms $\pi_\infty, \varrho_\infty \colon A \longrightarrow \ell(B_n, \mathbb{N})$ define the same class in $\mathrm{KK}_{\mathrm{nuc}}(A, B)$.

Proof. By the hypothesis imposed on the sequences $(\pi_1, \pi_2, \ldots)$ and $(\varrho_1, \varrho_2, \ldots)$, they induce the same element $\bar{\pi} = \bar{\varrho} \in \prod_n \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B_n))$. According to proposition 4.4.3(ii-iii), the morphism σ_* is injective. By construction, $\sigma_*(\pi_*) = \bar{\pi} = \bar{\varrho} = \sigma_*(\varrho_*)$, which thus forces $\pi_* = \varrho_*$. Functoriality of total K-theory thus ensures that the morphisms $\pi_* = \varrho_*$ give rise to the same element

$$q_* \pi_* = q_* \varrho_* \colon \underline{\mathrm{K}}(A) \longrightarrow \underline{\mathrm{K}}(B) \xrightarrow{q_*} \underline{\mathrm{K}}(\ell(B_n, \mathbb{N}))$$

inside $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\ell(B_n, \mathbb{N})))$. Let us denote these elements by π_* and ϱ_* once more.

According to proposition 4.4.3(iv) combined with the UCT-condition of A, the map $\Lambda_A^{\ell(B_n, \mathbb{N})}$ in (4.20) must be an isomorphism of abelian groups. Therefore, we may regard the corresponding classes $[\pi]$ and $[\varrho]$ in $\mathrm{KK}_{\mathrm{nuc}}(A, \ell(B_n, \mathbb{N}))$ as being equal. Notice that these belong to Skandalis' nuclear version of KK-theory due to A being separable and in the UCT-class, see proposition 4.4.5 for details, as desired. $\qquad\square$

Full $*$-homomorphisms will play a central role when appealing to the stable uniqueness result. However, we will require such structural properties to pass into limit algebras, which may falter. Dardalat and Eilers bypasses the issue by restricting to simple domains. Unfortunately, our domain will not be simple, hence an altered version of the stable uniqueness theorem must be installed. In order to track fullness, we introduce the notion of control functions.

Definition. Suppose A and B denote unital C*-algebras. A *control function* attached to B is a function $\Delta\colon (A_+)_1 \setminus \{0\} \longrightarrow \mathbb{N}$. Adopting this notion, a unital $*$-homomorphism $\gamma\colon A \longrightarrow B$ is called Δ-*full* if, for every nonzero positive contraction a in A, there exist contractions $b_1,\ldots,b_{\Delta(a)}$ belonging to B such that
$$1_B = \sum_{k=1}^{\Delta(a)} b_k^* \gamma(a) b_k.$$

Observe that a unital $*$-homomorphism $\gamma\colon A \longrightarrow B$ is full if and only if B admits a control function Δ such that γ is Δ-full. Certainly, fullness entails the existence of a control function Δ turning γ Δ-full due to lemma 1.2.5 (in particular, unital $*$-homomorphisms having simple target algebras are automatically Δ-full). For the converse, we must to guarantee that the unit of B belongs to the norm-closure of the ideal $I_{\mathrm{alg}}(\gamma(a))$ generated by $\gamma(a)$ for each nonzero element a. Due to Δ-fullness implying this for all nonzero positive contractions a in A, the assertion may be deduced from $\gamma(|a|)$ belonging to the closure of $I_{\mathrm{alg}}(\gamma(a))$.

Lemma 4.4.7. *Suppose $(\psi_n)_{n\geq 1}$ is a sequence of nuclear completely positive maps $\gamma_n\colon A \longrightarrow E_n$ between C*-algebras with A being exact. Under these premises, the induced completely positive map $\gamma\colon A \longrightarrow \ell^\infty(E_n, \mathbb{N})$ must be nuclear.*

Proof. We verify the local approximate factorization property of γ, see section 1.4. For brevity, write $E = \ell^\infty(E_n, \mathbb{N})$. Let some finite subset $F \subseteq A$ together with tolerance $\varepsilon > 0$ be given. Based on nuclearity of the involved morphisms, define for each $n \in \mathbb{N}$ a finite-dimensional factorization
$$A \xrightarrow{\ \varphi_n\ } \mathbb{M}_{k(n)} \xrightarrow{\ \psi_n\ } E_n$$
such that $\psi_n \varphi_n(\cdot) \approx_{\varepsilon/2} \gamma_n(\cdot)$ on the set F. Exactness of A yields the existence of some nuclear faithful representation $\pi\colon A \longrightarrow \mathrm{B}(\mathcal{H})$. Appealing to Arveson's extension theorem, the completely positive map φ_n extends to completely positive map $\varphi_n'\colon \pi(A) \longrightarrow \mathbb{M}_{k(n)}$. Consider next the composition
$$\sigma\colon \pi(A) \xrightarrow{\ \varphi':=(\varphi_n')\ } \ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N}) \xrightarrow{\ \psi:=(\psi_n)\ } E.$$

The map σ remains completely positive being composed by such. By construction, we furthermore have $\|\sigma\pi(a) - \gamma(a)\| = \sup_n \|\psi_n \varphi_n' \pi(a) - \gamma_n(a)\| < \varepsilon/2$ for every element a belonging to F. From nuclearity of π, there exists some finite-dimensional factorization
$$A \xrightarrow{\ \pi_0\ } \mathbb{M}_\ell \xrightarrow{\ \pi_1\ } \pi(A)$$
comprised of completely positive maps such that $\pi(\cdot) \approx_{\varepsilon/2} \pi_0 \pi_1(\cdot)$ on F. It follows that
$$\sigma_\gamma\colon A \xrightarrow{\ \pi_0\ } \mathbb{M}_\ell \xrightarrow{\ \sigma\pi_1\ } E$$
determines a finite-dimensional factorable completely positive map such that $\sigma_\gamma(\cdot) \approx_\varepsilon \gamma(\cdot)$ on the finite subset F, completing the proof. $\square$

Remark. Comparing the following theorem below to theorem 4.3.3, it may seem peculiar why a uniform choice of the integer n is necessary. Indeed, this is the sole reason why the UCT-condition enters. The obstacle that would appear is a construction necessary to upgrade the maps (π_0, π_1, θ) constructed during the previous chapter, which cannot depend on n at any cost whatsoever!

Theorem 4.4.8 (Tikuisis-White-Winter). *Suppose A denotes a separable unital exact C^*-algebra in the UCT-class admitting a control function Δ. Fix some finite subset $F \subseteq A$ together with tolerance $\delta > 0$. Under these premises, there exists a positive integer n satisfying the following property. For each admissible target algebra B of finite type, unital Δ-full $*$-homomorphism $\gamma \colon A \longrightarrow B$ and nuclear $*$-homomorphisms $\pi, \varrho \colon A \longrightarrow B$ such that*

- $\pi_* = \varrho_* \colon \underline{K}(A) \longrightarrow \underline{K}(B)$*; and* $\qquad\qquad$ *(s.1)*

- $\pi(1_A)$ *is unitarily equivalent to* $\varrho(1_A)$, $\qquad\qquad$ *(s.2)*

there exists a unitary u in $\mathrm{M}_{n+1}(B)$ subject to

$$u(\pi(a) \oplus \gamma^n(a))u^* \approx_\delta \varrho(a) \oplus \gamma^n(a)$$

for every element a in F.

Proof. Seeking a contradiction, assume the conclusion fails. Let Δ be a control function attached to B, $F \subseteq A$ be finite and let $\delta > 0$ fixed. Then, for every positive integer n we may determine some admissible target algebra B_n of finite type, a unital Δ-full map $\gamma_n \colon A \longrightarrow B_n$ together with nuclear $*$-homomorphisms $\pi_n, \varrho_n \colon A \longrightarrow B_n$ fulfilling (s.1)-(s.2) and for which

$$\max_{a \in F} \|u(\pi_n(a) \oplus \gamma_n^n(a))u^* - \varrho_n(a) \oplus \gamma_n^n(a)\| \geq \delta \qquad (4.21)$$

holds for every unitary u in $\mathrm{M}_{n+1}(B_n)$.

To ease the notational burden, let $\pi_\infty, \varrho_\infty, \gamma_\infty \colon A \longrightarrow \ell(B_n, \mathbb{N})$ be the induced $*$-homomorphisms. According to lemma 4.4.7, nuclearity of the morphisms π_n, ϱ_n implies nuclearity of the induced maps π_∞ and ϱ_∞. Upon collecting the unitaries implementing (s.2), in the respective order, into a unitary of $\ell(B_n, \mathbb{N})$ one obviously obtains unitary equivalence of $\pi_\infty(1_A), \varrho_\infty(1_A)$.

We claim that γ_∞ must be Δ-full. Invoking Δ-fullness of each coordinate map γ_n, one may for each nonzero positive contraction $a \in A$ find contractions $b_{1,n}, \ldots, b_{\Delta(a),n}$ in B_n such that

$$1_{B_n} = \sum_{k=1}^{\Delta(a)} b_{k,n}^* \gamma_n(a) b_{k,n}.$$

Let $q \colon \ell^\infty(B_n, \mathbb{N}) \longrightarrow \ell(B_n, \mathbb{N})$ be the canonical quotient map. Setting $b_k := q(b_{k,1}, b_{k,2}, \ldots)$ yields an element in $\ell(B_n, \mathbb{N})$. Since $1_{\ell(B_n, \mathbb{N})} = q(1_{B_n})_{n \geq 1}$ in conjunction with the quotient map q being a $*$-homomorphism, one has

$$1_{\ell(B_n, \mathbb{N})} = \sum_{k=1}^{\Delta(a)} b_k^* \gamma_\infty(a) b_k.$$

Hence $\gamma_\infty(a)$ is Δ-full. Appealing to theorem 4.3.3, quasidiagonality of the induced diagonal representation $d_{\gamma_\infty} \colon A \longrightarrow \mathcal{M}(B \otimes \mathbb{K})$ follows. Remember that the quasidiagonalization of d_{γ_∞} is $(\gamma_\infty^n)_{n \geq 1}$. The stable uniqueness theorem (specifically theorem 4.3.3), applicable upon proposition 4.4.6 removing any KK-theoretic obstructions between π_∞ and ϱ_∞, yields the existence of a positive integer m together with a unitary u in $\mathrm{M}_{m+1}(\ell(B_n, \mathbb{N}))$ such that

$$u(\pi_\infty(a) \oplus \gamma_\infty^m(a))u^* \approx_\delta \varrho_\infty(a) \oplus \gamma_\infty^m(a)$$

whenever a belongs to F. Lift the unitary u to some sequence $(u_n)_{n \geq 1}$ consisting of unitaries in $\mathrm{M}_{m+1}(B_n)$, achievable via proposition 2.3.3. Choosing n sufficiently large will permit us to force

$$u_n(\pi_n(a) \oplus \gamma_n^m(a))u_n^* \approx_\delta \varrho_n(a) \oplus \gamma_n^m(a) \qquad (4.22)$$

for each element a inside F. Upon passing to a sufficiently large choice of $n \geq m$, the diagonally embedded version of u_n in $\mathrm{M}_{n+1}(B_n)$, meaning the unitary $v := u_n \oplus 1_{B_n} \oplus \ldots \oplus 1_{B_n}$ having $n - m$ copies of 1_{B_n}, fulfills (4.22). Thus, v becomes an obstruction towards (4.21), as required. $\qquad\square$

Chapter 5

Achieving Quasidiagonality

The labor finally comes to fruition. Having build a duo of $*$-homomorphisms (π_0, π_1) fulfilling certain compatibility criteria in terms of a third one θ and with a stable uniqueness theorem, quasidiagonality of a faithful trace τ on our nuclear separable C*-algebra satisfying the UCT is within our reach. Tikuisis, White and Winter manage to produce completely positive maps through (π_0, π_1) that are approximately multiplicative and remember the trace, thereafter stitch these maps together to form the designated morphism, much alike the method at the end of chapter 3.

The underlying strategy concerns a "stable uniqueness across the interval produce". Due to the extensive length, the proof has been separated into two major sections, commencing with proving the so-called patching lemma. In an honest attempt to shed some light upon the idea, a flawed outline of the main proof will be depicted, whereby the strategy hopefully becomes more vivid.

5.1   Patching

Let A be a nuclear unital separable C*-algebra satisfying the UCT-condition. Let (π_0, π_1, θ) be the triple of $*$-homomorphisms granted by proposition 3.4.5. Let Λ_0, Λ_1 be the restrictions of π_0, π_1 onto the suspension $C_0(0,1) \otimes A$. We initially tread the idea under a stronger condition than the actual one. Consider the situation wherein one may determine a positive integer n together with unitaries $u, v \in \mathrm{M}_{n+1}(\mathcal{Q}_\omega)$ such that

$$\Lambda_0^n = u(\Lambda_0^{n-1} \oplus \Lambda_1)u^* \quad \text{while} \quad \Lambda_1^n = v(\Lambda_1^{n-1} \oplus \Lambda_0)v^*. \tag{5.1}$$

One may even weaken the condition by demanding the unitary equivalences to occur on restrictions to $C_0(I_n, A)$ for some relatively open intervals $I_n \subseteq [0,1]$. In this manner, we may cut $[0,1]$ into $2n$ equidistant subintervals I_n. Imagine having established (5.1) on the restrictions to the $2n$ intervals I_n, then one acquires the chain of equivalences

$$\Lambda_0^n \sim_u \Lambda_0^{n-1} \oplus \Lambda_1 \sim_u \Lambda_0^{n-2} \oplus \Lambda_1^2 \sim_u \ldots \sim_u \Lambda_0 \oplus \Lambda_1^{n-1} \sim_u \Lambda_1^n$$

on the respective restrictions to I_n. Mimicking the argument presented on page 64 to each equivalence, one may collect these $2n$ $*$-homomorphisms via a partition of unity into a single completely positive map $\Lambda \colon A \longrightarrow \mathbb{M}_{2n} \otimes \mathbb{M}_2 \otimes \mathcal{Q}_\omega \cong \mathcal{Q}_\omega$ witnessing quasidiagonality of τ. To our dismay, the underlying induced maps to larger matrices will rely on n, hence without a uniform choice of n the argument becomes circular. To overcome the dependence, we are compelled to control n; this is why the uniform version of the stable uniqueness theorem was needed.

Summarizing, under the presumption that one separates $[0,1]$ into $2n$ suitable open intervals without losing compatibility, the issue remaining revolves around obtaining the unitary equivalences in (5.1). However, supplying (5.1) on the nose seems demanding, but fortunately an approximation-esque version will suffice. The patching lemma to be described expresses the asymptotic multiplicative and trace-preserving properties arising from (5.1) up to any tolerance.

Definition. Suppose A denotes some C*-algebra. We will refer to a quadruple $(\pi, \varrho, \theta, E)$ consisting of a unital C*-algebra E and $*$-homomorphisms

$$\pi \colon C_0(0,1] \otimes A \longrightarrow E, \quad \varrho \colon C[0,1) \otimes A \longrightarrow E \quad \text{and} \quad \theta \colon C([0,1]) \longrightarrow E,$$

in which θ is unital, as a *compatible system of* A if π, ϱ are compatible with θ and tracially recover the trace τ in the sense that $\tau_E \circ \pi = \tau_E \circ \varrho = \tau_{\mathcal{L}} \otimes \tau$ for every trace τ_E acting on E.

Additionally, for every given family $\mathcal{J} = \{J_1, J_2, J_3\}$ consisting of mutually disjoint open intervals in $[0,1]$, a compatible system $(\pi, \varrho, \theta, E)$ will be called *patched via* $\mathcal{J}$ if there exists a completely positive map $\psi \colon C_0(0,1) \otimes A \longrightarrow \mathrm{M}_2(E)$ such that

- the restrictions of ψ and $\pi \oplus 0$ onto $C_0(J_1) \otimes A$ coincide; $\hspace{2cm}$ (p.1)

- the restrictions of ψ and $\varrho \oplus 0$ onto $C_0(J_3) \otimes A$ coincide; $\hspace{2cm}$ (p.2)

- one has $\tau \circ \pi = \tau \circ \varrho = (\tau_E \otimes \mathrm{Tr}_2) \circ \psi$ for every trace τ_E acting on E. $\hspace{1cm}$ (p.3)

The notion of patched compatible systems has no occurrence in the original proof. It has been added for emphasis on the first step in the proof, namely the design of (π_0, π_1, θ). The existence of compatible systems attached to nuclear unital separable C*-algebras was established in proposition 3.4.5, so essentially patchability hereof remains to be accounted for.

Lemma 5.1.1 (Patching Lemma)**.** *Let A be some unital C*-algebra. Let furthermore $J_1 = (0, 1/3)$, $J_2 = (1/3, 2/3)$ and $J_3 = (2/3, 1)$, then set $\mathcal{J} := \{J_1, J_2, J_3\}$. If so, there exists a partition of unity $\{f_0, f_1, f_2\} \subseteq C([0,1])$, with the support of f_1 being located on the open interval J_2, while fulfilling the following property.*

For every compatible system $(\pi, \varrho, \theta, E)$ of A and every unitary u inside E, there exists some completely positive map $\psi \colon C(0,1) \otimes A \longrightarrow \mathrm{M}_2(E)$ making $(\pi, \varrho, \theta, E)$ patched via the family $\mathcal{J}$. Furthermore, ψ satisfies

$$\|\psi(sg) - \theta^2(g)\psi(s)\| \leq \|s\| \cdot \left(\|[\theta(f_1), u]\| \cdot \|g\| + \|[\theta(gf_1), u]\|\right) \tag{5.2}$$

together with

$$\|\psi(st) - \psi(s)\psi(t)\| \leq \|s\| \cdot \|t\| \cdot \left(7\|[u, \theta(f_1)]\| + 2\|[u, \theta(f_0 f_1)]\| + 5\|[u, \theta(f_1 f_2)]\|\right)$$
$$+ \|t\| \cdot \left(\|\varrho(f_1 f_2 s)u - u\pi(f_1 f_2 s)\| + \|\varrho(f_1 s)u - u\pi(f_1 s)\|\right) \tag{5.3}$$

for every $s, t \in C_0(0,1) \otimes A$ and $g \in C([0,1])$.

