UNIVERSITY OF COPENHAGEN

Universal algebras and dynamical systems associated with separated graphs

Matias Lolk Andersen

Thesis for the Master degree in Mathematics

Department of Mathematical Sciences

University of Copenhagen

Advisor: Mikael Rørdam

Corrected version of April 14th, 2014

Abstract

We present a recently developed theory by Pere Ara and Ruy Exel on universal algebras and dynamical systems associated with so-called separated graphs. To each such graph (E, C) and field K with involution, we define the Leavitt path algebra $L_K(E, C)$ and the graph C^*-algebra $C^*(E, C)$, generalizing the definitions for ordinary graphs. We also define "abelianized" versions $L_K^{\mathrm{ab}}(E, C)$ and $\mathcal{O}(E, C)$ of these algebras, which exhibit crossed product descriptions

$$L_K^{\mathrm{ab}}(E, C) \cong C_K(\Omega(E, C)) \rtimes_{\theta^*} \mathbb{F} \quad \text{and} \quad \mathcal{O}(E, C) \cong C(\Omega(E, C)) \rtimes_{\theta^*} \mathbb{F},$$

where $\theta \colon \mathbb{F} \curvearrowright \Omega(E, C)$ is a canonical partial action of the free group with generating set E^1 on a zero-dimensional compact metrizable space $\Omega(E, C)$. The main construction associates a sequence of finite bipartite separated graphs (E_n, C^n) to each finite bipartite graph (E, C), such that

$$L_K^{\mathrm{ab}}(E, C) = \varinjlim L_K(E_n, C^n) \quad \text{and} \quad \mathcal{O}(E, C) = \varinjlim C^*(E_n, C^n)$$

for appropriate transit maps. As a byproduct of this, we are able to construct global actions of finitely generated free groups on Cantor spaces whose type semigroups lack almost unperforation. Finally, we obtain an alternative description of the canonical partial action $\mathbb{F} \curvearrowright \Omega(E, C)$ that enables us to characterize the graphs, for which this action is topologically free.

Resumé

Vi præsenterer en nyere teori om algebraer og dynamiske systemer hørende til såkaldt separerede grafer, der primært er udviklet af Pere Ara og Ruy Exel. For enhver sådan graf (E, C) og for ethvert legeme med involution K, definerer vi Leavitt path-algebraen $L_K(E, C)$ og graf-C^*-algebraen $C^*(E, C)$ hørende til (E, C), som begge generaliserer de tilsvarende objekter for sædvanlige grafer. Vi definerer også kanoniske kvotienter $L_K^{\mathrm{ab}}(E, C)$ og $\mathcal{O}(E, C)$ af disse algebraer og viser at

$$L_K^{\mathrm{ab}}(E, C) \cong C_K(\Omega(E, C)) \rtimes_{\theta^*} \mathbb{F} \quad \text{samt} \quad \mathcal{O}(E, C) \cong C(\Omega(E, C)) \rtimes_{\theta^*} \mathbb{F}$$

for en kanonisk partiel virkning $\theta \colon \mathbb{F} \curvearrowright \Omega(E, C)$ af den frie gruppe over E^1 på et nuldimensionelt, kompakt og metriserbart rum. For enhver endelig, todelt graf (E, C) giver hovedkonstruktionen en følge af endelige, todelte grafer (E_n, C^n), så

$$L_K^{\mathrm{ab}}(E, C) = \varinjlim L_K(E_n, C^n) \quad \text{og} \quad \mathcal{O}(E, C) = \varinjlim C^*(E_n, C^n)$$

for passende $*$-homomorfier. Som et biprodukt heraf er vi i stand til at vise, at der findes virkninger af endeligt genererede frie grupper på Cantorrum, hvis typesemigrupper ikke er næsten uperforerede. Endeligt giver vi en alternativ beskrivelse af den kanoniske virkning $\mathbb{F} \curvearrowright \Omega(E, C)$, som gør os i stand til at karakterisere de separerede grafer (E, C), for hvilken den kanoniske virkning er topologisk fri.

Contents

Introduction

In order to comprehend this thesis, the reader really only needs a basic understanding of algebra and topology, although knowledge of C^*-algebras might also be beneficial. In case the reader is not acquainted with C^*-algebras, a minor appendix has been added to cover the relevant definitions and results. In the following we will motivate and describe the contents of the thesis.

Over the last two decades much progress has been made in the field of universal algebras and C^*-algebras associated to directed graphs. Not only do the graph C^*-algebras provide a fairly rich and diverse class, but at the same time they are very well understood in terms of ideal structure and K-theory, which can be computed directly from the graph. As they are always nuclear and separable, and the simple graph C^*-algebras are either purely infinite or AF, they can be classified using either Elliott's classification theorem or the celebrated theorem by Eberhard Kirchberg and N. Christopher Phillips. These properties and many more make the class of graph C^*-algebras a natural test object for all sorts of conjectures, and in some cases a place to look for counterexamples.

In a number of recent articles (see [3],[4],[6],[5]), a considerably larger class of algebras and C^*-algebras associated to so-called *separated* graphs have been studied by Pere Ara, Ruy Exel, Ken R. Goodearl, and Takeshi Katsura. The hope is not at all that one can extend the results that hold for ordinary graphs, but rather to obtain algebras that behave wildly different from classical graph algebras. In this thesis we mainly present the results of [3].

Chapter 1 contains the relevant theory on partial actions and crossed products, including a globalization result for partial actions on topological space. We also give concrete realizations of universal C^*-algebras for partial representations of a group satisfying certain relations. In Chapter 2, we introduce the category of finitely separated graphs and the various algebras associated to such a graph. Specifically, we define Leavitt path algebras $L_K(E,C)$ and graph C^*-algebras $C^*(E,C)$ along with canonical quotients $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}(E,C)$ for each separated graph (E,C). We investigate the basic properties of these algebras and are able to determine the monoid $\mathcal{V}(L_K(E,C))$ of idempotents over $L_K(E,C)$ in terms of graph-theoretic data. We also produce a canonical partial action $\theta\colon \mathbb{F} \curvearrowright \Omega(E,C)$ of the free group with the edges of E as generators on a zero-dimensional compact metrizable space, such that

$$L_K^{\mathrm{ab}}(E,C) \cong C_K(\Omega(E,C)) \rtimes_{\theta^*} \mathbb{F} \quad \text{and} \quad \mathcal{O}(E,C) \cong C(\Omega(E,C)) \rtimes_{\theta^*} \mathbb{F},$$

where $C_K(\Omega(E,C))$ denotes the $*$-algebra of continuous function $\Omega(E,C) \to K$, when K is endowed with the discrete topology. The main point of the thesis is to gain an understanding of these partial actions and their associated crossed products, and to a large extend, we are able to carry out our investigations in both the purely algebraic and the analytic setting at once. Surely, many questions about the C^*-algebras $C^*(E,C)$ and $\mathcal{O}(E,C)$ can only be answered through an analytical investigation, but we shall not be bothered with such questions.

Chapter 3 contains the main construction. To every finite and bipartite graph (E,C), we construct a sequence of finite bipartite graphs (E_n,C^n) along with transit maps

$$L_K(E_n,C^n) \to L_K(E_{n+1},C^{n+1}) \quad \text{and} \quad C^*(E_n,C^n) \to C^*(E_{n+1},C^{n+1}),$$

such that

$$L_K^{\mathrm{ab}}(E,C) \cong \varinjlim L_K(E_n,C^n) \quad \text{and} \quad \mathcal{O}(E,C) \cong \varinjlim C^*(E_n,C^n).$$

This description of $L_K^{\mathrm{ab}}(E,C)$ allows us to prove that the quotient map induces a refinement $\mathcal{V}(L_K(E,C)) \to \mathcal{V}(L_K^{\mathrm{ab}}(E,C))$. We also apply this description of $L_K^{\mathrm{ab}}(E,C)$ in Chapter 4 to prove that the type semigroup $S(\Omega(E,C),\mathbb{F},\mathbb{K})$ is canonically isomorphic to $\mathcal{V}(L_K^{\mathrm{ab}}(E,C))$. Since we can realize arbitrary finitely generated conical monoids as $\mathcal{V}(L_K(E,C))$ for an appropriate finite bipartite graph (E,C), in particular any such monoid can be order embedded into a type semigroup $S(\Omega(E,C),\mathbb{F},\mathbb{K})$ for an appropriate graph. As a consequence, we are able to produce global actions of finitely generated free groups on Cantor spaces whose type semigroups lack almost unperforation.

In Chapter 5, we apply the techniques of Section 1.5 to obtain another description of the action $\mathbb{F} \curvearrowright \Omega(E,C)$. Using the concept of so-called E-functions, we are able to characterize the graphs for which the canonical partial action is topologically free. This is of particular importance, since a reduced crossed product $C(X) \rtimes_{r,\theta^*} \mathbb{F}$ is simple whenever the partial action is minimal and topologically free. One might hope that topologically free minimal orbits with respect to the canonical partial action $\mathbb{F} \curvearrowright \Omega(E,C)$ will produce simple C^*-algebras with exotic properties.

We end the main part of the thesis with a minor chapter of examples. Then follow appendices that cover the relevant theory on abelian monoids, rings, algebras, and C^*-algebras. Chapters 1 and 2 are based on various sources, while the results of Chapters 3–6 are almost exclusively based on [3].

Notation and terminology

The following table illustrates the standard notation throughout the thesis:

Object type	Objects	Elements
Abelian monoids	M or N	a or b
Groups	G	s or t
Algebras	A or B	a or b
C^*-algebras	$\mathcal{A}$ or $\mathcal{B}$	a or b
Topological spaces	X or Y	x or y
Graphs	$E=(E^0,E^1,r,s)$	Vertices: u, v or w
	or $F=(F^0,F^1,r,s)$	Edges: e or f
Separated graphs	(E,C) or (F,D)	Colors: X or Y

We always use Greek letters for homomorphisms, and if a,b are elements of a ring, we denote their commutator by $[a,b]$. Whenever we place a hat on some object, i.e. if we write $\widehat{x}$, it means that the reader should disregard that particular object. If, for instance, we are given an n-tuple $(x_1,\dots,x_n)$, we will write

$$(x_1,\dots,\widehat{x_k},\dots x_n) = (x_1,\dots,x_{k-1},x_{k+1},\dots,x_n).$$

Every abelian monoid M will be equipped with the algebraic preorder $\le$, i.e. we will write $a \le b$ for $a,b \in M$ if there is some $c \in M$, such that $a+c=b$.

Acknowledgements

I would like to thank Mikael for guiding me through yet another project and for his everyday-guidance on relevant seminars, conferences, new articles, Ph.d.-applications and you name it. I would also like to thank Jamie for many (I mean many!) hours of discussions and some very dedicated proofreading.

Chapter 1

Partial actions and crossed products

This chapter establishes the basics of partial actions on algebras, topological spaces and C^*-algebras. Throughout the section, G will denote an arbitrary discrete group with neutral element 1. Roughly speaking, a partial action of G on some kind of object is an assignment $s \mapsto \alpha_s$, where each α_s is an isomorphism of sufficiently nice subobjects, such that $\alpha_s \circ \alpha_t$ equals α_{st} on elements for which the composition is meaningful.

1.1 Partial actions on sets

While the definition of a partial action may vary greatly from category to category, the objects in question always have underlying sets. Rather than proving the same thing within a number of different categories, we shall therefore first prove what can be proved in the context of sets.

Definition 1.1.1. A *partial action* $\alpha\colon G \curvearrowright X$ on a set is a collection of subsets $\{X_s \mid s \in G\}$ and bijections $\alpha_s\colon X_{s^{-1}} \to X_s$ such that

(a) $\alpha_s(X_{s^{-1}} \cap X_t) \subset X_{st}$

(b) $\alpha_{st}(x) = \alpha_s(\alpha_t(x))$ for $x \in X_{t^{-1}} \cap X_{t^{-1}s^{-1}}$

(c) $X_1 = X$

for all $s, t \in G$. Condition (a) might look slightly odd at first sight, but the point is just to make sure that (b) makes sense, i.e. that α_t maps $X_{t^{-1}} \cap X_{t^{-1}s^{-1}}$ into $X_{s^{-1}}$. Given two partial actions $\alpha\colon G \curvearrowright X$ and $\alpha'\colon G \curvearrowright X'$, a map $\varphi\colon X \to X'$ will be called *equivariant* if

(a) $\varphi(X_s) \subset X'_s$

(b) $\alpha'_s \circ \varphi(x) = \varphi \circ \alpha_s(x)$ for all $x \in X_{s^{-1}}$

for all $s \in G$. If φ has an equivariant inverse, then the partial actions are called *equivalent*. Finally, a subset $Y \subset X$ is called α-invariant (or simply invariant) if $\alpha_s(Y \cap X_{s^{-1}}) \subset Y$ for all $s \in G$. Note that α restricts to a partial action $G \curvearrowright Y$ for such a subset Y.

From the above definition we immediately obtain a few basic properties:

Proposition 1.1.2. *If $\alpha\colon G \curvearrowright X$ is a partial action, then*

(a) $\alpha_1 = \mathrm{Id}_X$ *and* $\alpha_{s^{-1}} = \alpha_s^{-1}$

(b) $\alpha_s(X_{s^{-1}} \cap X_{s^{-1}t}) = X_s \cap X_t$

for all $s, t \in G$.

Proof. From $\alpha_1 \circ \alpha_1 = \alpha_1$ and $\alpha_1(X) = X$ we get $\alpha_1 = \mathrm{Id}_X$. We also note that

$$\alpha_s(\alpha_{s^{-1}}(x)) = \alpha_1(x) = x \quad \text{and} \quad \alpha_{s^{-1}}(\alpha_s(y)) = \alpha_1(y) = y$$

for all $x \in X_s \cap X_{ss^{-1}} = X_s$ and $y \in X_{s^{-1}} \cap X_{s^{-1}s} = X_{s^{-1}}$, hence $\alpha_{s^{-1}} = \alpha_s^{-1}$. For (b) we use the fact that

$$\alpha_s(X_{s^{-1}} \cap X_{s^{-1}t}) \subset X_s \cap X_{ss^{-1}t} = X_s \cap X_t$$

with s replaced by s^{-1} and t replaced by $s^{-1}t$ to get

$$\alpha_{s^{-1}}(X_s \cap X_t) \subset X_{s^{-1}} \cap X_{s^{-1}t}$$

as well. We conclude that $X_s \cap X_t = \alpha_s(X_{s^{-1}} \cap X_{s^{-1}t})$. □

Definition 1.1.3. Given a set X, a partial bijection on X is a bijective map $\alpha\colon Y \to Z$ with $Y, Z \subset X$. Given two partial bijections $\alpha\colon Y_1 \to Z_1$ and $\beta\colon Y_2 \to Z_2$, we can define a partial composition

$$\alpha \cdot \beta\colon \beta^{-1}(Y_1 \cap Z_2) \to \alpha(Y_1 \cap Z_2) \text{ given by } \alpha \cdot \beta(x) = \alpha(\beta(x)),$$

and clearly $\alpha \cdot \beta$ is another partial bijection.

Lemma 1.1.4. *The composition $\cdot$ is associative. In particular*

$$\alpha^n := \begin{cases} \underbrace{\alpha \cdots \alpha}_{n \text{ times}} & \text{if } n > 0 \\ \underbrace{\alpha^{-1} \cdots \alpha^{-1}}_{-n \text{ times}} & \text{if } n < 0 \end{cases}$$

is well-defined for a partial bijection α.

Proof. Given partial bijections $\alpha\colon Y_1 \to Z_1$, $\beta\colon Y_2 \to Z_2$ and $\gamma\colon Y_3 \to Z_3$, we simply need to check that the domains of $\alpha \cdot (\beta \cdot \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma$ agree. To that end, just note that

$$\begin{aligned} \mathrm{Dom}\left(\alpha \cdot (\beta \cdot \gamma)\right) &= (\beta \cdot \gamma)^{-1}(Y_1 \cap \beta(Y_2 \cap Z_3)) = \gamma^{-1}\left(\beta^{-1}(Y_1 \cap \beta(Y_2 \cap Z_3))\right) \\ &= \gamma^{-1}\left(\beta^{-1}(Y_1 \cap Z_2) \cap Z_3\right) = \mathrm{Dom}\left((\alpha \cdot \beta) \cdot \gamma\right). \end{aligned}$$

□

Now we can construct partial actions from partial bijections.

Lemma 1.1.5. *If $\alpha_1\colon Y \to Z$ is a partial bijection on X, then $\alpha_n := (\alpha_1)^n$ for $n \neq 0$ and $\alpha_0 = \mathrm{Id}_X$ defines a partial action $\alpha\colon \mathbb{Z} \curvearrowright X$.*

Proof. We should check that $\alpha_m(X_{-m} \cap X_n) \subset X_{m+n}$ for all $m, n \in \mathbb{Z}$, but it suffices to prove this inclusion for only $m \geq 0$, as the case $m < 0$ then follows by the same argument applied to the the partial bijection α_1^{-1}. If $m = 0$ or $n = 0$ the claim is trivial, and if $n > 0$ we have

$$X_{m+n} = \operatorname{Ran}(\alpha_{m+n}) = \operatorname{Ran}(\alpha_m \cdot \alpha_n) = \alpha_m(X_{-m} \cap X_n).$$

In case $-m \leq n < 0$ we have $m+n \geq 0$ and therefore $\alpha_m(X_{-m} \cap X_n) \subset X_m \subset X_{m+n}$. Finally, if $n < -m$ we have

$$X_n = \operatorname{Ran}(\alpha_n) = \operatorname{Ran}(\alpha_{-m} \cdot \alpha_{m+n}) = \alpha_{-m}(X_m \cap X_{m+n}),$$

hence

$$\alpha_m(X_{-m} \cap X_n) = \alpha_m(X_n) = X_m \cap X_{m+n} \subset X_{m+n}$$

as requested. □

When given two actions on the same set, we can form a combined action of the free product of the groups.

Lemma 1.1.6. *Given two partial actions $\alpha\colon G_1 \curvearrowright X$ and $\beta\colon G_2 \curvearrowright X$. Then*

$$(\alpha * \beta)_{s_1 t_1 \cdots s_n t_n} = \alpha_{s_1} \cdot \beta_{t_1} \cdots \alpha_{s_n} \cdot \beta_{t_n}$$

for a reduced word $s_1 t_1 \cdots s_n t_n \in G_1 * G_2$ with $s_i \in G_1$ and $t_i \in G_2$, where we allow $s_1 = 1$ and $t_n = 1$, defines a partial action $G_1 * G_2 \curvearrowright X$.

Proof. In order to ease the notation, we will simply write $\gamma = \alpha * \beta$. By construction we have

$$X_{w_1 w_2} = \gamma_{w_1}(X_{w_1^{-1}} \cap X_{w_2}) \quad \text{and} \quad \gamma_{w_1 w_2}(x) = \gamma_{w_1} \circ \gamma_{w_2}(x)$$

for $x \in X_{w_2^{-1}} \cap X_{w_2^{-1} w_1^{-1}} = X_{w_2^{-1} w_1^{-1}}$, whenever the last letter of w_1 and the first letter of w_2 belong to different groups. Therefore we shall assume that w_1 ends and w_2 starts with letters from the same group, say G_1. Thus we can write $w_1 = w_1' s_1$ and $w_2 = s_2 w_2'$, and for now we shall assume that $s_1 \neq s_2^{-1}$. In this case

$$\begin{aligned}
\alpha_{s_1}(X_{w_1^{-1}} \cap X_{w_2}) &= \alpha_{s_1}\left(\alpha_{s_1^{-1}}(X_{s_1} \cap X_{w_1'^{-1}}) \cap \alpha_{s_2}(X_{s_2^{-1}} \cap X_{w_2'})\right) \\
&\subset X_{w_1'^{-1}} \cap \alpha_{s_1}\left(X_{s_1^{-1}} \cap \alpha_{s_2}(X_{s_2^{-1}} \cap X_{w_2'})\right) \\
&\subset X_{w_1'^{-1}} \cap \alpha_{s_1 s_2}\left(X_{(s_1 s_2)^{-1}} \cap X_{w_2'}\right) \\
&= X_{w_1'^{-1}} \cap X_{s_1 s_2 w_2'},
\end{aligned}$$

hence

$$\gamma_{w_1}(X_{w_1^{-1}} \cap X_{w_2}) \subset \gamma_{w_1'}\left(X_{w_1'^{-1}} \cap X_{s_1 s_2 w_2'}\right) = X_{w_1' s_1 s_2 w_2'} = X_{w_1 w_2}.$$

Also

$$\gamma_{w_2'}(X_{w_2^{-1}} \cap X_{w_2^{-1} w_1^{-1}}) \subset \gamma_{w_2'}(\gamma_{w_2'}^{-1}(X_{w_2} \cap X_{s_2^{-1}})) \subset X_{s_2^{-1}},$$

so

$$\gamma_{w_1 w_2}(x) = \gamma_{w_1'} \circ \alpha_{s_1 s_2} \circ \gamma_{w_2}(x) = \gamma_{w_1'} \circ \alpha_{s_1} \circ \alpha_{s_2} \circ \gamma_{w_2}(x) = \gamma_{w_1} \circ \gamma_{w_2}(x)$$

for all $x \in X_{w_2^{-1}} \cap X_{w_2^{-1}w_1^{-1}}$. If, on the other hand, $s_1 = s_2^{-1}$, we may write $w_1 = w_3 w$ and $w_2 = w^{-1}w_4$ such that the last letter of w_3 is not the inverse of the first letter of w_4. Then

$$\begin{aligned}\gamma_{w_1}(X_{w_1^{-1}} \cap X_{w_2}) &= \gamma_{w_3} \circ \gamma_w \left(\gamma_{w^{-1}}(X_w \cap X_{w_3^{-1}}) \cap \gamma_{w^{-1}}(X_w \cap X_{w_4})\right) \\ &= \gamma_{w_3}(X_w \cap X_{w_3^{-1}} \cap X_{w_4}) \subset \gamma_{w_3}(X_{w_3^{-1}} \cap X_{w_4}) \\ &\subset X_{w_3 w_4} = X_{w_1 w_2},\end{aligned}$$

where the last inclusion follows from the first case. Also

$$X_{w_2^{-1}} \cap X_{w_2^{-1}w_1^{-1}} \subset X_{w_4^{-1}} \cap X_{w_4^{-1}w_3^{-1}}$$

and

$$\gamma_{w_4}(X_{w_2^{-1}} \cap X_{w_2^{-1}w_1^{-1}}) \subset \gamma_{w_4}(\gamma_{w_4^{-1}}(X_{w_4} \cap X_w)) \subset X_w,$$

hence

$$\gamma_{w_1 w_2}(x) = \gamma_{w_3 w_4}(x) = \gamma_{w_3} \circ \gamma_{w_4}(x) = \gamma_{w_3} \circ \gamma_w \circ \gamma_{w^{-1}} \circ \gamma_{w_4}(x) = \gamma_{w_1} \circ \gamma_{w_2}(x)$$

for all $x \in X_{w_2^{-1}} \cap X_{w_2^{-1}w_1^{-1}}$. This completes the proof. $\square$

Combining the above observations, we obtain a way to produce partial actions of free groups.

Corollary 1.1.7. *Given a set S and partial bijections $\alpha_s \colon Y_s \to Z_s$ on X for $s \in S$, we let $\mathbb{F}$ denote the free group on the set S. Then*

$$\alpha_{s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k}} = \alpha_{s_1}^{n_1} \cdot \alpha_{s_2}^{n_2} \cdots \alpha_{s_k}^{n_k},$$

where $s_i \neq s_{i+1}$ for each $i = 1, \ldots, k-1$ and $n_i \in \mathbb{Z} \setminus \{0\}$ for $i = 1, \ldots, k$, defines a partial action $\mathbb{F} \curvearrowright X$.

Proof. In case S is finite, the claim follows from Lemma 1.1.5 and Lemma 1.1.6 since

$$\mathbb{F}_m = \mathbb{F}_{m-1} * \mathbb{Z}.$$

In case S is infinite, we still only need to check the requirements of Definition 1.1.1 on finite words $s, t \in \mathbb{F}$, hence the claim follows from the finite case. $\square$

1.2 Partial actions on algebras

In this section we introduce the crossed product of a partial action on an algebra and investigate the basic relationship between partial representations and partial actions. The results are mostly taken from [8].

Definition 1.2.1. A *partial action* $\alpha\colon G \curvearrowright A$ on an algebra is a set partial action

$$\{\alpha_s\colon D_{s^{-1}} \to D_s\},$$

where each D_s is a two-sided ideal and α_s is an isomorphism. If A is a $*$-algebra, we will also assume that the ideals are self-adjoint. Finally, α will be called *unital* if each of the ideals D_s is unital.

Note that everything we proved in the context of partial actions on sets carries through to partial actions on algebras.

Definition 1.2.2 (The crossed product $A \rtimes_\alpha G$)**.** Given a partial action $\alpha\colon G \curvearrowright A$, we let $A \rtimes_\alpha G$ denote the set of formal sums $\sum_{s\in G} a_s\delta_s$ on symbols $\{\delta_s\}_{s\in G}$, such that $a_s \in D_s$ and $a_s = 0$ for all but finitely $s \in G$. Then we equip $A \rtimes_\alpha G$ with the obvious addition and scalar multiplication, whereas the multiplication is given by

$$\Big(\sum_{s\in G} a_s\delta_s\Big)\Big(\sum_{t\in G} b_t\delta_t\Big) = \sum_{s\in G}\Big(\sum_{t\in G} \alpha_s(\alpha_{s^{-1}}(a_s)b_t)\Big)\delta_{st}.$$

We shall refer to $A \rtimes_\alpha G$ as the *crossed product* of A and G by α. While one might expect this to be an algebra in itself, this is not always the case, for there is nothing guaranteeing associativity of the multiplication. Therefore our first challenge is to establish sufficient conditions for the multiplication to be associative.

If the reader is not acquainted with double centralizers, see Definition A.2.8.

Lemma 1.2.3. *Let $\alpha\colon G \curvearrowright A$ denote a partial action on an algebra, such that D_s is (L,R)-associative for all $s \in G$ (see Definition A.2.9). Then the multiplication on $A \rtimes_\alpha G$ is associative.*

Proof. We must show that

$$\Big((a\delta_r)\cdot(b\delta_s)\Big)\cdot(c\delta_t) = (a\delta_r)\cdot\Big((b\delta_s)\cdot(c\delta_t)\Big) \tag{1.1}$$

for all $r, s, t \in G$, $a \in D_r$, $b \in D_s$ and $c \in D_t$. The left-hand side equals

$$\alpha_{rs}\Big(\alpha_{s^{-1}r^{-1}}(\alpha_r(\alpha_{r^{-1}}(a)b))c\Big)\delta_{rst} = \alpha_{rs}\Big(\alpha_{s^{-1}}(\alpha_{r^{-1}}(a)b)c\Big)\delta_{rst}.$$

Since $\alpha_{r^{-1}}(a)b \in D_{r^{-1}} \cap D_s$ we have $\alpha_{s^{-1}}(\alpha_{r^{-1}}(a)b)c \in D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$, hence

$$\Big((a\delta_r)\cdot(b\delta_s)\Big)\cdot(c\delta_t) = \alpha_r\Big(\alpha_s\Big(\alpha_{s^{-1}}(\alpha_{r^{-1}}(a)b)c\Big)\Big)\delta_{rst}.$$

The right-hand side of equation (1.1) equals

$$\alpha_r\Big(\alpha_{r^{-1}}(a)\alpha_s(\alpha_{s^{-1}}(b)c)\Big)\delta_{rst},$$

so the claim holds if and only if

$$\alpha_s\Big(\alpha_{s^{-1}}(\alpha_{r^{-1}}(a)b)c\Big) = \alpha_{r^{-1}}(a)\alpha_s(\alpha_{s^{-1}}(b)c)$$

for all $r, s, t \in G$, $a \in D_r$, $b \in D_s$ and $c \in D_t$. Equivalently, since $\alpha_{r^{-1}}$ is an isomorphism, the claim holds if and only if

$$\alpha_s\Big(\alpha_{s^{-1}}(ab)c\Big) = a\alpha_s(\alpha_{s^{-1}}(b)c)$$

for all $r, s, t \in G$, $a \in D_{r^{-1}}$, $b \in D_s$ and $c \in D_t$. Now, considering L_a as a left multiplier on D_s and R_c as a right multiplier on $D_{s^{-1}}$, $\alpha_s \circ R_c \circ \alpha_{s^{-1}}$ is a right multiplier on D_s. Since D_s is assumed to be (L, R)-associative it follows that

$$\alpha_s\Big(\alpha_{s^{-1}}(ab)c\Big) = (\alpha_s \circ R_c \circ \alpha_{s^{-1}} \circ L_a)(b) = (L_a \circ \alpha_s \circ R_c \circ \alpha_{s^{-1}})(b) = a\alpha_s(\alpha_{s^{-1}}(b)c)$$

as desired. □

Corollary 1.2.4. *Suppose that $\alpha\colon G \curvearrowright A$ is a partial action, where each D_s is non-degenerate or idempotent (see Definition A.2.9). Then the multiplication on $A \rtimes_\alpha G$ is associative.*

Proof. This follows immediately from Proposition A.2.10 and Lemma 1.2.3. □

Throughout the rest of this section we shall assume that the crossed products are associative.