Observation. Notice that the bounds on the error of multiplicativity attached to ψ and its compatibility of θ^2 are all described in accordance with

$$\omega := \|\pi|_{C_0(J_2) \otimes A}(\cdot) - u\varrho|_{C_0(J_2) \otimes A}(\cdot)u^*\|.$$

Ergo such a patched system would give rise to an exact multiplicative completely positive map ψ provided that $\omega = 0$. Now, maintain the notation in the patching lemma. Suppose now that two finite subsets $F_0 \subseteq C_0(0,1) \otimes A$, $F \subseteq C([0,1])$ and some tolerance $\varepsilon > 0$ have been chosen. Under these premises, one may find some finite subset $G \subseteq C_0(J_2) \otimes A$ together with a $\delta > 0$ such that, whenever $u\pi(s)u^* \approx_\delta \varrho(s)$ for each s in G, one has

$$\psi(st) \approx_\varepsilon \psi(s)\psi(t) \quad \text{and} \quad \psi(sg) \approx_\varepsilon \theta^2(g)\psi(s)$$

for all $s, t \in F_0$ and $g \in F$. To achieve such a pair (G, δ), let G consist of all configurations of products emerging in (5.2)-(5.3) for which $s \in F_0$ and $g \in F$. Select thereafter some $\delta > 0$ small enough to guarantee that both

$$(M_0 + 1)M\delta \leq \varepsilon \quad \text{and} \quad 14M_0 M\delta + 2M_0\delta \leq \varepsilon$$

become valid, where $M_0 = \max\{\|s\| : s \in F_0\}$, respectively, $M = \{\|g\| : g \in F\}$.

Proof. We initially build our partition of unity. Let $f_0, f_2 \in C([0, 1])$ be given as

$$f_0|_{[0, \frac{1}{3}]} = 1, \qquad\qquad f_0|_{[\frac{1}{3}, \frac{2}{5}]} \text{ is linear}, \qquad\qquad f_0|_{[\frac{2}{5}, 1]} = 0;$$
$$f_2|_{[0, \frac{3}{5}]} = 0, \qquad\qquad f_2|_{[\frac{3}{5}, \frac{2}{3}]} \text{ is linear}, \qquad\qquad f_2|_{[\frac{2}{3}, 1]} = 1.$$

Setting $f_1 = 1_{C([0,1])} - f_0 - f_2$ thus gives a partition of unity. By the construction and location of supports, the support of f_1 must be located on $J_2 = (1/3, 2/3)$. The situations is displayed in the figure found beneath.
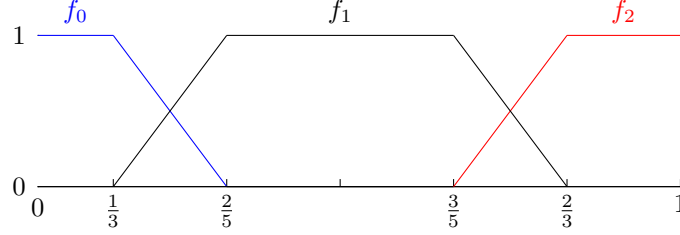
Suppose a compatible system $(\pi, \varrho, \theta, E)$ attached to A is given. Let some unitary u in E be fixed. Upon identifying $C([0, 1], \mathbb{M}_2)$ with $C([0, 1]) \otimes \mathbb{M}_2 \cong \mathrm{M}_2(C([0, 1]))$, choose your favourite unitary V hereon subject to the relation

$$V|_{[0, \frac{2}{5}]} = \begin{bmatrix} 1_{C([0,1])} & 0 \\ 0 & 1_{C([0,1])} \end{bmatrix} \quad \text{together with} \quad V|_{[\frac{3}{5}, 1]} = \begin{bmatrix} 0 & 1_{C([0,1])} \\ 1_{C([0,1])} & 0 \end{bmatrix}. \tag{5.4}$$

For instance, a reparametrized rotation matrix will work. Define a unitary w in $\mathrm{M}_2(E)$ by

$$w = \theta_2(V^*)\big(1_E \oplus u\big)\theta_2(V).$$

As usual, θ_2 denotes the 2-amplification of θ unlike the diagonal map θ^2. For future purposes, we record a repeatedly exploited observation. Let h be the generating element of $C([0, 1])$, i.e., the identity map onto $[0, 1]$. Since $h \oplus h$ lies within the center of $\mathrm{M}_2(C([0, 1]))$ and θ determines a $*$-homomorphism, $\theta^2(h)$ will commute with $\theta_2(V)$. Appealing to multiplicativity of θ in conjunction with the generating property of h therefore yields

$$\big[\mathrm{im}\, \theta^2, \theta_2(V)\big] = 0. \tag{5.5}$$

Keeping in this mind, we define $\psi_0, \psi_1, \psi_2 \colon C(0, 1) \otimes A \longrightarrow \mathrm{M}_2(E)$ by

$$\psi_0(s) = \pi(f_0 s) \oplus 0, \quad \psi_2(s) = \varrho(f_2 s) \oplus 0 \quad \text{and} \quad \psi_1(s) = w(\pi(f_1 s) \oplus 0)w^*.$$

Set $\psi = \psi_0 + \psi_1 + \psi_2$. Due to each ψ_k being a diagonal map induced from $*$-homomorphisms for $k = 0, 1$ whereas ψ_1 is the conjugation by a unitary of a $*$-homomorphism, each map ψ_k obviously becomes a $*$-homomorphism. Hence the map ψ must be completely positive. Justifying the criteria (p.1)-(p.3) is easy. Indeed the support of f_0 contains the open interval J_1 whereon $f_0 = 1$. However, the remaining elements f_1, f_2 vanish on the interval J_1, so that $\psi|_{J_1} = \pi(\cdot) \oplus 0$. An argument running parallel provides (p.2), replacing the role of ψ_0 with ψ_2.

Regarding the property (p.3), suppose τ denotes some trace on E such that $\tau \circ \pi = \tau \circ \varrho = \tau_{\mathcal{L}} \otimes \tau$. Then observe that this property yields

$$(\tau \otimes \mathrm{Tr}_2)(\psi(s)) = \tau(\pi(f_0 s) + \varrho(f_2 s) + \pi(f_1 s)) = (\tau \circ \pi)(f_0 s + f_1 s + f_2 s) = (\tau \circ \pi)(s).$$

The final equality stems from $\mathcal{P} = \{f_0, f_1, f_2\}$ comprising a partition of unity. For the two bounds (5.2)-(5.3), we recall the following consequence of compatibility:

$$\pi(sh) = \theta(h)\pi(s) = \pi(s)\theta(h) \quad \text{and} \quad \varrho(th) = \theta(h)\varrho(t) = \varrho(t)\theta(h) \tag{$*$}$$

holds for each $s \in C_0(0,1] \otimes A$, $t \in C_0[0,1) \otimes A$ and $h \in C([0,1])$. When referring to compatibility, we implicitly use $(*)$ throughout the proof. Let us proceed to establishing (5.2). Rescaling accordingly if necessary, verifying it on contractions $h \in C([0,1])$ and $s \in C_0(0,1) \otimes A$ suffices. We shall estimate the right-hand side of

$$\|\psi(hs) - \theta^2(h)\psi(s)\| \leq \sum_{k=0,1,2} \|\psi_k(hs) - \theta^2(h)\psi_k(s)\|.$$

Compatibility of the system $(\pi, \varrho, \theta, E)$ entails that $\psi_k(sh) = \theta^2(h)\psi_k(s)$ if $k = 0, 2$, thereby turning the two corresponding terms above into zero. We therefore need only concern ourselves with the case $k = 1$. For this, note that (5.5) implies that $\|[w, \theta^2(\cdot)]\| = \|[u, \theta(\cdot)]\|$. Compatibility thus yields

$$\begin{aligned}
\|\psi_1(hs) - \theta^2(h)\psi_1(s)\| &= \|w\theta^2(f_1 h)(\pi(s) \oplus 0)w^* - \theta^2(h)w\pi(f_1 s) \oplus 0)w^*\| \\
&\leq \|w\theta^2(f_1 h)(\pi(s) \oplus 0)w^* - \theta^2(hf_1)w(\pi(s) \oplus 0)w^*\| \\
&\quad + \|\theta^2(hf_1)w(\pi(s) \oplus 0)w^* - \theta^2(h)w\theta^2(f_1)(\pi(s) \oplus 0)w^*\| \\
&\leq \|[u, \theta^2(hf_1)]\| + \|[u, \theta(f_1)]\|. \quad (5.6)
\end{aligned}$$

In the scenario without contractions, one scale by these to acquire (5.2). Proving (5.3) is far more tedious, so prepare yourself for long-winded computations. Upon rescaling afterwards, we are only required to deduce the condition on contractions $s, t \in C_0(0,1) \otimes A$. It has been deemed optimal to consider each case separately. Indeed, we must derive the bound from

$$\|\psi(s)\psi(t) - \psi(st)\| \leq \sum_{i,j=0,1,2} \|\psi_i(s)\psi_j(t) - \theta^2(f_i)\psi_j(st)\| =: \omega(s,t).$$

Case 1. Here we tackle the endpoint cases, meaning the terms stemming from the indices $i, j = 0, 2$. For $i = j = 0$ compatibility of the system ensures that

$$\theta^2(f_0)\psi_0(st) = \theta(f_0)\pi(stf_0) \oplus 0 = (\pi(f_0 s) \oplus 0)(\pi(f_0 t) \oplus 0) = \psi_0(s)\psi_0(t).$$

Replacing π with ϱ and 0 with 2 provides the same conclusion for $i = j = 2$. If $i, j = 0, 2$ are distinct, compatibility in conjunction with $f_0 f_2 = 0$ by construction yields $\theta^2(f_i)\psi_j(st) = 0 = \psi_i(s)\psi_j(t)$. It follows that no contribution to $\omega(s,t)$ is supplied from the terms in which $i, j = 0, 2$.

Case 2. Consider the case where $i = j = 1$. Compatibility implies that

$$\psi_1(s)\psi_1(t) = w\theta^2(f_1)(\pi(f_1 st) \oplus 0)w^* \quad \text{and} \quad \theta^2(f_1)\psi_1(st) = \theta^2(f_1)w(\pi(f_1 st) \oplus 0)w^*,$$

whereby the contribution for $i = j = 1$ to $\omega(s,t)$ becomes

$$\|\psi_1(s)\psi_1(t) - \theta^2(f_1)\psi_1(st)\| \leq \|[\theta^2(f_1), w]\| = \|[\theta(f_1), u]\|.$$

Case 3. Here we settle the cases wherein $i, j \in \{0, 1\}$. The support of f_0 agrees with the interval upon which V subsumes the role of the unit by (5.4). Since θ defines a $*$-homomorphism, the unitary $\theta_2(V)$ must act as the unit on $\theta(f_0) \oplus 0$ and thus w, w^* act as the unit on $\theta(f_0) \oplus 0$. Compatibility guarantees that θ commutes with π pointwise, so exploiting this ensures

$$\begin{aligned}
\psi_1(s)\psi_0(t) &= w(\theta(f_1)\pi(s) \oplus 0)w^*(\theta(f_0)\pi(t) \oplus 0) \\
&= \theta^2(f_0 f_1)\pi(st) \oplus 0 \\
&= \theta^2(f_1)\psi_0(st).
\end{aligned}$$

Hence the contribution to $\omega(s,t)$ of the term associated to $i = 1$ and $j = 0$ is zero. For the reverse situation, ease the notational burden by letting $\varepsilon_1 := \|[u, \theta(f_1)]\|$ and $\varepsilon_{01} := \|[u, \theta(f_0 f_1)]\|$. Recall

that $\theta_2(V)$ acts as the unit on $\theta(f_0) \oplus 0$. Then

$$
\begin{aligned}
\theta^2(f_0)w\theta^2(f_1) &= \theta^2(f_0)(1_E \oplus u)\theta_2(V)\theta^2(f_1) \\
&\overset{(5.5)}{=} \left[\theta(f_0 f_1) \oplus \theta(f_0)u\theta(f_1)\right]\theta_2(V) \\
&\approx_{\varepsilon_1} \left[\theta(f_0 f_1) \oplus \theta(f_0 f_1)u\right]\theta_2(V) \\
&\approx_{\varepsilon_{01}} \left[\theta(f_0 f_1) \oplus u\theta(f_0 f_1)\right]\theta_2(V) \\
&= [\theta(f_0 f_1) \oplus u\theta(f_0 f_1)] \\
&\approx_{\varepsilon_1 + \varepsilon_{01}} [\theta(f_0 f_1) \oplus \theta(f_0)u\theta(f_1)].
\end{aligned}
$$

Declaring that $\mu := 2\varepsilon_1 + 2\varepsilon_{01}$, then using that w, w^* acts as the unit on $\theta(f_0) \oplus 0$ yields

$$
\begin{aligned}
\psi_0(s)\psi_1(t) &= (\theta(f_0 f_1)\pi(st) \oplus 0)w^* \\
&= (\theta(f_0 f_1) \oplus \theta(f_0)u\theta(f_1))(\pi(st) \oplus 0)w^* \\
&\approx_\mu \theta^2(f_0)w(\pi(f_1 st) \oplus 0)w^* \\
&= \theta^2(f_0)\psi_1(st).
\end{aligned}
$$

The total contribution to $\omega(s,t)$ stemming from the indices $i, j = \{0, 1\}$ is thus bounded by μ.

Case 4. Here we tackle the terms arising from $\{i, j\} = \{1, 2\}$. In a fashion resembling the previous case, put $\varepsilon_{21} := \|[u, \theta(f_1 f_2)]\|$. The relations (5.4)-(5.5) combined with compatibility ensure that

$$
\begin{aligned}
\theta^2(f_2)w\theta^2(f_1) &= \begin{bmatrix} \theta(f_2) & 0 \\ 0 & \theta(f_2) \end{bmatrix} \theta_2(V)^* \begin{bmatrix} 1_E & 0 \\ 0 & u \end{bmatrix} \theta_2(V) \begin{bmatrix} \theta(f_1) & 0 \\ 0 & \theta(f_1) \end{bmatrix} \\
&\overset{(5.4)}{=} \begin{bmatrix} 0 & \theta(f_2)u \\ \theta(f_2) & 0 \end{bmatrix} \theta_2(V) \begin{bmatrix} \theta(f_1) & 0 \\ 0 & \theta(f_1) \end{bmatrix} \\
&\overset{(5.5)}{=} \begin{bmatrix} 0 & \theta(f_2)u\theta(f_1) \\ \theta(f_1 f_2) & 0 \end{bmatrix} \theta_2(V) \\
&\approx_{\varepsilon_1} \begin{bmatrix} 0 & \theta(f_1 f_2)u \\ \theta(f_1 f_2) & 0 \end{bmatrix} \theta_2(V) \\
&\approx_{\varepsilon_{21}} \begin{bmatrix} 0 & u\theta(f_1 f_2) \\ \theta(f_1 f_2) & 0 \end{bmatrix} \theta_2(V) \\
&\overset{(5.4)}{=} \begin{bmatrix} u\theta(f_1 f_2) & 0 \\ 0 & \theta(f_1 f_2) \end{bmatrix} \\
&\approx_{\varepsilon_{21}} \begin{bmatrix} \theta(f_1 f_2)u & 0 \\ 0 & \theta(f_1 f_2) \end{bmatrix}.
\end{aligned}
\tag{5.7}
$$

The sixth equality is based on V determining the permuted unit matrix on the support of f_2, so that it rotates offdiagonal-matrices with values in the image of θ. Setting $\mu' := \varepsilon_1 + 2\varepsilon_{21}$, then exploiting that V rotates off-diagonal matrices on the support of f_2 therefore provides the estimates

$$
\begin{aligned}
\psi_2(s)\psi_1(t) &= \begin{bmatrix} \varrho(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(f_2) & 0 \\ 0 & \theta(f_2) \end{bmatrix} w \begin{bmatrix} \theta(f_1) & 0 \\ 0 & \theta(f_1) \end{bmatrix} \begin{bmatrix} \pi(t) & 0 \\ 0 & 0 \end{bmatrix} w^* \\
&\overset{(5.7)}{\approx_{\mu'}} \theta_2(V^*) \begin{bmatrix} 0 & 0 \\ \varrho(f_1 f_2 t)u\pi(t) & 0 \end{bmatrix} w^*,
\end{aligned}
$$

alongside

$$
\psi_1(f_2 st) = w \begin{bmatrix} \pi(f_1 f_2 st) & 0 \\ 0 & 0 \end{bmatrix} w^* \overset{(5.4)}{=} \theta_2(V^*) \begin{bmatrix} 0 & 0 \\ u\pi(f_1 f_2 s)\pi(t) & 0 \end{bmatrix} w^*.
$$

Applying (5.6) having f_2 replace h, the contribution to $\omega(s,t)$ in the cases stemming from the situations $\{i,j\} = \{1,2\}$ thus becomes

$$
\|\psi_2(s)\psi_1(t) - \theta^2(f_2)\psi_1(st)\| \overset{(5.6)}{\leq} \|\psi_2(s)\psi_1(t) - \psi_1(f_2st)\| + \|\psi_1(f_2st) - \theta^2(f_2)\psi_1(st)\|
$$
$$
\leq \left(\mu' + \|\varrho(f_1f_2s)u - u\pi(f_1f_2s)\|\right) + (\varepsilon_{21} + \varepsilon_1)
$$
$$
= \|\varrho(f_1f_2s)u - u\pi(f_1f_2s)\| + 3\varepsilon_{21} + 2\varepsilon_1.
$$

The reverse configuration amounts to calculations of the same nature, so details have been spared. First of all, the fact that V turns into the permuted unit on the support of f_2 forces

$$
w \begin{bmatrix} \theta(f_1f_2) & 0 \\ 0 & 0 \end{bmatrix} = \theta_2(V^*) \begin{bmatrix} 0 & 0 \\ 0 & u\theta(f_1f_2) \end{bmatrix} \tag{5.8}
$$
$$
\approx_{\varepsilon_{21}} \theta_2(V^*) \begin{bmatrix} 0 & 0 \\ 0 & \theta(f_1f_2)u \end{bmatrix}
$$
$$
= \begin{bmatrix} \theta(f_1f_2)u & 0 \\ 0 & 0 \end{bmatrix}.
$$

Let $\mu'' := \varepsilon_1 + \varepsilon_{21}$ and compute

$$
\psi_1(s)\psi_2(t) = w \begin{bmatrix} \pi(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(f_1) & 0 \\ 0 & \theta(f_1) \end{bmatrix} w^* \begin{bmatrix} \theta(f_2) & 0 \\ 0 & \theta(f_2) \end{bmatrix} \begin{bmatrix} \varrho(t) & 0 \\ 0 & 0 \end{bmatrix}
$$
$$
\overset{(5.7)}{\approx_{\mu''}} w \begin{bmatrix} \pi(s) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(f_1f_2)u^* & 0 \\ 0 & \theta(f_1f_2) \end{bmatrix} \begin{bmatrix} \varrho(t) & 0 \\ 0 & 0 \end{bmatrix}
$$
$$
\overset{(5.8)}{\approx_{\varepsilon_{21}}} \begin{bmatrix} \theta(f_1f_2)u\pi(s)u^*\varrho(t) & 0 \\ 0 & 0 \end{bmatrix}.
$$

On the other hand, notice that one has $\theta^2(f_1)\psi_2(st) = \theta(f_2)\varrho(f_1s)\varrho(t) \oplus 0$. Remember in addition that the members of $\mathcal{P}$ and both s,t are all contractions. Entering this alongside the above estimate into the corresponding term in $\omega(s,t)$ gives the bound:

$$
\|\psi_1(s)\psi_2(t) - \theta^2(f_1)\psi_2(st)\| \leq \mu'' + \varepsilon_{21} + \|\theta(f_1)u\pi(s)u^* - \varrho(f_1s)\|
$$
$$
\leq \varepsilon_1 + 2\varepsilon_{21} + \|\theta(f_1)u\pi(s)u^* - u\theta(f_1)\pi(s)u^*\| + \|u\pi(f_1s)u^* - \varrho(f_1s)\|
$$
$$
\leq \varepsilon_1 + 2\varepsilon_{21} + \|[u,\theta(f_1)]\| + \|u\pi(sf_1) - \varrho(sf_1)u\|
$$
$$
\leq 2\varepsilon_1 + 2\varepsilon_{21} + \|u\pi(sf_1) - \varrho(sf_1)u\|.
$$

This establishes all possible contributions to $\omega(s,t)$. Collecting all contributions of the distinct cases yield (5.3) modulo scaling by $\|s\|$ and $\|t\|$ throughout, completing the proof. $\qquad\square$

Remark. If one follows the proof with the assumption that π and ϱ are in fact unitarily equivalent, one basically reproduces the proof of multiplicativity of the map π constructed on page 64. The benefit of the patching lemma is its ability to tackle general compatible systems and approximate unitary equivalences (in the main theorem, the trio (π_0, π_1, θ) will not fully do the job).

The properties $(p.1)$-$(p.2)$ on the other hand may seem oddly placed. However, they will allow us to arrange the unitary equivalence on overlaps of several intervals; we apply the patching lemma more than once during the main proof.

The patching lemma permits one to "glue" an alteration of our compatible system conjured in proposition 3.4.5 via an approximately multiplicative and approximately θ^2-compatible completely positive map. This feature surely seems intriguing from a quasidiagonal point of view. During the process, the trace will be twisted, so we salvage this by adjusting it via corner algebras arising from projections attached to the intervals. No mischief occurs here, for $p\mathcal{Q}_\omega p \cong \mathcal{Q}_\omega$ while also $\mathcal{Q}_\omega \hookrightarrow p\mathcal{Q}_\omega p \otimes \mathbb{M}_2$ may be arranged.

Lemma 5.1.2. *Let a be some positive contraction in $\mathcal{Q}_\omega$ having Lebesgue spectral measure. For each relatively open interval $I \subseteq [0,1]$ there exists a projection p_I in $\mathcal{Q}_\omega$ fulfilling the following.*

(i) *p_I commutes with a.*

(ii) *$p_I h(a) = h(a) p_I = h(a)$ for each h in $C_0(I)$.*

(iii) *$h(a) p_I = p_I$ for every h in $C([0,1])$ which restricts to the identity on I.*

(iv) *One has $p_I p_J = 0$ for disjoint relatively open intervals $I, J \subseteq [0,1]$.*

(v) *One has $\tau_\omega(p_I) = |I|$.*

Proof. Let $\varrho_\omega \colon \ell^\infty(\mathcal{Q}) \longrightarrow \mathcal{Q}_\omega$ be the quotient map. Choose some positive contractive lift $(b_1, b_2, \ldots)$ in $\ell^\infty(\mathcal{Q})$ of a. Due to $\mathcal{Q}$ having real rank zero, each element b_n may be approximated by self-adjoint elements a_n of finite spectrum, see proposition B.0.7. The sequence $(a_1, a_2, \ldots)$ determines an element in $\ell^\infty(\mathcal{Q})$ and lifts a. Being elements of finite spectra, their spectra are discrete, thereby eliminating potential continuity issues ahead. Let $\chi_I \colon [0,1] \longrightarrow \{0,1\}$ denote the indicator map attached to I. Then set $p_{I,n} = \chi_I(a_n)$ for each n in $\mathbb{N}$ and put $p_I = \varrho_\omega(p_{I,1}, p_{I,2}, \ldots)$.

(i)-(iii): The continuous functional calculus commutes with $*$-homomorphisms, hence with ϱ_ω and every projection map $p_k \colon \ell^\infty(\mathcal{Q}) \longrightarrow \mathcal{Q}$. As an element in $\ell^\infty(\mathcal{Q})$ is uniquely in terms of the actions of p_k onto it, we deduce that

$$h(\varrho_\omega(a_1, a_2, \ldots)) = \varrho_\omega(h(a_1, a_2, \ldots)) = \varrho_\omega(h(a_1), h(a_2), \ldots)$$

for any h belonging to $C([0,1])$. Identifying a_n with $\mathrm{id}_{\sigma(a_n)}(a_n)$, it is evident that $p_{I,n}$ must commute with a_n, which combined with the preceding observations forces a to commute with p_I. Arguments heavily resembling this one will yield (ii)-(iii).

(iv)-(v): (iv) is an immediate consequence of $\chi_I \chi_J = 0$ for disjoint elements and the continuous functional calculus being of order zero. To verify (v), we evaluate $|I|$ in the following manner. Let X denote the subcollection in $C([0,1])$ consisting of positive contractions, $X_0 := X \cap C_0(I)$ and let X_I be the subcollection in X whose elements restrict to the identity on I. Due to a having Lebesgue spectral measure, we may infer that

$$|I| = \int_{[0,1]} \chi_I \, dm = \sup_{h \in X_0} \int_{[0,1]} h \, dm \stackrel{(ii)}{=} \sup_{h \in X_0} \tau_\omega(h(a) p_I) \leq \sup_{h \in X_0} \tau_\omega(h(a)^2)^{1/2} \tau(p_I)^{1/2} \leq \tau(p_I).$$

The first inequality stems from the Cauchy-Schwarz inequality (1.3) for positive functionals, whereas the latter is based on a being a positive contraction. For the reverse inequality, one appeals to (iii) whereby Lebesgue spectral measure of a entails that, for every h in X_I, we must have

$$\tau_\omega(p_I) \stackrel{(iii)}{\leq} \tau_\omega(h(a)^2)^{1/2} \tau_\omega(p_I)^{1/2} \leq \inf_{f \in X_I} \int_{[0,1]} f \, dm = |I|.$$

Altogether $|I| = \tau_\omega(p_I)$, granting (v). This proves the claim. $\qquad\qquad\square$

When attempting to enable the patching lemma, the unitary equivalence occurring in the observation underneath it must be arranged. Accomplishing this requires the stable uniqueness theorem, hence demands Δ-fullness. Acquiring Δ-fullness will be done via the following preliminary result.

Lemma 5.1.3. *Suppose A and B denote unital C^*-algebras with B having strict comparison of positive elements with respect to bounded traces. Let $\gamma \colon A \longrightarrow B$ be some unital $*$-homomorphism. If for each positive contraction $a \in A$ one has some $m_a \in \mathbb{N}$ satisfying $\tau(\gamma(a)) > 2m^{-1}$ whenever $\tau \in T(B)$, then γ becomes Δ-full with $\Delta(a) = m_a^2$ as control function.*

Proof. To ease the notation write $b = \gamma(a)$. Due to $T(B)$ being weak*-compact in the unital case, the real-number $\varepsilon = 2^{-1}\min_{\tau \in T(A)} \tau(b)$ exists. Choose some trace τ acting on B, should it exist. Define hereafter an element z in $\mathbb{M}_m \otimes A$ by

$$z = (b - \varepsilon)^2_+ \oplus (b - \varepsilon)^2_+ \oplus \ldots \oplus (b - \varepsilon)^2_+ \quad \text{(m-copies occuring)}.$$

For every positive contraction e, $d_\tau((e - \varepsilon))^2_+)$ exceeds $\tau(e)$. Since τ dominates d_τ, regarded as a dimension function induced on $W(A)$, one may deduce the inequality

$$\varepsilon \leq \tau(b - \varepsilon 1_B) \leq \tau\big((b - \varepsilon)_+\big) \leq d_\tau\big((b - \varepsilon)^2_+\big).$$

The element $1_B \oplus 0^{\oplus(m-1)}$ is Cuntz-equivalent to 1_B. Hence

$$d_\tau(1_B \oplus 0^{\oplus(m-1)}) = 1 < m\varepsilon \leq m d_\tau((b - \varepsilon)^2_+) = d_\tau(z).$$

Strict comparison thus forces $e = 1_B \oplus 0^{\oplus(m-1)}$ to be Cuntz subequivalent to z in the matrix algebra $A \otimes \mathbb{M}_m$. Our objective will be to compare 1_B with z without the use of diagonal matrices. This will be accomplished trough the following minor trick. According to proposition 3.3.2, given any $\delta > 0$ there exists some element r in $M_m(A)$ such that $(e - \delta)_+ = rzr^*$. Some functional calculus reveals that $(1 - \delta)e = (e - \delta)_+$. Declaring that $r_0 := (1 - \delta)^{-1/2}r$ thus ensures that

$$r_0 z r_0^* = e \quad \text{and} \quad r_0 e = e.$$

The latter condition forces r_0 to be a column-matrix $[b_1, b_2, \ldots, b_m]^t$ in $M_{1,m}(A)$. Entering the expressions for e, r_0 and z yields

$$(1_B \oplus 0^{m-1}) = r_0 z r_0 = \sum_{k=1}^m b_k(b - \varepsilon)^2_+ b_k^* \oplus 0^{m-1}.$$

The sum on the right-hand side needs to rearranged to omit using the cut-down. For this, note that the C*-identity ensures that $\|b_k(b - \varepsilon)_+\| \leq 1$ for each integer $k = 1, 2, \ldots, m$. Consider the function $h\colon [0,1] \longrightarrow \mathbb{R}^+$ defined by

$$h(t) = \begin{cases} t^{-1/2}, & \text{if } \varepsilon \leq t, \\ t\varepsilon^{-3/2}, & \text{if } 0 \leq t \leq \varepsilon. \end{cases}$$

For every t in $[0,1]$ exceeding ε one has $h(t)^2 t = 1$, whereby $(b - \varepsilon)_+ h(b)^2 b = (b - \varepsilon)_+$. Consider the element $x_k = m^{-1/2} b_k(b - \varepsilon)_+ h(b)$ belonging to A for each $k = 1, \ldots, m$. This must be a contraction, since the previous bound on $\|b_k(b - \varepsilon)_+\|$ forces $\|x_k\| \leq (m\varepsilon)^{-1/2}\|b_k(b - \varepsilon)_+\| < 1$. Hence, due to $h(b)$ commuting with b, one may infer that

$$\sum_{k=1}^{m^2} x_k b x_k^* = m \sum_{k=1}^m m^{-1} b_k(b - \varepsilon)_+ h(b) b h(b)(b - \varepsilon)_+ b_k^*$$

$$= \sum_{k=1}^m b_k(b - \varepsilon)_+ h(b)^2 b(b - \varepsilon)_+ b_k^*$$

$$= \sum_{k=1}^m b_k(b - \varepsilon)^2_+ b_k^* = 1_B,$$

proving the claim.

$\square$

5.2   Implementing Quasidiagonality

We arrive at the grand finale; the proof of the main theorem. Due to the taxing length, it has been split into three major parts. Moreover, motivations for each step will be supplied, hopefully building some intuition during the process while conveying the underlying strategy. For the record, the chronological order exhibited here deviates from the original version in [42]. During the entire proof, the *supports are all open unless specified otherwise*. Without further ado, the main theorem.

Theorem 5.2.1 (Tikuisis - White - Winter). *Faithful traces acting on nuclear separable C^*-algebras fulfilling the UCT-condition are automatically quasidiagonal.*

Proof. According to theorem 4.4.4, the class comprised of nuclear members in $\mathcal{N}$ is stable under restricting to subalgebras and extensions in the category of separable C^*-algebras with $*$-homomorphisms. Since any separable C^*-algebra B constitutes an ideal in its (separable) unitization B^+, B satisfies the UCT-condition if and only if B^+ does. In addition, the trace τ on A is quasidiagonal and faithful if and only if the induced trace on A^+ is according to the remark on page 46. Thus assuming the presence of a unit causes no hindrances in the proof.

Let therefore A be a unital separable nuclear C^*-algebra fulfilling the UCT-condition and fix some faithful trace τ hereon. Let furthermore some finite subset $F_A \subseteq A$ together with tolerance $\varepsilon > 0$ be given. Without loss of generality, we may assume that F_A contains the unit in A. The task amounts to finding a positive integer N and a completely positive map $\psi \colon A \longrightarrow \mathrm{M}_{2N}(\mathcal{Q}_\omega) \cong \mathcal{Q}_\omega$ subject to the condition

$$\psi(ab) \approx_\varepsilon \psi(a)\psi(b) \quad \text{and} \quad \frac{1}{2}\tau(e) = (\tau_\omega \otimes \tau_{2N})(\psi(e)) \tag{5.9}$$

for every $a, b \in F_A$ and each $e \in A$. Since each involved map is linear and positive, verifying (5.9) on self-adjoint contractions will suffice, so we further assume that $F_A \subseteq A_1$ is involutive.

Part 1. In the current part, we prescribe a setup that to each positive integer n_0 permits one to form a partition of unity attached to $C([0,1])$ subordinate to a collection of n_0 subintervals in $[0,1]$. There are many functions involved in the process, so illustrations have been made along the way. The reader is recommended to primarily focus on these

The procedure contains two intermediate steps. First we construct five continuous functions on $[0,1]$, where two of these arise as reflections of former ones. Define $f, g_\ell, g_r, h_\ell, h_r \in C([0,1])$ by

$$f|_{\left[0,\frac{1}{9}\right]} = 0, \qquad f|_{\left[\frac{1}{9},\frac{2}{9}\right]} \text{ is linear }, \qquad f|_{\left[\frac{2}{9},\frac{7}{9}\right]} = 1, \qquad f|_{\left[\frac{7}{9},\frac{8}{9}\right]} \text{ is linear }, \qquad f|_{\left[\frac{8}{9},1\right]} = 0;$$

and

$$g_\ell|_{\left[0,\frac{8}{9}\right]} = 1, \qquad\qquad g_\ell|_{\left[\frac{8}{9},1\right]} \text{ is linear }, \qquad\qquad g_\ell(1) = 0$$

$$g_r(0) = 0, \qquad\qquad g_r|_{\left[0,\frac{1}{9}\right]} \text{ is linear }, \qquad\qquad g_r|_{\left[\frac{1}{9},1\right]} = 1;$$

$$h_\ell|_{\left[0,\frac{2}{9}\right]} = 1, \qquad\qquad h_\ell|_{\left[\frac{2}{9},\frac{3}{9}\right]} \text{ is linear }, \qquad\qquad h_\ell|_{\left[\frac{3}{9},1\right]} = 0;$$

$$h_r|_{\left[0,\frac{6}{9}\right]} = 0, \qquad\qquad h_r|_{\left[\frac{6}{9},\frac{7}{9}\right]} \text{ is linear }, \qquad\qquad h_r|_{\left[\frac{7}{9},1\right]} = 1.$$

The subscripts r and ℓ are supposed to indicate whether the majority of the support lies on the left - or right hand side of the interval $[0,1]$. Furthermore, g_r, h_ℓ are reflections h_r, h_ℓ against the midpoint $t = 1/2$ in the respective order. In the language of the observation presented after lemma 5.1.1, set

$$F_0 = \{f \otimes a, f \otimes ab : a, b \in F_A\} \subseteq C_0(0,1) \otimes A \quad \text{and} \quad F = \{f, g_r, g_\ell, h_r, h_\ell\} \subseteq C([0,1]).$$

Fix some $\delta_0 < \varepsilon/24$. For an illustration displaying the functions, see the following figure.
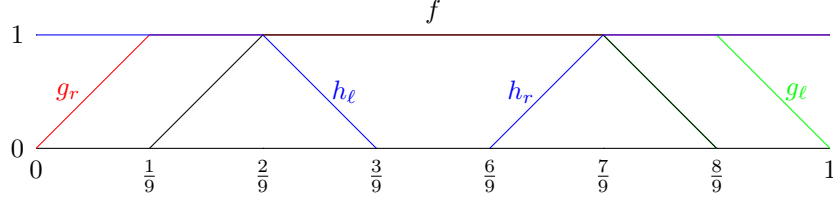
Let us consider a blueprint of the proof. It will fail, however, it displays how the patching lemma enters. Let $\mathcal{J} = \{J_1, J_2, J_3\}$ be the family of disjoint intervals of the lemma 5.1.1 and consider some compatible system (π, ϱ, μ, E) in which $E \cong \mathcal{Q}_\omega$. Invoking the observation following the patching lemma, one may construct a finite subset $G \subseteq C_0(J_2) \otimes A$ and $\delta > 0$, such that if

$$u\pi(s)u^* \approx_\delta \varrho(s) \tag{5.10}$$

for every s in G and some unitary u inside E, then one may determine the existence of a completely positive map $\varphi\colon C_0(0,1) \otimes A \longrightarrow \mathrm{M}_2(E)$ satisfying

- $(\tau_E \otimes \mathrm{Tr}_2) \circ \varphi = \tau_{\mathcal{L}} \otimes \tau;$ (cp.1)

- the map φ makes (π, ϱ, μ, E) a compatible system patched via $\mathcal{J}$;   (cp.2)

- one has $\varphi(st) \approx_{\delta_0} \varphi(s)\varphi(t)$ together with $\varphi(hs) \approx_{\delta_0} \mu^2(h)\varphi(s)$ for all $s, t \in F_0$ and $h \in F$. (cp.3)

As discussed previously (5.10) cannot be ensured for the compatible system $(\pi_0, \pi_1, \theta, \mathcal{Q}_\omega)$, otherwise φ would subsume[1] the role of the designated ψ in (5.9). Providing (5.10) is achieved at the cost of passing to larger matrices, whereby we extend the original compatible system by forming maps onto the diagonal parts. As such we must instead produce several maps resembling φ and stitch these together. The ability to stitch these maps is the crux of the procedure.