Proposition 1.2.5. *If α is a partial action on a $*$-algebra A, then*

$$\Big(\sum_{s\in G} a_s\delta_s\Big)^* = \sum_{s\in G} \alpha_s(a^*_{s^{-1}})\delta_s$$

defines an involution on $A \rtimes_\alpha G$.

Proof. Clearly, $*$ is a conjugate linear map of order two, so the only work lies in checking antimultiplicativity. To this end, it suffices to consider elements of the form $a\delta_s$ and $b\delta_t$ with $a \in D_s$, $b \in D_t$. The claim now follows from the computation

$$\begin{aligned}\Big((a\delta_s)\cdot(b\delta_t)\Big)^* &= \Big(\alpha_s(\alpha_{s^{-1}}(a)b)\delta_{st}\Big)^* = \alpha_{t^{-1}s^{-1}}\Big(\alpha_s(\alpha_{s^{-1}}(a)b)^*\Big)\delta_{t^{-1}s^{-1}}\\ &= \alpha_{t^{-1}}(b^*\alpha_{s^{-1}}(a^*))\delta_{t^{-1}s^{-1}} = \alpha_{t^{-1}}\Big(\alpha_t(\alpha_{t^{-1}}(b^*))\alpha_{s^{-1}}(a^*)\Big)\delta_{t^{-1}s^{-1}}\\ &= (\alpha_{t^{-1}}(b^*)\delta_{t^{-1}})\cdot(\alpha_{s^{-1}}(a^*)\delta_{s^{-1}}) = (b\delta_t)^*\cdot(a\delta_s)^*.\end{aligned}$$

□

Let us consider a few examples.

Example 1.2.6. Given a field K, we claim that there is a decomposition $M_n(K) \cong K^n \rtimes_\alpha \mathbb{Z}$. Write α_1 for the partial isomorphism of ideals $I = K^{n-1} \oplus 0$ and $J = 0 \oplus K^{n-1}$ given by shifting a tuple one to the right. Applying the notation of Lemma 1.1.5, we have

$$D_m = \begin{cases} K^{n+m} \oplus 0 \oplus \cdots \oplus 0 & \text{if } -n+1 \le m < 0 \\ 0 \oplus \cdots \oplus 0 \oplus K^{n-m} & \text{if } 0 \le m \le n-1 \\ 0 & \text{if } |m| \ge n \end{cases}$$

Then we can define matrix units

$$e_{i,j} = e_i \delta_{i-j}$$

for $1 \leq i, j \leq n$, where e_i is the i'th element of the standard basis on K^n. Indeed, the $e_{i,j}$'s form a basis of $K^n \rtimes_\alpha \mathbb{Z}$ and

$$\begin{aligned} e_{i,j}e_{k,l} &= (e_i\delta_{i-j}) \cdot (e_k\delta_{k-l}) = \alpha_{i-j}(\alpha_{j-i}(e_i)e_k)\delta_{i+k-j-l} = \alpha_{i-j}(e_je_k)\delta_{i+k-j-l} \\ &= \delta_{j,k} \cdot \alpha_{i-j}(e_j)\delta_{i-l} = \delta_{j,k} \cdot e_i\delta_{i-l} = \delta_{j,k}e_{i,l}. \end{aligned}$$

Thus $K^n \rtimes_\alpha \mathbb{Z}$ is isomorphic to $M_n(K)$.

Example 1.2.7. Given partial actions $\alpha \colon G \curvearrowright A$ and $\beta \colon G \curvearrowright B$, there is a combined action $\alpha \oplus \beta \colon G \curvearrowright A \oplus B$ given by $(\alpha \oplus \beta)_s = \alpha_s \oplus \beta_s$. Surely we have an isomorphism

$$(A \oplus B) \rtimes_{\alpha\oplus\beta} G \cong (A \rtimes_\alpha G) \oplus (B \rtimes_\beta G),$$

and applying this observation to Example 1.2.6 we obtain a crossed product structure

$$M_{n_1}(K) \oplus M_{n_2}(K) \oplus \cdots \oplus M_{n_r}(K) \cong K^N \rtimes_\alpha \mathbb{Z}$$

with $N = n_1+n_2+\ldots+n_r$ and $\alpha = \alpha_1 \oplus \ldots \oplus \alpha_r$, where α_i is the partial action of Example 1.2.6 with $n = n_i$.

Crossed products allow extensions of equivariant homomorphisms.

Proposition 1.2.8. *Given partial actions $\alpha \colon G \curvearrowright A$ and $\beta \colon G \curvearrowright B$ on $*$-algebras. If $\varphi \colon A \to B$ is equivariant, then it extends to a $*$-homomorphism $\widetilde{\varphi} \colon A \rtimes_\alpha G \to B \rtimes_\beta G$ given by $\widetilde{\varphi}(a\delta_s) = \varphi(a)\delta_s$.*

Proof. Using equivariance, we have

$$\begin{aligned} \widetilde{\varphi}((a\delta_s) \cdot (b\delta_t)) &= \widetilde{\varphi}(a\alpha_s(b)\delta_{st}) = \varphi(a)\varphi(\alpha_s(b))\delta_{st} = \varphi(a)\beta_s(\varphi(b))\delta_{st} \\ &= (\varphi(a)\delta_s) \cdot (\varphi(b)\delta_t) = \widetilde{\varphi}(a\delta_s)\widetilde{\varphi}(b\delta_t) \end{aligned}$$

and

$$\widetilde{\varphi}((a\delta_s)^*) = \widetilde{\varphi}(\alpha_{s^{-1}}(a^*)\delta_{s^{-1}}) = \varphi(\alpha_{s^{-1}}(a^*))\delta_{s^{-1}} = \beta_{s^{-1}}(\varphi(a)^*)\delta_{s^{-1}} = (\varphi(a)\delta_s)^* = \widetilde{\varphi}(a\delta_s)^*$$

for all $s, t \in G$, $a \in D_{s^{-1}}$ and $a' \in D_{t^{-1}}$. □

Definition 1.2.9. A *length function* $|\cdot|$ on G is a map $G \to \mathbb{R}_+$ such that $|1| = 0$ and $|st| \leq |s| + |t|$ for all $s, t \in G$.

Example 1.2.10 (The free group $\mathbb{F}$ on a set)**.** Given a set S, there is a canonical length function on the free group $\mathbb{F}$ of reduced words in S: For a reduced word $t = t_1t_2 \cdots t_n$ with $t_i \in S \cup S^{-1}$, we define $|t| := n$.

Definition 1.2.11. Let A denote a unital $*$-algebra. A *partial representation* of G on A is a map $\sigma \colon G \to A$ such that

(a) $\sigma(s)\sigma(t)\sigma(t^{-1}) = \sigma(st)\sigma(t^{-1})$

(b) $\sigma(s)^* = \sigma(s^{-1})$

(c) $\sigma(1) = 1$

for all $s, t \in G$. If G is equipped with a length function, then σ is called *semi-saturated* if

$$\sigma(st) = \sigma(s)\sigma(t)$$

whenever $|st| = |s| + |t|$. Finally, whenever $\sigma\colon G \to A$ is a partial representation, we define $\varepsilon_\sigma(s) = \sigma(s)\sigma(s)^*$ for $s \in G$. Sometimes there will be multiple partial representations in play at once, but otherwise we will usually skip the subscript and simply write $\varepsilon = \varepsilon_\sigma$.

Here are some immediate observations:

Proposition 1.2.12. *If $\sigma\colon G \to A$ is a partial representation, then the following holds:*

(a) $\sigma(s^{-1})\sigma(s)\sigma(t) = \sigma(s^{-1})\sigma(st)$ for all $s, t \in G$.

(b) For every $s \in G$, $\sigma(s)$ is a partial isometry.

(c) We have $\sigma(t)\varepsilon(s) = \varepsilon(ts)\sigma(t)$ for all $s, t \in G$.

(d) The final projections $\varepsilon(s)$ and $\varepsilon(t)$ commute for all $s, t \in G$.

Proof. For (a) we simply note that

$$\sigma(s^{-1})\sigma(s)\sigma(t) = \Big(\sigma(t^{-1})\sigma(s^{-1})\sigma(s)\Big)^* = \Big(\sigma(t^{-1}s^{-1})\sigma(s)\Big)^* = \sigma(s^{-1})\sigma(st).$$

(b) is seen by

$$\sigma(s)\sigma(s)^*\sigma(s) = \sigma(s)\sigma(s^{-1})\sigma(s) = \sigma(s)\sigma(s^{-1}s) = \sigma(s),$$

and (c) follows from

$$\sigma(t)\varepsilon(s) = \sigma(t)\sigma(s)\sigma(s^{-1}) = \sigma(ts)\sigma(s^{-1}) = \sigma(ts)\sigma\left((ts)^{-1}\right)\sigma(t) = \varepsilon(ts)\sigma(t).$$

As a consequence, we have

$$\varepsilon(s)\varepsilon(t) = \sigma(s)\sigma(s^{-1})\varepsilon(t) = \sigma(s)\varepsilon(s^{-1}t)\sigma(t^{-1}) = \varepsilon(ss^{-1}t)\sigma(s)\sigma(s^{-1}) = \varepsilon(t)\varepsilon(s).$$

□

We shall shortly consider the case, where G is a free group.

Proposition 1.2.13. *Suppose that A is a unital $*$-algebra. For a set S, denote by $\mathbb{F}$ the free group on S and consider a set of partial isometries $X = \{x_s \mid s \in S\}$ in A. Then the following are equivalent:*

(a) There exists a semi-saturated partial representation $\sigma\colon \mathbb{F} \to A$ such that $\sigma(s) = x_s$ for all $s \in S$.

(b) There exists a partial representation $\sigma\colon \mathbb{F} \to A$ such that $\sigma(s) = x_s$ for all $s \in S$.

(c) X is tame (see Definition A.2.2).

(d) For any $x, y \in \langle X \cup X^* \rangle$ we have $[xx^*, yy^*] = 0$.

Proof. $(a) \Rightarrow (b)$: This is trivial.

$(b) \Rightarrow (c)$: We first show that any element in the multiplicative semigroup generated by $X \cup X^*$ is a partial isometry, and writing $x = x_1 x_2 \cdots x_n$ for $x_i \in X \cup X^*$, we proceed by induction over n. The case $n = 1$ is vacuously true. Assuming the claim holds for $n - 1$ and taking $s_i \in S \cup S^{-1}$ such that $\sigma(s_i) = x_i$, we have $x = \sigma(s_1)\sigma(s_2)\cdots\sigma(s_n)$. Now by Proposition 1.2.12c and the induction hypothesis, we obtain

$$\begin{aligned} xx^*x &= \sigma(s_1)\cdots\sigma(s_{n-1})\varepsilon(s_n)\sigma(s_{n-1}^*)\cdots\sigma(s_1)^*\sigma(s_1)\cdots\sigma(s_n) \\ &= \varepsilon(s_1\cdots s_n)\sigma(s_1)\cdots\sigma(s_{n-1})\sigma(s_{n-1})^*\cdots\sigma(s_1)^*\sigma(s_1)\cdots\sigma(s_{n-1})^*\sigma(s_n) \\ &= \varepsilon(s_1\cdots s_n)\sigma(s_1)\cdots\sigma(s_n) = x. \end{aligned}$$

We move on to proving that the final projections xx^* and yy^* commute for $x, y \in U$. Writing $y = y_1 y_2 \cdots y_m$ with $y_i = \sigma(t_i)$ and $t_i \in S \cup S^{-1}$, we proceed by induction over $\max\{m, n\}$. In case $m = 0$ or $n = 0$ the claim is trivial, and if $m = n = 1$ it follows from Proposition 1.2.12d. Thus we may assume that $m, n \geq 1$ and that the claim holds for products of at most $\max\{m, n\} - 1$ generators. Writing $\bar{x} = \sigma(s_1)\cdots\sigma(s_{n-1})$ and $\bar{y} = \sigma(t_1)\cdots\sigma(t_{m-1})$, we now deduce that

$$\begin{aligned} xx^*yy^* &= \bar{x}\varepsilon(s_n)\bar{x}^*\bar{y}\varepsilon(t_m)\bar{y}^* = \varepsilon(t_1\cdots t_m)\bar{x}\bar{x}^*\bar{y}\bar{y}^*\varepsilon(s_1\cdots s_n) \\ &= \varepsilon(t_1\cdots t_m)\bar{y}\bar{y}^*\bar{x}\bar{x}^*\varepsilon(s_1\cdots s_n) = yy^*xx^* \end{aligned}$$

using Proposition 1.2.12c,d multiple times.

$(c) \Rightarrow (d)$: Trivial

$(d) \Rightarrow (a)$: Define $\sigma(s) = x_s$ and $\sigma(s^{-1}) = x_s^*$ for $s \in S$ along with $\sigma(1) = 1$. For general reduced $s = s_1 s_2 \cdots s_n$ with $s_i \in S \cup S^{-1}$, we let

$$\sigma(s) = \sigma(s_1)\sigma(s_2)\cdots\sigma(s_n).$$

Taking another reduced word $t = t_1 \cdots t_m$, we have $|st| = |s| + |t|$ if and only if st has reduced form $st = s_1 \cdots s_m t_1 \cdots t_n$, in which case $\sigma(s)\sigma(t) = \sigma(st)$. It remains to show that σ is indeed a partial representation, i.e. that $\sigma(s)\sigma(t)\sigma(t^{-1}) = \sigma(st)\sigma(t^{-1})$ for all $s, t \in \mathbb{F}$. We shall proceed by induction over $\max\{|s|, |t|\}$, noting that the claim is trivial in case $|s| = 0$ or $|t| = 0$. Therefore we may write $s = \tilde{s}s_0$ and $t = t_0\tilde{t}$ with $s_0, t_0 \in S \cup S^{-1}$, $|s| = |\tilde{s}| + 1$ and $|t| = |\tilde{t}| + 1$. In case $s_0 \neq t_0^{-1}$ we have $|st| = |s| + |t|$, so in particular $\sigma(s)\sigma(t)\sigma(t^{-1}) = \sigma(st)\sigma(t^{-1})$. Finally in case $s_0 = t_0{}^{-1}$, we obtain

$$\begin{aligned} \sigma(s)\sigma(t)\sigma(t^{-1}) &= \sigma(\tilde{s}s_0)\sigma(s_0{}^{-1}\tilde{t})\sigma(\tilde{t}^{-1}s_0) = \sigma(\tilde{s})\sigma(s_0)\sigma(s_0{}^{-1})\sigma(\tilde{t})\sigma(\tilde{t}^{-1})\sigma(s_0) \\ &= \sigma(\tilde{s})\sigma(\tilde{t})\sigma(\tilde{t}^{-1})\sigma(s_0)\sigma(s_0{}^{-1})\sigma(s_0) = \sigma(\tilde{s}\tilde{t})\sigma(\tilde{t}^{-1})\sigma(s_0) \\ &= \sigma(st)\sigma(\tilde{t}^{-1}t_0{}^{-1}) = \sigma(st)\sigma(t^{-1}). \end{aligned}$$

by the induction hypothesis. $\square$

Partial actions give rise to partial representations.

Lemma 1.2.14. *Given a partial action $\alpha\colon G \curvearrowright A$ on a $*$-algebra such that each D_s is unital with unity 1_s. Then $\sigma_\alpha\colon G \to A \rtimes_\alpha G$ given by $\sigma_\alpha(s) = 1_s\delta_s$ defines a partial representation.*

Proof. Obviously, σ_α maps the neutral element to the unit and

$$\sigma_\alpha(s)^* = (1_s\delta_s)^* = \alpha_{s^{-1}}(1_s^*)\delta_{s^{-1}} = \sigma_\alpha(s^{-1}).$$

Noting that

$$(1_s\delta_s)\cdot(1_t\delta_r) = \alpha_s(1_{s^{-1}}1_t)\delta_{sr} = 1_s 1_{st}\delta_{sr},$$

where we have applied Proposition 1.1.2b, the multiplicativity condition follows from

$$\begin{aligned}\sigma_\alpha(s)\sigma_\alpha(t)\sigma_\alpha(t^{-1}) &= (1_s\delta_s)\cdot(1_t\delta_t)\cdot(1_{t^{-1}}\delta_{t^{-1}}) = (1_s\delta_s)\cdot(1_t\delta_1)\\ &= 1_s 1_{st}\delta_s = (1_{st}\delta_{st})\cdot(1_{t^{-1}}\delta_{t^{-1}}) = \sigma_\alpha(st)\sigma_\alpha(t^{-1}).\end{aligned}$$

□

On the other hand, partial representations also give rise to partial actions.

Proposition 1.2.15. *Given a partial representation $\sigma\colon G \to A$, let B denote the commutative subalgebra of A generated by the $\varepsilon(s)$'s and set $D_s = \varepsilon(s)B$. Then $\alpha_s^\sigma\colon D_{s^{-1}} \to D_s$ given by*

$$\alpha_s^\sigma(a) = \sigma(s)a\sigma(s^{-1})$$

defines a partial action $\alpha^\sigma\colon G \curvearrowright B$. Moreover, if σ is semi-saturated, then

$$\alpha_{st}^\sigma = \alpha_s^\sigma \cdot \alpha_t^\sigma$$

for all $s,t \in G$ with $|st| = |s| + |t|$.

Proof. In order to ease the notation (and the TeX'ing) we write $\alpha = \alpha^\sigma$. Note that by commutativity of B, D_s is indeed a two-sided, self-adjoint ideal of B and clearly $D_1 = B$. We first check that the map $a \mapsto \sigma(s)a\sigma(s^{-1})$ is in fact invariant on B. Clearly, it suffices to consider elements of the form $a = \varepsilon(s_1)\cdots\varepsilon(s_k) \in B$, and then we even have

$$\sigma(s)a\sigma(s^{-1}) = \sigma(s)\varepsilon(s_1)\cdots\varepsilon(s_k)\sigma(s^{-1}) = \varepsilon(s)\varepsilon(ss_1)\varepsilon(ss_2)\cdots\varepsilon(ss_k) \in D_s \subset A$$

from Proposition 1.2.12c. Obviously α_s is linear and involutive, and for multiplicativity we note that

$$\alpha_s(a)\alpha_s(b) = \sigma(s)a\sigma(s^{-1})\sigma(s)b\sigma(s^{-1}) = \sigma(s)a\varepsilon(s^{-1})b\sigma(s^{-1}) = \sigma(s)ab\sigma(s^{-1}) = \alpha_s(ab)$$

for all $a,b \in D_{s^{-1}}$. Also, $\alpha_{s^{-1}}$ is clearly an inverse of α_s, hence each α_s is an isomorphism of ideals. Noting that

$$\alpha_s(\varepsilon(s^{-1})\varepsilon(t)) = \sigma(s)\varepsilon(s^{-1})\varepsilon(t)\sigma(s^{-1}) = \varepsilon(s)\varepsilon(st),$$

it follows that $\alpha_s(D_{s^{-1}} \cap D_t) \subset D_{st}$ and, since

$$\sigma(s)\sigma(t)a\sigma(t^{-1})\sigma(s^{-1}) = \sigma(st)a\sigma(t^{-1}s^{-1})$$

for all $a \in D_{t^{-1}}$, we have $\alpha_s(\alpha_t(a)) = \alpha_{st}(a)$ for all $a \in D_{t^{-1}} \cap D_{t^{-1}s^{-1}}$. We conclude that α^σ is indeed a partial action. Now, assuming that σ is semi-saturated and picking $s,t \in G$ with $|st| = |s| + |t|$, we have

$$\begin{aligned}\mathrm{Dom}(\alpha_s^\sigma \cdot \alpha_t^\sigma) &= \alpha_{t^{-1}}^\sigma(D_t \cap D_{s^{-1}}) = D_{t^{-1}} \cap D_{t^{-1}s^{-1}} = \varepsilon(t^{-1})\varepsilon(t^{-1}s^{-1})B\\ &= \varepsilon(t^{-1}s^{-1})B = D_{(st)^{-1}},\end{aligned}$$

hence $\alpha_{st}^\sigma = \alpha_s^\sigma \cdot \alpha_t^\sigma$. □

Lemma 1.2.16. *Consider a partial action $\alpha\colon G \curvearrowright A$, where each ideal D_s has unity 1_s, and write B for the commutative $*$-subalgebra of A generated by 1_s's. Then*

$$\alpha_s(b)\delta_1 = \alpha_s^{\sigma_\alpha}(b\delta_1)$$

for each $b \in 1_{s^{-1}}B$.

Proof. Simply note that

$$\alpha_s^{\sigma_\alpha}(b\delta_1) = \sigma_\alpha(s) \cdot (b\delta_1) \cdot \sigma_\alpha(s)^* = (1_s\delta_s) \cdot (b\delta_1) \cdot (1_{s^{-1}}\delta_{s^{-1}}) = (1_s\delta_s) \cdot (b\delta_{s^{-1}}) = \alpha_s(b)\delta_1.$$

$\square$

We end this section with a universal property for crossed products of unital partial actions.

Proposition 1.2.17. *Given a partial action $\alpha\colon G \curvearrowright A$ on a $*$-algebra, a $*$-homomorphism $\varphi\colon A \to B$ and a partial representation $\sigma\colon G \to B$. If the image of φ commutes with $\varepsilon(s)$ for all $s \in G$, and*

$$\varphi(\alpha_s(a)) = \sigma(s)\varphi(a)\sigma(s^{-1})$$

for all $a \in D_{s^{-1}}$ and $s \in G$, then there is a unique $*$-homomorphism $\varphi \times \sigma\colon A \rtimes_\alpha G \to B$ given by

$$\varphi \times \sigma\left(\sum_{s\in G} a_s\delta_s\right) = \sum_{s\in G} \varphi(a_s)\sigma(s).$$

Proof. Clearly, $\varphi \times \sigma$ is a well-defined linear map. Multiplicativity follows from the computation

$$\begin{aligned}\varphi \times \sigma\Big((a\delta_s) \cdot (b\delta_t)\Big) &= \varphi \times \sigma\Big(\alpha_s(\alpha_{s^{-1}}(a)b)\delta_{st}\Big) = \varphi(\alpha_s(\alpha_{s^{-1}}(a)b))\sigma(st)\\ &= \varepsilon(s)\varphi(a)\sigma(s)\varphi(b)\sigma(s^{-1})\sigma(st) = \varepsilon(s)\varphi(a)\sigma(s)\varphi(b)\varepsilon(s)\sigma(t)\\ &= \varphi(a)\sigma(s)\varphi(b)\sigma(t) = (\varphi \times \sigma)(a\delta_s) \cdot (\varphi \times \sigma)(b\delta_t),\end{aligned}$$

and it is involutive by

$$\begin{aligned}(\varphi \times \sigma)(a\delta_s)^* &= \sigma(s^{-1})\varphi(a^*) = \sigma(s^{-1})\varepsilon(s)\varphi(a^*) = \sigma(s^{-1})\varphi(a)\sigma(s)\sigma(s^{-1})\\ &= \varphi(\alpha_{s^{-1}}(a^*))\sigma(s^{-1}) = \varphi \times \sigma(\alpha_{s^{-1}}(a^*)\delta_{s^{-1}}) = \varphi \times \sigma\left((a\delta_s)^*\right).\end{aligned}$$

$\square$

Corollary 1.2.18. *Given a partial representation $\sigma\colon G \to A$, define $B \subset A$ and α^σ as in Proposition 1.2.15, and let ι denote the inclusion $B \hookrightarrow A$. Then there is a unique $*$-homomorphism $\iota \times \sigma\colon B \rtimes_{\alpha^\sigma} G \to A$ such that*

$$(\iota \times \sigma)\left(\sum_{s\in G} b_s\delta_s\right) = \sum_{s\in G} b_s\sigma(s).$$

Moreover we have $(\iota \times \sigma) \circ \sigma_{\alpha^\sigma} = \sigma$.

Proof. The first part of the claim follows immediately from Proposition 1.2.17. For the second part, we simply note that

$$(\iota \times \sigma) \circ \sigma_{\alpha^\sigma}(s) = \iota \times \sigma(\varepsilon(s)\delta_s) = \varepsilon(s)\sigma(s) = \sigma(s).$$

$\square$

1.3 Partial actions on topological spaces

The proper definition in the topological case is the following:

Definition 1.3.1. A partial action $\theta\colon G \curvearrowright X$ on a topological space X is a set partial action $\{\theta_s\colon X_{s^{-1}} \to X_s\}$, where each θ_s is a homeomorphism of open subsets of X. If all the sets X_s are also closed or compact, we shall simply refer to θ as a closed or compact partial action, respectively. Also, θ is called *minimal* if there are no non-trivial θ-invariant open subsets of X, and it is called *topologically free* if the set

$$F_s = \{x \in X_{s^{-1}} \mid \theta_s(x) = x\}$$

has empty interior for all $1 \neq s \in G$.

Definition 1.3.2. For a topological partial action $\theta\colon G \curvearrowright X$, we can define an induced partial action on the $*$-algebra $C_{c,K}(X)$ (denoted $C_K(X)$ if X is compact) of compactly supported continuous function $X \to K$, when K is endowed with the discrete topology. Specifically, we set

$$D_s = \{f \in C_{c,K}(X) \mid f(x) = 0 \text{ for all } x \notin X_s\}$$

and define $\alpha_s\colon D_{s^{-1}} \to D_s$ by $\alpha_s(f) = f \circ \theta_{s^{-1}}$. Usually we will write $\alpha = \theta^*$.

Example 1.3.3. Given a global action $\beta\colon G \curvearrowright Y$ and an open subset $X \subset Y$, we can define a partial action $\theta\colon G \curvearrowright X$ by $X_s = X \cap \beta_s(X)$ and $\theta_s(x) = \beta_s(x)$ for all $x \in X_{s^{-1}}$ and $s \in G$. Indeed $X_1 = X \cap \beta_1(X) = X$,

$$\begin{aligned}\theta_s(X_{s^{-1}} \cap X_t) &= \theta_s(\beta_{s^{-1}}(X) \cap \beta_t(X) \cap X) = \beta_s(\beta_{s^{-1}}(X) \cap \beta_t(X) \cap X)\\ &\subset X \cap \beta_{st}(X) = X_{st}\end{aligned}$$

and $\theta_s(\theta_t(x)) = \beta_s(\beta_t(x)) = \beta_{st}(x) = \theta_{st}(x)$ for all $x \in X_{t^{-1}} \cap X_{t^{-1}s^{-1}}$. We shall refer to θ as the *restriction* of β to X and write $\theta = \beta|_X$. We are led to the following definition.

Definition 1.3.4. Suppose that $\theta\colon G \curvearrowright X$ is a partial action. A *globalization* of θ is a global action $\beta\colon G \curvearrowright Y$ along with an equivariant injective open map $\iota\colon X \to Y$, such that $Y = \bigcup_{s\in G} \beta_s(\iota(X))$ and θ is equivalent to $\beta|_{\iota(X)}$ via ι. A globalization will be called *universal*, if for any global action $G \curvearrowright Z$ and any equivariant map $\psi\colon X \to Z$, there is a unique equivariant map $\widetilde{\psi}\colon Y \to Z$ such that $\widetilde{\psi} \circ \iota = \psi$.

Remark 1.3.5. If $\beta\colon G \curvearrowright Y$ is a globalization of $\theta\colon G \curvearrowright X$, then Y has the same local properties as X, since $Y = \bigcup_{s\in G} \beta_s(\iota(X))$ and ι is an open map. For instance, locally compactness and totally disconnectivity both pass to the globalization, but in general the Hausdorff property does not.

Globalizations have been studied by Fernando Abadie in [1], and the following results are due to him.