    Throughout the remainder of the part, select some positive integer n_0 and set $m = 2n_0 + 1$. Define accordingly $n_0 + 2$ relatively open intervals $I_k \subseteq [0,1]$ by

$$I_k = \left(\frac{2k-2}{m}, \frac{2k+1}{m} \right) \cap [0,1], \quad k = 0, \ldots, n_0 + 1.$$

Based on these intervals, we form our partition of unity as follows. Define for each $k = 1, \ldots, n_0$ a positive function $\alpha_k\colon [0,1] \longrightarrow [0,1]$ by stipulating that

$$\alpha_k\big|_{\left[0, \frac{2k-2}{m}\right]} = 0, \qquad \alpha_k\big|_{\left[\frac{2k-2}{m}, \frac{2k+1}{m}\right]} \text{ is linear}, \qquad \alpha_k\big|_{\left[\frac{2k+1}{m}, 1\right]} = 1.$$

Observe that α_k maps I_k homeomorphically onto $(0,1)$, stretching each endpoint by an $m/3$ factor. Using the α_k maps, we create a collection of maps $\mathcal{P} = \{f_k \colon k = 0, \ldots, n_0\}$ constituting a partition of unity for $C([0,1])$ subordinate to the collection $\mathcal{I}$ of intervals I_k for $k = 0, 1, \ldots n_0 + 1$. Write

$$f_k = f \circ \alpha_k, \quad g_{\ell,k} = g_\ell \circ \alpha_k \quad \text{and} \quad g_{r,k} = g_r \circ \alpha_k$$

for every $k = 1, \ldots, n_0$. To tackle the endpoint cases set $g_{\ell,0} = h_\ell \circ \alpha_1$ and $g_{n_0+1,r} = h_r \circ \alpha_{n_0}$. Consider for a brief moment the situation for some fixed index $k = 1, \ldots n_0$. The image $\alpha_k(I_k)$ is an identical copy of $\alpha_{k-1}(I_{k-1})$ shifted to the right-hand side by $2/m$. Ergo the ascending linear part

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[1] The idea is exhibited on page 64 would work, but multiplicativity will only be achieved approximately.

of f_k initiates once the descending one of f_{k-1} begins. If the setup strikes the reader as daunting, the figure below might depict the locations of supports and linear parts more vividly.
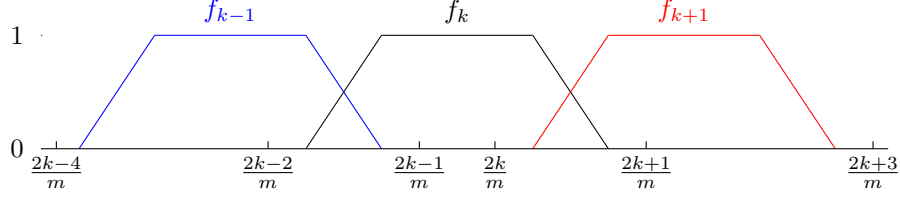
The support of $f_{k-1} + f_k + f_{k+1}$ equals the union of summands' support, meaning $I_{k-1} \cup I_k \cup I_{k+1}$. Moreover, the location of their supports entail that at most two terms contribute at a time. Hence

$$(f_{k-1} + f_k + f_{k+1})|_{I_k} = 1. \tag{5.11}$$

To manufacture the partition of unity, we are inclined to add the cases $k \in \{-1, 0, n_0 + 1, n_0 + 2\}$. Observe first that $I_0 = [0, 1/m)$ whereas $I_{n_0+1} = (1 - 1/m, 1]$. Keeping this in mind, recall that f vanishes on the closed interval $[0, 1/9]$. The intervals I_k all have length $3/m$, hence the map f_1 must necessarily vanish on $[0, 1/3m]$. The endpoint function f_0 should therefore restrict to 1 thereon. A symmetric argument ensures an analogues observation for f_{n_0}. As such stipulating that

$$f_0|_{\left[0, \frac{1}{3m}\right]} = 1, \qquad f_0|_{\left[\frac{1}{3m}, \frac{2}{3m}\right]} \text{ is linear}, \qquad f_0|_{\left[\frac{2}{3m}, 1\right]} = 0$$

$$f_{n_0+1}|_{\left[0, 1 - \frac{2}{3m}\right]} = 0, \qquad f_{n_0+1}|_{\left[1 - \frac{2}{3m}, 1 - \frac{1}{3m}\right]} \text{ is linear}, \qquad f_{n_0+1}|_{\left[1 - \frac{1}{3m}, 1\right]} = 1$$

ensures, in conjunction with (5.11), that the set $\mathcal{P} := \{f_k; k = 0, \ldots, n_0\}$ comprises a partition of unity on $C([0,1])$ subordinate to $\mathcal{I}$, where we add the convention $f_{-1} = f_{n_0+2} = 0$ to avoid index issues ahead. A few properties attached to $\mathcal{P}$ will be recorded, albeit not of current significance. Fix again some $k = 1, \ldots, n_0$. Then one checks that

$$g_{\ell, k-1} = h_\ell \circ \alpha_k \quad \text{and} \quad g_{r, k+1} = h_r \circ \alpha_k.$$

Consider next the product $f_k g_{\ell, k}$. The map g_ℓ restricts to $t \mapsto 1$ on the support of f. Due to $\alpha_k(I_k)$ being a homeomorphic copy of $(0, 1)$ and the support of f_k being properly contained in I_k, one may infer that $f_k g_{\ell, k} = f_k$ if $k = 0, \ldots, n_0$. Arguments running parallel provide the same for $g_{r, k}$.

Having established a partition of unity based on a given parameter n_0, we craft the associated compatible system - when passing to larger matrices, we must configure our compatible system thereto. On the merits of proposition 3.4.5, the quadruple $(\pi_0, \pi_1, \theta, \mathcal{Q}_\omega)$ constitutes a compatible system with respect to the unique trace on $\mathcal{Q}_\omega$, i.e.,

$$\tau_\omega \circ \pi_0 = \tau_\omega \circ \pi_1 = \tau_{\mathcal{L}} \otimes \tau. \tag{5.12}$$

In terms of n_0, define $n_0 + 1$ *-homomorphisms $\sigma_0, \sigma_1, \ldots, \sigma_{n_0}$ as follows. Remember that π_0 is defined on $C_0(0, 1] \otimes A$ whereas π_1 is defined on $C[0, 1) \otimes A$. Extend (π_0, π_1, θ) to n_0-matrices by

$$\sigma_0 = \pi_1^{n_0}: \ C_0[0, 1) \otimes A \longrightarrow \mathrm{M}_{n_0}(\mathcal{Q}_\omega);$$

$$\sigma_{n_0} = \pi_0^{n_0}: C_0(0, 1] \otimes A \longrightarrow \mathrm{M}_{n_0}(\mathcal{Q}_\omega);$$

$$\sigma_k = \pi_1^{n_0-k}|_{C_0(0,1)\otimes A} \oplus \pi_0^k|_{C_0(0,1)\otimes A}: C_0(0, 1) \otimes A \longrightarrow \mathrm{M}_{n_0}(\mathcal{Q}_\omega).$$

Notice that compatibility withstands: σ_0 and σ_{n_0} are compatible with the diagonal map θ^{n_0}. During the proof, we change the codmains to some corner algebras, thereby altering the trace. Luckily, there is a backdoor; lemma 5.1.2 will be used to rescale the trace in the corner algebras. Being a *-homomorphism the element $z := \theta(\mathrm{id}_{[0,1]})$ constitutes a positive contraction in $\mathcal{Q}_\omega$ having Lebesgue spectral measure, the latter property was proven during the proof of proposition 3.4.5. Furthermore, the intervals I_k are all of length $3/m$ unless $k = 0, n_0 + 1$, and $I_k \cap I_j = \emptyset$ holds whenever $|k - j| > 1$ for indices $k, j = 0, \ldots, n_0 + 1$. Applying lemma 5.1.2 onto z, one acquires projections $p_0, \ldots, p_{n_0+1}$ in $\mathcal{Q}_\omega$ fulfilling:

- $p_k p_j = 0$ whenever $|k - j| > 1$;   (p.1)

- p_k commutes with the image of θ for $k = 1, \ldots, n_0$;   (p.2)

- $\tau_\omega(p_k) = |I_k| = 3/m$ for $k = 1, \ldots, n_0$ and $\tau_\omega(p_0) = \tau_\omega(p_{n_0+1}) = 1/m$;   (p.3)

- for each $k = 0, \ldots, n_0 + 1$, the element $\theta(f_{k-1} + f_k + f_{k+1})$ acts as a unit on p_k;   (p.4)

- the projection p_0 acts as a unit on the images $\theta(C_0(I_0))$ and $\pi_1(C_0(I_0) \otimes A)$;

- the projection p_{n_0+1} acts as a unit on the images $\theta(C_0(I_{n_0+1}))$ and $\pi_0(C_0(I_{n_0+1}) \otimes A)$;

- for all $k = 1, \ldots, n_0$, p_k acts as the unit on $\theta(C_0(I_k))$, $\pi_0(C_0(I_k) \otimes A)$ and $\pi_1(C_0(I_k) \otimes A)$.   (p.5)

Before proceeding to the next part, we justify the application above. The property (p.1) is immediate from $I_k \cap I_j = \emptyset$ whenever $|k - j| > 1$ combined with the corresponding orthogonality occurring in lemma 5.1.2. The lemma ensures that p_k commutes with z. However, $\mathrm{id}_{[0,1]}$ generates $C([0,1])$, hence p_k must commute with the image of θ, whereupon (p.2) follows. The third property is a direct consequence of the lemma together with the measure of I_k whereas (p.4) may be deduced from (5.11) alongside unitality of θ.

The final conditions in $(p.5)$ require extra treatment. Fix some integer $k = 1, \ldots, n_0$. According to the lemma part (ii), the projection p_k act as the unit upon the C*-algebra $E \subseteq \mathcal{Q}_\omega$ generated by elements of the form $h(z)$ with h being any member of $C_0(I_k)$. Due to the continuous functional calculus commuting with $*$-homomorphisms, one has $E = \theta(C_0(I_k))$. Suppose B denotes the hereditary C*-subalgebra generated by $\theta(C_0(I_k))$. Compatibility of the system $(\pi_0, \pi_1, \theta, \mathcal{Q}_\omega)$ passes to restrictions of π_0, π_1 onto open subintervals in $(0, 1)$. In particular, compatibility ensures that

$$\pi_i\big(\{s \otimes a : s \in C_0(I_k),\ a \in A\}\big) \subseteq B, \quad i = 0, 1.$$

It follows that B contains $\pi_i(C_0(I_k) \otimes A)$ via linearity and continuity of the involved maps for each index $i = 0, 1$ and every choice of $k = 0, \ldots, n_0 + 1$, yielding the third part of (p.5). This completes the setup of part 1, which we employ with respect to the stable uniqueness theorem.

Part 2. We invoke the procedure in a specific manner, namely in terms of the stable uniqueness theorem. The unitary equivalence therein will be the fuel to enabling (5.10). To apply theorem 4.4.8 we introduce an auxiliary algebra C, which becomes $*$-isomorphic to the unitization of $C_0(J_2) \otimes A$. This will help us recover Δ-fullness, since $\tau_\mathcal{L} \otimes \tau$ will restrict a faithful trace on C. Let

$$C = \left\{ s \in C([0,1], A) : \text{there exists a } z \in \mathbb{C} \text{ such that } s|_{[0,\frac{1}{3}]} = s|_{[\frac{2}{3},1]} = z1_A \right\}.$$

Consider the map $\beta \colon C_0(J_2, A)^+ \longrightarrow C$ given by $s + d1 \mapsto s + d1_A$, where 1 denotes the unit attached to $C_0(J_2, A)^+$. The map is well-defined, since the s term vanishes outside J_2 by hypothesis. One checks that β must be a $*$-isomorphism, meaning $C \cong C_0(J_2, A)^+ \cong (C_0(J_2) \otimes A)^+$. The latter algebra remains nuclear. Due to A fulfilling the UCT-condition by hypothesis, it must be KK-equivalent to some abelian C*-algebra E. According to example 19.1.2(c) in [3], KK-equivalence passes to minimal tensor products factor-wise, whereof $C \cong (A \otimes C_0(J_2))^+$ becomes KK-equivalent to the abelian C*-algebra $(E \otimes C_0(J_2))^+$ and thus must fulfill the UCT-condition.

Keeping theorem 4.4.8 in mind, we must present a control function. Appealing to corollary 1.3.2, the trace $\tau_\mathcal{L} \otimes \tau$ restricted to C must be faithful. Define a control function $\Delta \colon (C_+)_1 \setminus \{0\} \longrightarrow \mathbb{N}$ by choosing, for each s in C, some square number $\Delta(s)$ such that

$$(\tau_\mathcal{L} \otimes \tau)(s) > \frac{2}{\Delta(s)^{1/2}}. \tag{5.13}$$

On the merits of proposition 4.4.3, the ultrapower $\mathcal{Q}_\omega$ is an admissible target of finite type. Moreover, nuclearity of C turns completely positive maps having C as domain into nuclear maps. Altogether, theorem 4.4.8 may be accessed to provide the following property.

Observation 5.2.2. *The triple (G, δ, Δ) consisting of the control function Δ above, the finite subset $G \subseteq (C_0(J_2) \otimes A)^+ \cong C$ and tolerance $\delta > 0$ prescribed previously on page 97 admits a positive integer n satisfying the following property*

Suppose $\gamma \colon C \longrightarrow \mathcal{Q}_\omega$ denotes a Δ-full map and let $\pi, \varrho \colon C \longrightarrow \mathcal{Q}_\omega$ be unital $*$-homomorphisms inducing the same morphisms in total K-theory. Under these premises, one may extract some unitary u in $\mathrm{M}_{n+1}(\mathcal{Q}_\omega)$ such that, for every s in G, one has

$$u(\pi(s) \oplus \gamma^n(s))u^* \approx_\delta \varrho(s) \oplus \gamma^n(s).$$

Repeat the construction exhibited in part 1 with $n_0 = 2n$ (so $m = 4n+1$) and maintain the notation therein. Deviating slightly from the original proof, we will finish the proof under certain assumptions. Afterwards we arrange the assumed properties. The reader is therefore asked to humor the following train of thought. For every $k = 0 \ldots, 2n - 1$, let

$$\Lambda_k := \sigma_k|_{C_0\left(\frac{2k}{m}, \frac{2k+1}{m}\right) \otimes A} \oplus 0_{2n} \colon \qquad C_0\left(\frac{2k}{m}, \frac{2k+1}{m}\right) \otimes A \longrightarrow \mathbb{M}_{4n} \otimes \mathcal{Q}_\omega;$$

$$\Lambda_{2n} := \sigma_{2n}|_{C_0(I_{2n+1}) \otimes A} \oplus 0_{2n} \colon \qquad C_0(I_{2n+1}) \otimes A \longrightarrow \mathbb{M}_{4n} \otimes \mathcal{Q}_\omega;$$

$$\Lambda_0 := \sigma_0|_{C_0(I_0) \otimes A} \oplus 0_{2n} \colon \qquad C_0(I_0) \otimes A \longrightarrow \mathbb{M}_{4n} \otimes \mathcal{Q}_\omega.$$

Formulated in words, Λ_k denotes the restriction of σ_k onto the right-hand third of I_k for $k = 1, \ldots, 2n$. To reduce the notation, for each integer $k = 1, \ldots, 2n - 1$ set

$$F_0^k = \{f_k \otimes a, f_k \otimes ab : a, b \in F_A\} \quad \text{and} \quad F^k = \{f_k, g_{\ell,k}, g_{r,k}, h_{r,k}, h_{\ell,k}\}.$$

If one has no qualms with minor abuse of notation, one may write $F_0^k = F_0 \circ \alpha_k$ together with $F^k = F \circ \alpha_k$ for every such integer k. Now, suppose we were able to derive the these tools:

Claim. There exists a completely positive map $\psi_k \colon C_0(I_k) \otimes A \longrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_2 \otimes \mathbb{M}_{2n}$ for each positive integer $k = 0, \ldots, n_0 + 1$ satisfying the properties below.

- One has $(\tau_\omega \otimes \tau_{4n}) \circ \psi_k = \frac{1}{2}\tau_{\mathcal{L}} \otimes \tau$; $\hspace{6cm}$ (cp$_n$.1)

- one has $\psi_0 = \Lambda_0$ and $\psi_{2n+1} = \Lambda_{2n}$; $\hspace{6cm}$ (cp$_n$.2)

- the restriction of ψ_k onto $C_0\left(\frac{2k-2}{m}, \frac{2k-1}{m}\right) \otimes A$ agrees with Λ_{k-1} and

- the restriction of ψ_k onto $C_0\left(\frac{2k}{m}, \frac{2k+1}{m}\right) \otimes A$ agrees with Λ_k for all $k = 1, \ldots, 2n$; $\hspace{1cm}$ (cp$_n$.3)

- for each positive integer $k = 1, \ldots, 2n$, all $s, t \in F_0^k$ and every $h \in F^k$ one has

$$\psi_k(st) \approx_{\delta_0} \psi_k(s)\psi_k(t) \quad \text{and} \quad \psi_k(s)\theta^{4n}(h) \approx_{\delta_0} \psi_k(hs) \approx_{\delta_0} \theta^{4n}(h)\psi_k(s)$$

while ψ_0 together with ψ_{2n+1} are $*$-homomorphisms compatible with θ^{4n}. $\hspace{2cm}$ (cp$_n$.4)

Finalizing the proof modulo justifying the claim may be accomplished by summing the completely positive maps therein. Certainly, it seems plausible that property (cp$_n$.1) will allow us to pick up half the trace in this manner whereas (cp$_n$.4) ought to supply approximate multiplicativity. The remaining properties are to ensure that stitching the maps together does not cause obstructions on overlaps. This is somewhat the vague idea, so let us dive into the delicate and delicious technicalities. Consider the map $\psi \colon A \longrightarrow \mathcal{Q}_\omega \otimes \mathbb{M}_{2n} \otimes \mathbb{M}_2$ expressed as

$$\psi(a) = \sum_{k=0}^{2n+1} \psi_k(f_k \otimes a).$$

Recall that the support of f_k lies in I_k, whereby one may regard f_k as an element of $C_0(I_k)$. It follows that the above map is meaningful and completely positive being the pointwise sum of completely positive maps combined with $f_k \geq 0$ for each $k = 0, \ldots, 2n + 1$.

The point of decorating with the partition of unity will be presented first. Let us establish the latter property in (5.9) regarding the trace. Due to $\mathcal{P} = \{f_k : k = 0, \ldots, 2n\}$ forming a partition of unity, one may deduce that

$$(\tau_\omega \otimes \tau_{4n})(\psi(a)) = \sum_{k=0}^{2n+1} (\tau_\omega \otimes \tau_{4n})\big(\psi_k(f_k \otimes a)\big) \overset{(\mathrm{cp}_n.1)}{=} \frac{\tau(a)}{2} \sum_{k=0}^{2n+1} \tau_{\mathcal{L}}(f_k) = \frac{1}{2}\tau(a)$$

for every a belonging to A, granting the second half of (5.9). In order to show the first half of (5.9), we truncate the task into a manageable problem. Let elements a, b in F_A be given. According to (p.1), the projections p_k, p_j are orthogonal whenever $|k - j| > 1$. Upon ψ_k attaining values in $p_k\mathcal{Q}_\omega p_k$, the images of ψ_k, ψ_j must be orthogonal provided that $|k - j| > 1$. Simplifying the notation by setting $\psi_{-1} = \psi_{2n+2} = 0$ to match the convention $f_{-1} = f_{2n+2} = 0$, one may conclude that

$$\psi(a)\psi(b) = \sum_{k,j=0}^{2n+1} \psi_k(f_k \otimes a)\psi_j(f_j \otimes b)$$

$$= \sum_{j=-1}^{1} \sum_{k=0}^{2n+1} \psi_k(f_k \otimes a)\psi_{k+j}(f_{k+j} \otimes b).$$

On the other hand,

$$\psi(ab) = \sum_{k=0}^{2n+1} \psi_k(f_k \otimes ab) \overset{(p.4)}{=} \sum_{k=0}^{2n+1} \psi_k(f_k \otimes ab) \sum_{j=-1}^{1} \theta^{4n}(f_{k+j})$$

$$= \sum_{j=-1}^{1} \sum_{k=0}^{2n+1} \psi_k(f_k \otimes ab)\theta^{4n}(f_{k+j}).$$

Let N_e and N_o denote the subcollections of $\{0, \ldots, 2n + 1\}$ consisting of all even and odd integers therein, respectively. Then

$$\psi(a)\psi(b) - \psi(ab) = \sum_{j=-1}^{1} \sum_{k \in N_e} \psi_k(f_k \otimes a)\psi_{k+j}(f_{k+j} \otimes b) - \psi_k(f_k \otimes ab)\theta^{4n}(f_{k+j})$$

$$+ \sum_{j=-1}^{1} \sum_{k \in N_0} \psi_k(f_k \otimes a)\psi_{k+j}(f_{k+j} \otimes b) - \psi_k(f_k \otimes ab)\theta^{4n}(f_{k+j}).$$

We shall investigate the norm of each term arising in both the odd and even parts. To this end, let $e_{k,j}$ denote the element in $\mathcal{Q}_\omega \otimes \mathbb{M}_{4n}$ given by the expression

$$e_{k,j} = \psi_k(f_k \otimes a)\psi_{k+j}(f_{k+j} \otimes b) - \psi_k(f_k \otimes ab)\theta^{4n}(f_{k+j}).$$

Let q_k be the projection $p_k \oplus \ldots \oplus p_k$ inside $\mathbb{M}_{4n}$ for each integer $k = 0, \ldots, 2n + 1$. The property (p.5) guarantees that $e_{k,j} = q_k e_{k,j} q_{k+j}$. The finite sequences $(p_k)_{k \in N_e}, (p_k)_{k \in N_0}$ consist of mutually orthogonal projection according to (p.1), whereupon

$$\Big\| \sum_{k \in N_e} e_{k,j} \Big\| = \Big\| \sum_{k \in N_e} q_k e_{k,j} q_{k+j} \Big\| = \max_{k \in N_e} \|q_k e_{k,j} q_{k+j}\| = \max_{k \in N_e} \|e_{k,j}\|$$

becomes valid for each $j = -1, 0, 1$. Repeating the argument, mutatis mutantis, the same bound may be supplied for the odd part. Entering these estimates yields

$$\|\psi(a)\psi(b) - \psi(ab)\| \leq 2 \sum_{j=-1}^{1} \max_k \|e_{k,j}\|. \tag{5.14}$$

Ergo we are only required to estimate $\|e_{k,j}\|$ for each $k = 0, \ldots, 2n + 1$ and $j = -1, 0, 1$. The crucial feature of $\|e_{k,j}\|$ revolves around it being expressed solely in terms of how multiplicative ψ_k is and how compatible ψ_k is with θ^{4n}, each approximation emerging in $(\mathrm{cp}_n.4)$. The estimates will be established in separate scenarios. Suppose at first $j = 0$. At the endpoint cases $k = 0, 2n + 1$, the corresponding maps ψ_k are $*$-homomorphisms compatible with θ^{4n} by $(\mathrm{cp}_n.4)$. This entails that $e_{0,0} = e_{2n+1,0} = 0$, and for the remaining choices of k, $(\mathrm{cp}_n.4)$ implies that

$$\psi_k(f_k \otimes a)\psi_k(f_k \otimes b) \approx_{\delta_0} \psi_k(f_k^2 \otimes ab) \approx_{\delta_0} \psi(f_k \otimes ab)\theta^{4n}(f_k).$$

Thus the total contribution of the term attached to $j = 0$ will be $4\delta_0$. We proceed to handling the case wherein $j = 1$ fulfills $k + j = 0, \ldots, 2n + 1$. Since the case $j = -1$ will supply the same estimate via minor configurations, we confine ourselves to $j = 1$ and discuss the substantial adjustments. Recall that $g_{\ell,k}$ acts as the unit on f_k whenever $k = 0, \ldots, 2n$ (see page 98). Meanwhile $g_{r,k}$ acts as the unit on f_k for integers $k = 1, \ldots, 2n + 1$. Therefore

$$f_k g_{\ell,k} = f_k, \quad \text{respectively}, \quad g_{r,k+1} f_{k+1} = f_{k+1}. \tag{5.15}$$

Consider the former term arising in $e_{k,1}$. We will rewrite the expression, modulo some error depending on δ_0, using $(\mathrm{cp}_n.4)$. Obtaining this will be accomplished by invoking the compatibility inherited by the morphisms σ_k. According to $(\mathrm{cp}_n.3)$, the maps ψ_k agree with the restrictions Λ_{k-1}, Λ_k to the left (resp. right) -hand side of I_k, so we employ these maps through a trick. Now,

$$\psi_k(f_k \otimes a)\psi_{k+1}(f_{k+1} \otimes b) \stackrel{(5.15)}{=} \psi_k(f_k g_{\ell,k} \otimes a)\psi_{k+1}(g_{r,k+1} f_{k+1} \otimes b)$$
$$\stackrel{(\mathrm{cp}_n.4)}{\approx_{2\delta_0}} \psi_k(f_k \otimes a)\theta^{4n}(g_{\ell,k} g_{r,k+1})\psi_{k+1}(f_{k+1} \otimes b)$$
$$\stackrel{(\mathrm{cp}_n.4)}{\approx_{2\delta_0}} \psi_k(f_k g_{r,k+1} \otimes a)\psi_{k+1}(g_{\ell,k} f_{k+1} \otimes b).$$

As such one may swap the placements of $g_{r,k+1}$ and $g_{\ell,k}$ by paying an error of $4\delta_0$. Fix momentarily an integer $k = 1, \ldots, 2n$. If one compares the supports of each factor in $f_{k+1} g_{\ell,k}$ and similarly for $f_k g_{r,k+1}$, then one may verify that

$$f_{k+1} g_{\ell,k} \in C_0\left(\frac{2k-2}{m}, \frac{2k-1}{m}\right), \quad \text{respectively}, \quad f_k g_{r,k+1} \in C_0\left(\frac{2k}{m}, \frac{2k+1}{m}\right).$$

Then compatibility once more ensures that

$$\psi_k(f_k \otimes a)\psi_{k+1}(f_{k+1} \otimes b) \approx_{4\delta_0} \psi_k(f_k g_{r,k+1} \otimes a)\psi_{k+1}(g_{\ell,k} f_{k+1} \otimes b)$$
$$\stackrel{(\mathrm{cp}_n.3)}{=} \Lambda_k(f_k g_{r,k+1} \otimes a)\Lambda_k(g_{\ell,k} f_{k+1} \otimes b)$$
$$= \Lambda_k(f_k g_{r,k+1} \otimes a)\Lambda_k(g_{\ell,k} \otimes b)\theta^{4n}(f_{k+1})$$
$$\stackrel{(5.15)}{=} \Lambda_k(f_k g_{r,k+1} \otimes ab)\theta^{4n}(f_{k+1})$$
$$\stackrel{(\mathrm{cp}_n.3)}{=} \psi_k(f_k g_{r,k+1} \otimes ab)\theta^{4n}(f_{k+1})$$
$$\stackrel{(\mathrm{cp}_n.4)}{\approx_{\delta_0}} \psi_k(f_k \otimes ab)\theta^{4n}(f_{k+1}).$$

The total contribution of the term attached to $j = 1$ will then be $10\delta_0$. Reiterating the trick for $j = -1$, where $g_{\ell,k}$ is replaced with $g_{r,k}$ and $g_{r,k+1}$ replaces $g_{\ell,k-1}$ leads to the same locations of support in terms of the interval I_{k-1}. The computations preceding and following the trick will supply the exact same estimates, albeit one must replace Λ_k with Λ_{k-1} throughout. Altogether the contribution of the cases $j = \pm 1$ becomes $10\delta_0$ a piece, whereupon (5.9) must be valid due to

$$\|\psi(a)\psi(b) - \psi(ab)\| \leq 4\delta_0 + 10\delta_0 + 10\delta_0 = 24\delta_0 < \varepsilon.$$

In summary, the proof has been reduced into deducing the claim.

Part 3. We finalize the proof by proving the claim. The maps ψ_0, ψ_{2n+1} are straightforward; letting $\psi_0 = \Lambda_0$ and $\psi_{2n+1} = \Lambda_{2n}$ will work. Indeed the property (p.5) implies that $p_0 \oplus \ldots \oplus p_0$ acts as the unit onto the image of Λ_0. Hence the map Λ_0 attains values inside the corner $p_0 \mathcal{Q}_\omega p_0 \otimes \mathbb{M}_{4n}$ and fulfills (cp_n.1) automatically through (5.12). The required compatibility of Λ_0 stems from (cp_n.2) being automatic in conjunction with the established compatibility of the $*$-homomorphism π_1. The same argument may be applied to $\psi_{2n+1} = \Lambda_{2n+1}$ to provide the same properties. Since Λ_k is a $*$-homomorphism for each index k, the choice of ψ_0, ψ_{2n+1} do the job.

For the remaining cases, choose some integer $k = 1, \ldots, 2n$. In order to invoke observation 5.2.2 in part 2, we encode the $*$-homomorphisms (π_0, π_1) into morphisms from C into the corner $p_k \mathcal{Q}_\omega p_k$ as follows. Regarding the nuclear unital C*-algebra C as an isomorphic copy of $(C_0(J_2) \otimes A)^+$, one may define maps $\gamma_0, \gamma_1 \colon C \longrightarrow p_k \mathcal{Q}_\omega p_k$ by

$$\gamma_0(s + z1_C) = zp_k + \pi_0(s \circ \alpha_k) \quad \text{and} \quad \gamma_1(s + z1_C) = zp_k + \pi_1(s \circ \alpha_k).$$

Due to p_k being the unit of $p_k \mathcal{Q}_\omega p_k$, both maps must be unital. We assert that they even constitute $*$-homomorphisms. Linearity and preservation of involution are easily verified, while

$$\gamma_0(s + z1_C)\gamma_0(t + z'1_C) = zz'p_k + z'\pi_0(s \circ \alpha_k)p_k + zp_k\pi_0(t \circ \alpha_k) + \pi_0\big((s \circ \alpha_k)(t \circ \alpha_k)\big)$$

$$\stackrel{(p.5)}{=} zz'p_k + z'\pi_0(s \circ \alpha_k) + z\pi_0(t \circ \alpha_k) + \pi_0(st \circ \alpha_k)$$

$$= \gamma_0\big((st + zt + z's) + zz'1_C\big).$$

Multiplicativity of γ_1 may be deduced verbatim. From this point onwards, there are two separate situations to cover, namely for $k \leq n$ and $k > n$. Dealing with $k \leq n$ during the computations, we shall concurrently describe the corresponding method for the case $k > n$. Stipulate that

$$\gamma = \gamma_1 \text{ if } k \leq n \quad \text{and} \quad \gamma = \gamma_0 \text{ if } k > n.$$

Suppose $k \leq n$. The unital $*$-homomorphism γ will serve as the Δ-full morphism in observation 5.2.2. Therefore we must verify Δ-fullness, which is where lemma 5.1.3 benefits us. For simplicity, denote the restriction of τ_ω onto the corner $p_k \mathcal{Q}_\omega p_k$ by $\tau_{\omega,k}$. Notice that uniqueness of trace on $p_k \mathcal{Q}_\omega p_k \cong \mathcal{Q}_\omega$ entails that $\tau_{\omega,k}(\cdot) = z\tau_\omega(\cdot)$ for some scalar z. As such $1 = z\tau_\omega(p_k)$ implies that

$$\tau_{\omega,k}(a) = \frac{\tau_\omega(a)}{\tau_\omega(p_k)} \stackrel{(p.3)}{=} \frac{m}{3}\tau_\omega(a) \tag{5.16}$$

for every element a belonging to $p_k \mathcal{Q}_\omega p_k$. Furthermore, the map α_k maps I_k homeomorphically onto $(0, 1)$ stretching by a $3/m$-factor as $|I_k| = 3/m$, so

$$(\tau_{\mathcal{L}} \otimes \tau)(e \circ \alpha_k) = \frac{3}{m}(\tau_{\mathcal{L}} \otimes \tau)(e) \tag{5.17}$$

for any e inside $C_0(0,1) \otimes A$. Let $c = s + z1_C$ be some nonzero positive element in C. If $k \leq n$, then

$$\tau_{\omega,k}(\gamma(c)) = \tau_{\omega,k}\big(zp_k + \pi_1(s \circ \alpha_k)\big)$$

$$\stackrel{(5.16)}{=} z + \frac{m}{3}\tau_\omega(\pi_1(s \circ \alpha_k))$$

$$\stackrel{(5.12)}{=} z + \frac{m}{3}(\tau_{\mathcal{L}} \otimes \tau)(s \circ \alpha_k)$$

$$\stackrel{(5.17)}{=} z + (\tau_{\mathcal{L}} \otimes \tau)(s)$$

$$= (\tau_{\mathcal{L}} \otimes \tau)(c) > \frac{2}{\Delta(c)^{1/2}}$$

with the latter estimate being a consequence of (5.13). Invoking lemma 5.1.3 reveals that γ must be Δ-full holds regardless of whether $k \leq n$ or $k > n$ (recall that $\Delta(c)$ was chosen to be a square).

The lacking ingredient is control of the induced total K-theory maps $(\gamma_0)_*, (\gamma_1)_* \colon \underline{K}(C) \longrightarrow \underline{K}(\mathcal{Q}_\omega)$, where we identify $p_k \mathcal{Q}_\omega p_k$ with $\mathcal{Q}_\omega$. Remember that π_0, π_1 defines $*$-homomorphisms on cones of A, that is, contractible spaces. Every $*$-homomorphism having a contractible domain is automatically contractible itself. The restriction of π_0, π_1 onto $C_0(I_k) \otimes A$ thus determine contractible morphisms into $\mathcal{Q}_\omega$. Based on $\tau_\omega(p_k) > 0$ due to $(p.3)$, extract via proposition 3.2.2 some positive integer N and $*$-monomorphism $\mathcal{Q}_\omega \hookrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N$ such that $p_k a p_k \mapsto p_k a p_k \otimes e_{11}$. Consider thereafter the $*$-homomorphisms $\tilde{\gamma}_0, \tilde{\gamma}_1 \colon C_0(J_2) \otimes A \longrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_M$ defined as

$$\tilde{\gamma}_0(s) = \pi_0(s \circ \alpha_k) \otimes e_{11} \quad \text{and} \quad \tilde{\gamma}_1(s) = \pi_1(s \circ \alpha_k) \otimes e_{11}.$$

Notice that the embedding $\mathcal{Q}_\omega \hookrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N$ permits us to regard $\tilde{\gamma}_0, \tilde{\gamma}_1$ having the alleged codomain. Surely, the images of the restrictions of π_0, π_1 to $C_0(I_k)$ attain values in $p_k \mathcal{Q}_\omega p_k$ according to $(p.5)$. We next examine their unitizations. Let now $\gamma_{j,C}$ denote the induced unitized map from $C \cong (C_0(J_2) \otimes A)^+$ into $p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N$. The scenario is visualized in the commutative diagram beneath wherein $j = 0, 1$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(J_2) \otimes A & \overset{\iota}{\longrightarrow} & C & \overset{\varrho}{\longrightarrow} & p_k \mathcal{Q}_\omega p_k & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \tilde{\gamma}_j} & & \Big\downarrow{\scriptstyle \gamma_{j,C}} & & \Big\downarrow{\scriptstyle z \mapsto z \otimes 1_N} & & \\
0 & \longrightarrow & p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N & \overset{\mathrm{id}}{\longrightarrow} & p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N & \overset{\varrho \otimes \mathrm{id}}{\longrightarrow} & p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N & \longrightarrow & 0
\end{array}
$$

Here ϱ represents the $*$-epimorphism given via $s + z 1_C \mapsto z p_k$, ι is the canonical embedding into the unitization (see page 6). Commutativity of the diagram guarantees that

$$
\begin{aligned}
\gamma_{j,C}(s + z 1_C) &= \tilde{\gamma}_j(s) + \big(z p_k \otimes 1_N\big) \\
&= \big((\pi_j(s \circ \alpha_k) + z p_k) \otimes e_{11}\big) \oplus 0_{N-1} + 0 \oplus \varrho^{N-1}(s + z 1_C) \\
&= \gamma_j(s + z 1_C) \oplus \varrho^{N-1}(s + z 1_C)
\end{aligned}
$$

Due to π_0, π_1 restricting to homotopic maps in the space of $*$-homomorphisms from $C_0(I_k) \otimes A$ into $\mathcal{Q}_\omega \hookrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_N$ and α_k being a homeomorphism, $\tilde{\gamma}_0 \sim_h \tilde{\gamma}_1$ follows.