Theorem 1.3.6. *Suppose that $\theta\colon G \curvearrowright X$ is a partial action of a discrete group on a topological space. Then θ has a (up to canonical equivariant isomorphism) unique universal globalization*

$$\iota\colon X \to X^{\mathrm{e}} \quad \textit{and} \quad \theta^{\mathrm{e}}\colon G \curvearrowright X^{\mathrm{e}}.$$

Proof. Define a global action $\gamma\colon G \curvearrowright G \times X$ by $\gamma_s(t,x) = (st,x)$, and note that it respects the equivalence relation

$$(s,x) \sim (t,y) \overset{\text{def}}{\Leftrightarrow} x \in X_{s^{-1}t} \text{ and } \theta_{t^{-1}s}(x) = y.$$

Letting X^{e} denote the quotient $\frac{G\times X}{\sim}$ with quotient map $q\colon G \times X \to X^{\mathrm{e}}$, γ drops to an action $\theta^{\mathrm{e}}\colon G \curvearrowright X^{\mathrm{e}}$ satisfying $\theta^{\mathrm{e}}_s(q(t,x)) = q(st,x)$. Now define $\iota(x) = q(1,x)$ and note that

$$\iota(\theta_s(x)) = q(1,\theta_s(x)) = q(s,x) = \theta^{\mathrm{e}}_s(\iota(x))$$

for all $x \in X_{s^{-1}}$, thereby verifying equivariance of ι. Also $q(s,x) = \theta^{\mathrm{e}}_s(\iota(x))$ holds for any $x \in X$, hence X^{e} is indeed the θ^{e}-orbit of $\iota(X)$. Injectivity of ι is trivial since

$$q(1,x) = \iota(x) = \iota(y) = q(1,y)$$

holds exactly if $(1,x) \sim (1,y)$, and this precisely means that $x = y$. It is also clear that θ is equivalent to $\theta^{\mathrm{e}}|_X$ via ι. For θ^{e} to be a globalization, it now only remains to check openness. To this end, assume that $U \subset X$ is open — then by definition of the quotient topology, we should show that $q^{-1}(\iota(U))$ is open in $G \times X$. But since

$$q^{-1}(\iota(U)) = \{(s,x) \in G \times X \mid (s,x) \sim (1,y) \text{ for some } y \in U\} = \bigcup_{s\in G} \{s\} \times \theta_{s^{-1}}(U \cap X_s),$$

this is clearly the case. We move on to proving universality of $(\theta^{\mathrm{e}},\iota)$, so let $\beta\colon G \curvearrowright Y$ denote a global action and assume that $\psi\colon X \to Y$ is equivariant. Then we define $\psi'\colon G \times X \to Y$ by $\psi'(s,x) = \beta_s(\psi(x))$ and note that

$$\psi'(\gamma_s(t,x)) = \psi'(st,x) = \beta_{st}(\psi(x)) = \beta_s(\beta_t(\psi(x))) = \beta_s(\psi'(t,x)),$$

i.e. ψ' is equivariant. If $(s,x) \sim (t,y)$ we have

$$\beta_{t^{-1}}(\psi'(s,x)) = \beta_{t^{-1}s}(\psi(x)) = \psi(\theta_{t^{-1}s}(x)) = \psi(y),$$

hence $\psi'(s,x) = \beta_t(\psi(y)) = \psi'(t,y)$. Thus ψ' drops to an equivariant map $\psi^{\mathrm{e}}\colon X^{\mathrm{e}} \to Y$ satisfying $\psi^{\mathrm{e}}(q(s,x)) = \beta_s(\psi(x))$, so in particular

$$\psi^{\mathrm{e}}(\iota(x)) = \psi^{\mathrm{e}}(q(e,x)) = \beta_1(\psi(x)) = \psi(x).$$

Uniqueness of ψ^{e} is immediate from equivariance, $\psi^{\mathrm{e}} \circ \iota = \psi$ and the identity

$$X^{\mathrm{e}} = \bigcup_{s\in G} \theta^{\mathrm{e}}_s(\iota(X)),$$

while uniqueness of $(\theta^{\mathrm{e}},\iota)$ up to canonical equivariant isomorphism follows as usual. □

Having proved the existence and uniqueness of universal globalizations, we turn to the question of when the Hausdorff property is preserved.

Proposition 1.3.7. *Given a partial action $\theta\colon G \curvearrowright X$ on a Hausdorff space. Then X^{e} is Hausdorff if the graph of θ_s is closed in $X \times X$ for all $s \in \Gamma$. Moreover, the converse implication holds as well if θ is closed.*

Proof. Regard X as an open subspace of X^{e}, such that θ is the restriction of θ^{e} to X. Suppose that the graph of each θ_s is closed and take $x^{\mathrm{e}}, y^{\mathrm{e}} \in X^{\mathrm{e}}$. We shall prove that if any pair of open neighbourhoods of these two points intersect, then in fact $x^{\mathrm{e}} = y^{\mathrm{e}}$. Since each α_s^{e} is a homeomorphism of X^{e} and $X^{\mathrm{e}} = \bigcup_{s\in G} \theta_s^{\mathrm{e}}(X)$, we may assume that $x^{\mathrm{e}} = x \in X$. Take $y \in X$ and $s \in G$ such that $\alpha_s^{\mathrm{e}}(y) = y^{\mathrm{e}}$. Now, for any pair of open neighbourhoods (U, V) of x and y respectively, by assumption there is some $x_{U,V} \in U \cap \alpha_s(V)$, and we write $x_{U,V} = \alpha_s^{\mathrm{e}}(y_{U,V})$ for $y_{U,V} \in V$. Then the net $(y_{U,V}, x_{U,V})$ converges to (y, x), so by closedness of the graph $\mathrm{Gr}(\theta_s)$ we have $y^{\mathrm{e}} = \alpha_s(y) = x$ as promised.

Assuming that θ is closed and X^{e} is Hausdorff, the graph $\mathrm{Gr}(\theta_s)$ is clearly closed for all $s \in G$. $\square$

We end this investigation of globalizations with a crucial observation. It turns out that for sufficiently nice partial actions, on the level of algebras, the universal globalization does not affect the Morita equivalence class.

Proposition 1.3.8. *Given a closed partial action $\theta\colon G \curvearrowright X$ on a compact Hausdorff space. Then the crossed products*

$$C_K(X) \rtimes_{\theta^*} G \quad \textit{and} \quad C_{c,K}(X^{\mathrm{e}}) \rtimes_{(\theta^{\mathrm{e}})^*} G$$

are Morita equivalent (see Definition A.2.3).

Proof. Regard X as a subset of X^{e}. Since X is assumed to be compact, so is $\mathrm{Gr}(\theta_s)$ inside $X \times X$ for all $s \in G$, and from the Hausdorff assumption on X, we deduce that $\mathrm{Gr}(\theta_s)$ is closed in $X \times X$. It now follows from Proposition 1.3.7 that X^{e} is Hausdorff as well. In particular X is clopen in X^{e}, hence so is $\theta_s^{\mathrm{e}}(X)$ for any $s \in G$. We claim that $p = 1_X\delta_1$ is a full projection in $C_{c,K}(X^{\mathrm{e}}) \rtimes_{\theta^{\mathrm{e}}} G$. To see this, assume that I is an ideal in $C_{c,K}(X^{\mathrm{e}}) \rtimes_{\theta^{\mathrm{e}}} G$ containing p — then we must prove that $f\delta_t \in I$ for all $t \in G$ and $f \in C_{c,K}(X^{\mathrm{e}})$. Since f is compactly supported and $X^{\mathrm{e}} = \bigcup_{s\in G} \theta_s^{\mathrm{e}}(X)$ is an open covering, there is some finite subset $F \subset G$ such that $\mathrm{supp}(f) \subset Z := \bigcup_{s\in F} \theta_s^{\mathrm{e}}(X)$. Now observe that for all $s \in G$ we have

$$1_{\theta_s^{\mathrm{e}}(X)}\delta_1 = (1_{\theta_s^{\mathrm{e}}(X)}\delta_s) \cdot (1_X\delta_1) \cdot (1_X\delta_{s^{-1}}) \in I$$

from which it easily follows that $1_Z\delta_1 \in I$ as well. Thus we may conclude that

$$(1_Z\delta_1) \cdot (f\delta_t) = f\delta_t \in I,$$

which proves that $I = C_{c,K}(X^{\mathrm{e}}) \rtimes_{(\theta^{\mathrm{e}})^*} G$. Since θ is the restriction of θ^{e}, we have an inclusion $C_K(X) \rtimes_{\theta^*} G \hookrightarrow C_{c,K}(X^{\mathrm{e}}) \rtimes_{(\theta^{\mathrm{e}})^*} G$ and we claim that

$$p(C_{c,K}(X^{\mathrm{e}}) \rtimes_{(\theta^{\mathrm{e}})^*} G)p = C_K(X) \rtimes_{\theta^*} G.$$

This simply follows from the observation that

$$p(f\delta_t)p = (1_X\delta_1) \cdot (f\delta_t) \cdot (1_X\delta_1) = f1_{X\cap\theta_t^{\mathrm{e}}(X)}\delta_t = f1_{X_t}\delta_t \in C_K(X) \rtimes_{\theta^*} G.$$

We conclude that $C_K(X) \rtimes_{\theta^*} G$ sits as a full corner inside $C_{c,K}(X^{\mathrm{e}}) \rtimes_{(\theta^{\mathrm{e}})^*} G$, hence they are Morita equivalent. $\square$

1.4 Partial actions on C^*-algebras

Definition 1.4.1. A partial action $\alpha\colon G \curvearrowright \mathcal{A}$ on a C^*-algebra $\mathcal{A}$ is an algebraic partial action, such that each ideal D_s is closed.

Note that this definition coincides with the topological one for locally compact Hausdorff spaces. Indeed, if $\theta\colon G \curvearrowright X$ is such an action, then one can define a C^*-algebraic partial action on $C_0(X)$ exactly as in the purely algebraic case.

Crossed products by partial actions of arbitrary discrete groups were first studied by K. McClanahan in [12], and most of the constructions in this section are due to him. As for tensor products, we can equip the algebraic crossed product $\mathcal{A} \rtimes_{\mathrm{alg},\alpha} G$ with both a minimal and maximal norm — and possibly even some intermediate norm. As is the standard for group C^*-algebras and classical crossed products, we refer to the minimal, respectively, maximal crossed product as the *reduced*, respectively, the *universal* crossed product. First, we shall define with the universal one.

Definition 1.4.2. Consider a partial action $\alpha\colon G \curvearrowright \mathcal{A}$ on a C^*-algebra. The crossed product $\mathcal{A} \rtimes_\alpha G$ is the universal enveloping C^*-algebra of the algebraic crossed product $\mathcal{A} \rtimes_{\mathrm{alg},\alpha} G$, which is easily seen to exist.

Strictly speaking, we shall never need any properties of the C^*-algebraic crossed product that cannot be derived in the purely algebraic setting, but for the sake of overview we include some basic results on representations of C^*-crossed products and crossed products coming from topologically free actions. As for proper actions we have the concept of a covariant representation.

Definition 1.4.3. A *covariant representation* of $(\mathcal{A}, G, \alpha)$ is a pair (π, σ) consisting of a non-degenerate representation $\pi\colon \mathcal{A} \to B(H)$ and a partial representation $\sigma\colon G \to B(H)$, such that $\varepsilon_\sigma(s)$ is the projection onto $\overline{span}(\pi(D_s)H)$ and

$$\sigma(s)\pi(a)\sigma(s)^* = \pi(\alpha_s(a))$$

for all $s \in G$ and $a \in D_{s^{-1}}$.

It is not hard to see that any covariant representation satisfies the conditions of Proposition 1.2.17 and induces a non-degenerate representation $\pi \times \sigma\colon \mathcal{A} \rtimes_\alpha G \to B(H)$. In fact, any non-degenerate representation is of this form.

Proposition 1.4.4. *The map $(\pi, \sigma) \mapsto \pi \times \sigma$ defines a bijective correspondence between the covariant representations of $(\mathcal{A}, G, \alpha)$ and the non-degenerate representations of $\mathcal{A} \rtimes_\alpha G$.*

Proof. See [9, Theorem 1.4]. □

As for crossed products of global actions, there is a canonical way of constructing representations of $\mathcal{A} \rtimes_{\mathrm{alg},\alpha} G$ from representations of $\mathcal{A}$. Given a representation $\pi\colon \mathcal{A} \to B(H)$ one defines a twisted representation $\pi_s\colon D_s \to B(H)$ by $\pi_s = \pi \circ \alpha_{s^{-1}}$. Then there is a unique extension $\pi'_s\colon \mathcal{A} \to B(H)$ which annihilates $(\pi_s(D_s)H)^\perp$, and one defines $\tilde{\pi}\colon \mathcal{A} \to B(\ell^2(G, H))$ by

$$\tilde{\pi}(a)(\xi)(s) = \pi'_s(a)(\xi(s)).$$

Now let $P_s \in B(\ell^2(G, H))$ denote the projection onto $\overline{span}\{\tilde{\pi}(D_s)(\ell^2(G, H))\}$ and define a partial representation $\tilde{\lambda}\colon G \to B(\ell^2(G, H))$ by $\tilde{\lambda}_s = \lambda_s P_{s^{-1}}$, where λ denotes the left regular representation. Then $(\tilde{\pi}, \tilde{\lambda})$ is a covariant representation, and one can prove the following rather difficult proposition.

Proposition 1.4.5. *If $\pi\colon \mathcal{A} \to B(H)$ is faithful, then so is $\tilde{\pi} \times \tilde{\lambda}$. Thus the expression*

$$\|a\|_r = \sup\{\|\tilde{\pi} \times \tilde{\lambda}(a)\| \quad | \quad \pi\colon \mathcal{A} \to B(H) \text{ is a representation}\}$$

defines a norm on $\mathcal{A} \rtimes_{alg,\alpha} G$.

Proof. See [12, Proposition 3.4]. □

This allows the following unambiguous definition of the reduced crossed product.

Definition 1.4.6. Given a partial action $\alpha\colon G \curvearrowright \mathcal{A}$. The *reduced* crossed product $\mathcal{A} \rtimes_{r,\alpha} G$ is the completion of $\mathcal{A} \rtimes_{\mathrm{alg},\alpha} G$ with respect to $\|\cdot\|_r$.

In the last chapter of this thesis, we will pay a particular interest to topologically free actions. The following theorem provides the motivation for this endeavour.

Theorem 1.4.7. *Suppose that $\theta\colon G \curvearrowright X$ is a topologically free partial action of a discrete group on a locally compact Hausdorff space. If $\mathcal{I}$ is an ideal in $C_0(X) \rtimes_{r,\theta^*} G$ and*

$$\mathcal{I} \cap C_0(X) = \{0\},$$

then $\mathcal{I} = \{0\}$. In particular, a representation of $C_0(X) \rtimes_{r,\theta^*} G$ is faithful if and only if, it is faithful on $C_0(X)$.

Proof. See [9, Theorem 2.6]. □

Assuming minimality, one obtains simplicity of the reduced crossed product.

Corollary 1.4.8. *Suppose that $\theta\colon G \curvearrowright X$ is a topologically free and minimal partial action on a locally compact Hausdorff space. Then the crossed product $C_0(X) \rtimes_{r,\theta^*} G$ is simple.*

Proof. See [9, Corollary 2.9]. □

Finally, one can often expect the reduced crossed product to be exact.

Theorem 1.4.9. *If $\alpha\colon G \curvearrowright \mathcal{A}$ is a partial action of an exact discrete group on an exact C^*-algebra, then the reduced crossed product $\mathcal{A} \rtimes_{r,\alpha} G$ is exact.*

Proof. See [4, Corollary 5.3]. □

1.5 Universal C^*-algebras for partial representations

In this section, we shall construct concrete realizations of universal C^*-algebras for partial representations of G subject to relations $\mathcal{R}$. Later on, this will allow us to obtain a rather nice description of the partial action $\mathbb{F} \curvearrowright \Omega(E, C)$, which we will associate to the graph (E, C). The results here are taken from [9].

Definition 1.5.1. The universal C^*-algebra for partial representations of G is a C^*-algebra $C_p^*(G)$ and a partial representation $\sigma\colon G \to C_p^*(G)$, such that for every partial representation $\tau\colon G \to \mathcal{A}$ into a C^*-algebra, there exists a unique $*$-homomorphism $\varphi\colon C_p^*(G) \to A$ making the diagram

$$\begin{tikzcd} G \arrow[r, "\tau"] \arrow[d, "\sigma"'] & A \\ C_p^*(G) \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

commute. It is easily seen that this characterizes $C_p^*(G)$ up to canonical isomorphism.

In order to construct a realization of $C_p^*(G)$, we shall consider the space

$$X_G = \{\omega \in \mathcal{P}(G) \mid 1 \in \omega\}$$

given the topology that identifies it with a subspace of $\{0,1\}^G$. Define clopen subsets of X_G by

$$X_s = \{\omega \in X_G \mid s \in \omega\}$$

and equip X_G with a partial action θ^u given by $\theta_s^u(\omega) = s\omega$. For $s \in G$, we let 1_s denote the characteristic function on X_s, and for any finite $H \subset G \setminus \{1\}$ we define

$$1_H = \prod_{s \in H} 1_s.$$

Letting $\alpha^u = (\theta^u)^*$, we note that $\alpha_s^u(1_H 1_{s^{-1}}) = 1_{sH} 1_s$.

Proposition 1.5.2. *$C(X_G) \rtimes_{\alpha^u} G$ is universal for partial representations of G.*

Proof. Due to Lemma 1.2.14, there is a canonical partial representation

$$\sigma\colon G \to C(X_G) \rtimes_{\theta^{u*}} G$$

given by $\sigma(s) = 1_s \delta_s$. Considering any partial representation $\tau\colon G \to \mathcal{A}$, one easily checks that $\rho_\tau(1_s) = \varepsilon_\tau(s)$ defines a $*$-homomorphism $\rho_\tau\colon C(X_G) \to \mathcal{A}$. Then

$$\begin{aligned} \tau(s)\rho_\tau(1_{s^{-1}} \cdot 1_H)\tau(s)^* &= \tau(s) \cdot \Big(\prod_{t \in H} \varepsilon_\tau(t)\Big) \cdot \tau(s)^* = \varepsilon_\tau(s) \cdot \prod_{t \in H} \varepsilon_\tau(st) \\ &= \rho_\tau(1_s 1_{sH}) = \rho_\tau(\alpha_s^u(1_{s^{-1}} 1_H)), \end{aligned}$$

for any finite $H \subset G \setminus \{1\}$, hence $\tau(s)\rho_\tau(g)\tau(s)^* = \rho_\tau(\alpha_s^u(g))$ for all $g \in C(X_G)$. From Proposition 1.2.17 we deduce that there is an induced $*$-homomorphism

$$\rho_\tau \times \tau\colon C(X_G) \rtimes_{\alpha^u} G \to \mathcal{A}.$$

Observe that

$$\rho_\tau \times \tau(\sigma(s)) = \rho_\tau \times \tau(1_s\delta_s) = \varepsilon_\tau(s)\tau(s) = \tau(s)$$

for all $s \in G$, and since the $\sigma(s)$'s generate all of $C(X_G) \rtimes_{\alpha^u} G$ as a C^*-algebra, $\rho_\tau \times \tau$ is the unique $*$-homomorphism with this property. $\square$

Now we shall start invoking relations:

Definition 1.5.3. A set of relations on $C(X_G)$ is a subset of $\mathcal{R} \subset C(X_G)$. A partial representation $\tau\colon G \to \mathcal{A}$ is said to *satisfy* $\mathcal{R}$, if $\rho_\tau(\mathcal{R}) = \{0\}$. A universal C^*-algebra for partial relations satisfying $\mathcal{R}$ is a C^*-algebra $C_p^*(G;\mathcal{R})$ and a partial representation $\sigma\colon G \to C_p^*(G;\mathcal{R})$ satisfying $\mathcal{R}$, such that for every partial representation $\tau\colon G \to A$ satisfying $\mathcal{R}$, there is a unique $*$-homomorphism $\varphi\colon C_p^*(G;\mathcal{R}) \to A$ making the diagram

$$\begin{tikzcd} G \arrow[r, "\tau"] \arrow[d, "\sigma"'] & A \\ C_p^*(G;\mathcal{R}) \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

commute.

Proposition 1.5.4. *Let $\mathcal{R}$ be a set of relations. Then the smallest α^u-invariant ideal of $C(X_G)$ containing $\mathcal{R}$, denoted I, is generated by the set*

$$\{\alpha_s^u(g1_{s^{-1}}) \mid s \in G, g \in \mathcal{R}\}$$

as an ideal. Moreover,

$$\Omega_{\mathcal{R}} = \{\omega \in X_G \mid g(s^{-1}\omega) = 0 \textit{ for all } s \in \omega, g \in \mathcal{R}\}$$

is a compact invariant subset of X_G, such that $I = C_0(X_G \setminus \Omega_{\mathcal{R}})$, and the quotient $C(X_G)/I$ is canonically isomorphic to $C(\Omega_{\mathcal{R}})$ by restriction.

Proof. Temporalily denote the ideal generated by the $\alpha_s^u(g1_{s^{-1}})$'s by J. First we shall see that J is truly invariant. By definition, J is the closure of linear combinations of elements of the form $\alpha_s^u(g1_{s^{-1}})h$ for $s \in G$, $g \in \mathcal{R}$ and $h \in C(X_G)$, so it suffices to show that

$$\alpha_t^u(\alpha_s^u(g1_{s^{-1}})h1_{t^{-1}}) \in J$$

for any $t \in G$. This follows from the calculation

$$\begin{aligned}\alpha_t^u(\alpha_s^u(g1_{s^{-1}})h1_{t^{-1}}) &= \alpha_t^u\Big(\alpha_s^u(g\alpha_{s^{-1}}^u(h1_{t^{-1}}1_s))\Big) = \alpha_{ts}^u\Big(g\alpha_{s^{-1}}^u(h1_{t^{-1}}1_s)\Big) \\ &= \alpha_{ts}^u(g1_{(ts)^{-1}})h' \in J\end{aligned}$$

with $h' = \alpha_t^u(h1_{t^{-1}}1_s)$. Now, note that $I \subset J$ simply because $\alpha_1^u(1_{1^{-1}}g) = g$. On the other hand $J \subset I$, for we must certainly have $\alpha_s^u(g1_{s^{-1}}) \in I$ for any $s \in G$ and $g \in \mathcal{R}$.

Being an invariant ideal, there is some open θ^u-invariant $U \subset X_G$, such that $I = C_0(U)$ and $C(X_G)/I \cong C(X_G \setminus U)$ canonically — so we need only show that $X_G \setminus U = \Omega_{\mathcal{R}}$. But by duality and the above observations, $X_G \setminus U$ is the set of $\omega \in X_G$ such that $\alpha^u_s(g1_{s^{-1}})(\omega) = 0$ for all $g \in \mathcal{R}$ and $s \in G$, and since

$$\alpha^u_s(g1_{s^{-1}})(\omega) = \begin{cases} g(s^{-1}\omega) & \text{if } t \in \omega \\ 0 & \text{if } t \notin \omega \end{cases},$$

it follows that $X_G \setminus U = \Omega_{\mathcal{R}}$. Being the complement of an invariant open subspace, $\Omega_{\mathcal{R}}$ is a compact invariant subspace. □

Definition 1.5.5. For $s \in G$ we define $\Omega_s = \Omega_{\mathcal{R}} \cap X_s$ and let 1_s denote the characteristic function on Ω_s. We will say that $\omega \in X_G$ satisfies $\mathcal{R}$, if $g(s^{-1}\omega) = 0$ for all $s \in \omega$ and $g \in \mathcal{R}$.

Since $\Omega_{\mathcal{R}}$ is invariant, θ^u drops to a partial action $G \curvearrowright \Omega_{\mathcal{R}}$, which we shall also just denote by θ^u. Likewise, we shall not distinguish notationally between α^u and its restriction to $C(\Omega_{\mathcal{R}})$. Now we can we give a concrete realization of $C^*(G;\mathcal{R})$.

Proposition 1.5.6. *The crossed product $C(\Omega_{\mathcal{R}}) \rtimes_{\alpha^u} G$ is the universal C^*-algebra for partial representations satisfying $\mathcal{R}$.*

Proof. By Lemma 1.2.14, there is a partial representation $\sigma\colon G \to C(\Omega_{\mathcal{R}}) \rtimes_{\alpha^u} G$ given by $\sigma(s) = 1_s\delta_s$. The induced $*$-homomorphism $\rho_\sigma\colon C(X_G) \to C(\Omega_{\mathcal{R}}) \rtimes_{\alpha^u} G$ is simply the composition of the restriction map with the inclusion $C(\Omega_{\mathcal{R}}) \hookrightarrow C(\Omega_{\mathcal{R}}) \rtimes_{\alpha^u} G$, hence it vanishes on I. In particular $\rho_\sigma(\mathcal{R}) = \{0\}$. Now let $\tau\colon G \to \mathcal{A}$ denote any partial representation satisfying $\mathcal{R}$. Then ρ_τ vanishes on I, hence it drops to a $*$-homomorphism $\pi_\tau\colon C(\Omega_{\mathcal{R}}) \to \mathcal{A}$. The pair (π_τ, τ) satisfies the requirements of Proposition 1.2.17, because (ρ_τ, τ) does so, hence there is an induced $*$-homomorphism $\pi_\tau \times \tau\colon C(\Omega_{\mathcal{R}}) \rtimes_{\alpha^u} G \to \mathcal{A}$. We note that

$$(\pi_\tau \times \tau) \circ \sigma(s) = \pi_\tau(1_s)\tau(s) = \tau(s)$$

for all $s \in G$, and since the $\sigma(s)$'s generate $C(\Omega_{\mathcal{R}}) \rtimes_{\alpha^u} G$ as a C^*-algebra, $\pi_\tau \times \tau$ is the unique $*$-homomorphism with this property. □

Chapter 2

Finitely separated graphs and graph algebras

In this chapter we introduce the the category of finitely separated graphs along with the algebras associated to such graphs. Sections 2.1–2.3 are based on both [5] and [3], while the results of Section 2.4 are exclusively based on [3].

2.1 Finitely separated graphs

A directed graph E is a tuple (E^0, E^1, r, s) consisting of two sets $E^0 \neq \emptyset$ and E^1 along with functions $r, s\colon E^1 \to E^0$. The elements of E^0 are called *vertices* and will usually be denoted by v, while the elements of E^1 are called *edges* and are most commonly denoted by e or f. For $e \in E^1$ we call $r(e)$ the *range* of e and $s(e)$ the *source* of e. A vertex v is called a *source* if $r^{-1}(v) = \emptyset$. If both E^0 and E^1 are finite, we shall refer to E as a *finite* graph. A homomorphism of graphs $\varphi\colon E \to F$ is a pair of maps $\varphi^0\colon E^0 \to F^0$ and $\varphi^1\colon E^1 \to F^1$, such that $r_F(\varphi^1(e)) = \varphi^0(r_E(e))$ and $s_F(\varphi^1(e)) = \varphi^0(s_E(e))$. Here, we have added subscripts to indicate, in which graph we are applying the range/source map, but usually this will not be the case, and we will simply write r or s. A *finitely separated* graph is a pair (E, C) consisting of a graph E and a collection

$$C = \bigsqcup_{v \in E^0} C_v,$$

where each C_v is a partition of $r^{-1}(v)$ into non-empty finite subsets. However, we will usually just write *separated* when we really mean finitely separated. We shall refer to $X \in C$ as a *color*, and for an edge e we denote by $[e]$ its color, i.e. the set $X \in C$ such that $e \in X$. Finally, if $C_v = \{r^{-1}(v)\}$ for each v, then we will call (E, C) *trivially separated.*

We shall be working inside the following category:

Definition 2.1.1 (The category **FSGr**)**.** The objects of **FSGr** are the finitely separated graphs, and a morphism $\varphi\colon (E, C) \to (F, D)$ is a graph homomorphism $E \to F$ such that

(a) $\varphi^0\colon E^0 \to F^0$ is injective.

(b) for each $v \in E^0$ and each $X \in C_v$ there is $Y \in D_{\varphi^0(v)}$ such that φ^1 restricts to a bijection $X \to Y$. The associated map $X \mapsto Y$ will be denoted $\widetilde{\varphi}$.

A *complete subobject* of (E, C) is an object (F, D), where F is a subgraph of E and

$$D = \{X \in C \mid X \cap F^1 \neq \emptyset\}.$$

That is, a complete subobject is precisely the kind of subgraph for which the inclusion is a morphism in **FSGr**. Finally, an object (E, C) is *finite* if E is finite.