Homotopy invariance ($\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are both contractible, hence so must their unitizations be) of total K-theory forces $\gamma_{0,C}, \gamma_{1,C}$ to induce the same map in total K-theory. Functoriality in conjunction with the previous computation ensures that

$$(\gamma_0)_* \oplus (\varrho^{N-1})_* = (\gamma_1)_* \oplus (\varrho^{N+1})_*$$

It follows that $(\gamma_0)_* + (N-1)\varrho_* = (\gamma_1)_* + (N-1)\varrho_*$, whence $(\gamma_0)_* = (\gamma_1)_*$ as desired. Unitally identifying the corner $p_k \mathcal{Q}_\omega p_k$ with $\mathcal{Q}_\omega$, observation 5.2.2 then grants us some unitary u_0 inside $\mathrm{M}_{n+1}(\mathcal{Q}_\omega)$ fulfilling the estimate

$$u_0(\gamma_1(s) \oplus \gamma^n(s))u_0^* \approx_\delta (\gamma_0(s) \oplus \gamma^n(s)) \tag{5.18}$$

for each s in G. Since $k \leq n$ by hypothesis, the Δ-full $*$-homomorphism γ agrees with γ_1. Adding the element $\gamma_1^{n-k}(s) \oplus \gamma_0^{k-1}(s)$ on each side of (5.18), within the matrix algebra $\mathrm{M}_{2n}(p_k \mathcal{Q}_\omega p_k)$, retains the approximation (5.18). One therefore has

$$(u_0 \gamma_1^{n+1}(s) u_0^*) \oplus \gamma_1^{2n-k-n}(s) \oplus \gamma_0^{k-1} \approx_\delta \gamma_0(s) \oplus \gamma_1^n(s) \oplus \gamma_1^{2n-k-n}(s) \oplus \gamma_0^{k-1}(s)$$

for all s in G. Permuting the diagonal entries corresponds to conjugation by a unitary. Hence conjugating by suitable unitaries allows one to find some unitary u in $\mathrm{M}_{2n}(p_k \mathcal{Q}_\omega p_k)$ such that

$$u(\gamma_1^{2n+1-k}(s) \oplus \gamma_0^{k-1}(s))u^* \approx_\delta \gamma_1^{2n-k}(s) \oplus \gamma_0^k(s) \tag{5.19}$$

for every s in G. One may repeat this trick for $k > n$, instead adding $\gamma_1^{2n-k}(s) \oplus \gamma_0^{k-n-1}(s)$ to both sides of (5.18). Having established a unitary equivalence between maps essentially of the form σ_k,

we may form our compatible system $(\pi, \varrho, \mu, \mathcal{Q}_\omega)$ implementing (5.10) and patch it to enable (cp.1)-(cp.3), then translate this into the properties in the claim. Define $\pi, \varrho \colon C_0(0,1) \otimes A \longrightarrow \mathrm{M}_{2n}(p_k \mathcal{Q}_\omega p_k)$ via the compositions

$$\pi(s) = \sigma_{k-1}(s \circ \alpha_k), \quad \text{respectively,} \quad \varrho(s) = \sigma_k(s \circ \alpha_k).$$

The domains are meaningful due to $s \circ \alpha_k$ having support on I_k and $\alpha_k(I_k) \cong (0,1)$ as topological spaces. The codomains are correct, for $q_k := p_k \oplus p_k \oplus \ldots \oplus p_k$, with $2n$-copies occurring, subsumes the role of the unit on the images of the maps σ_{k-1}, σ_k restricted to I_k, according to property (p.5). Since γ_j is the unitized $*$-homomorphism of the assignment $s \mapsto \pi_j(s \circ \alpha_k)$ on J_2,

$$\pi|_{C_0(J_2) \otimes A} = \sigma_{k-1}|_{C_0(J_2) \otimes A} \stackrel{\text{def}}{=} \left(\gamma_1^{2n-k+1} \oplus \gamma_0^{k-1} \right)|_{C_0(J_2) \otimes A};$$

$$\varrho|_{C_0(J_2) \otimes A} = \sigma_k|_{C_0(J_2) \otimes A} \stackrel{\text{def}}{=} \left(\gamma_1^{2n-k} \oplus \gamma_0^k \right)|_{C_0(J_2) \otimes A}.$$

The missing map μ should simply be θ viewed as a map on the image of α_k, albeit we should compress by p_k to match the codomains. As such letting $\mu \colon C([0,1]) \longrightarrow \mathrm{M}_{2n}(p_k \mathcal{Q}_\omega p_k)$ be defined as

$$\mu(h) = \left(p_k \theta(h \circ \alpha_k) p_k \right)^{2n}$$

provides the required unital $*$-homomorphism; according to (p.1), the projection p_k commutes with the image of θ, whereof $\mu(1_{C_{([0,1])}}) = p_k \oplus p_k \oplus \ldots \oplus p_k$. Let now $E = \mathrm{M}_2(p_k \mathcal{Q}_\omega p_k) \cong \mathcal{Q}$. Then the quadruple (π, ϱ, μ, E) becomes a compatible system due to the established compatibility attached to $(\pi_0, \pi_1, \theta, \mathcal{Q}_\omega)$ combined with

$$\tau_{\omega,k}(\pi(s)) \stackrel{(5.16)}{=} \frac{m}{3}(\tau_\omega \otimes \tau_{2n})(\sigma_{k-1}(s \circ \alpha_k))$$
$$= \frac{m}{3 \cdot 2n} \tau_\omega \left((2n-k+1)\pi_1(s \circ \alpha_k) + (k-1)\pi_0(s \circ \alpha_k) \right)$$
$$\stackrel{(5.12)}{=} \frac{2n \cdot m}{3 \cdot 2n} (\tau_{\mathcal{L}} \otimes \tau)(s \circ \alpha_k)$$
$$\stackrel{(5.17)}{=} (\tau_{\mathcal{L}} \otimes \tau)(s)$$

for all s inside $C_0(0,1) \otimes A$. The same computation yields the analogue condition for ϱ. The equivalence (5.19) holds for every element in $G \subseteq C_0(0,1) \otimes A$, so in particular the restrictions of π and ϱ onto $C_0(J_2) \otimes A$ must obey (5.19), granting the equivalence (5.10) to the compatible system (π, ϱ, μ, E). In conclusion, the properties (cp.1)-(cp.3) are fulfilled with respect to (π, ϱ, μ, E) via some completely positive map $\psi_0 \colon C_0(0,1) \otimes A \longrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_{4n}$. To supply ψ_0 with the designated domain, the completely positive map $\psi_k \colon C_0(I_k) \otimes A \longrightarrow p_k \mathcal{Q}_\omega p_k \otimes \mathbb{M}_{4n}$ given by

$$\psi_k(s) = \psi_0(s \circ \alpha_k|_{I_k}^{-1})$$

will work. We finalize the proof by describing how these properties imply those occurring in the claim. For $(\mathrm{cp}_n.1)$, one computes:

$$((\tau_{\omega,k} \otimes \tau_{4n}) \circ \psi_k)(s) \stackrel{(5.16)}{=} \frac{m}{3}((\tau_\omega \otimes \tau_{4n}) \circ \psi_0)(s \circ \alpha_k|_{I_k}^{-1})$$
$$\stackrel{(cp.1)}{=} \frac{m}{3}(\tau_{\mathcal{L}} \otimes \tau)(s \circ \alpha_k|_{I_k}^{-1})$$
$$\stackrel{(5.17)}{=} (\tau_{\mathcal{L}} \otimes \tau)(s)$$

for each s in $C_0(I_k) \otimes A$. Since $\alpha_k|_{I_k}^{-1}$ is a homeomorphism, the properties $(\mathrm{cp}_n.2)$-$(\mathrm{cp}_n.3)$ are easily seen to correspond to the patching condition of (π, ϱ, μ, E), meaning (cp.2). Upon F_0^k, F^k becoming F_0, F, respectively, under the image of $\alpha_k|_{i_k}^{-1}$ (see page 100 for their definition), the approximations (cp.3) grant the ones in $(\mathrm{cp}_n.4)$, except for δ_0-almost commutativity of θ^{4n} and ψ_k. However, applying the involution on (cp.3) will provide this (each involved map is positive, hence preserves the involution) without hindrances due to F_A consisting of self-adjoints. This complete the proof. $\qquad \square$

5.3   Consequences

The main theorem has a wide range of applications. We exhibit a select few of these. The reader should be warned, the amount of references will be vast and one ought not expect a completely self-contained exposition. For the majority of results, save the Rosenberg conjecture, there will be missing details and the section therefore primarily serves as a survey of understanding the results with an added emphasis on where the main theorem enters.

Let us initiate the section with the Rosenberg theorem and his conjecture. For completeness, we provide a proof of Rosenberg's theorem and its converse. To avoid notational confusion, a brief recap of the reduced group C*-algebras and amenable groups is supplied.

The Rosenberg Problem

Let G be a discrete group. Let $\ell^2(G)$ be the associated Hilbert space consisting of square summable functions $\xi\colon G \longrightarrow \mathbb{C}$, whose point-image $\xi(s)$, for $s \in G$, will be written as ξ_s for notational purposes. Let further $\{\delta_s\}_{s \in G}$ represent the canonical orthonormal bases therein and let $\lambda\colon \mathbb{C}G \longrightarrow \mathcal{U}(\ell^2(G))$ be the left regular representation associated to the group ring $\mathbb{C}G$. Here $\lambda(\mathbb{C}G)$ is implicitly endowed with the ordinary $*$-algebraic structure having

$$C_r^*(G) := \overline{\lambda(\mathbb{C}G)}^{\|\cdot\|} \subseteq \mathrm{B}(\ell^2(G))$$

be its minimal completion inducing a C*-algebraic structure. The resulting C*-algebra is called *the reduced group C*-algebra* associated to G. On the opposite end of the scope, the *full group C*-algebra* associated to G is the C*-algebra is the completion of $\mathbb{C}G$ under the universal norm $\|\cdot\|_u$, that is,

$$\|a\|_u = \sup\{\|\pi(a)\| : \pi \text{ is a non-degenerate representation of } \mathbb{C}G\}.$$

The full group C*-algebra is denoted by $C^*(G)$. Every unitary representation $u_0\colon G \longrightarrow \mathcal{U}(\ell^2(G))$ onto some Hilbert space induces a unital representation $u\colon \mathbb{C}G \longrightarrow \mathrm{B}(\ell^2(G))$. The induced representation will automatically be $\|\cdot\|_u$-contractive, so that u extends to a unital $*$-homomorphism $\pi_u\colon C^*(G) \longrightarrow \mathrm{B}(\ell^2(G))$, uniquely even. This property is referred to as *the universal property* of the full group C*-algebra.

Another pivotal property attached to $C_r^*(G)$ is its inherited faithful trace, the so-called canonical trace. We often forego mentioning additional traces or their whereabouts, if present at all, due to the subject being a large field of study in its own right. However, existence of a faithful one is obviously crucial to us. The canonical trace $\tau\colon C_r^*(G) \longrightarrow \mathbb{C}$ is the functional given by

$$\tau(a) = \langle a\delta_{1_G}, \delta_{1_G}\rangle$$

The proposition found below is well-known.

Proposition 5.3.1 (and definition)**.** *For any discrete group G, the following are equivalent.*

(i) *G admits a left-translation invariant finitely additive probability measure, meaning a finitely additive measure $\mu\colon G \longrightarrow \mathbb{C}$ such that $\mu(G) = 1$ and $\mu(s.A) = \mu(A)$ for any subset $A \subseteq G$, where $s.A$ represents the left-translation action onto A.*

(ii) *There exists a left-translation invariant state on $\ell^\infty(G)$, the mentioned translation being the induced one on $\ell^\infty(G)$, meaning $s.\xi(t) = \xi(s^{-1}t)$ for each ξ in $\ell^\infty(G)$ and s, t in G.*

If G satisfies either, hence both, of the conditions one calls G amenable.

Amenability is the group-theoretic interpretation of nuclearity and vice versa. This assertion may be justified through the following statement, whose proof may be uncovered as *theorem* 2.6.8 in [9]. For the record, there are countless characterizations of amenability differing from those supplied here.

Theorem 5.3.2. *For a discrete group G, the following are equivalent.*

(i) *G is amenable.*

(ii) *$C_r^*(G) \cong C^*(G)$.*

(iii) *$C_r^*(G)$ is nuclear.*

We shall address a resembling question in terms of quasidiagonality. Does amenability translate into quasidiagonality of the reduced group C*-algebra? One implication was proven by Rosenberg, namely that quasdiagonality forces amenability of the group in question. Rosenberg himself conjectured the converse to be valid as well. However, the posed questioned remained unanswered for several years until recently. We provide the full argument here in two steps. We initially verify an equivalent characterization of amenability in terms of the reduced group C*-algebra, then modify the proof to deduce Rosenberg's theorem[2]. Afterwards, we apply the main theorem to assemble the equivalence of amenability and quasidiagonality.

Proposition 5.3.3. *A discrete group G is amenable if and only if $C_r^*(G)$ admits a finite dimensional representation.*

Proof. The "only if " part is trivial, for amenability entails $C_r^*(G) = C^*(G)$, whereupon universality of $C^*(G)$ induces the sought finite dimensional representation via $s \mapsto 1$.

For the converse, suppose one has a finite dimensional representation $\pi \colon C_r^*(G) \longrightarrow \mathbb{M}_n$. The proof is essentially encapsulated within the upcoming commutative diagram. Recall that $\ell^\infty(G)$ faithfully represents into $B(\ell^2(G))$ as multiplication operators. Denote the representation by ϱ, then use Arveson's extension theorem to produce a commutative diagram

$$
\begin{array}{ccc}
B(\ell^2(G)) & \xleftarrow{\;\varrho\;} & \ell^\infty(G) \\
{\scriptstyle\iota}\Big\uparrow \quad & {\scriptstyle\psi}\nearrow & \Big\downarrow{\scriptstyle\varphi = \psi|_{\ell^\infty(G)}} \\
C_r^*(G) & \xrightarrow[\pi]{} \mathbb{M}_n & \xrightarrow{\;\tau_n\;} \mathbb{C}
\end{array}
$$

Here ι represents the inclusion map and ψ is the unital completely positive from Arveson's theorem. Consider the composed map $\tau := \tau_n \circ \psi$. Due to ψ restricting to the unital *-homomorphism π on $C_r^*(G)$, the C*-algebra $C_r^*(G)$ must belong to its multiplicative domain. Ergo,

$$\tau(uau^*) = \tau_n(\psi(u)\psi(a)\psi(u)^*) \overset{(1.10)}{=} \tau_n(\psi(uu^*)\psi(a)) = \tau(a). \tag{5.20}$$

One checks that $\lambda_s \xi \lambda_s^* = s.\xi$ whenever $s \in G$ and $\xi \in \ell^\infty(G)$. Applying the trace, it follows that the relation $\tau(s.\xi) = \tau(\lambda_s \xi \lambda_s^*) = \tau(\xi)$ must be valid by (5.20). As such τ constitutes a left-translation invariant state on $\ell^\infty(G)$, since ψ being unital completely positive forces $\|\psi\| = \|\psi(1)\| = 1$. $\qquad\square$

Theorem 5.3.4. *Suppose G denotes any discrete, not necessarily countable, group. Under this hypothesis, the following conditions are equivalent.*

(i) *G is amenable.*

(ii) *$C_r^*(G) \cong C^*(G)$.*

(iii) *$C_r^*(G)$ is nuclear.*

(iv) *$C_r^*(G)$ is quasidiagonal.*

---

[2]Rosenberg himself used a different strategy relying on Hilbert-Schmidt operators.

Proof. The equivalence (i)⇔(iv) is the sole one missing. We commence by proving (iv)⇒(i). Let once more ϱ be the faithful representation of $\ell^\infty(G)$. Consider the scenario in which G is countable. It is apparent that $C_r^*(G)$ must be separable thereby. Suppose $(\varphi_n)_{n\geq 1}$ denotes the sequence unital completely positive maps $\varphi_n\colon C_r^*(G) \longrightarrow \mathbb{M}_{k(n)}$ detecting quasidiagonality. In the spirit of the previous proof, consider the commutative diagram below.

$$
\begin{array}{ccc}
\ell^\infty(G) & \xrightarrow{\;\varrho\;} & \mathrm{B}(\ell^2(G)) \\
& {\scriptstyle \iota}\uparrow & \;\;\searrow{\scriptstyle \psi_n} \\
C_r^*(G) & \xrightarrow{\;\varphi_n\;} & \mathbb{M}_{k(n)} \subseteq \mathcal{Q} \xrightarrow{\;\delta_\omega\;} \mathcal{Q}_\omega \xrightarrow{\;\tau_\omega\;} \mathbb{C}
\end{array}
$$

Here ι denotes the natural inclusion. The map ψ_n is the unital completely positive map arising from Arveson's extension theorem. In accordance with the diagram, let $\varphi\colon C_r^*(G) \longrightarrow \ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N})$ denote the map induced from the sequence $(\varphi_n)_{n\geq 1}$. Let likewise ψ be the unital completely positive one associated to the sequence $(\psi_n)_{n\geq 1}$. By commutativity of the above diagram in conjunction with the inclusions $\mathbb{M}_{k(n)} \subseteq \mathcal{Q}$ providing an inclusion $\ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N}) \subseteq \ell^\infty(Q)$, we acquire the new larger commutative diagram

$$
\begin{array}{ccccccc}
\ell^\infty(G) & \xrightarrow{\;\varrho\;} & \mathrm{B}(\ell^2(G)) & \xrightarrow{\;\psi\;} & \ell^\infty(\mathcal{Q}) & \xrightarrow{\;\varrho_\omega\;} & \mathcal{Q}_\omega \xrightarrow{\;\tau_\omega\;} \mathbb{C} \\
& {\scriptstyle \iota}\uparrow & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
C_r^*(G) & \xrightarrow{\;\varphi\;} & \ell^\infty(\mathcal{Q}) & \xrightarrow{\;\varrho_\omega\;} & \mathcal{Q}_\omega & \xrightarrow{\;\tau_\omega\;} & \mathbb{C}
\end{array}
$$

The composed map $\pi_r := \varrho_\omega \circ \varphi$ defines a $*$-monomorphism due to asymptotic multiplicativity - and isometry properties of the sequence $(\varphi_n)_{n\geq 1}$, while $\pi := \varrho_\omega \circ \psi$ must be unital completely positive. Upon π restricting to the $*$-homomorphism π_r on $C_r^*(G)$, the algebra $C_r^*(G)$ must belong to the multiplicative domain of π. Thus the composition $\tau := \tau_\omega \circ \pi$ satisfies a property resembling (5.20) through an analogues argument.

For the converse implication, let us treat the countable case as well. Since $C_r^*(G)$ admits a faithful trace and is separable via countability, it fulfills every requirement of theorem 5.2.1 due to Tu's theroem (granting the UCT-condition, see his original article [45] on the matter). Ergo, the canonical faithful trace must be quasidiagonal, upon which $C_r^*(G)$ must be quasidiagonal due to the final part in proposition 3.2.3. We emphasize on separability being an indispensable ingredient for this argument, i.e., countability of G.

The countable case entails the general one as follows. The group G is the inductive, non-sequential, limit of its subgroups generated by finite subsets having inclusions as connecting morphisms, each of which must be amenable, say $G = \varinjlim G_\alpha$. Since inductive limits of amenable groups remain amenable while the same remains true for inductive limits of quasidiagonal C*-algebras with faithful connecting morphisms, the assertion stems from the countable case in conjunction with

$$
C_r^*(G) = C_r^*\big(\varinjlim G_\alpha\big) \cong \varinjlim C_r^*(G_\alpha).
$$

For a proof of the latter isomorphism, we refer to *proposition* 1.5.2 in [30].   □

The Blackadar-Kirchberg Problem

The next aim will be to establish sufficient criteria to deduce quasidiagonality. The converse consideration has been addressed prior, namely that stably finiteness and existence of trace are necessary conditions to sustain quasidiagonality in the unital case. Based on these necessities, Blackadar and Kirchberg posed the following question: Are separable nuclear stably finite C*-algebras automatically quasidiagonal? An affirmative answer would entail that separable nuclear C*-algebras are quasidiagonal if and only if they are stably finite. Stably finite unital nuclear separable C*-algebras do admit at bare minimum one trace due to the following result.

Theorem 5.3.5 (Blackadar-Handlemann, Haagerup)**.** *Every unital stably finite C*-algebra admits a quasitrace. Therefore every exact stably finite C*-algebra admits a trace.*

The latter contribution was accomplished by Haagerup in the unital case, who verified that quasitraces on exact C*-algebras are full-fledged traces. For references of the above, see *theorem* 1.1.4 in [35]. Since we restrict ourselves to nuclear C*-algebras, we omit providing the formal definition of quasitraces. We turn our gaze towards an affirmative answer to Blackader-Kirchberg's problem for the subclass consisting of simple C* algebras in the UCT-class. For the proof, we require a result of Pedersen, whose proof may be found in [34] as *theorem* 5.6.1.

Proposition 5.3.6. *Within every C*-algebra A there exists a minimal norm-dense algebraic two-sided ideal, symbolically denoted by* $\mathrm{Ped}(A)$ *and commonly referred to as the Pedersen ideal.*

Theorem 5.3.7. *Let A be a separable nuclear C*-algebras fulfilling the UCT-condition.*

(i) *If A admits a faithful trace, then A must be quasidiagonal. In particular, A must be quasidiagonal should it admit a trace and be simple.*

(ii) *If A is stably finite and simple, then it must be quasidiagonal.*

Proof. (i): Suppose τ denotes any faithful trace acting on A. According to theorem 5.2.1, τ must be quasidiagonal, whereby proposition 3.2.3 provides quasidiagonality. The second statement stems from the closed left-ideal $\mathcal{L}_\tau = \{a \in A : \tau(a^*a) = 0\}$ being two-sided for traces, hence must be trivial as $\tau \neq 0$. It follows that τ is automatically faithful, whence the first part applies.

(ii): For the second statement, suppose A is separable, nuclear, stably finite, and satisfies the UCT-condition. The proof requires several involved results. As such we record their statements in brevity during the proof. Consider the stabilization $A \otimes \mathbb{K}$ of A. In the event of $A \otimes \mathbb{K}$ admitting a nonzero projection p, the corner $B := p(A \otimes \mathbb{K})p$ must be hereditary. We wish to invoke L. Brown's theorem to acquire a stable isomorphism between A and B, then transfer quasidiagonality therefrom via theorem 5.3.5 and the Tikuisis-White-Winter theorem. However, this requires B to be full. This may be deduced from the correspondence between left-ideals and hereditary subalgebras; any hereditary subalgebra of a simple C*-algebra is simple. By simplicity, any element is automatically full, permitting the use of L. Brown's theorem. Ergo we have

$$p(A \otimes \mathbb{K})p \otimes \mathbb{K} = B \otimes \mathbb{K} \cong A \otimes \mathbb{K}.$$

According to theorem 5.3.5, B admits a trace τ. Upon B being hereditary in the nuclear C*-algebra $A \otimes \mathbb{K}$, it must be nuclear itself. The trace τ is automatically faithfulness by simplicity of B whereas the UCT-condition passes to hereditary subalgebras, whereupon B inherits the UCT-condition through A. The Tikuisis-White-Winter theorem thus ensures quasidiagonality of τ, hence quasidiagonality of B according to (i). The quasidiagonality is inherited to A via the canonical embedding $A \hookrightarrow A \otimes \mathbb{K}$ given by $a \mapsto a \otimes e$ with e being any rank one projection.

Assume now that $A \otimes \mathbb{K}$ is projectionless. In [4], Blackadar-Cuntz managed to verify that for stable simple C*-algebras, the existence of an infinite projection is the sole obstruction towards the existence of lower-semicontinuous dimension functions defined on the Pedersen ideal. The stabilization of A is always stable while it cannot contain infinite projections by hypothesis. Since simplicity stems from simplicity of $A, \mathbb{K}$, then $A \otimes \mathbb{K}$ must admit some lower-semicontinuous dimension function. According to Haagerup's theorem, applicable upon $A \otimes \mathbb{K}$ being nuclear, combined with the correspondence between traces and lower-semicontinuous dimension functions, $A \otimes \mathbb{K}$ must admit some trace τ defined on $\mathrm{Ped}(A \otimes \mathbb{K})$ in the sense that τ is bounded on positive elements thereon. By separability of A, the stabilization becomes σ-unital. Due to minimality, $\mathrm{Ped}(A \otimes \mathbb{K})$ belongs to

$$J_\tau := \{e \in A \otimes \mathbb{K} : \|\tau(e)\| < \infty\} \lhd_{\mathrm{alg}} A \otimes \mathbb{K}.$$

Thus the claim below (for algebraic simplicity, we refer to *corollary* 2.2 in [43]) reveals that τ is a trace on $A \otimes \mathbb{K}$ and the Tikuisis-White-Winter theorem applies thereof to grant (ii).

Claim. For every σ-unital algebraically simple C*-algebra E, each lower semi-continuous trace defined on $\mathrm{Ped}(E)$ is a trace acting on E.

Proof of claim. Choose an increasing approximate unit $(e_n)_{n\geq 1}$ of E such that $e_n e_{n+1} = e_{n+1}$ for each positive integer n[3]. Fix some positive element a in E. Set accordingly $\mu_n = a^{1/2} e_n a^{1/2}$ to acquire an increasing sequence consisting of positive contractions converging in norm to a. Then

$$\tau(a) = \sup_n \tau(\mu_n) = \sup_n \tau(e_n^{1/2} a e_n^{1/2}) \overset{(1.2)}{\leq} \sup_n \tau(e_n)\|a\|.$$

Lower semi-continuity of τ was exploited during the first equality. Therefore $\|\tau\| = \sup_n \tau(e_n)$ must hold. Seeking a contradiction, assume that τ were unbounded. Based on the just established formula for $\|\tau\|$, one may extract an increasing subsequence $(m_k)_{k\geq 1}$ of integers in $\mathbb{N}$ such that $\tau(z_k) \geq 1$ for all k in $\mathbb{N}$, where $z_k := e_{m_{k+1}} - e_{m_k}$. Due to e_n acting as a unit on e_{n-1}, the sequence $(z_k)_{k\geq 1}$ consists of mutually orthogonal positive contractions. Hence the series $\sum_{k=1}^\infty z_k k^{-1}$ converges absolutely to some positive contraction z. Since τ is densely defined and lower semi-continuous, one has $\sum_{k=1}^\infty k^{-1} \leq \sum_{k=1}^\infty \tau(z_k) k^{-1} = \tau(z) < \infty$, a contradiction. $\square$

Spreading Quasidiagonality

Before attending classification natured results, we strengthen the main theorem in the unital case. It shall be revealed that the imposed faithfulness on a single existing trace ensures quasidiagonality of every not necessarily faithful trace. The essential component entering the scene is a lemma in [8] by N. P. Brown, which describes how no distinction occurs between the trace simplex and quasidiagonal traces for certain classes of C*-algebras.

Proposition 5.3.8. *Suppose $\mathscr{A}$ is a class of C*-algebras containing $\mathbb{C}$ such that $\mathscr{A}$ is closed under performing inductive limits with connecting $*$-monomorphisms, taking extensions by members in $\mathscr{A}$ and tensoring with $\mathbb{M}_n$ for each n. Consider the three following conditions.*

(i) Every trace on A is quasidiagonal, for every simple separable unital member A in $\mathscr{A}$.

(ii) Every trace on A is quasidiagonal, for every residually finite member A in $\mathscr{A}$.

(iii) Every trace on A is quasidiagonal, for every quasidiagonal member A in $\mathscr{A}$.

The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are all valid.

Proof. We restrict ourselves to proving (ii)$\Rightarrow$(iii). The remaining implication may be derived from *lemma* 6.1.20 in [8], nothing that the hypothesis concerning Popa algebras in property (ii) is avoidable. Suppose A denotes a quasidiagonal member in the class $\mathscr{A}$, the assumed quasidiagonality being implemented via the sequence $(\psi_n)_{n\geq 1}$ of unital completely positive maps $\psi_n\colon A \longrightarrow \mathbb{M}_{k(n)}$. The idea becomes more transparent after establishing some setup. Consider the unital completely positive map $\psi\colon A \longrightarrow \ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N})$ induced via the sequence $(\psi_n)_{n\geq 1}$ and let $E = C^*(\psi(A))$. We shall investigate the obtained sequence

$$0 \longrightarrow c_0(\mathbb{M}_{k(n)}, \mathbb{N}) \longrightarrow E + c_0(\mathbb{M}_{k(n)}, \mathbb{N}) \overset{\varrho_\infty}{\longrightarrow} A \longrightarrow 0,$$

where $\varrho_\infty\colon \ell^\infty(\mathbb{M}_{k(n)}, \mathbb{N}) \longrightarrow \ell(\mathbb{M}_{k(n)}, \mathbb{N})$ denotes the quotient map. Due to $(\psi_n)_{n\geq 1}$ being asymptotically isometric - and multiplicative, the associated induced map ψ composed with ϱ_∞ turns into a $*$-monomorphism. Therefore the above sequence is meaningful and short-exact, regarding ϱ_∞ as its restriction onto $E + c_0(\mathbb{M}_{k(n)}, \mathbb{N})$. The point here is that $c_0(\mathbb{M}_{k(n)}, \mathbb{N})$ is residually finite dimensional and it ought to belong to $\mathscr{A}$. After having settled these properties, our hypothesis allows us to determine the designated unital completely positive maps.

---

[3]This may be arranged using the continuous functional calculus. We omit dwelling into details.

Having showcased the strategy we dive into the rigorous execution. Upon passing to a suitable subsequence of $(\psi_n)_{n \geq 1}$, we may arrange that ψ becomes multiplicative up to any tolerance. Suppose some finite subset $F \subseteq A$ together with tolerance $\varepsilon > 0$ has been selected. Let furthermore τ be some fixed trace acting on A. As discussed, we may assume without loss of generality that

$$\psi(a)\psi(b) \approx_\varepsilon \psi(ab), \quad a, b \in F. \tag{5.21}$$

Our objective will be to enable (ii) on $E + c_0(\mathbb{M}_{k(n)}, \mathbb{N})$. For this, observe that the containment $\mathbb{C} \in \mathscr{A}$ in conjunction with $\mathscr{A}$ being stable under tensoring with matrix-algebras entails $\mathbb{M}_\ell \in \mathscr{A}$ for every positive integer ℓ. Due to the class being closed under extensions, finite direct sums of elements in $\mathscr{A}$ remain in $\mathscr{A}$. Exploiting this particular property along with $\mathscr{A}$ being closed under performing inductive limits, one may deduce that

$$c_0(\mathbb{M}_{k(n)}, \mathbb{N}) = \varinjlim \left(\bigoplus_{i=1}^n \mathbb{M}_{k(i)} \right) \in \mathscr{A}.$$

The preceding short-exact sequence thus guarantees that $B := E + c_0(\mathbb{M}_{k(n)}, \mathbb{N})$ determines a residually finite dimensional member in $\mathscr{A}$, which permits the use of (ii). Passing to the quotient we acquire the induced trace $\tau_B \colon B \longrightarrow \mathbb{C}$ given by

$$\tau_B(\cdot) = (\tau \circ \varrho_\infty)(\cdot).$$

According to the hypothesis (ii), τ_B must be quasidiagonal. Let $(\varphi_n)_{n \geq 1}$ denote the sequence detecting quasidiagonality of τ_B, meaning for some stage n there exists a unital completely positive map $\varphi := \varphi_n \colon B \longrightarrow \mathbb{M}_k$ fulfilling

$$\varphi(\psi(a)\psi(b)) \approx_\varepsilon \varphi(\psi(a))\varphi(\psi(b)), \tag{5.22}$$

$$\tau_k \circ \varphi \approx_\varepsilon \tau_B. \tag{5.23}$$

One thereof defines $\gamma \colon A \longrightarrow \mathbb{M}_k$ by $a \mapsto \varphi(\psi(a))$ to produce a unital completely positive map subject to the estimates

$$\gamma(ab) = \varphi(\psi(ab)) \overset{(5.21)}{\approx_\varepsilon} \varphi(\psi(a)\psi(b)) \overset{(5.22)}{\approx_\varepsilon} \varphi(\psi(a))\varphi(\psi(b)) = \gamma(a)\gamma(b).$$

An analogues computation leaning on (5.23) ensures that γ recovers the trace τ up to an ε-based error, completing the proof. $\qquad\square$

Combining this result of N. Brown with theorem 5.3.7(i), we deduce the following.

Corollary 5.3.9. *Any trace acting on a separable unital nuclear quasidiagonal C*-algebra satisfying the UCT-condition is quasidiagonal.*

Proof. According to theorem 4.4.4, the $\mathcal{N}$ whose members are separable C*-algebras in the UCT-class, contains $\mathbb{C}$ and is stable under tensoring with $\mathbb{M}_k$ for any positive integer k, extensions and performing inductive limits with monic connecting morphisms. The class consisting of nuclear C*-algebras is likewise closed under performing these operations, see proposition 1.4.6, hence the subclass $\mathcal{N}_{\mathrm{nuc}}$ consisting of nuclear members in $\mathcal{N}$ must be subject to these properties as well. Invoking the Tikuisis-White-Winter theorem, every trace acting on a simple member in $\mathcal{N}_{\mathrm{nuc}}$ must be quasidiagonal. The preceding proposition thereby entails that every trace acting on a quasidiagonal member in $\mathcal{N}_{\mathrm{nuc}}$ must be quasidiagonal itself, proving the claim.

$\qquad\square$

5.4   Connections to the Classification Program

The primary goal behind deriving the main theorem was arguably to provide an affirmative answer to a question posed when the classification of nuclear unital separable simple C*-algebras in the UCT class was within reach. This final section will attempt to convey an overview of how the main theorem affects the classification program. For the record, the work is a culmination of a myriad of involved results from several participants. At first, let us discuss what "classification" means.

Suppose $\mathscr{C}$ denotes some class of C*-algebras, for instance the class of AF-algebras[4]. The class $\mathscr{C}$ is *classifiable* if there is a collection of invariants for members in $\mathscr{C}$, suggestively denoted by $\mathrm{K}(\cdot)$, such that one acquires

$$A \cong B \text{ in } \mathscr{C} \quad \Longleftrightarrow \quad \mathrm{K}(A) \cong \mathrm{K}(B).$$

The isomorphism on the right-hand side depends on the collection of invariants, or "data". To visualize the principle, consider the class $\mathscr{U}$ comprised of UHF-algebras. These are classifiable via their K-theoretic data and placement of unit, that is, one has

$$A \cong B \text{ in } \mathscr{U} \quad \Longleftrightarrow \quad (\mathrm{K}_0(A), [1_A]_0) \cong (\mathrm{K}_0(B), [1_B]_0),$$

where the right-hand isomorphism ought to be read thus: There exists some abelian group isomorphism $\varphi \colon \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(B)$ satisfying $\varphi([1_A]_0) = [1_B]_0$ and φ arises as the group homomorphism induced by a unital $*$-isomorphism $\pi \colon A \longrightarrow B$. K-theory is a prime candidate to encapsulate classification. However, it tends to be insufficient when pursuing classification of say nuclear separable simple unital C*-algebras, despite this originally being conjectured by Elliott.

   In an effort to mend the wound, the K-theoretic data was updated and transformed into what in modern terms is called the *Elliott invariant*. Let $\mathcal{N}$ be the class of unital simple separable nuclear C*-algebras fulfilling the UCT-condition. The Elliott invariant consists of six invariants all of which are collected into a 6-tuple, namely

$$\mathrm{Ell}(A) := (\mathrm{K}_0(A), \mathrm{K}_0(A)^+, \mathrm{K}_1(A), [1_A]_0, T(A), r_A).$$

The invariants, save the latter r_A, have been accounted for already. The map r_A is defined in the following manner. Recall that for any preordered abelian group (G, G^+, u) with order unit u, the set $S(G)$ denotes the weak*-compact convex set of unit-preserving states on G. Let τ be some trace acting on A and define $\mathrm{K}_0(\tau) \colon \mathrm{K}_0(A) \longrightarrow \mathbb{R}$ by

$$\mathrm{K}_0(\tau)([p]_0 - [q]_0) = \tau(p - q)$$

for any pair of projections p, q in $\mathcal{P}_\infty(A)$. Here one adopts the standard picture of $\mathrm{K}_0(A)$ in the unital scenario. The above is independent on the choice of representatives due to traces lacking the ability to distinguish Murray - von Neumann equivalent elements. The assignment $r_A \colon T(A) \longrightarrow S(\mathrm{K}_0(A))$ is then $\tau \mapsto \mathrm{K}_0(\tau)$. This becomes weak* to weak*-continuous and affine.

   For two members A, B in $\mathcal{N}$, one declares that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ should the following conditions be met. Firstly, the preordered abelian groups $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0)$ and $(\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0)$ admit some unit-preserving group isomorphism $\varphi \colon \mathrm{K}_0(A) \longrightarrow \mathrm{K}_0(B)$ such that $\varphi(\mathrm{K}_0(A)^+) = \mathrm{K}_0(B)^+$ becomes valid. Secondly, there must exist some group isomorphism $\psi \colon \mathrm{K}_1(A) \longrightarrow \mathrm{K}_1(B)$ and, lastly, the existence of an affine homeomorphism $\alpha \colon T(A) \longrightarrow T(B)$ making the diagram

$$
\begin{array}{ccc}
T(A) & \xrightarrow{\ \ \alpha\ \ } & T(B) \\
{\scriptstyle r_A}\big\downarrow & & \big\downarrow{\scriptstyle r_B} \\
S(\mathrm{K}_0(A)) & \xrightarrow{\ \varphi_*\ } & S(\mathrm{K}_0(B))
\end{array}
$$

commute, where φ_* is the induced morphism $\varphi_*(f) = \varphi \circ f$, is demanded. The greatest achievement so far is exhibited in the following.

---

[4]The classification of AF-algebras somewhat serves as a narrative example here, see the first chapter in [35].

Theorem 5.4.1. *The subclass in $\mathcal{N}$ whose members consists of those having finite nuclear dimension and in which every trace is quasidiagonal is classified via the Elliott invariant. More precisely, every isomorphism between Elliott invariants lifts to an isomorphism of the C^*-algebras.*

The theorem was proven in [20] by Elliott, Gong, Niu and Lin, heavily relying on older results and especially on the large paper [19]. Let us shed some light upon the new entity: Nuclear dimension. We bring forth this notion alongside its stronger version, finite decomposition rank, without further elaboration outside those specifications we require to understand the aforementioned result. However, let us initially apply the main theorem. Indeed it makes the assumption regarding quasidiagonality of traces obsolete, viz.:

Corollary 5.4.2. *The subclass in $\mathcal{N}$ whose members consists of those having finite nuclear dimension is classified via the Elliott invariant. More precisely, every isomorphism between two Elliott invariants lifts to an isomorphism of the C^*-algebras.*

So what is nuclear dimension? Finite nuclear dimension (and its big brother finite decomposition rank) may be regarded as a refined version of nuclearity.