Proposition 2.1.2. *Direct limits always exist in* ***FSGr****. Moreover, the underlying graph of a direct limit of finitely separated graphs is just the limit in the category of directed graphs.*

Proof. Given a directed system $\{\varphi_{i,j} \colon (E_i, C_i) \to (E_j, C_j)\}_{i,j \in I}$ in **FSGr**, we define

$$E^0 := \varinjlim E_i^0 \quad \text{and} \quad E^1 := \varinjlim E_i^1$$

as direct limits in the category of sets. From the diagrams

$$\begin{tikzcd} E_i^1 \arrow[r, "\lambda_i^1"] \arrow[d, "r_i"'] & E^1 \arrow[d, dashed, "r"] \\ E_i^0 \arrow[r, "\lambda_i^0"'] & E^0 \end{tikzcd} \qquad \begin{tikzcd} E_i^1 \arrow[r, "\lambda_i^1"] \arrow[d, "s_i"'] & E^1 \arrow[d, dashed, "s"] \\ E_i^0 \arrow[r, "\lambda_i^0"'] & E^0 \end{tikzcd}$$

we obtain range and source maps, and we can define $E = (E^0, E^1, r, s)$. By definition of r and s, the λ_i's become graph homomorphisms. Before making E into a separated graph, let us see that E is a direct limit of the E_i's in the category of graphs and graph homomorphisms. Taking graph homomorphisms $\psi_i \colon E_i \to F$ such that $\psi_j = \psi_i \circ \varphi_{i,j}$ for each $i \leq j$ (and briefly assuming that the transit morphisms are regular graph homomorphisms), we have limit maps $\psi^0 \colon E^0 \to F^0$ and $\psi^1 \colon E^1 \to F^1$ such that $\psi^0 \circ \lambda_i^0 = \psi_i^0$ and $\psi^1 \circ \lambda_i^1 = \psi_i^1$ for all i. For $e \in E^0$ we pick $e' \in E_i^0$ such that $\lambda_i^0(e') = e$ and note that

$$\begin{aligned} r(\psi^1(e)) &= r(\psi^1(\lambda_i^1(e'))) = r(\psi_i^1(e')) = \psi_i^0(r_i(e')) = \psi^0(\lambda_i^0(r_i(e'))) \\ &= \psi^0(r(\lambda_i^1(e'))) = \psi^0(r(e)). \end{aligned}$$

Similarly we of course have $s(\psi^1(e)) = \psi^0(r(e))$, hence $\psi = (\psi^0, \psi^1)$ defines a graph homomorphism such that $\psi \circ \lambda_i = \psi_i$ for all i, and clearly it is unique with this property. Now we shall separate E as follows: For every $i \leq j$ we have a diagram of the form

$$\begin{tikzcd} E_i^1 \arrow[r, "\varphi_{i,j}^1"] \arrow[d, two heads] & E_j^1 \arrow[d, two heads] \\ C_i \arrow[r, "\widetilde{\varphi_{i,j}}"'] & C_j \end{tikzcd},$$

where the vertical maps are simply the quotient maps. Letting $C := \varinjlim C_i$ by the above maps, we gain an induced surjective map $E^1 \to C$ as in

$$\begin{tikzcd} E_i^1 \arrow[r, "\varphi_{i,j}^1"] \arrow[d, two heads] & E_j^1 \arrow[r, "\lambda_j^1"] \arrow[d, two heads] & E^1 \arrow[d, dashed, two heads] \\ C_i \arrow[r, "\widetilde{\varphi_{i,j}}"'] & C_j \arrow[r, "\widetilde{\lambda_j}"'] & C \end{tikzcd},$$

and separate E by this map. Taking $e, f \in E^1$ such that $r(e) \neq r(f)$, we should have $[e] \neq [f]$. This holds if and only if $[e'] \neq [f']$ for all $e', f' \in E^1_i$ and all $i \in I$ such that $[e] = [\lambda^1_i(e')]$ and $[f] = [\lambda^1_i(f')]$. And for such e', f' we would indeed have

$$\lambda^0_i(r(e')) = r(\lambda^1_i(e')) = r(e) \neq r(f) = r(\lambda^1_i(f')) = \lambda^0_i(r(f')),$$

hence $r(e') \neq r(f')$ and thus $[e'] \neq [f']$. Note that the $\widetilde{\lambda}_i(X) = \varinjlim_j \varphi_{i,j}(X)$, since the fiber of a map of direct limits is the limit of the fibers. In particular each $X \in C$ is finite. We conclude that C is a separation of E, and we need to check that the λ_i's are morphisms in **FSGr**. Injectivity of λ^0_i is immediate by injectivity of the transit maps (and was already used above), and for every $X \in C_i$, λ^1_i restricts to a map $X \to \widetilde{\lambda}_i(X)$ in view of the above diagram. Since every transit map $\varphi_{j,k}$ restricts to a bijection $\varphi_{i,j}(X) \to \varphi_{i,k}(X)$, so does the limit maps, hence λ_i is indeed a morphism. We move on to proving that ψ is a morphism in **FSGr**, if the ψ_i's are morphisms of **FSGr**. Injectivity of ψ^0 is immediate from injectivity of the ψ^0_i's. Furthermore, for any $i \in I$ we have the commutative diagram

$$\begin{tikzcd} E^1_i \arrow[r, "\psi^1_i"] \arrow[d, two heads] & F^1 \arrow[d, two heads] \\ C_i \arrow[r] & D \end{tikzcd},$$

which gives the commutative diagram

$$\begin{tikzcd} E^1 \arrow[r, "\psi^1"] \arrow[d, two heads] & F^1 \arrow[d, two heads] \\ C \arrow[r, dashed] & D \end{tikzcd}$$

of limits. We conclude that for any $X \in C_v$ there is $\widetilde{\psi}(X) \in D_{\psi^0(v)}$ such that ψ^1 restricts to a map $X \to \widetilde{\psi}(X)$. However, we should also prove that this restriction is a bijection. Taking $X_k \in C_k$ such that $\widetilde{\lambda}_k(X_k) = X$ and letting $X_i = \varphi_{k,i}(X_k)$ for $i \geq k$, the directed system $\{\varphi^1_{i,j} \colon X_i \to X_j\}$ for $i, j \geq k$ has limit $(X, \{\lambda_i\}_{i \geq j})$. From the diagram

$$\begin{tikzcd} X_i \arrow[r, "\varphi^1_{i,j}"] \arrow[drr, bend right, "\psi^1_i"'] & X_j \arrow[r, "\lambda^1_j"] \arrow[dr, "\psi^1_j"] & X \arrow[d, dashed, "\psi^1"] \\ & & \widetilde{\psi}(X) \end{tikzcd}$$

it then follows that ψ^1 is a bijection $X \to \widetilde{\psi}(X)$, since all the ψ_i's restrict to bijections. $\square$

We shall study a number of functors out of **FSGr**, and as they will all be continuous, the following observation essentially allows us to consider only finite graphs.

Proposition 2.1.3. *Every object (E, C) of **FSGr** admits arbitrarily large finite complete subobjects (F, D). In particular, the finite complete subobjects form a directed system with (E, C) as direct limit.*

Proof. Take any finite $A \subset E^0 \cup E^1$; then we will construct a finite complete subobject (F, D) such that $A \subset F^0 \cup F^1$. To this end, let E_1 be the subgraph of E generated by A, i.e set $E_1^1 = A \cap E^1$ and $E_1^0 = (A \cap E^0) \cup s(E_1^1) \cup r(E_1^1)$. For $v \in E_1^0$ we then define

$$F_v = r_{E^1}^{-1}(v) \cup \bigcup_{X \in C_v, X \cap A \neq \emptyset} X,$$

and let F be the subgraph of E generated by $E_1^0 \cup \bigcup_{v \in E_1^0} F_v$. Since E_1 is finite and $X \cap A \neq \emptyset$ for only finitely many $X \in C$, we observe that F_v is finite, and thus so is F. By construction we have $r_F^{-1}(v) = F_v$ for $v \in E_1^0$ and $r_F^{-1}(v) = \emptyset$ for $v \in F^0 \setminus E_1^0$. For any $v \in F^0$ we put

$$D_v = \{Y \cap F^1 \mid Y \in C_v, Y \cap F^1 \neq \emptyset\}$$

as required. Setting $D = \sqcup_{v \in F^0} D_v$, (F, D) is clearly a finite complete subobject of (E, C) with $A \subset F^0 \cup F^1$. Thus the finite complete subgraphs with morphisms induced by inclusion form a directed system, and in view of Proposition 2.1.2, (E, C) is the limit. □

Definition 2.1.4 (The functor M)**.** For each finitely separated graph (E, C), we define an abelian monoid $M(E, C)$ as the free abelian monoid on E^0 modulo the relations

$$v = \mathbf{s}(X) := \sum_{e \in X} s(e)$$

for all $v \in E^0$ and $X \in C_v$. Given a morphism $\varphi\colon (E, C) \to (F, D)$ we can define a monoid homomorphism $M(\varphi)\colon M(E, C) \to M(F, D)$ by $M(\varphi)(v) = \varphi^0(v)$. This is well defined since

$$M(\varphi)(\mathbf{s}(X)) = \sum_{e \in X} \varphi^0(s(e)) = \sum_{e \in X} s(\varphi^1(e)) = \sum_{e \in \widetilde{\varphi}(X)} s(e) = \mathbf{s}(\widetilde{\varphi}(X)),$$

which is equivalent to $\varphi^0(v)$. Obviously, M now defines a functor from the category of finitely separated graphs to the category of abelian monoids, and it is easily seen to be continuous. Observe that $M(E, C)$ is always non-zero and conical, i.e. $a + b = 0$ implies $a = b = 0$. As for any other monoid throughout this thesis, $M(E, C)$ is equipped with the algebraic preorder.

2.2 Leavitt path algebras

Throughout this section K will denote an arbitrary field with involution, possibly the trivial involution.

Definition 2.2.1. The *Leavitt path algebra* of the separated graph (E, C) with coefficients in K is the $*$-algebra $L_K(E, C)$ with generators $\{v, e : v \in E^0, e \in E^1\}$ subject to relations

(V) $vv' = \delta_{v,v'}v$ and $v = v^*$ for all $v, v' \in E^0$

(E) $r(e)e = es(e) = e$ for all $e \in E^1$

(SCK1) $e^*e' = \delta_{e,e'}s(e)$ for all $e, e' \in X$ and $X \in C$

(SCK2) $v = \sum_{e \in X} ee^*$ for every $X \in C_v$.

Thus, we do not distinguish notationally between the edges or vertices of the graph and the corresponding elements in the Leavitt path algebra. In case $K = \mathbb{C}$ is equipped with the usual involution, we shall just write $L_K(E, C) = L(E, C)$, and we use the notation $L_K(E)$ if E is trivially separated.

For clarity we explicitly mention the basic algebraic consequences that can be derived from these axioms.

Proposition 2.2.2. *If (E, C) is a separated graph, then*

(a) e, e^* are partial isometries, while ee^*, e^*e and v are projections

(b) $ve = e^*v = 0$ if $v \neq r(e)$

(c) $ev = ve^* = 0$ if $v \neq s(e)$

(d) $ef = 0$ and $f^*e^* = 0$ if $r(f) \neq s(e)$

(e) $ef^* = 0$ if $s(e) \neq s(f)$

(f) $ee^* \leq v$ if $r(e) = v$

for all $v \in E^0$ and $e, f \in E^1$.

Proof. For (a) we note that $e = es(e) = ee^*e$, hence e is a partial isometry and thus ee^* and e^*e are projections. v is clearly a projection, being both idempotent and self adjoint by (V). (b) and (c) follows from

$$ve = vr(e)e = 0 \text{ for } v \neq r(e) \quad \text{and} \quad ev = es(e)v = 0 \text{ for } v \neq s(e),$$

while (d) is seen by

$$ef = es(e)r(f)f = 0$$

if $r(f) \neq s(e)$. (e) is due to

$$ef^* = es(e)s(f)f^* = 0$$

and (f) follows from $vee^* = r(e)ee^* = ee^*$. $\square$

Proposition 2.2.3. *The assignment* $(E, C) \mapsto L_K(E, C)$ *extends to a continuous functor* $\boldsymbol{FSGr} \to \boldsymbol{K\text{-}alg}$*, such that*

$$L_K(\varphi)(v) = \varphi^0(v) \quad \text{and} \quad L_K(\varphi)(e) = \varphi^1(e)$$

for all morphisms $\varphi\colon (E, C) \to (F, D)$*,* $v \in E^0$ *and* $e \in E^1$*. In particular,* $L_K(E, C)$ *is the direct limit of* $L_K(F, D)$ *over all finite complete subobjects* (F, D)*.*

Proof. First, we need check that $L_K(\varphi)$ is well defined — also this will bring meaning to the definition of **FSGr**. Formally one first defines $L_K(\varphi)$ to be the $*$-homomorphism on the free $*$-algebra generated by $E^0 \cup E^1$ acting as specified above, and then we must check that it respects the defining relations of $L_K(E, C)$. For (V) we note that

$$L_K(\varphi)(vv') = \varphi^0(v)\varphi^0(v') = \delta_{\varphi^0(v),\varphi^0(v')}\varphi^0(v) = \delta_{v,v'}\varphi^0(v) = L_K(\varphi)(\delta_{v,v'}v)$$

by injectivity of φ^0. $L_K(\varphi)$ respects (E) simply because φ is a graph homomorphism, and for (SCK1) we take $e, e' \in X \in C_v$ for some $v \in E^0$. Then

$$L_K(\varphi)(e^*e') = \varphi^1(e)^*\varphi^1(e') = \delta_{\varphi^1(e),\varphi^1(e')}s(\varphi^1(e)) = \delta_{e,e'}\varphi^0(s(e)) = L_K(\varphi)(\delta_{e,e'}s(e)),$$

where we have used injectivity of φ^1 on X. Finally for (SCK2) we have

$$L_K(\varphi)\Big(\sum_{e\in X} ee^*\Big) = \sum_{e\in X} \varphi^1(e)\varphi^1(e)^* = \sum_{e\in\tilde{\varphi}(X)} ee^* = \varphi^0(v)$$

for each $X \in C_v$, since φ^1 is a bijection $X \to \tilde{\varphi}(X)$ and $\tilde{\varphi}(X) \in D_{\varphi^0(v)}$. Having checked that $L_K(\varphi)$ respects all the defining relations of $L_K(E, C)$, it drops to a $*$-homomorphism $L_K(E, C) \to L_K(F, D)$. Functoriality is now perfectly clear, and it remains only to check continuity.

Given a directed system

$$\varphi_{i,j}\colon (E_i, C_i) \to (E_j, C_j)\ ,\ i \leq j \text{ with } i, j \in I$$

in **FSGr** with limit (E, C) and limit morphisms $\lambda_i\colon (E_i, C_i) \to (E, C)$, we shall prove that $L_K(E, C)$ has the universal property of the limit of the directed system

$$L_K(\varphi_{i,j})\colon L_K(E_i, C_i) \to L_K(E_j, C_j)\ ,\ i \leq j \text{ with } i, j \in I.$$

So for any $i \in I$, let $\mu_i\colon L_K(E_i, C^i) \to A$ denote $*$-homomorphism such that $\mu_j = \mu_i \circ \varphi_{i,j}$ for all $i \leq j$. Then for any $v \in E^0$, there is some $i \in I$ and $v' \in E_i^0$ such that $L_K(\lambda_i)(v') = \lambda_i^0(v') = v$. Then we simply define $\mu(v) = \mu_i(\lambda_i(v'))$ — clearly this is independent by the choice of i and v'. We define μ similarly on element e, e^* for $e \in E^1$, and it straightforward to verify that μ satisfies the defining relations of $L_K(E, C)$. Thus μ defines a $*$-homomorphism $L_K(E, C) \to A$ such that $\mu \circ \lambda_i = \mu_i$ for all $i \in I$, and clearly μ is unique with this property. This finishes the proof. □

It turns out that the monoid $\mathcal{V}$ of idempotents over a Leavitt path algebra is very well understood in terms of graph-theoretic data.

Theorem 2.2.4. *There is a natural isomorphism of functors* $\Gamma\colon M \Rightarrow \mathcal{V} \circ L_K$*.*

Proof. Given a finitely separated graph (E, C), we define $\Gamma(E, C)\colon M(E, C) \to \mathcal{V}(L_K(E, C))$ in the slightly ambiguous way $\Gamma(E, C)(v) = [v]$ for $v \in E^0$. This is well defined since

$$[v] = \sum_{e \in X} [ee^*] = \sum_{e \in X} [e^* e] = \sum_{e \in X} [s(e)] = [\mathbf{s}(X)]$$

for any $X \in C_v$. Γ is clearly a natural transformation, and since $\mathcal{V}$ is continuous, so is $\mathcal{V} \circ L_K$. Furthermore, since any object (E, C) of **FSGr** is a direct limit of its finite complete subobjects, it suffices to prove that $\Gamma(E, C)$ is an isomorphism for finite objects, and we shall do this by induction over $|C|$. If $|C| = 0$ then $E^1 = \emptyset$, and so $M(E, C) = \mathbb{Z}_+(E^0)$. On the other hand, $L_K(E, C) = K^{E^0}$ and $\mathcal{V}(K^{E^0}) \cong \mathbb{Z}_+(E^0)$ by the isomorphism $[v] \mapsto v$, hence $\Gamma(E, C)$ is indeed an isomorphism in this case. For the induction step, assume that (E, C) is finite, take any $X \in C_v$ for some $v \in E^0$ and define $F^0 = E^0$, $F^1 = E^1 \setminus X$, $r_F = r|_{F^1}$, $s_F = s|_{F^1}$ and $D = C \setminus \{X\}$. Then (F, D) is a complete subobject of (E, C) with $|D| = |C| - 1$, so by the induction hypothesis $\Gamma(F, D)$ is an isomorphism. By definition of M we have $M(E, C) = \frac{M(F,D)}{v \sim \mathbf{s}(X)}$, which is mapped isomorphically onto $\frac{\mathcal{V}(L_K(F,D))}{[v] \sim [\mathbf{s}(X)]}$ by $\Gamma(F, D)$. Thus it remains only to prove that $\mathcal{V}(L_K(E, C)) = \frac{\mathcal{V}(L_K(F,D))}{[v] \sim [\mathbf{s}(X)]}$. Letting $A = L_K(F, D)$,

$$P = Av \quad \text{and} \quad Q = \oplus_{e \in X} As(e)$$

are finitely generated projective left A-modules, and we claim that the Bergman algebra of A with an isomorphism adjoint between P and Q is simply $B = L_K(E, C)$ (see Definition A.2.6). In that case, the proof is complete by Theorem A.2.7. Surely B has an A-module structure coming from the inclusion $A \hookrightarrow B$, and we have canonical isomorphisms

$$B \otimes_A P \cong Bv \quad \text{and} \quad B \otimes_A Q \cong \bigoplus_{e \in X} Bs(e).$$

We can define an isomorphism of B-modules $\mu\colon Bv \to \bigoplus_{e \in X} Bs(e)$ by $\mu(b) = (be)_{e \in X}$, indeed the homomorphism $(b_e)_{e \in X} \mapsto \sum_{e \in X} b_e e^*$ defines an inverse. Suppose that C is a unital K-algebra with an A-module structure coming from a unital K-algebra homomorphism $\Phi\colon A \to C$, and note that we have canonical isomorphisms of C-modules

$$C \otimes_A P = C \otimes_A Av \cong C\Phi(v) \quad \text{and} \quad C \otimes_A Q = C \otimes_A \left(\bigoplus_{e \in X} As(e) \right) \cong \bigoplus_{e \in X} C\Phi(s(e)).$$

So rather than assuming we have an isomorphism of C-modules $C \otimes_A P \to C \otimes_A Q$, we consider an isomorphism of C-modules $\varphi\colon C\Phi(v) \to \bigoplus_{e \in X} C\Phi(s(e))$. Our job is therefore to construct a unital K-algebra homomorphism $\Psi\colon B \to C$ giving C a B-module structure, such that the diagram

$$\begin{tikzcd}
C \otimes_B Bv \arrow[r, "\mathrm{Id}_C \otimes \mu"] \arrow[d, "\cong"'] & C \otimes_B \left(\bigoplus_{e \in X} Bs(e) \right) \arrow[d, "\cong"] \\
C\Phi(v) \arrow[r, "\varphi"'] & \bigoplus_{e \in X} C\Phi(s(e))
\end{tikzcd}$$

commutes. Since C is not involutive, we need to define elements $\Psi(v), \Psi(e)$ and $\Psi(e^*)$ for all $v \in E^0$ and $e \in E^1$. Write $\varphi = (\varphi_e)_{e \in X}$ and

$$\varphi_e^{-1}\colon C\Phi(s(e)) \hookrightarrow \bigoplus_{f \in X} C\Phi(s(f)) \xrightarrow{\varphi^{-1}} C\Phi(v).$$

Then we set

$$\Psi(v) = \Phi(v), \quad \Psi(e) = \begin{cases} \Phi(e) & e \notin X \\ \varphi_e(\Phi(v)) & e \in X \end{cases} \quad \text{and} \quad \Psi(e^*) = \begin{cases} \Phi(e^*) & e \notin X \\ \varphi_e^{-1}(\Phi(s(e))) & e \in X \end{cases}.$$

We must check that these elements satisfy the defining relations of $L_K(E,C)$. (V) is trivial, and so is (E) for element e and e^* with $e \notin X$. When $e \in X$ we have

$$\Psi(r(e))\Psi(e) = \Phi(r(e))\varphi_e(\Phi(v)) = \varphi_e(\Phi(v)) = \Psi(e),$$

and $\Psi(e)\Psi(s(e)) = \Psi(e)$ is trivial. Likewise we have

$$\Psi(s(e))\Psi(e^*) = \Phi(s(e))\varphi_e^{-1}(\Phi(s(e)) = \varphi_e^{-1}(\Phi(s(e))) = \Psi(e),$$

and $\Psi(e^*)\Psi(r(e)) = \Psi(e^*)$ is trivial. (SCK1) and (SCK2) are trivial for $Y \in C$ with $Y \neq X$, and in case $e, f \in X$ we have

$$\begin{aligned} \Psi(e^*)\Psi(f) &= \varphi_e^{-1}\Big(\Phi(s(e))\Big)\varphi_f(\Phi(v)) = \varphi_f\Big(\varphi_e^{-1}(\Phi(s(e)))\Phi(v)\Big) \\ &= \varphi_f\Big(\varphi_e^{-1}(\Phi(s(e)))\Big) = \delta_{e,f}\Phi(s(e)) = \delta_{e,f}\Psi(s(e)), \end{aligned}$$

thereby verifying (SCK1). Finally

$$\begin{aligned} \sum_{e\in X} \Psi(e)\Psi(e^*) &= \sum_{e\in X} \varphi_e(\Phi(v))\varphi_e^{-1}(\Phi(s(e))) = \sum_{e\in X} \varphi_e^{-1}\Big(\varphi_e(\Phi(v))\Phi(s(e))\Big) \\ &= \varphi^{-1}\Big(\oplus_{e\in X}\, \varphi_e(\Phi(v))\Big) = \Phi(v) = \Psi(v), \end{aligned}$$

which verifies (SCK2). Thus Ψ defines a K-algebra homomorphism, and it remains only to check that the above diagram commutes. Take $c \in C$ and $b \in Bv$. Going around the above diagram clockwise, $c \otimes b$ is mapped to $(c\Psi(b)\Psi(e))_{e\in X}$, and going around counter-clockwise, $c \otimes b$ is mapped to $c\varphi(\Psi(b))$. Thus we should verify that $\varphi_e(\Psi(b)) = \Psi(b)\Psi(e)$ for $e \in X$. But this is clearly the case as $\Psi(b) = \Psi(bv) = \Psi(b)\Psi(v)$ and $\varphi_e(\Psi(v)) = \Psi(e)$. □

Excluding graphs with isolated vertices, we can translate the defining relations of the Leavitt path algebra into relations only imposed on the final projections. In the following we use the notation $p(s) = ss^*$.

Proposition 2.2.5. *Given a finitely separated graph (E,C) without isolated vertices. Then $L_K(E,C)$ is the universal $*$-algebra generated by a set of partial isometries $e \in E^1$ satisfying the following:*

(PI1) $p(e)p(f) = \delta_{e,f}p(e)$ *for all* $e, f \in X \in C$.

(PI2) $p(e^*) = p(f^*)$ *for all* $e, f \in E^1$ *such that* $s(e) = s(f)$.

(PI3) $\sum_{e\in X} p(e) = \sum_{e\in Y} p(e)$ *for all* $X, Y \in C_v$ *and* $v \in E^0$.

(PI4) Identify each vertex $v \in E^0$ with a projection in $L_K(E,C)$ by the formulas

$$s(e) := p(e^*) \quad \text{and} \quad r(e) := \sum_{f\in[e]} p(f)$$

for $e \in E^1$, and note that this is well-defined by (PI1)-(PI3). Then $vw = 0$ for distinct $v, w \in E^0$.

Proof. One simply applies universality to obtain mutually inverse $*$-homomorphisms. The assumption on isolated vertices is necessary, as such a vertex is not generated by any set of edges. $\square$

It is not hard to see that $E^1 \subset L_K(E,C)$ is not a tame set of partial isometries in general (see for instance Example 6.1.4). So rather than attacking the full Leavitt path algebra, we will actually be studying a certain quotient in which we force E^1 to be tame.

Definition 2.2.6. Denote by $U = U(E,C)$ the multiplicative semigroup of $L_K(E,C)$ generated by $E^1 \cup (E^1)^*$ and write $p(s) = ss^*$ for $s \in U$. Then the *abelianized Leavitt path algebra* associated with a separated graph (E,C) is the quotient

$$L_K^{\mathrm{ab}}(E,C) = L_K(E,C)/J,$$

where $J = J(E,C)$ is the ideal generated by all the commutators $[p(s),p(t)]$ for $s,t \in U$. For an element $a \in L_K(E,C)$ we denote by $\underline{a}$ the associated element in the quotient, and in case $K = \mathbb{C}$ with the usual involution, we just write $L_K^{\mathrm{ab}}(E,C) = L^{\mathrm{ab}}(E,C)$.

Proposition 2.2.7. *The assignment $(E,C) \mapsto L_K^{ab}(E,C)$ extends to a continuous functor $\boldsymbol{FSGr} \to \boldsymbol{K\text{-}alg}$ such that*

$$L_K^{ab}(\varphi)(\underline{v}) = \underline{\varphi^0(v)} \quad \text{and} \quad L_K^{ab}(\varphi)(\underline{e}) = \underline{\varphi^1(e)}$$

for all morphisms $\varphi\colon (E,C) \to (F,D)$, $v \in E^0$ and $e \in E^1$. In particular, $L_K^{ab}(E,C)$ is the direct limit of $L_K^{ab}(F,D)$ over all finite complete subobjects (F,D).

Proof. We start by checking that $L_K^{\mathrm{ab}}(\varphi)$ is well-defined. Note that for $s,t \in U(E,C)$ we have $L_K(\varphi)(s) = \tilde{s} \in U(F,D)$ and $L_K(\varphi)(t) = \tilde{t} \in U(F,D)$, hence

$$L_K(\varphi)([p(s),p(t)]) = [p(\tilde{s}),p(\tilde{t})].$$

Thus $L_K(\varphi)$ drops to a $*$-homomorphism $L_K^{\mathrm{ab}}(E,C) \to L_K^{\mathrm{ab}}(F,D)$ acting as specified above. It is clear from the definition, that the assignment is functorial. For continuity we consider a directed system

$$\varphi_{i,j}\colon (E_i,C_i) \to (E_j,C_j)\ ,\ i \leq j \text{ with } i,j \in I$$

in **FSGr** with limit (E,C) and limit morphisms $\lambda_i\colon (E_i,C_i) \to (E,C)$. Suppose that for any $i \in I$ we have a $*$-homomorphism $\mu_i\colon L_K(E_i,C^i) \to A$ such that $\mu_j = \mu_i \circ \varphi_{i,j}$ for all $i \leq j$. Then from continuity of L_K, there is a unique $*$-homomorphism $\mu'\colon L_K(E,C) \to A$ making the diagram

$$\begin{tikzcd}
L_K(E_i,C_i) \arrow[r] \arrow[d, "L_K(\lambda_i)"'] & L_K^{\mathrm{ab}}(E_i,C_i) \arrow[d, "L_K^{\mathrm{ab}}(\lambda_i)"] \arrow[dr, bend left, "\mu_i"] & \\
L_K(E,C) \arrow[r] \arrow[rr, dashed, bend right, "\mu'"'] & L_K^{\mathrm{ab}}(E,C) & A
\end{tikzcd}$$

commute for each $i \in I$, and we should simply check that it vanishes on the ideal J. Taking $s, t \in U(E, C)$ there are $\tilde{s}, \tilde{t} \in U(E_i, C_i)$ such that $L_K(\lambda_i)(\tilde{s}) = s$ and $L_K(\lambda_i)(\tilde{t}) = t$ for sufficiently large i, and

$$\mu'([p(s), p(t)]) = \mu' \circ L_K(\lambda_i)([p(\tilde{s}), p(\tilde{t})]) = \mu_i([p(\underline{\tilde{s}}), p(\underline{\tilde{t}})]) = 0.$$

Thus we obtain a unique $*$-homomorphism $\mu\colon L_K^{\mathrm{ab}}(E, C) \to A$ such that $\mu \circ L_K^{\mathrm{ab}}(\lambda_i) = \mu_i$ for all $i \in I$. □

Definition 2.2.8. Denote by $\mathbb{F}$ the free group on E^1. A partial representation $\sigma\colon \mathbb{F} \to A$ on a unital $*$-algebra is said to satisfy (PI1)-(PI4), if the partial isometries $\sigma(e)$, $e \in E^1$, satisfy (PI1)-(PI4).

For finite graphs (E, C), we can give an alternative definition of $L_K^{\mathrm{ab}}(E, C)$ in terms of universal properties concerning partial representations satisfying certain relations.