Definition. A C^*-algebra E has *nuclear dimension* n, written $\dim_{\mathrm{nuc}}(E) = n$, if there exists some net $(\varphi_\alpha, \psi_\alpha, F_\alpha)_{\alpha \in J}$ comprised of completely positive maps $\varphi_\alpha \colon E \longrightarrow F_\alpha$, $\psi_\alpha \colon F_\alpha \longrightarrow E$ with φ_α being contractive and finite-dimensional C^*-algebras F_α such that

- $\psi_\alpha \varphi_\alpha(\cdot) \to \mathrm{id}(\cdot)$ in the point-norm topology;

- each of the algebras F_α decompose into n-ideals $F_\alpha = \bigoplus_{k=0}^n I_k^\alpha$ and ψ_α restricts to an order zero map on each summand.

If the morphism ψ_α of the approximating net may be chosen as contractions, then A is said to have *decomposition rank* n, written $\mathrm{dr}(A) = n$. For both notions, if such an integer n cannot be found, then one declares the nuclear dimension and decomposition rank to be infinite.

We confine ourselves to supplying a few remarks. It is apparent that $\mathrm{dr}(A) \leq \dim_{\mathrm{nuc}}(A)$. Furthermore, a C^*-algebra must be nuclear in the event of either dimension being finite. Some additional notable observations regarding both notions is that they generalize the topological dimension in the sense that $\dim_{\mathrm{nuc}}(C_0(\Omega)) = \mathrm{dr}(C_0(\Omega)) = \dim(\Omega)$ for any locally compact second countable space X, see [50], [27]. Both notions also enjoy some ordinary permanence properties concerning computing the nuclear dimension (resp. decomposition rank). These "dimension" invariants of C^*-algebras have been predicted by Toms and Winter to be the topological formulations of notions established hitherto. We address in brevity the Toms-Winter conjecture.

Conjecture 5.4.1 (Toms-Winter)**.** *Let A be a separable, simple, unital, nuclear, and infinite dimensional C^*-algebras. Then the following are equivalent.*

(i) A is $\mathcal{Z}$-stable, meaning $A \otimes \mathcal{Z} \cong \mathcal{Z}$.

(ii) A has finite nuclear dimension.

(iii) A has strict comparison.

Moreover, (i) may be replaced by finite decomposition rank if A is stably finite.

Combining the results in [37], [48], the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) hold in full generality, whereas (ii)$\Rightarrow$(i) together with (iii)$\Rightarrow$(ii) hold should the extreme boundary of $T(A)$ be compact and finite-dimensional, all of which are consequences of results found in [26], [38], [44] and *Theorem B* in [23], respectively. In particular, the additional condition is fulfilled in the monotracial case and one may combine the main theorem with *Theorem F* in [23] to obtain the equivalence of all four conditions. Hence the Toms-Winter conjecture is valid for monotracial C^*-algebras.

Appendix A

Multiplier Algebras and Tensor Products of Adjointable Operators

Multiplier algebras occur non-stop throughout the thesis. The definition alongside existence (and uniqueness) are assumed familiar, albeit two realizations of multiplier algebras come into play throughout the thesis. As such the concept has received a spot in the appendix together with certain handy permanence properties. First and foremost, the general notion.

Definition. Let E be some C*-algebra. A *multiplier algebra M* of E is a unital C*-algebra containing E as an essential ideal, meaning E non-trivially intersects every ideal in M, fulfilling the following universal property. For any additional multiplier algebra N, there is a $*$-monomorphism $\sigma \colon N \hookrightarrow M$ such that the diagram

$$
\begin{array}{ccc}
N & & \\
\uparrow{\scriptstyle \iota} & \searrow{\scriptstyle \sigma} & \\
E & \xrightarrow{\ \mathrm{id}_E\ } & E
\end{array}
$$

commutes, where ι denotes the ordinary inclusion map.

The multiplier algebra exists and the universal property applied twice allows on to effortlessly deduce uniqueness up to $*$-isomorphism. Therefore, one often refers to *the multiplier algebra* of A, commonly denoted by $\mathcal{M}(A)$. A few characterizations are listed beneath.

Proposition A.0.3. *Suppose A is some C*-algebra faithfully represented on A, say $A \subseteq \mathrm{B}(\mathcal{H})$.*

(i) *The idealizer I_A of A, meaning*

$$
I_A = \big\{ x \in \mathrm{B}(\mathcal{H}) : ax, xa \in A \text{ for all } a \in A \big\}
$$

constitutes a copy of $\mathcal{M}(A)$.

(ii) *The space of double centralizers, meaning*

$$
\{(L, R) \in B(A) \times B(A) : aL(b) = R(a)b, \ a, b \in A\},
$$

with $B(A)$ denoting the space of bounded operators on A, constitutes a copy of $\mathcal{M}(A)$.

Proof. For the realization (i) we refer to *lemma A*.2.1 in [30], whereas the realization (ii) may be recovered in [47] as *proposition* 2.2.11. $\qquad\square$

Computing $\mathcal{M}(A)$ in general may be plainly unachievable. However, a couple of important instances are $\mathcal{M}(C_0(\Omega)) = C_b(\Omega)$ for any locally compact Hausdorff space Ω; and $\mathcal{M}(\mathcal{K}(\mathcal{H})) = \mathrm{B}(\mathcal{H})$. The latter is especially instructive to keep in mind when examining KK-theory. We close the discussion of multiplier algebras by deducing a few permanence properties. It requires some aid from Hilbert modules, so a minor detour is taken.

Definition. Suppose E, F denote Hilbert A, B-modules, respectively. Form the $\mathbb{C}$-algebraic tensor product $E \odot F$. Endow $E \odot F$ with the $A \odot B$ action defined on elementary tensors by

$$(a \otimes b)(\xi \otimes \eta) = a\xi \otimes b\eta$$

for all $a \in A$, $b \in B$, $\xi \in E$ and $\eta \in F$. Equip $E \odot F$ with an $A \odot B$-valued inner product[1] by declaring that for each $\xi, \xi_0 \in E$ together with $\eta, \eta_0 \in F$,

$$\langle \xi \otimes \eta, \xi_0 \otimes \eta_0 \rangle = \langle \xi, \xi_0 \rangle_E \otimes_{\mathbb{C}} \langle \eta, \eta_0 \rangle_F.$$

Being bounded in the minimal tensor norm on $A \odot B$, it extends to $A \otimes B$. We define $E \otimes F$ to be Hilbert-$A \otimes B$ obtained from applying the double completion procedure.

Remark. For Hilbert spaces $\mathcal{H}, \mathcal{K}$ there exists a $*$-monomorphism $\mathrm{B}(\mathcal{H}) \otimes \mathrm{B}(\mathcal{K}) \hookrightarrow \mathrm{B}(\mathcal{H} \otimes \mathcal{K})$. Tensor products of adjointables give rise to a similar inclusion. Verifying this requires extra meticulous care. A detailed exposition of the fact may be recovered in fourth chapter 4 of [28]. Let E, F be Hilbert modules over a common C*-algebra A. Then the map $\beta_0 \colon \mathcal{L}_A(E) \times \mathcal{L}_A(F) \longrightarrow \mathcal{L}_A(E \otimes F)$ given by $(s, t) \mapsto s \otimes t$, wherein

$$(s \otimes t)(\xi \otimes \eta) := s\xi \otimes t\eta$$

for each $\xi \in E$ and $\eta \in F$, induces a $*$-monomorphism $\beta \colon \mathcal{L}_A(E) \otimes \mathcal{L}_A(F) \hookrightarrow \mathcal{L}_A(E \otimes F)$.

Proposition A.0.4. *Let A, B be a C*-algebra. If so, the following hold.*

   (i) $A \cong \mathcal{M}(A)$ whenever A admits a units.

   (ii) One has $\mathrm{M}_n(\mathcal{M}(A)) \cong \mathcal{M}(\mathrm{M}_n(A))$ for all n in $\mathbb{N}$.

   (iii) One has $\mathcal{M}(A_1 \oplus \ldots \oplus A_n) \cong \mathcal{M}(A_1) \oplus \ldots \oplus \mathcal{M}(A_n)$ for C*-algebras $A_1, A_2, \ldots, A_n$.

   (iv) There exists a unital $*$-monomorphism $\beta_0 \colon \mathcal{M}(A) \otimes \mathcal{M}(B) \longrightarrow \mathcal{M}(A \otimes B)$.

Proof. (i): By hypothesis. one must have $(1_A - 1_{\mathcal{M}(A)})A = \{0\}$. Since A is an essential ideal inside $\mathcal{M}(A)$, it follows that $1_A = 1_{\mathcal{M}(A)}$, whereupon $A = \mathcal{M}(A)$.

(ii): Suppose x belongs to the idealizer $\mathcal{M}(\mathrm{M}_n(A))$ of $\mathrm{M}_n(A)$. Representing A faithfully into some Hilbert space $\mathcal{H}$, we may assume without loss of generality that $A \subseteq \mathrm{B}(\mathcal{H})$. By hypothesis, the element $x = [x_{ij}]$ must satisfy $xa, ax \in \mathrm{M}_n(A)$ for any element $a = [a_{ij}]$ inside $\mathrm{M}_n(A)$. In particular, this must be valid for the column-matrix $[a, 0, \ldots, 0]^t \in A^n$, which translates into

$$[x_{11}a, x_{21}a, \ldots, x_{n1}a]^t = xa \in A^n,$$

so that each entry must belong to A. Due to the choice of a being arbitrary, each of the entries x_{k1} for $k \leq n$ must determine an element in $\mathcal{M}(A)$. Similar arguments apply to ensure the containments $x_{ij} \in \mathcal{M}(A)$ for all remaining pairs of indices $i, j \leq n$. The idealizer of $\mathrm{M}_n(A)$ must thereby coincide with $\mathrm{M}_n(\mathcal{M}(A))$, granting $\mathcal{M}(\mathrm{M}_n(A)) \hookrightarrow \mathrm{M}_n(\mathcal{M}(A))$. The reverse inclusion is obvious.

(iii)+(v): The proof of (iii) proceed in the same fashion as (ii), although easier. We omit the details. To prove (iv) one appeals to the embedding $\mathcal{L}(A) \otimes \mathcal{L}(B) \hookrightarrow \mathcal{L}(A \otimes B)$. The $*$-isomorphisms $\mathcal{K}(A) \cong A$ and $\mathcal{M}(\mathcal{K}(A)) \cong \mathcal{L}(A)$ from section 4.1 thus imply that $\mathcal{M}(A) \otimes \mathcal{M}(B) \hookrightarrow \mathcal{M}(A \otimes B)$. $\square$

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[1]Separation of points is quite difficult to ensure. We refer to [28] chapter 6 for a detailed survey.

Appendix B

Stable Rank One and Real Rank Zero

Stable rank one and real rank zero are two invariants of C*-algebras that tend to appear in the thesis. Stable rank one is used solely to invoke Elliott and Ciuperca's classification of *-homomorphisms defined on $C_0(0, 1]$ in terms of their Cuntz semigroup. Real rank zero is primarily used to control K-theoretic behavior and therefore the current appendix was written to supply an overview.

Definition. A unital C*-algebra A has *stable rank one*, written $\mathrm{sr}(A) = 1$, if the topological group $\mathrm{GL}(A)$ consisting of invertible elements in A is norm dense in A.

Remark. Invertible elements in a C*-algebra A admit unitary polar decompositions. Moroever, elements in A admitting a unitary polar decomposition belong to $\overline{\mathrm{GL}(A)}$. Letting $\mathrm{UP}(A)$ denote the collection of unitary polar decomposable elements in A yields

$$\mathrm{GL}(A) \subseteq \mathrm{UP}(A) \subseteq \overline{\mathrm{GL}(A)}.$$

In particular, stable rank one becomes equivalent to $\mathrm{UP}(A)$ being norm-dense in A.

Proposition B.0.5. *The following hold.*

 (i) *Finite dimensional C*-algebras have stable rank one.*

 (ii) *If $(A_n)_{n \geq 1}$ denotes a sequence of unital C*-algebras with stable rank one, then $\ell^\infty(A_n, \mathbb{N})$ attains stable rank one.*

 (iii) *Stable rank one passes to quotients of C*-algebras. Thus, stable rank passes to ultrapowers along a free ultrafilter ω on $\mathbb{N}$.*

 (iv) *Inductive sequences of stable rank one C*-algebras with unital connecting morphisms have stable rank one. Hence UHF-Algebras have stable rank one.*

Proof. (i): The spectrum of an element a inside a finite dimensional C*-algebra is necessarily finite. As such, one may choose an element λ of length at most $\delta > 0$ belonging to the resolvent. This entails that $b = a - 1_A\lambda$ becomes invertible and $\|a - b\| < \delta$. Thus, we may determine an invertible element b within any ε-distance of a (choosing δ sufficiently small) as desired.

(ii): To please the eyes, abbreviate $\ell^\infty(A_n, \mathbb{N})$ by A. Suppose $a = (a_1, a_2, \ldots)$ belongs to A so that each a_n belongs to A_n for $n \in \mathbb{N}$ and fix some tolerance $\varepsilon > 0$. Upon each A_n having stable rank one, there exists some unitary u_n satisfying $a_n \approx_\varepsilon u_n|a_n|$ in A_n. The tuple $(u_1, u_2, \ldots)$ clearly defines a unitary in A, whereof $(u_n|a_n|)$ becomes an element in A such that

$$\|(a_n) - (u_n) \cdot |(a_n)|\| = \sup_{n \in \mathbb{N}} \|a_n - u_n|a_n|\| < \varepsilon.$$

Therefore $\mathrm{UP}(A)$ lies norm-densely inside A, granting (ii).

(iii): Suppose I denotes an ideal inside a stable rank one C*-algebra A. Fix some tolerance $\varepsilon > 0$ and let $\pi\colon A \longrightarrow A/I$ be the canonical $*$-epimorphism. An element b in A/I admits a lift a in A under π. Due to A having stable rank one, there exists a unitary u in A subject to the relation $a \approx_\varepsilon u|a|$. The $*$-homomorphism π is unital, so that $\pi(u)$ must be a unitary inside A/I fulfilling $\pi(a) \approx_\varepsilon \pi(u|a|) = \pi(u)|\pi(a)|$, for the continuous functional calculus commutes with $*$-homomorphisms. It follows that $\mathrm{UP}(A/I)$ is norm-dense in A/I as desired.

(iv): Suppose $(A_n, \varphi_n)_{n \geq 1}$ is an inductive sequence consisting of stable rank one C*-algebras and unital $*$-homomorphisms. Write A for the inductive limit and and let $\varphi_n^\infty\colon A_n \longrightarrow A$ be the associated maps φ_n^∞ per usual. The inductive limit may be identified with the norm-closure of $\bigcup_n \varphi_n^\infty(A_n)$. Any element a in A must therefore be the norm-limit of a sequence $(\varphi_{n(m)}^\infty(a_{n(m)}))_{m \geq 1}$. Let some tolerance $\varepsilon > 0$ be fixed and exploit that each A_n is of stable rank one to produce untaries $u_{n(m)}$ in A_n fulfilling $a_{n(m)} \approx_{\varepsilon/2} u_{n(m)}|a_{n(m)}|$ for each m in $\mathbb{N}$. Select hereafter some m in $\mathbb{N}$ such that $\varphi_{n(m)}^\infty(a_{n(m)}) \approx_{\varepsilon/2} a$. Since each connecting map is unital, $\varphi_{n(m)}^\infty(u_{n(m)})$ becomes a unitary. Moreover, one has

$$\varphi_{n(m)}^\infty(u_{n(m)})|\varphi_{n(m)}^\infty(a_{n(m)})| = \varphi_{n(m)}^\infty(u_{n(m)}|a_{n(m)}|) \approx_\varepsilon a.$$

Again one exploits that the continuous functional calculus commutes with $*$-homomorphisms. Altogether, the set of unitary polar decomposable elements is norm-dense, yielding stable rank one. $\quad\square$

A notion resembling stable rank one is real rank zero. Real rank zero will primarily enter to control unpredictable K-theoretic behavior once we to modify a stable uniqueness theorem by Dadarlat and Eilers. Although we avoid dwelling deep into the theory of real rank zero, it does pose a pivotal notion to us. To remedy the absence of using real rank zero, examples are given.

Definition. A unital C*-algebra A has *real rank zero*, symbolically represented by $\mathrm{RR}(A) = 0$, provided that the set of invertible self-adjoint elements therein is dense in A_{sa}.

For non-unital C*-algebras, one requires the unitization to be of real rank zero. Real rank zero demands that a vast amount of projections must exist. This interpretation may be justified through the theorem below, due to Pedersen and L. Brown in [7].

Theorem B.0.6. *For a unital C*-algebra A, the following conditions are equivalent.*

   (i) A has real rank zero.

  (ii) The set of self-adjoint elements having finite spectra are norm-dense in A_{sa}.

 (iii) Every hereditary subalgebra in A admits an approximate unit (not necessarily increasing) consisting of projections.

We present some fundamental properties attached to real rank zero, again omitting proofs entirely. Verifications are carried out in full detail in [7]. We present these permanence properties, thereby supplying a larger framework of when real rank zero may be expected. Additionally, they reveal real rank zero of $\mathcal{Q}_\omega$.

Proposition B.0.7. *Let $A, A_1, A_2, \ldots$ some C*-algebras. Then:*

  (i) real rank zero passes to quotients;

 (ii) real rank zero passes to inductive limits;

(iii) real rank zero passes to hereditary subalgebras;

(iv) real rank zero passes to matrix algebras, meaning if $RR(A) = 0$, then $RR(M_n(A)) = 0$.

The reader is warned that stable rank one and real rank zero do not imply one another, despite their deceptively resembling appearances. Regardless, the standard examples occurring through the thesis of stable rank one prevail.

Examples.

- Every finite dimensional C*-algebra must have real rank zero. Indeed, every self-adjoint element therein has a finite spectrum, whereby an argument running parallel (i) in proposition B.0.5 implies the second condition of the preceding theorem.

- In light of finite dimensional C*-algebras having real rank zero, all UHF-algebras and ultraproducts thereof must fulfill the same condition by the above permanence properties.

- Every von Neumann algebra has real rank zero.

- The Cuntz-algebra $\mathcal{O}_n$, i.e., the universal C*-algebra generated by n partial isometries $v_1, \ldots v_n$ for which $v_1 v_1^* + \ldots + v_n v_n^* = 1$, has real rank zero for each positive integer n.

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