Proposition 2.2.9. *If (E, C) is a finite graph without isolated vertices, then $L_K^{ab}(E, C)$ is the universal $*$-algebra for semi-saturated partial representations of $\mathbb{F}$ satisfying (PI1)-(PI4). That is, there is a semi-saturated partial representation $\tau\colon \mathbb{F} \to L_K^{ab}(E, C)$ satisfying (PI1)-(PI4), and for any other semi-saturated partial representation $\sigma\colon \mathbb{F} \to A$ satisfying (PI1)-(PI4), there is a unique $*$-homomorphism $\varphi\colon L_K^{ab}(E, C) \to A$ such that $\varphi \circ \tau = \sigma$.*

Proof. By construction, $[p(\underline{s}), p(\underline{t})] = 0$ for all $\underline{s}, \underline{t} \in \langle \underline{E^1} \cup \underline{E^{1*}} \rangle$, so from Proposition 1.2.13 we obtain a semi-saturated partial representation $\tau\colon \mathbb{F} \to L_K^{\mathrm{ab}}(E, C)$ satisfying $\tau(e) = \underline{e}$ for all $e \in E^1$. Now, given a semi-saturated partial representation $\tau\colon \mathbb{F} \to A$ satisfying (PI1)-(PI4), we obtain a $*$-homomorphism $\varphi\colon L_K(E, C) \to A$ with $\varphi(e) = \sigma(e)$ for all $e \in E^1$ by Proposition 2.2.5. Using Proposition 1.2.13, we observe that

$$\varphi([p(s), p(t)]) = 0$$

for all $s, t \in \langle E^1 \cup (E^1)^* \rangle$, hence φ drops to a $*$-homomorphism $L_K^{\mathrm{ab}}(E, C) \to A$ satisfying $\varphi(\tau(e)) = \varphi(\underline{e}) = \sigma(e)$ for all $e \in E^1$. Since the $\tau(e)$'s generate $L_K^{\mathrm{ab}}(E, C)$, φ is unique with this property. Moreover, since both $\varphi \circ \tau$ and σ are semi-saturated, and they agree on E^1, we must in fact have $\varphi \circ \tau = \sigma$. This finishes the proof. □

2.3 Graph C^*-algebras

Definition 2.3.1. The *graph C^*-algebra* $C^*(E,C)$ associated with a separated graph (E,C) is the universal C^*-algebra generated by elements $\{v, e \mid v \in E^0, e \in E^1\}$ satisfying (V),(E), (SCK1) and (SCK2). That is, $C^*(E,C)$ is the universal enveloping C^*-algebra of $L(E,C)$, so for any $*$-homomorphism $L(E,C) \to \mathcal{A}$ into a C^*-algebra $\mathcal{A}$, there is a unique $*$-homomorphism $C^*(E,C) \to \mathcal{A}$ making the diagram

$$\begin{tikzcd} L(E,C) \arrow[r] \arrow[dr] & C^*(E,C) \arrow[d, dashed] \\ & \mathcal{A} \end{tikzcd}$$

commute.

In particular, we obtain the following corollaries from the prior results on Leavitt path algebras.

Corollary 2.3.2. *The assignment $(E,C) \mapsto C^*(E,C)$ extends to a continuous functor*

$$C^* \colon \boldsymbol{FSGr} \to \boldsymbol{C^*\text{-}alg}.$$

In particular, $C^*(E,C)$ is the direct limit of $C^*(F,D)$ over all finite complete subobjects (F,D) of (E,C).

Proof. This is immediate by Proposition 2.2.3 with $K = \mathbb{C}$ and universality of $C^*(E,C)$. □

Corollary 2.3.3. *Given a finitely separated graph without isolated vertices. Then $C^*(E,C)$ is the universal C^*-algebra generated by a set of partial isometries $e \in E^1$ satisfying (PI1)-(PI4).*

Proof. This is immediate by Proposition 2.2.5 with $K = \mathbb{C}$ and universality of $C^*(E,C)$. □

The following theorem provides deep information about the involved algebras and should be mentioned, even though we will never make any real use of it. Indeed, we shall only apply it once in the main construction, and in that case a much weaker result would also do the trick. The proof is fairly comprehensive and is skipped in order to maintain focus.

Theorem 2.3.4. *Suppose that (E,C) is a finitely separated graph. Then the canonical $*$-homomorphism $L(E,C) \to C^*(E,C)$ is injective.*

Proof. See [6, Theorem 3.8(1)]. □

As for the purely algebraic case, our primary interest is not the full graph C^*-algebra, but rather the quotient where all the commutators $[p(s), p(t)]$ have been annihilated.

Definition 2.3.5. Denote by $U = U(E,C)$ the semigroup in $C^*(E,C)$ generated by $E^1 \cup (E^1)^*$. Then the *abelianized graph C^*-algebra* $\mathcal{O}(E,C)$ associated with the separated graph (E,C) is the quotient

$$\mathcal{O}(E,C) = C^*(E,C)/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by all the commutators $[p(s), p(t)]$ for $s,t \in U$. Clearly, $\mathcal{O}(E,C)$ is the universal enveloping C^*-algebra of $L_{\mathbb{C}}^{\mathrm{ab}}(E,C)$, and for $a \in C^*(E,C)$ we denote by $\underline{a}$ the associated element in $\mathcal{O}(E,C)$.

Applying universality, we obtain the following corollaries from the results about $L_K^{\mathrm{ab}}(E,C)$.

Corollary 2.3.6. *The assignment $(E,C) \mapsto \mathcal{O}(E,C)$ extends to a continuous functor*

$$\mathcal{O}\colon \boldsymbol{FSGr} \to \boldsymbol{C^*\text{-}alg}.$$

In particular, $\mathcal{O}(E,C)$ is the direct limit of $\mathcal{O}(F,D)$ over all finite complete subobjects (F,D).

Proof. This is immediate by Proposition 2.2.7 with $K = \mathbb{C}$ and universality of $\mathcal{O}(E,C)$. □

Corollary 2.3.7. *If (E,C) is a finite graph without isolated vertices, then $\mathcal{O}(E,C)$ is the universal C^*-algebra for semi-saturated partial representations satisfying (PI1)-(PI4). That is, there is a semi-saturated partial representation*

$$\tau\colon \mathbb{F} \to \mathcal{O}(E,C)$$

satisfying (PI1)-(PI4), such that for any semi-saturated partial representation $\sigma\colon \mathbb{F} \to \mathcal{A}$ into a C^*-algebra $\mathcal{A}$ satisfying (PI1)-(PI4), there is a unique $*$-homomorphism $\varphi\colon \mathcal{O}(E,C) \to \mathcal{A}$ such that $\sigma = \varphi \circ \tau$.

Proof. This is immediate by Proposition 2.2.9 with $K = \mathbb{C}$ and universality of $\mathcal{O}(E,C)$. □

for all $s, t \in G$

2.4 A dynamical interpretation

In this section we shall obtain a description of $L_K^{\mathrm{ab}}(E,C)$ and $\mathcal{O}(E,C)$ as crossed products of commutative algebras with the free group $\mathbb{F}$ on E^1.

Definition 2.4.1. A partial (E,C)-preaction α on a commutative, unital $*$-algebra A is a set of projections $\{p_\alpha(s) \mid s \in E^1 \cup (E^1)^{-1}\}$ satisfying (PI1)-(PI4) when $p_\alpha(e^*) := p_\alpha(e^{-1})$, along with isomorphisms $\alpha_e \colon p_\alpha(e^{-1})A \to p_\alpha(e)A$ for $e \in E^1$. Given partial (E,C)-preactions α and β on unital, commutative $*$-algebras A and B, a $*$-homomorphism $\varphi \colon A \to B$ is called *equivariant* if $\varphi(p_\alpha(s)) = p_\beta(s)$ for all $s \in E^1 \cup (E^1)^{-1}$, and the diagram

$$\begin{tikzcd} p_\alpha(e^{-1})A \arrow[r, "\varphi"] \arrow[d, "\alpha_e"'] & p_\beta(e^{-1})B \arrow[d, "\beta_e"] \\ p_\alpha(e)A \arrow[r, "\varphi"'] & p_\beta(e)B \end{tikzcd}$$

commutes for all $e \in E^1$. A commutative, unital $*$-algebra A with a partial (E,C)-preaction is called *universal* for partial (E,C)-preactions, if given any partial (E,C)-preaction on a unital, commutative $*$-algebra B, there is a unique equivariant $*$-homomorphism $\varphi \colon A \to B$. Finally, a *partial* (E,C)*-action* is the canonical extension of a partial (E,C)-preaction α on A to a partial action $\mathbb{F} \curvearrowright A$ given by Lemma 1.1.7. Notationally, we shall not distinguish between the partial preactions and their induced partial actions, and replacing "$*$-algebra" with "C^*-algebra" in the above definition, one obtains the C^*-algebraic analogue.

While the above definition might seem slightly artificial, the lemma below provides us with some well known examples.

Lemma 2.4.2. *If $\sigma \colon \mathbb{F} \to A$ is a semi-saturated partial representation satisfying (PI1)-(PI4), then α^σ is a partial (E,C)-action.*

Proof. Clearly, α^σ restricts to a partial (E,C)-preaction, and by Proposition 1.2.15 we observe that α^σ is the canonical extension to a partial (E,C)-action. □

Definition 2.4.3 (The canonical partial (E,C)-action)**.** Denote by τ the canonical semi-saturated partial representation

$$\mathbb{F} \to L_K^{\mathrm{ab}}(E,C) \quad \text{or} \quad \mathbb{F} \to \mathcal{O}(E,C).$$

From Stone duality there is a zero-dimensional compact metrizable space $\Omega(E,C)$, such that the $*$-algebra, respectively, C^*-algebra generated by the final projections $\varepsilon_\tau(s) = \tau(s)\tau(s)^*$ is isomorphic to $C_K(\Omega(E,C))$, respectively, $C(\Omega(E,C))$. Thus $\alpha = \alpha^\tau$ defines partial (E,C)-actions

$$\mathbb{F} \curvearrowright C_K(\Omega(E,C)) \quad \text{and} \quad \mathbb{F} \curvearrowright C(\Omega(E,C)).$$

Each of these will be referred to as the *canonical partial* (E,C)*-action*.

Lemma 2.4.4. *Let α denote a partial (E,C)-action on a commutative, unital $*$-algebra A. Then there are projections $p_\alpha(s)$ in A such that $D_s = p_\alpha(s)A$ for every $s \in \mathbb{F}$, and the canonical partial representation $\sigma\colon \mathbb{F} \to A \rtimes_\alpha \mathbb{F}$ given by $\sigma(s) = p_\alpha(s)\delta_s$ is semi-saturated. Moreover, if β is a partial (E,C)-action on B and $\varphi\colon A \to B$ is equivariant in the sense of Definition 2.4.1, then $\varphi(p_\alpha(s)) = p_\beta(s)$ for all $s \in \mathbb{F}$, and φ is equivariant in the usual sense. In particular, a partial (E,C)-preaction is universal for partial (E,C)-preactions if and only if its extension is universal for partial (E,C)-actions.*

Proof. It is clear from the construction of α that each ideal D_s is unital. If $|st| = |s| + |t|$, then obviously $D_{st} \subset D_s$, hence $p_\alpha(st) \leq p_\alpha(s)$. We deduce that

$$\sigma(s)\sigma(t) = (p_\alpha(s)\delta_s)\cdot(p_\alpha(t)\delta_t) = p_\alpha(s)p_\alpha(st)\delta_{st} = p_\alpha(st)\delta_{st} = \sigma(st),$$

so σ is indeed semi-saturated. Now let

$$\varphi\colon (A,\alpha) \to (B,\beta)$$

denote an equivariant $*$-homomorphism in the sense of Definition 2.4.1. Then we shall prove that $p_\alpha(s) = p_\beta(s)$ for any $s \in \mathbb{F}$, and we proceed by induction over the length of s. The induction start holds by assumption, so assume that the claim holds for words of length n. Assuming that $|s| = n+1$, we write $s = s_1 s_n$ with $|s_1| = 1$ and $|s_n| = n$. Then

$$\begin{aligned}\varphi(p_\alpha(s)) &= \varphi\left(\alpha_{s_1}(p_\alpha(s_1^{-1})p_\alpha(s_n))\right) = \beta_{s_1}\left(\varphi(p_\alpha(s_1^{-1})p_\alpha(s_n))\right)\\ &= \beta_{s_1}(p_\beta(s_1^{-1})p_\beta(s_n)) = p_\beta(s_1 s_n) = p_\beta(s).\end{aligned}$$

Equivariance is now completely trivial. $\square$

Theorem 2.4.5. *Given a finitely separated graph (E,C), denote by $\mathbb{F}$ the free group on E^1. If $\sigma\colon \mathbb{F} \to A$ is universal for semi-saturated partial representations satisfying (PI1)-(PI4), then $\alpha^\sigma\colon \mathbb{F} \curvearrowright B$ is a universal partial (E,C)-action on the commutative, unital $*$-algebra B generated by the $\varepsilon_\sigma(s)$'s, and $A \cong B \rtimes_{\alpha^\sigma} \mathbb{F}$. Conversely, if $A = B \rtimes_\alpha \mathbb{F}$ where α is a universal partial (E,C)-action on a commutative, unital $*$-algebra, then $\sigma_\alpha\colon \mathbb{F} \to A$ is universal for semi-saturated partial representations satisfying (PI1)-(PI4).*

Proof. Assume first that (A,σ) is universal with respect to semi-saturated partial representations satisfying (PI1)-(PI4), and let B denote the $*$-subalgebra generated by all the final projections $\varepsilon(s) = \sigma(s)\sigma(s)^*$. From Lemma 2.4.2 we obtain the partial (E,C)-action $\alpha^\sigma\colon \mathbb{F} \curvearrowright B$, and we claim that in fact $A \cong B \rtimes_{\alpha^\sigma} \mathbb{F}$. First of all, by Lemma 1.2.14 there is a partial representation $\tau = \sigma_{\alpha^\sigma}\colon \mathbb{F} \to B \rtimes_{\alpha^\sigma} \mathbb{F}$, given by $\tau(s) = \varepsilon(s)\delta_s$ for each $s \in \mathbb{F}$. Obviously, τ is semi-saturated and satisfies (PI1)-(PI4), because σ have these properties. From universality of (A,σ), we obtain a $*$-homomorphism $\varphi\colon A \to B \rtimes_{\alpha^\sigma} \mathbb{F}$ satisfying $\varphi(\sigma(s)) = \tau(s) = \varepsilon(s)\delta_s$. On the other hand, from Corollary 1.2.18 we have a $*$-homomorphism $\psi\colon B \rtimes_{\alpha^\sigma} \mathbb{F} \to A$ satisfying $\psi(\varepsilon(s)\delta_s) = \varepsilon(s)\sigma(s) = \sigma(s)$. Now $\psi \circ \varphi(\sigma(s)) = \sigma(s)$, so $\psi \circ \varphi = \mathrm{Id}$ by uniqueness. Likewise, $\varphi \circ \psi(\varepsilon(s)\delta_s) = \varepsilon(s)\delta_s$, and since the $\varepsilon(s)\delta_s$'s generate $B \rtimes_{\alpha^\sigma} \mathbb{F}$, $\varphi \circ \psi = \mathrm{Id}$ as well. It remains to prove that α^σ is universal for partial (E,C)-actions, so let β denote another partial (E,C)-action on a commutative, unital $*$-algebra B'. The canonical partial representation $\rho\colon \mathbb{F} \to B' \rtimes_\beta \mathbb{F}$ is semi-saturated due to Lemma 2.4.4, so by the first part of the proof there is a unique $*$-homomorphism $\varphi\colon B \rtimes_{\alpha^\sigma} \mathbb{F} \to B' \rtimes_\beta \mathbb{F}$ such that $\varphi(\tau(s)) = \varphi(\varepsilon(s)\delta_s) = p_\beta(s)\delta_s$. We observe that

$$\varphi(\varepsilon(s)\delta_1) = \varphi(\tau(s))\varphi(\tau(s))^* = (p_\beta(s)\delta_s)\cdot(p_\beta(s)\delta_s)^* = p_\beta(s)\delta_1$$

for all $s \in \mathbb{F}$, so in particular it restricts to a $*$-homomorphism $B \to B'$. Equivariance follows from the calculation

$$\begin{aligned}\varphi(\alpha_s^\sigma(b)\delta_1) &= \varphi(\sigma(s)b\sigma(s)^*\delta_1) = \varphi(\tau(s)\cdot(b\delta_1)\cdot\tau(s)^*)\\ &= \rho(s)\varphi(b\delta_1)\rho(s)^* = \alpha_s^{\sigma_\beta}(\varphi(b))\delta_1 = \beta_s(\varphi(b))\delta_1\end{aligned}$$

for all $s \in \mathbb{F}$ and $b \in p_{\alpha^\sigma}(s^{-1})B$, where we have made use of Lemma 1.2.16. Clearly, φ is unique with these properties.

Now assume that $A = B \rtimes_\alpha \mathbb{F}$ for a universal partial (E, C)-action α on a unital, commutative $*$-algebra B and let $\sigma = \sigma_\alpha : \mathbb{F} \to B \rtimes_\alpha \mathbb{F}$ denote the canonical partial representation, which is semi-saturated due to Lemma 2.4.4. Obviously, σ satisfies (PI1)-(PI4) since α is a partial (E, C)-action. Letting $\rho: \mathbb{F} \to A'$ be a semi-saturated partial representation satisfying (PI1)-(PI4), denote by B' the commutative, unital $*$-algebra generated by all the final projections $\varepsilon_\rho(s)$ and write $\beta = \alpha^\rho$ for the induced partial (E, C)-action on B'. Then by assumption, there is a unique equivariant $*$-homomorphism $\varphi: B \to B'$, and it extends to a $*$-homomorphism $\widetilde{\varphi}: B \rtimes_\alpha \mathbb{F} \to B' \rtimes_\beta \mathbb{F}$ by Proposition 1.2.8. Composing this with the $*$-homomorphism $\iota \times \rho: B' \rtimes_\beta \mathbb{F} \to A'$ of Corollary 1.2.18, we obtain a $*$-homomorphism $\psi: B \rtimes_\alpha \mathbb{F} \to A'$ acting by $\psi(a\delta_s) = \varphi(a)\rho(s)$ for $a \in D_{s^{-1}}$. In particular we have $\psi(\sigma(s)) = \rho(s)$ for all $s \in \mathbb{F}$, and as the $\sigma(s)$'s generate A, ψ is unique with this property. □

In a completely similar manner, we obtain the C^*-analogue.

Theorem 2.4.6. *Given a finitely separated graph (E, C), denote by $\mathbb{F}$ the free group on E^1. If $\sigma: \mathbb{F} \to \mathcal{A}$ is universal for partial representations satisfying (PI1)-(PI4), then $\alpha^\sigma: \mathbb{F} \curvearrowright \mathcal{B}$ is a universal partial (E, C)-action on the commutative, unital C^*-algebra $\mathcal{B}$ generated by the $\varepsilon_\sigma(s)$'s, and $\mathcal{A} \cong \mathcal{B} \rtimes_{\alpha^\sigma} \mathbb{F}$. Conversely, if $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{F}$ where α is a universal partial (E, C)-action on a commutative, unital C^*-algebra, then $\sigma_\alpha: \mathbb{F} \to \mathcal{A}$ is universal for semi-saturated partial representations satisfying (PI1)-(PI4).*

From the above general facts, we now obtain the main results of this section:

Corollary 2.4.7. *Assume that (E, C) is a finite graph without isolated vertices. Then the canonical partial (E, C)-action $\alpha: \mathbb{F} \curvearrowright C_K(\Omega(E, C))$ is universal and*

$$L_K^{ab}(E, C) \cong C_K(\Omega(E, C)) \rtimes_\alpha \mathbb{F}.$$

Proof. This is immediate from Proposition 2.2.9 and Theorem 2.4.5. □

Corollary 2.4.8. *Assume that (E, C) is a finite graph without isolated vertices. Then the canonical partial (E, C)-action $\alpha: \mathbb{F} \curvearrowright C(\Omega(E, C))$ is universal and*

$$\mathcal{O}(E, C) \cong C(\Omega(E, C)) \rtimes_\alpha \mathbb{F}.$$

Proof. This is immediate from Corollary 2.3.7 and Theorem 2.4.6. □

In the C^*-case we can apply duality to obtain another description of a partial (E, C)-action.

Definition 2.4.9. A partial (E, C)-preaction θ on a compact Hausdorff space Ω is a family of clopen subsets $\{\Omega_v \mid v \in E^0\}$ such that

$$\Omega = \bigsqcup_{v \in E^0} \Omega_v,$$

and a family of clopen subsets $\{\Omega_e \mid e \in r^{-1}(v)\}$ for all $v \in E^0$, such that

$$\Omega_v = \bigsqcup_{e \in X} \Omega_e$$

for all $X \in C_v$, together with homeomorphisms $\theta_e \colon \Omega_{s(e)} \to \Omega_e$ for all $e \in E^1$. If (Ω', θ') is another such pair, a continuous map $f \colon \Omega' \to \Omega$ is called *equivariant* if $f(\Omega'_v) \subset \Omega_v$ for all $v \in E^0$, $f(\Omega'_e) \subset \Omega_e$ for all $e \in E^1$ and $f(\theta'_e(x)) = \theta_e(f(x))$ for all $x \in \Omega'_{s(e)}$. Also, θ is called *universal* if, given another partial (E, C)-preaction θ' on Ω', there is a unique equivariant map $(\Omega', \theta') \to (\Omega, \theta)$. Finally, a partial (E, C)-action is the canonical extension of the θ_e's to a partial action $\mathbb{F} \curvearrowright \Omega$, and we shall refer to the pair (Ω, θ) as an (E, C)-dynamical system.

It is easily seen that the above definitions correspond to the one for C^*-algebras under the contravariant equivalence of unital commutative C^*-algebras and compact Hausdorff spaces. In particular, there is a canonical partial (E, C)-action $\theta \colon \mathbb{F} \curvearrowright \Omega(E, C)$. We obtain the following corollaries.

Corollary 2.4.10. *A continuous map of (E, C)-dynamical systems is equivariant in the sense of Definition 2.4.9, if and only if it is equivariant in the usual sense.*

Proof. This is immediate by Lemma 2.4.4. □

Corollary 2.4.11. *Assume that (E, C) is a finite graph without isolated vertices. Then the canonical partial (E, C)-action on $\Omega(E, C)$ is universal.*

Proof. This is immediate by Corollary 2.4.8. □

Chapter 3

The main construction

In this chapter we shall introduce a sequence of finite bipartite graphs (E_n, C^n) to every finite bipartite graph (E, C). On the level of algebras and C^*-algebras, they will provide an important approximation result (Theorem 3.3.11 and Theorem 3.3.12) from which it will follow that $\mathcal{V}(L_K(E,C)) \to \mathcal{V}(L_K^{\mathrm{ab}}(E,C))$ is a refinement. We also produce an explicit description of the space $\Omega(E,C)$ associated to (E,C), which will be vital to us in Chapter 4.

3.1 Multiresolutions

In order to define the graphs (E_n, C^n), we first need to define and investigate the properties of a graph-theoretic construction, which we will call a *multiresolution*.

Definition 3.1.1. Define a relation $\rightsquigarrow_1$ on the free abelian monoid $\mathbb{Z}_+(E^0)$ as follows: $a \rightsquigarrow_1 b$ if there are $v_1, \dots, v_k \in E^0 \setminus \mathrm{Source}(E)$ and $v_{k+1}, \dots, v_n \in E^0$ such that

$$a = \sum_{i=1}^{k} v_i + \sum_{i=k+1}^{n} v_i \quad \text{and} \quad b = \sum_{i=1}^{k} \mathbf{s}(X_i) + \sum_{i=k+1}^{n} v_i$$

for some $X_i \in C_{v_i}$, where $\mathbf{s}(X) = \sum_{e \in X} s(e)$. Repetitions of the same vertices in the above sums are of course allowed. Taking $k = 0$ it follows that $a \rightsquigarrow_1 a$, and we obviously have

$$a_1 \rightsquigarrow_1 b_1 \quad \text{and} \quad a_2 \rightsquigarrow_1 b_2 \Rightarrow a_1 + a_2 \rightsquigarrow_1 b_1 + b_2.$$

However, besides these properties, $\rightsquigarrow_1$ does not behave particularly well.

Definition 3.1.2 (Assumption $(*)$)**.** A separated graph (E,C) is said to satisfy Assumption $(*)$ at $v \in E^0$, if for any $X, Y \in C_v$ there is some $\gamma \in \mathbb{Z}_+(E^0)$ such that $s(X) \rightsquigarrow_1 \gamma$ and $s(Y) \rightsquigarrow_1 \gamma$. Note that if $|C_v| \leq 1$, then the assumption is vacuously satisfied at v. Finally, if (E,C) satisfies Assumption $(*)$ at every vertex, then (E,C) is simply said to satisfy Assumption $(*)$.

The importance of this property is witnessed by the lemma just below, which will provide the main motivation for the multiresolution construction. However, we skip the fairly comprehensive proof (see Remark 4.1.12 for a reason why).

Lemma 3.1.3. *Let (E,C) be a finitely separated graph. If (E,C) satisfies Assumption $(*)$, then $M(E,C)$ is a refinement monoid (see Definition A.1.15).*

Proof. See [5, Theorem 5.15]. □

Definition 3.1.4. Given a separated graph (E, C) and a vertex v such that $|r^{-1}(v)| < \infty$, write $C_v = \{X_1, \ldots, X_k\}$. Then the *multiresolution* (E_v, C^v) of (E, C) at v is the separated graph defined as follows: Set

$$E_v^0 = E^0 \sqcup \Lambda_v^0 \text{ with } \Lambda_v^0 := \{v(x_1, \ldots, x_k) \mid x_i \in X_i, i = 1, \ldots, k\}$$

and

$$E_v^1 = E^1 \sqcup \Lambda_v^1 \text{ with } \Lambda_v^1 := \{e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_k) \mid x_j \in X_j, i = 1, \ldots, k\}.$$

Now extend r, s to E_v^1 by

$$r\Big(e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_k)\Big) = s(x_i) \quad \text{and} \quad s\Big(e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_k)\Big) = v(x_1, \ldots, x_k),$$

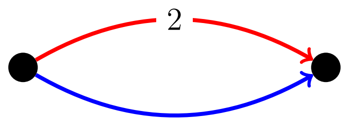
and note that this makes the new vertices into sources. Finally, given any $w \in E^0$ we extend the separation by letting

$$X(x_i) = \{e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_k) \mid x_j \in X_j, j \neq i\}$$

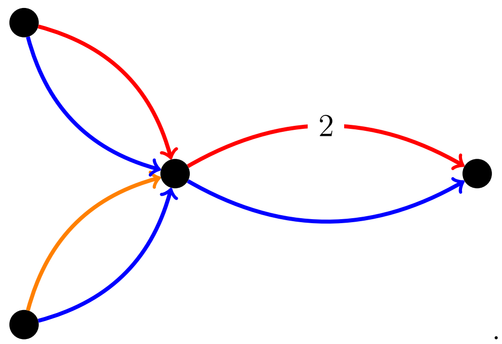
for $x_i \in X_i$ and

$$(C^v)_w = C_w \sqcup \{X(x_i) \mid x_i \in X_i, s(x_i) = w, i = 1, \ldots, k\}.$$

Example 3.1.5. Given the graph



we can consider the multiresolution at the right vertex. Since the separation consists of two subsets with 2 and 1 edges, we should add $n = 2 \cdot 1 = 2$ vertices. Furthermore, for each of the red edges we should add $n/2 = 1$ edge, and for the blue we should add $n/1 = 2$ edges. The result is as follows



.

Definition 3.1.6. Given a separated graph (E, C) and a set of vertices V such that $|r^{-1}(v)| < \infty$ for all $v \in V$, we can form a simultaneous multiresolution at all $v \in V$. To be more precise, we set $E_V^0 := E^0 \sqcup \bigsqcup_{v \in V} \Lambda_v^0$ and $E_V^1 := E^1 \sqcup \bigsqcup_{v \in V} \Lambda_v^1$ and extend r and s to E_V just as above. As for the multiresolution at a single vertex, all the new vertices are sources. Finally, we separate E_V by letting

$$(C^V)_w = C_w \sqcup \bigsqcup_{v \in V} \{X(x_i^v) \mid x_i^v \in X_i^v, s(x_i^v) = w, i = 1, \ldots, k_v\}$$

for each $w \in E^0$, thereby defining another separated graph (E_V, C^V) known as the *multiresolution at* V. We shall only use the v-superscripts, when several vertices are in play at once. Finally, note that the inclusion $E \hookrightarrow E_V$ gives a morphism in **FSGr**.

The multiresolution is constructed precisely such that the following proposition holds. The reader should note that while the requirement $s(r^{-1}(V)) \cap V = \emptyset$ was not included in [3], it certainly is necessary.

Proposition 3.1.7. *Let V be a set of vertices such that $s(r^{-1}(V)) \cap V = \emptyset$ and $|r^{-1}(v)| < \infty$ for all $v \in V$. Then the multiresolution (E_V, C^V) satisfies Assumption $(*)$ at all $v \in V$.*

Proof. Note that the assumption $s(r^{-1}(V)) \cap V = \emptyset$ implies that $(C^V)_v = C_v$ for every $v \in V$. Now let $v \in V$ and write $C_v = \{X_1, \dots, X_{k_v}\}$. Then we have

$$s(x_i) \rightsquigarrow_1 \mathbf{s}(X(x_i)) = \sum_{\substack{x_j \in X_j \\ j \neq i}} v(x_1, \dots, x_{k_v})$$

for any $x_i \in X_i$, hence

$$\mathbf{s}(X_i) \rightsquigarrow_1 \sum_{x_i \in X_i} \mathbf{s}(X(x_i)) = \sum_{\substack{x_j \in X_j \\ j=1,\dots,k_v}} v(x_1, \dots, x_{k_v}) =: \gamma$$

for any $i = 1, \dots, k_v$. We thus obtain Assumption $(*)$ at $v \in V$. $\square$

Finally, we prove that multiresolutions act nicely on the level of monoids.

Lemma 3.1.8. *Let (E, C) be a separated graph with a set of vertices V such that $|r^{-1}(v)| < \infty$ for all $v \in V$, and let ι denote the inclusion $(E, C) \to (E_V, C^V)$. Then*

$$M(\iota) \colon M(E, C) \to M(E_V, C^V)$$

is a unitary embedding (see Definition A.1.10).

Proof. For $v \in V$ we write $C_v = \{X_1^v, \dots, X_{k_v}^v\}$ and let M denote the quotient of the free abelian monoid on generators

$$\bigsqcup_{\substack{v \in V \\ i=1,\dots,k_v}} X_i^v \quad \text{by relations} \quad \sum_{x_i^v \in X_i^v} x_i^v = \sum_{x_j^v \in X_j^v} x_j^v$$

for every $v \in V$ and $1 \leq i, j \leq k_v$. Note that M is canonically isomorphic to $\bigoplus_{v \in V} M_v$, where M_v is the monoid on generators

$$\bigsqcup_{\substack{v \in V \\ i=1,\dots,k_v}} X_i^v \quad \text{by relations} \quad \sum_{x_i^v \in X_i^v} x_i^v = \sum_{x_j^v \in X_j^v} x_j^v$$

for $1 \leq i, j \leq k_v$ and a particular $v \in V$. Letting $S^v = \prod_{i=1}^{k_v} X_i^v$, $S_i^v = X_1^v \times \ldots \times \widehat{X_i^v} \times \ldots \times X_{k_v}^v$ and $S = \bigsqcup_{v \in V} S^v$, there is a canonical isomorphism of monoids $\mathbb{Z}_+(S) \cong \bigoplus_{v \in V} \mathbb{Z}_+(S^v)$. Now it follows by Lemma A.1.13 that there are unitary embeddings $\psi_v \colon M_v \to \mathbb{Z}_+(S^v)$ given by

$$\psi_v(x_i^v) = \sum_{(x_1, \dots, \widehat{x_i^v}, \dots, x_{k_v}^v) \in S_i^v} (x_1^v, \dots, x_{k_v}^v)$$

for all $x_i^v \in X_i^v$. Thus $\bigoplus_{v \in V} \psi_v$ defines a unitary embedding $\bigoplus_{v \in V} M_v \to \bigoplus_{v \in V} \mathbb{Z}_+(S_v)$, and under the canonical isomorphisms, this corresponds to a unitary embedding $\psi \colon M \to \mathbb{Z}_+(S)$ acting on x_i^v as specified above. Also consider the homomorphisms

$$\eta \colon M \to M(E, C) \quad \text{and} \quad \nu \colon \mathbb{Z}_+(S) \to M(E_V, C^V)$$

acting by
$$\eta(x_i^v) = s(x_i^v) \quad \text{and} \quad \nu(x_1^u, \ldots, x_{k_u}^u) = v(x_1^u, \ldots, x_{k_u}^u),$$
and note that η is well defined by definition of $M(E,C)$. It is straightforward to verify that the diagram

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[d, "\eta"'] & \mathbb{Z}_+(S) \arrow[d, "\nu"] \\
M(E,C) \arrow[r, "M(\iota)"'] & M(E_V, C^V)
\end{tikzcd}$$

commutes — in fact, we claim that it is a pushout diagram in the category of abelian monoids. To see this, assume that α and β are homomorphisms making the diagram

$$\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[d, "\eta"'] & \mathbb{Z}_+(S) \arrow[d, "\nu"] \arrow[ddr, bend left, "\beta"] & \\
M(E,C) \arrow[r, "M(\iota)"'] \arrow[drr, bend right, "\alpha"'] & M(E_V, C^V) & \\
& & N
\end{tikzcd}$$

commute, and define a homomorphism $\rho\colon M(E_V, C^V) \to N$ by

$$\rho(v) = \begin{cases} \alpha(v) & \text{if} \quad v \in E^0 \\ \beta(x_1^u, \ldots, x_{k_u}^u) & \text{if} \quad v = v(x_1^u, \ldots, x_{k_u}^u) \end{cases}.$$

Since the vertices in $E_V^0 \setminus E^0$ are all sources, we only need to check that ρ is well defined on $v \in E^0$. But for any $X \in C_v$, we simply have $\alpha(v) = \alpha(\mathbf{s}(X))$ by definition of $M(E,C)$. Evidently $\rho \circ M(\iota) = \alpha$ as well as $\rho \circ \nu = \beta$, and clearly ρ is the unique homomorphism with this property. It finally follows from Lemma A.1.14 that $M(\iota)$ is a unitary embedding. $\square$

We shall record a consequence of the above proof for future references:

Corollary 3.1.9. *Let (E,C) be a separated graph with a set of vertices V, such that $|r^{-1}(v)| < \infty$ for all $v \in V$. If $\varphi\colon M(E,C) \to N$ is a homomorphism, and for all $u \in V$ there is a refinement $\{a(x_1^u, \ldots, x_{k_u}^u) \mid x_i^u \in X_i^u\}$ of the equation system (see Definition A.1.15)*

$$\sum_{x_1^u \in X_1^u} \varphi(s(x_1^u)) = \sum_{x_2^u \in X_2^u} \varphi(s(x_2^u)) = \ldots = \sum_{x_{k_u}^u \in X_{k_u}^u} \varphi(s(x_{k_u}^u)),$$

then there is a unique homomorphism $\widetilde{\varphi}\colon M(E_V, C^V) \to N$ such that

$$\widetilde{\varphi}(v) = \varphi(v) \text{ for } v \in E^0 \quad \text{and} \quad \widetilde{\varphi}(v(x_1^u, \ldots, x_{k_u}^u)) = a(x_1^u, \ldots, x_{k_u}^u).$$

In particular, $\widetilde{\varphi} \circ M(\iota) = \varphi$.

Proof. Apply the same notation as in the proof of Lemma 3.1.8, and define a homomorphism $\beta\colon Z_+(S) \to N$ by

$$\beta(x_1^u, \ldots, x_{k_u}^u) = a(x_1^u, \ldots, x_{k_u}^u)$$

for all $u \in V$ and $x_i^u \in X_i^u$. Then by the refinement property we have

$$\varphi \circ \eta(x_i^u) = \varphi(s(x_i^u)) = \sum_{j \neq i} \sum_{x_j^u \in X_j^u} a(x_1^u, \ldots, x_{k_u}^u) = \beta \circ \psi(x_i^u),$$

hence there is a unique homomorphism $\widetilde{\varphi}\colon M(E_V, C^v) \to N$ satisfying $\widetilde{\varphi}(v) = \varphi(v)$ for $v \in E^0$ and

$$\widetilde{\varphi}(v(x_1^u, \ldots, x_{k_u}^u)) = \widetilde{\varphi}(\nu(x_1^u, \ldots, x_{k_u}^u)) = \beta(x_1^u, \ldots, x_{k_u}^u) = a(x_1^u, \ldots, x_{k_u}^u).$$

□

3.2 Bipartite graphs

Definition 3.2.1. A separated graph (E, C) is called *bipartite* if $E^0 = E^{0,0} \sqcup E^{0,1}$ with $s(E^1) \subset E^{0,1}$ and $r(E^1) \subset E^{0,0}$. However, in order to avoid trivialities we shall always assume that $s(E^1) = E^{0,1}$ and $r(E^1) = E^{0,0}$, i.e. that E does not have any isolated vertices.

The following two propositions show that bipartiteness is not a very restrictive assumption.

Proposition 3.2.2. *Let (E, C) denote a separated graph. Then there is a bipartite separated graph $(\tilde{E}, \tilde{C})$ such that*

$$L_K(\tilde{E}, \tilde{C}) \cong M_2(L_K(E, C)) \quad , \quad C^*(\tilde{E}, \tilde{C}) \cong M_2(C^*(E, C))$$

and

$$L_K^{ab}(\tilde{E}, \tilde{C}) \cong M_2(L_K^{ab}(E, C)) \quad , \quad \mathcal{O}(\tilde{E}, \tilde{C}) \cong M_2(\mathcal{O}(E, C)).$$

Moreover, if (E, C) is finite, then so is $(\tilde{E}, \tilde{C})$.

Proof. We start by proving the claim for $L_K(E, C)$. The graph $(\tilde{E}, \tilde{C})$ will be obtained by doubling up the vertices of E and adding some appropriate new edges. Specifically, define $\tilde{E}^0 = \tilde{E}^{0,0} \sqcup \tilde{E}^{0,1}$, where

$$\tilde{E}^{0,0} = \{v_0 \mid v \in E^0\} \quad \text{and} \quad \tilde{E}^{0,1} = \{v_1 \mid v \in E^0\},$$

and set

$$\tilde{E}^1 = \{f_v \mid v \in E^0\} \cup \{e_0 \mid e \in E^1\}$$

with

$$\tilde{r}(f_v) = v_0, \quad \tilde{s}(f_v) = v_1, \quad \tilde{r}(e_0) = r(e)_0 \quad \text{and} \quad \tilde{s}(e_0) = s(e)_1.$$

For $X \in C_v$ we define $X_0 = \{e_0 \mid e \in X\}$ and then separate $\tilde{E}$ by

$$\tilde{C}_{v_0} = \{X_0, \{f_v\} \mid X \in C_v\}.$$

Note that for a finite complete subobject(F, D) of (E, C), $(\tilde{F}, \tilde{D})$ is a finite complete subobject of $(\tilde{E}, \tilde{C})$. Moreover, we can obtain arbitrarily big finite complete subobjects in this way. Thus, assuming the claim holds for finite graphs (and the appropriate diagram commutes), for any graph (E, C) we have

$$M_2(L_K(E, C)) \cong M_2(\varinjlim L_K(F, D)) \cong \varinjlim M_2(L_K(F, D)) \cong \varinjlim L_K(\tilde{F}, \tilde{D}) \cong L_K(\tilde{E}, \tilde{C}),$$

where the direct limit is taken over all finite complete subobject (F, D) of (E, C). We may therefore assume that (E, C) is finite. Now write $e_{i,j}$ with $1 \leq i, j \leq 2$ for the (i, j)'th standard matrix unit in $M_2(K)$ and define $\varphi \colon L_K(\tilde{E}, \tilde{C}) \to M_2(L_K(E, C))$ by

$$\varphi(v_i) = v \otimes e_{i+1,i+1}, \quad \varphi(f_v) = v \otimes e_{1,2} \quad \text{and} \quad \varphi(e_0) = e \otimes e_{1,2}$$

for $v \in E^0$, $e \in E^1$ and $i = 0, 1$. It is straightforward to check that φ respects the defining relations of $L_K(\tilde{E}, \tilde{C})$. In order to construct an inverse, we define $\psi_1 \colon M_2(K) \to L_K(\tilde{E}, \tilde{C})$ by

$$\psi_1(e_{1,2}) = \sum_{v \in E^0} f_v, \quad \psi_1(e_{2,1}) = \sum_{v \in E^0} f_v^*, \quad \psi_1(e_{1,1}) = \sum_{v \in E^0} v_0 \quad \text{and} \quad \psi(e_{2,2}) = \sum_{v \in E^0} v_1.$$

It is clear that these elements form matrix units, hence ψ_1 is a well defined $*$-homomorphism. Now define $\psi_2\colon L_K(E,C) \to L_K(\tilde{E},\tilde{C})$ by

$$\psi_2(v) = v_0 + v_1 \quad \text{and} \quad \psi_2(e) = e_0 f^*_{s(e)} + f^*_{r(e)} e_0.$$

It is not hard to check that ψ_2 respects the defining relations of $L_K(E,C)$, and that the images of ψ_1 and ψ_2 commute. Now we obtain a $*$-homomorphism

$$\psi = \psi_1 \times \psi_2\colon M_2(L_K(E,C)) \to L_K(\tilde{E},\tilde{C})$$

given by $\psi(a \otimes b) = \psi_1(a)\psi_2(b)$. Surely, ψ provides us with an inverse of φ, thereby finishing the proof for $L_K(E,C)$. For finite (E,C), universality now proves the claim for $C^*(E,C)$, and the general case follows by taking limits.

In order to prove the claim for $L_K^{\mathrm{ab}}(E,C)$, we may once again assume that (E,C) is finite. Define U and J as in Definition 2.2.6, and also write $\tilde{U} = U(\tilde{E},\tilde{C})$ and $\tilde{J} = J(\tilde{E},\tilde{C})$. Then we need to show that $\varphi(\tilde{J}) = M_2(J)$. Surely $\varphi(p(\tilde{s}))$ is of the form

$$\begin{pmatrix} p(s) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & p(s) \end{pmatrix}$$

for some $s \in U$, hence $\varphi(\tilde{J}) \subset M_2(J)$. On the other hand, since $\psi(e \otimes e_{1,1}) = e_0 f^*_{s(e)}$ for every $e \in E^1$, we see that $\psi([p(s),p(t)] \otimes e_{1,1}) \in \tilde{J}$ for each $s,t \in U$. We deduce that $\psi(M_2(J)) \subset \tilde{J}$, hence $M_2(J) = \varphi(\tilde{J})$ and therefore

$$L_K^{\mathrm{ab}}(\tilde{E},\tilde{C}) = L_K(\tilde{E},\tilde{C})/\tilde{J} \cong M_2(L_K(E,C))/M_2(J) \cong M_2(L_K^{\mathrm{ab}}(E,C)).$$

The claim finally follows for $\mathcal{O}(E,C)$ as well by universality. $\square$

Proposition 3.2.3. *If M is any finitely generated, conical abelian monoid, then there is a finite bipartite separated graph (E,C), such that $M \cong M(E,C)$.*

Proof. By Corollary A.1.9 we have a finite presentation $\langle X \mid \{\mathbf{r}_j\}_{j\in J}\rangle$ with

$$\mathbf{r}_j\colon \quad \sum_{x\in X} m_{x,j} x = \sum_{x\in X} n_{x,j} x,$$

such that $\sum_{x\in X} m_{x,j}, \sum_{x\in X} n_{x,j} > 0$ for all $j \in J$ and $\sum_{j\in J} m_{x,j} + n_{x,j} > 0$ for all $x \in X$. Now, define a finite bipartite graph (E,C) by

- $E^0 = E^{0,0} \sqcup E^{0,1}$ with $E^{0,0} = \{u_j\}_{j\in J}$ and $E^{0,1} = \{v_x\}_{x\in X}$,
- $E^1 = \bigsqcup_{x\in X, j\in J} \{e^i_{x,j} \mid i = 1,\dots,m_{x,j}+n_{x,j}\}$,
- $r(e^i_{x,j}) = u_j$ and $s(e^i_{x,j}) = v_x$ for all $x \in X$, $j \in J$ and $i = 1,\dots,m_{x,j}+n_{x,j}$,
- $C_{u_j} = \left\{ \bigsqcup_{x\in X} \bigsqcup_{i=1}^{m_{x,j}} \{e^i_{x,j}\}, \bigsqcup_{x\in X} \bigsqcup_{i=m_{x,j}+1}^{m_{x,j}+n_{x,j}} \{e^i_{x,j}\} \right\}$ for all $j \in J$.

Then $M \cong M(E,C)$ is immediate from the assumption that $\sum_{x\in X} m_{x,j}, \sum_{x\in X} n_{x,j} > 0$ for all $j \in J$. Finally, note that the condition $\sum_{j=1}^n m_{x,j} + n_{x,j} > 0$ precisely implies $r(E^1) = E^{0,0}$ and $s(E^1) = E^{0,1}$. $\square$

Construction 3.2.4. Given a finite bipartite separated graph (E, C), we shall construct an increasing sequence of finite separated graphs (F_i, D^i) by first setting $(F_0, D^0) = (E, C)$ and then defining the others inductively. Assume that

$$(F_0, D^0) \subset (F_1, D^1) \subset \ldots \subset (F_n, D^n)$$

is an increasing sequence of finite bipartite graphs such that

- $F_i^0 = \bigsqcup_{j=0}^{i+1} F^{0,j}$ for all $0 \leq i \leq n$,
- $F_i^1 = \bigsqcup_{j=0}^{i} F^{1,j}$ for all $0 \leq i \leq n$,
- $r(F^{1,j}) = F^{0,j}$ and $s(F^{1,j}) = F^{0,j+1}$ for all $0 \leq j \leq n$,
- (F_i, D^i) satisfies Assumption $(*)$ at all $v \in \bigsqcup_{j=0}^{i-1} F^{0,j}$.

Then we define $V_n = F^{0,n}$ and let (F_{n+1}, D^{n+1}) be the multiresolution of (F_n, D^n) at V_n. Denote by $F^{0,n+2}$ and $F^{1,n+1}$ the added vertices and the added edges, respectively. By definition of the multiresolution we have $r(F^{1,n+1}) = s(r^{-1}(V_n)) = F^{0,n+1}$ and $s(F^{1,n+1}) = F^{0,n+2}$. Since no edges into $\bigsqcup_{j=0}^{n-1} F^{0,j}$ have been added, (F_{n+1}, D^{n+1}) still satisfies Assumption $(*)$ at these vertices. Finally, since $s(r^{-1}(V_n)) \cap V_n = \emptyset$, it follows from Proposition 3.1.7 that (F_{n+1}, D^{n+1}) also satisfies Assumption $(*)$ at V_n. This finishes the inductive construction. We can visualize the construction as below

$$\begin{tikzcd} \boxed{F^{0,0}} & \boxed{F^{0,1}} \arrow[l, shift right=2, "F^{1,0}"'] \arrow[l] \arrow[l, shift left=2] & \boxed{F^{0,2}} \arrow[l, shift right=2, "F^{1,1}"'] \arrow[l] \arrow[l, shift left=2] & \cdots \arrow[l, shift right=2, "F^{1,2}"'] \arrow[l] \arrow[l, shift left=2] \end{tikzcd}.$$

Finally, define finite bipartite separated graphs (E_n, C^n) by

- $E_n^0 = E_n^{0,0} \sqcup E_n^{0,1}$ with $E_n^{0,0} = F^{0,n}$ and $E_n^{0,1} = F^{0,n+1}$
- $E_n^1 = F^{1,n}$
- range and source maps are restrictions of the range and source maps of F_n
- $C_v^n = D_v^n$ for all $v \in E_n^{0,0}$.

The graph E_n can be depicted as

$$\begin{tikzcd} \boxed{F^{0,n}} & \boxed{F^{0,n+1}} \arrow[l, shift right=2, "F^{1,n}"'] \arrow[l] \arrow[l, shift left=2] \end{tikzcd}$$

and we shall refer to $\{(E_n, C^n)\}_n$ as the *canonical sequence of finite bipartite graphs* associated to (E, C).

Corollary 3.2.5. *Consider the separated graph* $(F_\infty, D^\infty) := \varinjlim_n (F_n, D^n) = (\bigcup_n F_n, \bigcup_n D^n)$ *and let* $\iota\colon (E, C) \to (F_\infty, D^\infty)$ *denote the inclusion. Then* $M(\iota)\colon M(E, C) \to M(F_\infty, D^\infty)$ *is a refinement (see Definition A.1.19).*

Proof. It is clear from Construction 3.2.4 that (F^{∞}, D_{∞}) satisfies Assumption $(*)$ at all vertices, hence $M(F_{\infty}, D^{\infty})$ is a refinement monoid by Lemma 3.1.3. Moreover,

$$M(F_{\infty}, D^{\infty}) \cong \varinjlim M(F_n, D^n)$$

by continuity of M, and each of the homomorphisms $M(F_n, D^n) \to M(F_{n+1}, D^{n+1})$ is unitary by Lemma 3.1.8, so $M(\iota)$ is unitary as well by Lemma A.1.11. Finally, given a refinement monoid P and a homomorphism $\varphi\colon M(E, C) \to P$, it follows from Corollary 3.1.9 and Lemma A.1.18 that there are homomorphisms $\varphi_n\colon M(F_n, D^n) \to P$ making each of the diagrams

$$\begin{tikzcd} M(F_n, D^n) \arrow[r] \arrow[dr, "\varphi_n"'] & M(F_{n+1}, D^{n+1}) \arrow[d, "\varphi_{n+1}"] \\ & P \end{tikzcd}$$

commute. We thus obtain a homomorphism of the limit $\widetilde{\varphi}\colon M(F_{\infty}, D^{\infty}) \to P$ such that $\widetilde{\varphi} \circ M(\iota) = \varphi$. This concludes the proof. $\square$

Lemma 3.2.6. *Given a finite bipartite separated graph (E, C), define (E_n, C^n), V_n, and (F_{∞}, D^{∞}) as in Construction 3.2.4. Then*

(a) The inclusion $i_n\colon (E_{n+1}, C^{n+1}) \to ((E_n)_{V_n}, (C^n)^{V_n})$ induces an isomorphism

$$M(i_n)\colon M(E_{n+1}, C^{n+1}) \to M((E_n)_{V_n}, (C^n)^{V_n}).$$

(b) There is a canonical unitary embedding $\iota_n\colon M(E_n, C^n) \to M(E_{n+1}, C^{n+1})$.

(c) The inclusion $j_n\colon (E_n, C^n) \to (F_n, D^n)$ induces an isomorphism

$$M(j_n)\colon M(E_n, C^n) \to M(F_n, D^n).$$

(d) There is an isomorphism of abelian monoids $M(F_{\infty}, D^{\infty}) \cong \varinjlim(M(E_n, C^n), \iota_n)$. Consequently, the limit homomorphism $M(E, C) \to \varinjlim M(E_n, C^n)$ is a refinement.

Proof. **(a):** For simplicity we write $E_{V_n} = (E_n)_{V_n}$ and $C^{V_n} = (C^n)^{V_n}$. We can depict E_{V_n} as follows:

$$\begin{tikzcd} \boxed{F^{0,n}} & \boxed{F^{0,n+1}} \arrow[l, "F^{1,n}"'] \arrow[l, shift left=2] \arrow[l, shift right=2] & \boxed{F^{0,n+2}} \arrow[l, "F^{1,n+1}"'] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

In order to construct an inverse of $M(i_n)$, define $\psi_n\colon \mathbb{Z}_+(E^0_{V_n}) \to M(E_{n+1}, C^{n+1})$ by

$$\psi_n(v) = \begin{cases} v & \text{if } \ v \in E^0_{n+1} = F^{0,n+1} \cup F^{0,n+2} \\ \mathbf{s}(X) \text{ for some } X \in (C^n)_v & \text{if } \ v \in E^{0,0}_n = F^{0,n} \end{cases}.$$

It is not a priori clear that ψ_n is well defined at vertices $v \in E^{0,0}_n$, i.e. that $\mathbf{s}(X)$ is independent of the choice of $X \in (C^n)_v$. However, writing $(C^n)_v = \{X_1, \ldots, X_{k_v}\}$ we have

$$\mathbf{s}(X_i) = \sum_{x_i \in X_i} s(x_i) = \sum_{x_i \in X_i} \sum_{\substack{j \neq i \\ x_j \in X_j}} v(x_1, \ldots, x_{k_v}) = \sum_{(x_1, \ldots, x_{k_v}) \in \prod_{j=1}^{k_v} X_j} v(x_1, \ldots, x_{k_v})$$

in $M(E_{n+1}, C^{n+1})$, hence $\mathbf{s}(X)$ does not depend on the choice of X. Now, we claim that ψ_n respects the relations in $M(E_{V_n}, C^{V_n})$ and thus drops to a homomorphism

$$M(E_{V_n}, C^{V_n}) \to M(E_{n+1}, C^{n+1}).$$

For $v \in E^{0,1}_{n+1}$, there is nothing to check, as these v are sources in E_{V_n}, and for $v \in E^{0,0}_{n+1}$ it simply follows from $(C^{n+1})_v = (C^{V_n})_v$. Finally, taking $v \in E^{0,0}_n$ we should check that $\psi_n(v) = \psi_n(\mathbf{s}(X))$ for any $X \in (C^{V_n})_v$, and by definition

$$\psi_n(v) = \mathbf{s}(X) = \psi_n(\mathbf{s}(X)).$$

Clearly, ψ_n is an inverse of $M(i_n)$, so $M(i_n)$ is indeed an isomorphism.

(b): Let ι_{V_n} denote the inclusion $(E_n, C^n) \hookrightarrow (E_{V_n}, C^{V_n})$ and define ι_n as the composition

$$M(E_n, C^n) \xrightarrow{M(\iota_{V_n})} M(E_{V_n}, C^{V_n}) \xrightarrow{M(i_n)^{-1}} M(E_{n+1}, C^{n+1}).$$

By Lemma 3.1.8, $M(\iota_{V_n})$ is a unitary embedding, hence so is ι_n.

(c): We shall argue by induction over n. In case $n = 0$ the claim is trivial, since $(E_0, C^0) = (F_0, C^0)$. Now, assuming the claim holds for some n, we recall that

$$(F_{n+1}, C^{n+1}) = ((F_n)_{V_n}, (C^n)^{V^n}).$$

From the inclusion $(E_n, C^n) \hookrightarrow (F_n, D^n)$ inducing an isomorphism, it follows that the inclusion

$$(E_{V_n}, C^{V_n}) \hookrightarrow ((F_n)_{V_n}, (C^n)^{V^n}) = (F_{n+1}, D^{n+1})$$

induces an isomorphism as well: Surjectivity is trivial and injectivity is ensured from the fact that all the new vertices are sources and thus have no relations. Now consider the diagram

$$\begin{tikzcd}
M(E_n, C^n) \arrow[r, "M(\iota_{V_n})"] \arrow[d, "M(j_n)"'] & M(E_{V_n}, C^{V_n}) \arrow[d, "\cong"'] & M(E_{n+1}, C^{n+1}) \arrow[l, "M(i_n)"'] \arrow[d, "M(j_{n+1})"] \\
M(F_n, D^n) \arrow[r] & M(F_{n+1}, D^{n+1}) \arrow[r, "\mathrm{Id}"'] & M(F_{n+1}, D^{n+1})
\end{tikzcd},$$

where all the homomorphisms are induced from inclusions, hence it commutes by functoriality of M. By (a) and the above argument, every homomorphism except $M(j_{n+1})$ in the right square is an isomorphism, hence so is $M(j_{n+1})$.

(d): From the outer rectangle in the above diagram, we get the commutative diagram

$$\begin{tikzcd}
M(E_n, C^n) \arrow[r, "\iota_n"] \arrow[d, "M(j_n)"'] & M(E_{n+1}, C^{n+1}) \arrow[d, "M(j_{n+1})"] \\
M(F_n, D^n) \arrow[r] & M(F_{n+1}, D^{n+1})
\end{tikzcd}$$

with vertical isomorphisms for every n. Thus there is an induced isomorphism of limits $\varinjlim M(E_n, C^n) \to M(F_\infty, D^\infty)$. In particular, the limit map $M(E, C) \to \varinjlim M(E_n, C^n)$ is a refinement by Corollary 3.2.5. $\square$

3.3 Algebras of bipartite graphs

In this highly technical section, we shall relate the algebras associated to the canonical sequence of bipartite graphs (E_n, C^n) with each other. As one might expect, this requires a fair amount of bookkeeping.

Theorem 3.3.1. *Given a finite bipartite separated graph (E, C), we let B_n denote the finite dimensional commutative $*$-algebra of $L_K(E_n, C^n)$ generated by E_n^0. Then there is a surjective $*$-homomorphism*

$$\phi_n \colon L_K(E_n, C^n) \to L_K(E_{n+1}, C^{n+1})$$

such that

(a) $\ker(\phi_n)$ *is the ideal I_n of $L(E_n, C^n)$ generated by all the commutators $[ee^*, ff^*]$ with $e, f \in E_n^1$. In particular $L(E_{n+1}, C^{n+1}) \cong L(E_n, C^n)/I_n$.*

(b) *ϕ_n restricts to an injection $B_n \to B_{n+1}$.*

(c) *The diagram*

$$\begin{tikzcd} M(E_n, C^n) \arrow[r, "\iota_n"] \arrow[d, "\cong"'] & M(E_{n+1}, C^{n+1}) \arrow[d, "\cong"] \\ \mathcal{V}(L_K(E_n, C^n)) \arrow[r, "\mathcal{V}(\phi_n)"'] & \mathcal{V}(L_K(E_{n+1}, C^{n+1})) \end{tikzcd},$$

where the vertical isomorphisms are due to Theorem 2.2.4, commutes.

Proof. Taking $u \in E_n^0$ with $(C^n)_u = \{X_1, \dots, X_{k_u}\}$ we define

$$\phi_n(u) = \begin{cases} \sum_{(x_1,\dots,x_{k_u}) \in \prod_{i=1}^{k_u} X_i} v(x_1, \dots, x_{k_u}) & \text{if } \ u \in E_n^{0,0} \\ u & \text{if } \ u \in E_n^{0,1} \end{cases}$$

along with

$$\phi_n(x_i) = \sum_{j \neq i} \sum_{x_j \in X_j} e^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})^* = \sum_{e \in X(x_i)} e^*$$

for $x_i \in X_i$. We should check that it respects the defining relations of $L_K(E_n, C^n)$. While (V) is trivially satisfied, for (E) we pick $\overline{x_i} \in X_i$. Then

$$\begin{aligned} \phi_n(u)\phi_n(\overline{x_i}) &= \Bigg(\sum_{(x_1,\dots,x_{k_u}) \in \prod_{i=1}^{k_u} X_i} v(x_1, \dots, x_{k_u}) \Bigg) \Bigg(\sum_{j \neq i} \sum_{x_j \in X_j} e^{\overline{x_i}}(x_1, \dots, \widehat{\overline{x_i}}, \dots, x_{k_u})^* \Bigg) \\ &= \sum_{j \neq i} \sum_{x_j \in X_j} e^{\overline{x_i}}(x_1, \dots, \widehat{\overline{x_i}}, \dots, x_{k_u})^* = \phi_n(\overline{x_i}), \end{aligned}$$

since

$$s(e^{\overline{x_i}}(x_1, \dots, \widehat{\overline{x_i}}, \dots, x_{k_u})) = v(x_1, \dots, \overline{x_i}, \dots, x_{k_u}),$$

and

$$\begin{aligned}\phi_n(\overline{x_i})\phi_n(s(\overline{x_i})) &= \sum_{j\neq i}\sum_{x_j\in X_j} e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})^* s(\overline{x_i})\\ &= \sum_{j\neq i}\sum_{x_j\in X_j} e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})^* = \phi_n(\overline{x_i})\end{aligned}$$

since $r\left(e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})\right) = s(\overline{x_i})$. Moving on to (SCK1), we take $\overline{x_i},\widetilde{x_i}\in X_i$ and note that

$$\left(e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})\right)\left(e^{\widetilde{x_i}}(y_1,\dots,\widehat{\widetilde{x_i}},\dots,y_{k_u})\right)^* = 0$$

unless $(x_1,\dots,\overline{x_i},\dots,x_{k_u}) = (y_1,\dots,\widetilde{x_i},\dots,y_{k_u})$. In particular

$$\phi_n(\overline{x_i})^*\phi_n(\widetilde{x_i}) = \left(\sum_{j\neq i}\sum_{x_j\in X_j} e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})\right)\left(\sum_{j\neq i}\sum_{y_j\in X_j} e^{\widetilde{x_i}}(y_1,\dots,\widehat{\widetilde{x_i}},\dots,y_{k_u})^*\right) = 0$$

if $\overline{x_i}\neq\widetilde{x_i}$. On the other hand, if $\overline{x_i}=\widetilde{x_i}$, then

$$\begin{aligned}\phi_n(\overline{x_i})^*\phi_n(\widetilde{x_i}) &= \left(\sum_{j\neq i}\sum_{x_j\in X_j} e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})\right)\left(\sum_{j\neq i}\sum_{y_j\in X_j} e^{\overline{x_i}}(y_1,\dots,\widehat{\overline{x_i}},\dots,y_{k_u})^*\right)\\ &= \sum_{j\neq i}\sum_{x_j\in X_j} e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})e^{\overline{x_i}}(x_1,\dots,\widehat{\overline{x_i}},\dots,x_{k_u})^*\\ &= \sum_{e\in X(\overline{x_i})} ee^* = s(\overline{x_i}) = \phi_n(s(\overline{x_i})).\end{aligned}$$

It remains only to check (SCK2), so take $u\in E_n^{0,0}$ and note that

$$\begin{aligned}\sum_{x_i\in X_i}\phi_n(x_i)\phi_n(x_i)^* &= \sum_{x_i\in X_i}\left(\sum_{\substack{j\neq i\\ x_j\in X_j}} e^{x_i}(x_1,\dots,\widehat{x_i},\dots,x_{k_u})^*\right)\left(\sum_{\substack{j\neq i\\ x_j\in X_j}} e^{x_i}(x_1,\dots,\widehat{x_i},\dots,x_{k_u})\right)\\ &= \sum_{x_i\in X_i}\sum_{\substack{j\neq i\\ x_j\in X_j}} s\left(e^{x_i}(x_1,\dots,\widehat{x_i},\dots,x_{k_u})\right)\\ &= \sum_{(x_1,\dots,x_{k_u})\in\prod_{j=1}^{k_u}X_j} v(x_1,\dots,x_{k_u}) = \phi_n(u).\end{aligned}$$

We finally conclude that ϕ_n drops to a well defined $*$-homomorphism

$$L_K(E_n,C^n)\to L_K(E_{n+1},C^{n+1}).$$

In order to prove surjectivity of ϕ_n, we need only check that it hits every canonical generator of $L_K(E_{n+1},C^{n+1})$, and this is trivially true for all $u\in E_{n+1}^{0,0}$. As in the above calculation we have

$$\phi_n(x_ix_i^*) = \sum_{\substack{j\neq i\\ x_j\in X_j}} v(x_1,\dots,x_{k_u}) \tag{3.1}$$

for any $x_i\in X_i$, so given $v(x_1,\dots,x_{k_u})\in E_{n+1}^{0,1}$ we note that

$$\phi_n\left(\prod_{i=1}^{k_u} x_ix_i^*\right) = \prod_{i=1}^{k_u}\sum_{\substack{j\neq i\\ y_j\in X_j}} v(y_1,\dots,y_{i-1},x_i,y_{i+1},\dots,y_{k_u}) = v(x_1,\dots,x_{k_u}). \tag{3.2}$$

Finally $e^{x_i}(x_1,\dots,\widehat{x_i},\dots,x_{k_u}) \in \phi_n(L_K(E_n,C^n))$ since

$$e^{x_i}(x_1,\dots,\widehat{x_i},\dots,x_{k_u})^* = v(x_1,\dots,x_{k_u})\phi_n(x_i) \in \phi_n(L_K(E_n,C^n)) \tag{3.3}$$

for all $(x_1,\dots,x_{k_u})$. This finishes the proof of surjectivity.

(a): It follows from equation 3.1 that $\phi_n(ee^*)$ and $\phi_n(ff^*)$ are commuting projections for $e,f \in E_n^1$, so we indeed have $I_n \subset \ker(\phi_n)$. For the converse inclusion we define a $*$-homomorphism $\gamma_n\colon L_K(E_{n+1},C^{n+1}) \to L_K(E_n,C^n)/I_n$ by

$$\gamma_n(v) = \begin{cases} [v] & \text{if } \ v \in E_{n+1}^{0,0} \\ \left[\prod_{i=1}^{k_u} x_i x_i^*\right] & \text{if } \ v = v(x_1,\dots,x_{k_u}) \end{cases}$$

and

$$\gamma_n\left(e^{x_i}(x_1,\dots,\widehat{x_i},\dots,x_{k_u})\right) = \left[x_i^* \cdot \prod_{j=1}^{k_u} x_j x_j^*\right].$$

Using commutativity of the $[x_i x_i^*]$'s, it is straightforward to verify that γ_n respects the defining relations of $L_K(E_{n+1},C^{n+1})$. Furthermore, it follows immediately from the above proof of surjectivity of ϕ_n, that γ_n is an inverse of the induced map

$$\overline{\phi_n}\colon L_K(E_n,C^n)/I_n \to L_K(E_{n+1},C^{n+1}),$$

so we may conclude that $\ker(\phi_n) = I_n$.

(b): This is immediate by definition of ϕ_n.

(c): Recall that $\iota_n = \psi_n \circ M(\iota_{V_n})$ with the notation as in the proof of Lemma 3.2.6. Now, commutativity is clear on vertices $v \in E_n^{0,1}$, as they are simply mapped as illustrated in the following diagram of elements

$$\begin{tikzcd} v \arrow[r, "\iota_n"] \arrow[d, "\Gamma(E_n,C^n)"'] & v \arrow[d, "\Gamma(E_{n+1},C^{n+1})"] \\ {[v]} \arrow[r, "\mathcal{V}(\phi_n)"'] & {[v]} \end{tikzcd}.$$

Moving on to the case of $v \in E_n^{0,1}$, write $(C^n)_v = \{X_1^v,\dots,X_{k_v}^v\}$ and take $X \in (C^n)_v$. Then the situation is as follows

$$\begin{tikzcd} v \arrow[r, "\iota_n"] \arrow[d, "\Gamma(E_n,C^n)"'] & \mathbf{s}(X) \arrow[d, "\Gamma(E_{n+1},C^{n+1})"] \\ {[v]} \arrow[r, "\mathcal{V}(\phi_n)"'] & \sum_{(x_1,\dots,x_{k_v})\in\prod_{i=1}^{k_v} X_i^v}[v(x_1,\dots,x_{k_v})] \end{tikzcd},$$

since we even have

$$\mathbf{s}(X) = \sum_{(x_1,\dots,x_{k_v})\in\prod_{i=1}^{k_v} X_i^v} v(x_1,\dots,x_{k_v})$$

in $M(E_{n+1},C^{n+1})$. This concludes the proof. $\square$

The C^*-algebra case now follows easily:

Corollary 3.3.2. *Given a finite bipartite separated graph (E, C), we let B_n denote the finite dimensional commutative C^*-algebra of $C^*(E_n, C^n)$ generated by E_n^0. Then there is a surjective $*$-homomorphism*

$$\phi_n \colon C^*(E_n, C^n) \to C^*(E_{n+1}, C^{n+1})$$

such that

1. $\ker(\phi_n)$ *is the ideal $\mathcal{I}_n$ of $C^*(E_n, C^n)$ generated by all the commutators $[ee^*, ff^*]$ with $e, f \in E_n^1$. In particular $C^*(E_{n+1}, C^{n+1}) \cong C^*(E_n, C^n)/\mathcal{I}_n$.*

2. ϕ_n *restricts to an injection $B_n \to B_{n+1}$.*

Proof. Applying Theorem 3.3.1 in the case $K = \mathbb{C}$, ϕ_n extends to a $*$-homomorphism of C^*-algebras by universality. Surjectivity is preserved due to the closedness of the image.

(a): We obtain an induced $*$-homomorphism $\overline{\phi_n} \colon C^*(E_n, C^n)/\mathcal{I}_n \to C^*(E_{n+1}, C^{n+1})$, and the extension to C^*-algebras $\gamma_n \colon C^*(E_{n+1}, C^{n+1}) \to C^*(E_n, C^n)/\mathcal{I}_n$ is an inverse on the dense $*$-subalgebras, hence it is a global inverse. We conclude that $\ker(\phi_n) = \mathcal{I}_n$.

(b): As each B_n is finite dimensional, this follows from Theorem 3.3.1 and Theorem 2.3.4. $\square$

Definition 3.3.3. Recall that $V_n = F^{0,n}$. For $n \geq 2$ and $u \in V_n$, write $C_u = \{X_1, \ldots, X_{k_u}\}$. Then we can define a map $r_n \colon V_n \to V_{n-2}$ by $r_n(v(x_1, \ldots, x_{k_u})) = u$. In particular, for each $n \geq 1$ we gain maps $\mathfrak{r}_{2n} \colon V_{2n} \to V_0$ and $\mathfrak{r}_{2n+1} \to V_1$ given by

$$\mathfrak{r}_{2n} = r_{2n} \circ r_{2n-2} \circ \ldots \circ r_2 \quad \text{and} \quad \mathfrak{r}_{2n+1} = r_{2n+1} \circ r_{2n-1} \circ \ldots \circ r_3.$$

Assembling all of these into a single map, we simply write $\mathfrak{r}$ for $\bigsqcup_{n=2}^{\infty} \mathfrak{r}_n \colon \bigsqcup_{n=2}^{\infty} V_n \to V_0 \sqcup V_1$. We will refer to $\mathfrak{r}(v)$ as the *root* of v.

Lemma 3.3.4. *For $n \geq 2$ and $v \in V_n$, there are canonical bijections*

$$D^{\infty}_{\mathfrak{r}(v)} \leftrightarrow s^{-1}(v) \leftrightarrow D^{\infty}_v.$$

Proof. It suffices to prove that there are canonical bijections $D^{\infty}_{r_n(v)} \leftrightarrow s^{-1}(v)$ for each $n \geq 2$ and a canonical bijection $s^{-1}(v) \leftrightarrow D^{\infty}_v$. For the first part let $u = r_n(v)$, write

$$D^{\infty}_u = \{X_1, \ldots, X_{k_u}\}$$

and $v = v(x_1, \ldots, x_{k_u})$. Then we can define a map by

$$X_i \mapsto e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{k_u}),$$

and as $s^{-1}(v) = \{e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{k_u}) \mid 1 \leq i \leq k_u\}$, this is indeed a bijection. For the second part we simply note that

$$D^{\infty}_v = \{X(e) \mid e \in s^{-1}(v)\},$$

so the bijection is plainly given by $e \mapsto X(e)$. $\square$

Definition 3.3.5 (The canonical enumeration)**.** Fix an enumeration

$$C_u = \{X_1^u, \ldots, X_{k_u}^u\}$$

for all $u \in V_0 = E^{0,0}$. Then by the above lemma, there is a canonical enumeration of the elements of D_v^∞ for all $v \in \mathfrak{r}^{-1}(u)$ by simply making the canonical bijections order preserving (considering the enumeration as an ordering). We shall refer to this as the *canonical enumeration.*

Definition 3.3.6. In both the purely algebraic and in the C^*-algebra context, we define

$$\Phi_n = \phi_{n-1} \circ \phi_{n-2} \circ \ldots \circ \phi_0.$$

Thus Φ_n is a surjective $*$-homomorphism $L_K(E, C) \to L_K(E_n, C^n)$ or $C^*(E, C) \to C^*(E_n, C^n)$.

The next few technical lemmas will provide a better understanding of the structural maps Φ_n.

Lemma 3.3.7. *Fix $u \in E^{0,0} = V_0$ and an enumeration $C_u = \{X_1^u, \ldots, X_{k_u}^u\}$. Given $v \in \mathfrak{r}_{2n}^{-1}(u)$, we equip D_v^∞ with the canonical enumeration and write $D_v^\infty = \{X_1^v, \ldots, X_{k_v}^v\}$. Then for every $1 \leq i \leq k_u$ there is a partition*

$$\mathfrak{r}_{2n}^{-1}(u) = \bigsqcup_{x_i \in X_i^u} Z_{2n}(x_i)$$

such that

$$\Phi_{2n}(x_i) = \sum_{v \in Z_{2n}(x_i)} \sum_{x \in X_i^v} x$$

for $x_i \in X_i^u$.

Proof. We argue by induction over n. For $n = 1$, we define

$$Z_2(x_i) = \{s(y) \mid y \in X(x_i)\} = s(X(x_i))$$

for every $x_i \in X_i^u$ and $i = 1, \ldots, k_u$. Clearly $\mathfrak{r}_2^{-1}(u) = \bigsqcup_{x_i \in X_i^u} Z_2(x_i)$ for every i, and we note that $s(y) \neq s(y')$ for $y, y' \in X(x_i)$ with $y \neq y'$. It follows that

$$\begin{aligned}
\Phi_2(x_i) &= \sum_{y \in X(x_i)} \phi_1(y)^* = \sum_{y \in X(x_i)} \sum_{x \in X(y)} x = \sum_{\substack{j \neq i \\ x_j \in X_j^u}} \sum_{x \in X(e^{x_i}(x_1, \ldots, \widehat{x_i}, \ldots, x_{k_u}))} x \\
&= \sum_{\substack{j \neq i \\ x_j \in X_j^u}} \sum_{x \in X_i^{v(x_1, \ldots, x_{k_u})}} x = \sum_{v \in Z_2(x_i)} \sum_{x \in X_i^v} x
\end{aligned}$$

as desired. For the induction step, let $n \geq 1$ and assume that the claim holds for n. Setting

$$Z_{2n+2}(x_i) = r_{2n+2}^{-1}(Z_{2n}(x_i))$$

for $x_i \in X_i^u$ we indeed have

$$\mathfrak{r}_{2n+2}^{-1}(u) = r_{2n+2}^{-1}(\mathfrak{r}_{2n}^{-1}(u)) = \bigsqcup_{x_i \in X_i^u} Z_{2n+2}(x_i)$$

for each $i = 1, \dots, k_u$. Furthermore,

$$r_{2n+2}^{-1}(v) = \bigsqcup_{x_i' \in X_i^v} \{s(y) \mid y \in X(x_i')\}$$

and $s(y) \neq s(y')$ for $y, y' \in X(x_i')$ with $y \neq y'$, hence

$$\begin{aligned}
\Phi_{2n+2}(x_i) = \phi_{2n+1}(\phi_{2n}(\Phi_{2n}(x_i))) &= \sum_{v \in Z_{2n}(x_i)} \sum_{x_i' \in X_i^v} \phi_{2n+1}(\phi_{2n}(x_i')) \\
&= \sum_{v \in Z_{2n}(x_i)} \sum_{x_i' \in X_i^v} \sum_{y \in X(x_i')} \sum_{x \in X(y)} x = \sum_{v \in Z_{2n}(x_i)} \sum_{x_i' \in X_i^v} \sum_{w \in s(X(x_i'))} \sum_{x \in X_i^w} x \\
&= \sum_{v \in Z_{2n}(x_i)} \sum_{w \in r_{2n+2}^{-1}(v)} \sum_{x \in X_i^w} x = \sum_{w \in Z_{2n+2}(x_i)} \sum_{x \in X_i^w} x,
\end{aligned}$$

thereby finishing the proof. $\square$

Lemma 3.3.8. *For $x \in E^1$, $n \geq 0$, and $b \in B_n$ we have*

$$\Phi_{n+1}(x)\phi_n(b)\Phi_{n+1}(x)^* \in B_{n+1} \quad \text{and} \quad \Phi_{n+1}(x)^*\phi_n(b)\Phi_{n+1}(x) \in B_{n+1}.$$

Moreover, if b is a projection, then so are both of the above products.

Proof. Note that for the first part of the claim, it suffices to consider the case $b = v$ for $v \in E_n^0$. We shall argue separately for $n = 0$, $n = 1$, even $n \geq 2$ and odd $n \geq 2$. In case $n = 0$ we have

$$\Phi_1(x_i)\phi_0(v)\Phi_1(x_i^*) = \begin{cases} \phi_0(x_i x_i^*) = \sum_{j \neq i} \sum_{x_j \in X_j^u} v(x_1, \dots, x_{k_u}) & \text{if} \quad s(x_i) = v \\ 0 & \text{if} \quad s(x_i) \neq v \end{cases}$$

and

$$\Phi_1(x_i^*)\phi_0(v)\Phi_1(x_i) = \phi_0(x_i^* v x_i) = \begin{cases} \phi_0(x_i^* x_i) = s(x_i) & \text{if} \quad v = u \\ 0 & \text{if} \quad v \neq u \end{cases},$$

so in either case the result is a projection in B_1, and by the above identities the second part of the claim is trivial. In case $n = 1$ we have

$$\Phi_1(x_i)v\Phi_1(x_i)^* = \phi_0(x_i)v\phi_0(x_i)^* = \begin{cases} \phi_0(x_i x_i^*) & \text{if} \quad v = s(x_i) \\ 0 & \text{if} \quad v \neq s(x_i) \end{cases}$$

and

$$\begin{aligned}
\Phi_1(x_i)^* v \Phi_1(x_i) &= \phi_0(x_i)^* v \phi_0(x_i) \\
&= \begin{cases} e^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u}) e^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})^* & \text{if } v = v(x_1, \dots, x_{k_u}) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

for some $x_j \in X_j^u$ for $j \neq i$. Due to equation (3.1), we conclude that

$$\Phi_2(x_i)\phi_1(v)\Phi_2(x_i)^* = \phi_1(\Phi_1(x_i)v\Phi(x_i)^*) \quad \text{and} \quad \Phi_2(x_i)^*\phi_1(v)\Phi_2(x_i) = \phi_1(\Phi_1(x_i)^* v\Phi(x_i))$$

are both projections in B_2. Varying $v \in E_1^0$ results in mutually orthogonal projections, which verifies the second part of the claim. Now we shall assume that $n = 2m$ for some $m \geq 1$, so from Lemma 3.3.7 we obtain

$$\Phi_{2m}(x_i)v\Phi_{2m}(x_i)^* = \Big(\sum_{w\in Z_{2m}(x_i)}\sum_{x\in X_i^w} x\Big)v\Big(\sum_{w\in Z_{2m}(x_i)}\sum_{x\in X_i^w} x^*\Big).$$

Note that $s(x) \neq s(y)$ for different $x, y \in \bigcup_{w\in Z_{2m}(x_i)} X_i^w$. Therefore the above term either vanishes or reduces to a single xx^* for some x such that $s(x) = v$. Either way

$$\Phi_{n+1}(x_i)\phi_n(v)\Phi_{n+1}(x_i)^* = \phi_{n+1}(\Phi_{2m}(x_i)v\Phi_{2m}(x_i)^*)$$

is a projection in B_{n+1} by equation (3.2), and the second part of the claim follows as above. Similarly

$$\Phi_{2m}(x_i)^*v\Phi_{2m}(x_i) = \Big(\sum_{w\in Z_{2m}(x_i)}\sum_{x\in X_i^w} x^*\Big)v\Big(\sum_{w\in Z_{2m}(x_i)}\sum_{x\in X_i^w} x\Big)$$

vanishes unless $v \in Z_{2m}(x_i)$, and in that case

$$\Phi_{2m}(x_i)^*v\Phi_{2m}(x_i) = \Big(\sum_{x\in X_i^v} x^*\Big)\Big(\sum_{x\in X_i^v} x\Big) = \sum_{x\in X_i^v} s(x) \tag{3.4}$$

is a projection in B_n. As we noted earlier in this proof, $s(x) \neq s(y)$ for different $x, y \in \bigcup_{w\in Z_{2m}(x_i)} X_i^w$, hence the second part of the claim is verified in this case as well. Finally, we consider the case $n = 2m+1$ for some $m \geq 1$. The part about $\Phi_{2(m+1)}(x_i)^*\phi_{2m+1}(v)\Phi_{2(m+1)}(x_i)$ follows immediately from the observations right above, since $\phi_{2m+1}(v) \in B_{2(m+1)}$ is a projection. For the part about $\Phi_{2(m+1)}(x_i)\phi_{2m+1}(v)\Phi_{2(m+1)}(x_i)^*$ we have

$$\Phi_{2m+1}(x_i)v\Phi_{2m+1}(x_i)^* = \Big(\sum_{w\in Z_{2m}(x_i)}\sum_{x_i'\in X_i^w}\sum_{y\in X(x_i')} y^*\Big)v\Big(\sum_{w\in Z_{2m}(x_i)}\sum_{x_i'\in X_i^w}\sum_{y\in X(x_i')} y\Big)$$

due to Lemma 3.3.7. Since $r(y) = s(x_i')$ for $y \in X(x_i')$, the above term will either vanish entirely or reduce to

$$\Big(\sum_{y\in X(x_i')} y^*\Big)\Big(\sum_{y\in X(x_i')} y\Big) = \sum_{y\in X(x_i')} s(y)$$

for some particular $w \in Z_{2m}(x_i)$ and $x_i' \in X_i^w$ such that $s(x_i') = v$. Either way the result is once again a projection in B_n, hence $\Phi_{2(m+1)}(x_i)\phi_{2m+1}(v)\Phi_{2(m+1)}(x_i)^*$ is a projection in B_{n+1}. Once more varying v obviously gives mutually orthogonal projections, so the second claim also follows in this last case. $\square$

Definition 3.3.9. Recall that U denotes the multiplicative semigroup of $L_K(E, C)$ generated by $\{e, e^* \mid e \in E^1\}$ and write U_n for the subset of U consisting of products of less than n generators. Also recall that $p(s) = ss^*$ for $s \in U$ and let J_n, respectively, $\mathcal{J}_n$ be the ideal of $L_K(E, C)$, respectively, $C^*(E, C)$ generated by the commutators $[p(s), p(t)]$ for $s, t \in U_n$.

Lemma 3.3.10. *The following holds:*

(a) $\Phi_n(p(s))$ *is a projection in* B_n *for all* $s \in U_n$ *and* $n \geq 1$.

(b) *If* $n \geq 0$ *and* $e \in E^1_{2n}$, *then there exist* $f \in E^1$ *and* $s_1, \dots, s_k \in U_{2n}$ *such that*

$$e = \Phi_{2n}(p(s_1)p(s_2)\cdots p(s_k)f).$$

(c) *If* $n \geq 0$ *and* $e \in E^1_{2n+1}$, *then there exist* $f \in E^1$ *and* $s_1, \dots, s_k \in U_{2n+1}$ *such that*

$$e = \Phi_{2n+1}(p(s_1)p(s_2)\cdots p(s_k)f^*).$$

Proof. **(a):** We shall argue by induction. The case $n = 1$ follows directly from Lemma 3.3.8 since $\Phi_1(p(s)) = \Phi_1(s)\varphi_0(1)\Phi_1(s^*)$. For the induction step, assume that the claim holds for some $n \geq 1$ and take $s \in U_{n+1}$. Then writing $s = s_1 s_n$ with $s_1 \in U_1$ and $s_n \in U_n$,

$$\Phi_{n+1}(s)\Phi_{n+1}(s)^* = \Phi_{n+1}(s_1)\phi_n(\Phi_n(s_n s_n^*))\Phi_{n+1}(s_1)^*$$

is a projection in B_{n+1} by Lemma 3.3.8.

(b) and (c): First observe that by part (a) we have

$$\Phi_n(s_1 p(s_n)) = \Phi_n(s_1 s_1^* s_1 p(s_n)) = \Phi_n(s_1 p(s_n) s_1^* s_1) = \Phi_n(p(s_1 s_n) s_1)$$

for all $s_1 \in U_n$, $s_n \in U_n$ and $n \geq 1$. For the problem at hand, we argue by induction once more: To be more precise, we shall show that (b) holds when $n = 0$ and that, for a given n, (b) implies (c), while (c) implies (b) for $n+1$. Note that (b) holds trivially in case $n = 0$ by simply taking $f = e$. Now assume that (b) holds for some $n \geq 0$ and take $e \in E^1_{2n+1}$. We may write $e = e^{x_i}(x_1, \dots, \widehat{x_i}, \dots, x_{k_u})$ for some $u \in V_{2n}$, hence

$$e = \phi_{2n}(x_i^*)v(x_1, \dots, x_{k_u})$$

by equation (3.3). From the inductive assumption there are $f_j \in E^1$ and $s^j_1, \dots, s^j_{k_j} \in U_{2n}$ such that

$$x_j = \Phi_{2n}(p(s^j_1)p(s^j_2)\cdots p(s^j_{k_j})f_j)$$

for all $j = 1, \dots, k_u$. In particular

$$\phi_{2n}(x_j x_j^*) = \Phi_{2n+1}(p(s^j_1)\cdots p(s^j_{k_j})p(f_j))$$

for all $j = 1, \dots, k_u$, hence

$$v(x_1, \dots, x_{k_u}) = \prod_{j=1}^{k_u} \Phi_{2n+1}(p(s^j_1)\cdots p(s^j_{k_j})p(f_j)) \tag{3.5}$$

by equation (3.2). Finally, letting $f = f_i$ and applying the above observation, we get

$$\begin{aligned} e &= \Phi_{2n+1}(f^* p(s^i_1)\cdots p(s^i_{k_i})) \cdot \prod_{j=1}^{k_u} \Phi_{2n+1}(p(s^j_1)\cdots p(s^j_{k_j})p(f_j)) \\ &= \Phi_{2n+1}\Bigg(p(f^* s^i_1)\cdots p(f^* s^i_{k_i}) \cdot \Bigg(\prod_{j=1}^{k_u} p(f^* s^j_1)\cdots p(f^* s^j_{k_j})p(f^* f_j)\Bigg) f^*\Bigg), \end{aligned}$$

thereby proving (c) for n. A completely similar argument shows that if (c) holds for n, then (b) holds for $n+1$, completing the induction step. ∎

We are finally able to prove the main theorem of this chapter in the algebraic setting.

Theorem 3.3.11. *Suppose that (E, C) is a finite bipartite separated graph and define*

$$\Phi_n \colon L_K(E, C) \to L_K(E_n, C^n)$$

as in Definition 3.3.6 for all $n \geq 1$. Then $\ker(\Phi_n) = J_n$ with J_n as in Definition 3.3.9. In particular, $L_K^{ab}(E, C)$ is the direct limit of the $L_K(E_n, C^n)$'s.

Proof. The case $n = 1$ is precisely the content of Theorem 3.3.1a. For general $n \geq 1$, the inclusion $J_n \subset \ker(\Phi_n)$ follows immediately from Lemma 3.3.10a. Indeed $\Phi_n([p(s), p(t)]) = 0$ for every $s, t \in U_n$, since B_n is commutative, hence $J_n \subset \ker(\Phi_n)$. We argue by induction over n for the reverse inclusion, so assume that $\ker(\Phi_n) = J_n$ for some n. Note that by surjectivity of Φ_n, it drops to an isomorphism

$$\frac{\ker(\Phi_{n+1})}{J_n} = \frac{\ker(\Phi_{n+1})}{\ker(\Phi_n)} \to \ker(\phi_n),$$

and since $J_n \subset J_{n+1} \subset \ker(\Phi_{n+1})$, it suffices to prove that $\Phi_n(J_{n+1}) = \ker(\phi_n)$. By Theorem 3.3.1a, $\ker(\phi_n)$ is the ideal in $L_K(E_n, C^n)$ generated by the commutators $[e_1e_1^*, e_2e_2^*]$ for $e_1, e_2 \in E_n^1$, and since J_{n+1} is an ideal of $\ker(\Phi_{n+1})$, it is enough to check that

$$[e_1e_1^*, e_2e_2^*] \in \Phi_n(J_{n+1})$$

for all $e_1, e_2 \in E_n^1$. Assuming first that $n = 2m$, we can apply Lemma 3.3.10b to obtain $f_1, f_2 \in E^1$ and $s_1, \ldots, s_k, t_1, \ldots, t_l \in U_{2m+1}$ such that

$$e_1 = \Phi_{2m}(p(s_1) \cdots p(s_k) f_1) \quad \text{and} \quad e_2 = \Phi_{2m}(p(t_1) \cdots p(t_l) f_2^*).$$

Then

$$[e_1e_1^*, e_2e_2^*] = \Phi_{2m}\left([p(s_1) \cdots p(s_k)p(f_1)p(s_k) \cdots p(s_1), p(t_1) \cdots p(t_l)p(f_2^*)p(t_l) \cdots p(s_1)]\right),$$

so the claim follows from Lemma A.2.11. A completely similar argument using Lemma 3.3.10c does the job for odd n. This finishes the proof of the first statement.

For the second statement, we note note that $L_K^{\mathrm{ab}}(E, C) = \varinjlim L(E, C)/J_n$ with respect to the quotient maps. Then, from the above proof, we have isomorphisms on each level of the sequences below, thereby inducing an isomorphism of the direct limits

$$\begin{tikzcd}
L_K(E,C) \arrow[r, "\phi_0"] \arrow[d, equal] & L_K(E_1,C^1) \arrow[r, "\phi_1"] \arrow[d, "\cong"] & L_K(E_2,C^2) \arrow[r, "\phi_2"] \arrow[d, "\cong"'] & \cdots \arrow[r] & \varinjlim L_K(E_n,C^n) \arrow[d, "\cong"] \\
L_K(E,C) \arrow[r] & L_K(E,C)/J_1 \arrow[r] & L_K(E,C)/J_2 \arrow[r] & \cdots \arrow[r] & L_K^{\mathrm{ab}}(E,C)
\end{tikzcd}.$$

∎

Theorem 3.3.12. *Suppose that (E, C) is a finite bipartite separated graph and define*

$$\Phi_n \colon C^*(E, C) \to C^*(E_n, C^n)$$

as in Definition 3.3.6 for all $n \geq 1$. Then $\ker(\Phi_n) = \mathcal{J}_n$ with $\mathcal{J}_n$ as in Definition 3.3.9. In particular, $\mathcal{O}(E, C)$ is the direct limit of the $C^*(E_n, C^n)$'s.

Proof. Referring to Corollary 3.3.2 rather than Theorem 3.3.1, the proof of the first statement is identical to that of Theorem 3.3.11. For the second statement, we note that

$$\mathcal{O}(E,C) = \varinjlim C^*(E,C)/\mathcal{J}_n$$

with respect to the quotient maps due to Lemma A.3.5. Arguing as above, we then obtain an isomorphism $\varinjlim C^*(E_n, C^n) \cong \mathcal{O}(E,C)$. □

We obtain some immediate corollaries.

Corollary 3.3.13. *The canonical $*$-homomorphism $L^{ab}(E,C) \to \mathcal{O}(E,C)$ is injective.*

Proof. The canonical $*$-homomorphism $L(E_n, C^n) \to C^*(E_n, C^n)$ is injective for all n by Theorem 2.3.4. In particular, the induced $*$-homomorphism of the limits is injective — but this is simply the canonical $*$-homomorphism $L^{\mathrm{ab}}(E,C) \to \mathcal{O}(E,C)$. □

Corollary 3.3.14. *The quotient map $L_K(E,C) \to L_K^{ab}(E,C)$ induces a refinement*

$$\mathcal{V}(L_K(E,C)) \to \mathcal{V}(L_K^{ab}(E,C)).$$

Proof. This follows directly from Lemma 3.2.6d and Theorem 3.3.1c. □

Definition 3.3.15. Define a $*$-subalgebra of $L_K^{\mathrm{ab}}(E,C)$ by $B_\infty = \varinjlim B_n$. In case $K = \mathbb{C}$, B_∞ embeds into $\mathcal{O}(E,C)$ by the above corollary, and we define $\mathcal{B}_0$ to be the completion of B_∞ inside $\mathcal{O}(E,C)$.

While it might not be apparent at first, B_∞ and $\mathcal{B}_0$ are well known algebras.

Proposition 3.3.16. *Denote by τ the canonical semi-saturated partial representation*

$$\mathbb{F} \to L_K^{ab}(E,C) \quad \text{or} \quad \mathbb{F} \to \mathcal{O}(E,C),$$

and write $\varepsilon(s) = \tau(s)\tau(s)^* = p(\underline{s})$ for $s \in \mathbb{F}$. Then B_∞ is the $*$-subalgebra of $L_K^{ab}(E,C)$ generated by the $\varepsilon(s)$'s, and $\mathcal{B}_0$ is the C^*-subalgebra of $\mathcal{O}(E,C)$ generated by the $\varepsilon(s)$'s.

Proof. Note that $\varepsilon(s)$ is indeed an element of B_∞ by Lemma 3.3.10a. We should to check that each $v \in V_n$ is a sum of products of elements of the form $\Phi_n(p(s))$ for $s \in U$. In case $n = 0$ this is clear by (SCK2), and for $n = 1$ it follows from (SCK1). Taking $v \in V_n$ for $n \geq 2$, we have $v = v(x_1, \ldots, x_{k_u})$ for some $u \in V_{n-2}$. Arguing as in the proof of Lemma 3.3.10b,c, the claim follows for v by an analogue of equation (3.5). □

This gives a description of the space $\Omega(E,C)$ in terms of graph-theoretic data.

Proposition 3.3.17. *Defining maps*

$$\nu_n = \mathrm{Id} \sqcup r_{n+1} \colon E_n^0 = V_n \sqcup V_{n+1} \to V_n \sqcup V_{n-1} = E_{n-1}^0,$$

there is a homeomorphism $\Omega(E,C) \cong \varprojlim(E_n^0, \nu_n)$. In particular, there is a basis of clopen subsets $\{\Omega(E,C)_v \mid v \in F_\infty^0\}$, such that any clopen subset of $\Omega(E,C)$ is a finite disjoint union of such basis elements.

Proof. Taking $K = \mathbb{C}$, we may regard B_n as the C^*-algebra $C(E_n^0)$ of continuous functions on the finite set E_n^0, equipped with the discrete topology. Then by Proposition 3.3.16, we have

$$C(\Omega(E,C)) \cong \varinjlim(C(E_n^0), \phi_n),$$

and since ϕ_n is induced from ν_{n+1}, $\Omega(E,C)$ is homeomorphic to $\varprojlim(E_n^0, \nu_n)$. By definition of the limit topology, any vertex $v \in F_\infty^0$ corresponds in a canonical way to a clopen set $\Omega(E,C)_v \subset \Omega(E,C)$, and $\{\Omega(E,C)_v \mid v \in F_\infty^0\}$ forms a basis of the topology. Furthermore, by compactness any clopen subset of $\Omega(E,C)$ is a finite union of basis elements, and clearly they can be chosen as disjoint. □

Chapter 4

The type semigroup

In this chapter we shall apply the previous results in a somewhat surprising way, enabling us to answer a question raised by several authors. Fix a set X and a collection $\mathbb{D}$ of subsets of X, which contains the empty set and is closed under finite unions and finite intersections. The elements of $\mathbb{D}$ will be referred to as *admissible*. Then we shall consider an *admissible* partial action $\theta\colon G \curvearrowright X$, i.e. a partial action with $X_s \in \mathbb{D}$ and $\theta_s(A) \in \mathbb{D}$ for all $s \in G$ and $A \subset X_{s^{-1}}$ such that $A \in \mathbb{D}$.

Definition 4.1.1. Two subsets $A, B \subset X$ are called *equidecomposable*, if there are disjoint admissible unions

$$A = \bigsqcup_{i=1}^{n} A_i \quad \text{and} \quad B = \bigsqcup_{i=1}^{n} B_i$$

along with group elements $s_1, \ldots, s_n \in G$, such that $A_i \subset X_{s_i^{-1}}$ and $B_i = \theta_{s_i}(A_i)$ for each i. That is, if A can be made into B by decomposing it into admissible pieces and applying the partial action on these pieces. A subset $E \subset X$ is then called *paradoxical*, if it has disjoint admissible subsets $A, B \subset E$ that are both equidecomposable with E, i.e. if E can be decomposed into admissible pieces from which one can obtain two copies of E by applying the partial action.

We can investigate these concepts in a systematic way by forming the so-called *type semigroup*.

Definition 4.1.2. We define the (relative) type semigroup $S(X, G, \mathbb{D})$ as the set

$$\left\{ \bigcup_{i=1}^{n} A_i \times \{i\} \mid A_i \in \mathbb{D}, n \in \mathbb{Z}_+ \right\}$$

modulo the following equivalence relation: We write $A \sim_S B$ if there are $l \in \mathbb{Z}_+$, $C_k \in \mathbb{D}$, $s_k \in G$ and (not necessarily distinct) $n_k, m_k \in \mathbb{N}$ such that $C_k \subset X_{s_k^{-1}}$ for all $k = 1, \ldots, l$, and

$$A = \bigsqcup_{k=1}^{l} C_k \times \{n_k\}, \quad B = \bigsqcup_{k=1}^{l} \theta_{s_k}(C_k) \times \{m_k\}.$$

Before we move on, we should check that $\sim_S$ is indeed an equivalence relation.

Lemma 4.1.3. *The relation $\sim_S$ is indeed an equivalence relation, and $S(X, G, \mathbb{D})$ forms an abelian conical refinement monoid with neutral element $[\emptyset]$, when equipped with the binary relation*

$$\left[\bigcup_{i=1}^{n} A_i \times \{i\}\right] + \left[\bigcup_{j=1}^{m} B_j \times \{j\}\right] = \left[\left(\bigcup_{i=1}^{n} A_i \times \{i\}\right) \cup \left(\bigcup_{j=1}^{m} B_j \times \{n+j\}\right)\right].$$

Proof. Reflexivity and symmetry of $\sim_S$ are trivial. For transitivity we assume that $A \sim_S B$ and $B \sim_S C$, i.e.

$$A = \bigsqcup_{k=1}^{l} C_k \times \{n_k\} \quad , \quad C = \bigsqcup_{p=1}^{q} \theta_{t_p}(D_p) \times \{m'_p\}$$

and

$$B = \bigsqcup_{k=1}^{l} \theta_{s_k}(C_k) \times \{m_k\} = \bigsqcup_{p=1}^{q} D_p \times \{n'_p\}$$

for $l, p \in \mathbb{Z}_+$, $C_k \subset X_{s_k^{-1}}$, $D_p \subset X_{t_p^{-1}}$ and $n_k, m_k, n'_p, m'_p \in \mathbb{Z}_+$. Define

$$E_{k,p} = \theta_{s_k^{-1}}(\theta_{s_k}(C_k) \cap D_p) \in \mathbb{D}$$

for $k = 1, \ldots, l$ and $p = 1, \ldots, q$. Then we have

$$\bigsqcup_{p=1}^{q} E_{k,p} = \theta_{s_k^{-1}}\left(\bigsqcup_{p=1}^{q} \theta_{s_k}(C_k) \cap D_p\right) = C_k$$

and

$$\bigsqcup_{k=1}^{l} \theta_{t_p s_k}(E_{k,p}) = \theta_{t_p}\left(\bigsqcup_{k=1}^{l} \theta_{s_k}(C_k) \cap D_p\right) = \theta_{t_p}(D_p),$$

hence

$$A = \bigsqcup_{k=1}^{l} \bigsqcup_{p=1}^{q} E_{k,p} \times \{n_k\} \quad \text{and} \quad C = \bigsqcup_{k=1}^{l} \bigsqcup_{p=1}^{q} \theta_{t_p s_k}(E_{k,p}) \times \{m'_p\},$$

from which we conclude that $A \sim_S C$. The addition is easily seen to be well-defined, abelian with $[\emptyset]$ as a neutral element, and it is evident that this makes $S(X, G, \mathbb{D})$ into a conical abelian monoid. The refinement property is less obvious, so take $A = \bigcup_{i=1}^{n} A_i \times \{i\}$, $B = \bigcup_{j=1}^{m} B_j \times \{j\}$, $A' = \bigcup_{i=1}^{n'} A'_i \times \{i\}$ and $B' = \bigcup_{j=1}^{m'} B'_j \times \{j\}$, and assume that

$$\left[\left(\bigcup_{i=1}^{n} A_i \times \{i\}\right) \cup \left(\bigcup_{j=1}^{m} B_j \times \{n+j\}\right)\right] = \left[\left(\bigcup_{i=1}^{n'} A'_i \times \{i\}\right) \cup \left(\bigcup_{j=1}^{m'} B'_j \times \{n'+j\}\right)\right].$$

Then there are $l \in \mathbb{Z}$, $C_k \in \mathbb{D}$, $s_k \in G$ and $n_k, m_k \in \mathbb{N}$ such that $C_k \subset X_{s_k^{-1}}$ for all $k = 1, \ldots, l$,

$$\left(\bigcup_{i=1}^{n} A_i \times \{i\}\right) \cup \left(\bigcup_{j=1}^{m} B_j \times \{n+j\}\right) = \bigsqcup_{k=1}^{l} C_k \times \{n_k\}$$

and

$$\left(\bigcup_{i=1}^{n'} A'_i \times \{i\}\right) \cup \left(\bigcup_{j=1}^{m'} B'_j \times \{n'+j\}\right) = \bigsqcup_{k=1}^{l} \theta_{s_k}(C_k) \times \{m_k\}.$$

Now define

$$A^1 = \bigsqcup_{\substack{k=1 \\ 1 \le n_k \le n \\ 1 \le m_k \le n'}}^{l} C_k \times \{n_k\} \qquad \text{and} \qquad A^2 = \bigsqcup_{\substack{k=1 \\ 1 \le n_k \le n \\ n'+1 \le m_k \le n'+m'}}^{l} C_k \times \{n_k\}$$

along with

$$B^1 = \bigsqcup_{\substack{k=1 \\ n+1 \leq n_k \leq n+m \\ 1 \leq m_k \leq n'}}^{l} C_k \times \{n_k\} \qquad \text{and} \qquad B^2 = \bigsqcup_{\substack{k=1 \\ n+1 \leq n_k \leq n+m \\ n'+1 \leq m_k \leq m'+n'}}^{l} C_k \times \{n_k\}.$$

Then clearly $[A] = [A^1] + [A^2]$, $[B] = [B^1] + [B^2]$, $[A'] = [A^1] + [B^1]$ and $[B'] = [A^2] + [B^2]$, hence $S(X, G, \mathbb{D})$ is indeed a refinement monoid. $\square$

We are now able to restate the definition of paradoxicality.

Definition 4.1.4. For an admissible subset $E \subset X$, we write $[E] := [E \times \{1\}]$. Then E is called *paradoxical* if $[E]$ is properly infinite in $S(X, G, \mathbb{D})$, i.e. if $2[E] \leq [E]$ with respect to the algebraic preorder.

Remark 4.1.5. Note that if the collection $\mathbb{D}$ is in fact an algebra, i.e. if $X \in \mathbb{D}$ and $\mathbb{D}$ is closed under complements, then the monoid homomorphisms $\mu\colon S(X, G, \mathbb{D}) \to [0, \infty]$ exactly correspond to the finitely additive θ-invariant measures $\mathbb{D} \to [0, \infty]$.

Several authors (see for instance [13, Theorem 5.4 and page 13] and [11, Question 3.10]) have raised a particular question regarding the type semigroup.

Question 4.1.6. Suppose that $\theta\colon G \curvearrowright X$ is a global action on a Cantor space and denote by $\mathbb{K}$ the collection of compact-open subsets of X. Is the type semigroup $S(X, G, \mathbb{K})$ *almost unperforated*, i.e. does it satisfy

$$\Big(\exists n \in \mathbb{N}\colon (n+1) \cdot a \leq n \cdot b\Big) \quad \Rightarrow \quad a \leq b$$

for any $a, b \in S(X, G, \mathbb{K})$ with respect to the algebraic preorder?

Example 4.1.7. There are two canonical classes of examples of abelian monoids lacking almost unperforation with respect to the algebraic preorder. First, let $2 \leq m < n$ and consider the monoid

$$M = \langle a \mid m \cdot a = n \cdot a \rangle.$$

Then $M = \{0, a, \dots, (n-1) \cdot a\}$, so we clearly have $2a \not\leq a$. On the other hand, it is easily seen that $(m+1) \cdot (2a) \leq m \cdot a$, hence M does indeed not have almost unperforation.

Secondly, consider positive integers $m < n$ such that $m \nmid n$, and let M denote the submonoid of $\mathbb{Z}_+$ generated by m and n. Clearly $m \not\leq n$ in M for otherwise $n - m \in M$, and then $n - m$ would be a multiple of m, hence so would n. Nonetheless we have $(m+1) \cdot m \leq m \cdot n$, so once again M is not almost unperforated.

While almost unperforation may seem like a fairly peculiar property at first sight, it has some important consequences. Here follows one of them.

Theorem 4.1.8. *Suppose that M is an almost unperforated abelian monoid, or that it satisfies the following two conditions:*

- *Antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$.*
- *n-cancellation: If $na = nb$, then $a = b$ for all $n \geq 2$.*

Then the following are equivalent for all $a \in M$:

- *a is not properly infinite, i.e. $2a \not\leq a$.*
- *There is a monoid homomorphism $\mu\colon M \to [0,\infty]$ satisfying $\mu(a) = 1$.*

Proof. See [13, Theorem 5.4 (v) $\Rightarrow$ (i)] and [14, Theorem 9.1] $\square$

For global actions $\theta\colon G \curvearrowright X$ on a set with $\mathbb{D} = \mathcal{P}(X)$, it turns out that the type semigroup $S(X, G, \mathcal{P}(X))$ has antisymmetry and n-cancellation. From Remark 4.1.5 and Theorem 4.1.8 we thus obtain the famous result known as *Tarski's theorem*

Theorem 4.1.9. *Given a global action $\theta\colon G \curvearrowright X$ on a set X. A subset $E \subset X$ is non-paradoxical (with respect to $\mathcal{P}(X)$) if and only if there exists a finitely additive θ-invariant measure $\mu\colon \mathcal{P}(X) \to [0,\infty]$ such that $\mu(E) = 1$.*

Proof. For the proofs of antisymmetry and n-cancellation, see [14, Theorem 3.5, Theorem 8.7]. $\square$

Hopefully the above discussion has motivated the study of cancellation and order-cancellation properties of the type semigroup. In the following we will basically show that arbitrary cancellation properties can fail in $S(X, G, \mathbb{D})$, using the machinery we have developed over the past chapters. We shall also consider a particular partial action $\theta\colon \mathbb{F} \curvearrowright \Omega(E, C)$ for which the analogue of Tarski's theorem fails.

The first step is to establish a homomorphism between the type semigroup for an action and the associated crossed product.

Lemma 4.1.10. *Given a compact partial action $\theta\colon G \curvearrowright X$ on a locally compact Hausdorff space, let $\mathbb{K}(X)$ denote the algebra of compact-open subsets of X. Then there is a canonical monoid homomorphism $\Phi\colon S(X, G, \mathbb{K}(X)) \to \mathcal{V}(C_{c,K}(X) \rtimes_{\theta^*} G)$ given by*

$$\Phi([A]) = [1_A \delta_1]$$

for every $A \in \mathbb{K}(X)$.

Proof. We only need to check that Φ is indeed well-defined, i.e. we should check that

$$1_A \delta_1 = 1_{\theta_s(A)} \delta_1$$

whenever $A \in \mathbb{K}(X)$ with $A \subset X_{s^{-1}}$. To this end, let $x = 1_A \delta_{s^{-1}}$. Then

$$xx^* = 1_A \delta_1 \quad \text{and} \quad x^*x = 1_{\theta_s(A)} \delta_1$$

as promised. $\square$

In our particular case, we can actually prove that the above map is an isomorphism.

Theorem 4.1.11. *Let (E, C) denote a finite bipartite finitely separated graph, denote by θ the canonical partial (E, C)-action on $\Omega(E, C)$ and set $\mathbb{K} = \mathbb{K}(\Omega(E, C))$. Then the map*

$$\Phi\colon S(\Omega(E, C), \mathbb{F}, \mathbb{K}) \to \mathcal{V}(C_K(\Omega(E, C)) \rtimes_{\theta^*} \mathbb{F})$$

is an isomorphism.

Proof. We have a series of isomorphisms

$$\begin{aligned}\mathcal{V}(C_K(\Omega(E,C)) \rtimes_{\theta^*} \mathbb{F}) \cong \mathcal{V}(L_K^{\mathrm{ab}}(E,C)) &\cong \varinjlim_n \mathcal{V}(L_K(E_n, C^n)) \\ &\cong \varinjlim_n M(E_n, C^n) \cong M(F^\infty, D_\infty),\end{aligned}$$

and composing Φ with the above composition, we obtain a monoid homomorphism

$$\Phi' \colon S(\Omega(E,C), \mathbb{F}, \mathbb{K}) \to M(F_\infty, D^\infty)$$

given by $\Phi'([\Omega(E,C)_v]) = v$ for $v \in F_\infty^0$. Since any clopen subset of $\Omega(E,C)$ is a finite disjoint union of basis elements $\Omega(E,C)_v$, this completely describes Φ'. Rather than directly proving that Φ is an isomorphism, we shall prove that Φ' is an isomorphism, and we do this by building an inverse. To this end, define a monoid homomorphism

$$\Psi \colon M(F_\infty, D^\infty) \to S(\Omega(E,C), \mathbb{F}, \mathbb{K})$$

by $\Psi(v) = [\Omega(E,C)_v]$. In order for Ψ to be well-defined, we should check that it respects the relation $v = \mathbf{s}(X)$ for each $X \in C_v$ and $v \in V_n$, i.e. that $[\Omega(E,C)_v] = \sum_{e \in X} [\Omega(E,C)_{s(e)}]$. Assuming n is even, for each $e \in X$ there are $s_1, \ldots, s_k \in U_n$ and $f \in E^1$ such that

$$e = \Phi_n(p(s_1) \cdots p(s_k) f)$$

due to Lemma 3.3.10b. Then

$$ee^* = \Phi_n(p(f) p(s_1) \cdots p(s_k))$$

since $\Phi_n(p(s)) \in B_n$ for all $s \in U_n$ and

$$e^*e = \Phi_n(f)^* \left(\Phi_n(p(f) p(s_1) \cdots p(s_k)) \right) \Phi_n(f) = \Phi_n(f)^* (ee^*) \Phi_n(f),$$

hence

$$\underline{e^*e} = \underline{f}^* \underline{ee^*}\, \underline{f} = \alpha_{f^{-1}}(\underline{ee^*}) \in L_K^{\mathrm{ab}}(E,C).$$

We conclude that $[\Omega(E,C)_{s(e)}] = [\Omega(E,C)_e]$, where $\Omega(E,C)_e$ denotes the clopen subset of $\Omega(E,C)$ corresponding to $p(\underline{e})$. Now, since $v = \sum_{e \in X} ee^*$ in $L_K(E_n, C^n)$ we have

$$[\Omega(E,C)_v] = \sum_{e \in X} [\Omega(E,C)_e] = \sum_{e \in X} [\Omega(E,C)_{s(e)}]$$

as desired. Clearly, one can argue completely similarly in the odd case using Lemma 3.3.10c. Since Ψ is an inverse of Φ', it follows that Φ is an isomorphism. □

Remark 4.1.12. Note that Theorem 4.1.11 provides another proof of the fact that the monoids

$$M(F_\infty, D^\infty) \cong \varinjlim_n M(E_n, C^n) \cong \mathcal{V}(L_K^{\mathrm{ab}}(E,C)) \cong \mathcal{V}(C_K(\Omega(E,C)) \rtimes_{\theta^*} \mathbb{F})$$

have refinement, since the type semigroup always has refinement. As this has been our only application of Lemma 3.1.3, it doesn't really matter that we skipped the proof of it.

Now we have all the ingredients needed to prove the main theorem in the case of a partial action.

Theorem 4.1.13. *Let M denote any finitely generated, conical abelian monoid. Then there exists a partial action $\mathbb{F} \curvearrowright X$ of a finitely generated free group on a zero-dimensional, metrizable compact space and a refinement*

$$\iota\colon M \to S(X, \mathbb{F}, \mathbb{K}(X)).$$

Proof. By Proposition 3.2.3, there is a finite bipartite graph (E, C) such that $M \cong M(E, C)$, and the limit morphism $M(E, C) \to \varinjlim_n M(E_n, C^n)$ is a refinement by Lemma 3.2.6. The result then follows, since we have isomorphisms

$$\varinjlim_n M(E_n, C^n) \cong \mathcal{V}(L_K^{\mathrm{ab}}(E, C)) \cong \mathcal{V}(C_K(\Omega(E, C)) \rtimes_\alpha \mathbb{F}) \cong S(\Omega(E, C), \mathbb{F}, \mathbb{K}(\Omega(E, C))).$$

□

In particular we obtain a (strong) falsification of Tarski's theorem for partial actions.

Corollary 4.1.14. *There exists a partial action $\theta\colon \mathbb{F} \curvearrowright Z$ of a finitely generated free group on a Cantor space and a non-paradoxical (w.r.t. $\mathbb{K}$) clopen subset $A \subset Z$, such that $\mu(A) = \infty$ for every finitely additive, θ-invariant and non-zero measure $\mu\colon \mathbb{K} \to [0, \infty]$.*

Proof. Let $2 \leq m < n$ and consider the graph $(E, C) := (E(m, n), C(m, n))$ defined as follows: Set

$$E^0 = \{u, v\}, \quad E^1 = \{e_i, f_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}, \quad r(e) = v, \quad \text{and} \quad s(e) = u$$

for all $e \in E^1$ along with $C_v = \{X, Y\}$, where $X = \{e_1, \dots, e_m\}$ and $Y = \{f_1, \dots, f_n\}$. Or represented graphically, (E, C) is the following graph

$$\begin{tikzcd} u \arrow[r, bend left, "m"] \arrow[r, bend right, "n"'] & v \end{tikzcd}.$$

Then define $Z = \Omega(E, C)$ and let θ denote the canonical action $\mathbb{F} \curvearrowright Z$. From Theorem 4.1.13 we have a unitary embedding

$$\iota\colon M(E, C) = \langle u \mid m \cdot u = n \cdot u \rangle \to S(Z, \mathbb{F}, \mathbb{K}),$$

and we define $A := \Omega(E, C)_u$. Then $2u \not\leq u$ and $\iota(u) = [A]$, hence A is not paradoxical. Note that $[A]$ is an *order unit* of $S(Z, \mathbb{F}, \mathbb{K})$, meaning that for any $a \in S(Z, \mathbb{F}, \mathbb{K})$ there is some $k \in \mathbb{N}$ such that $a \leq k \cdot [A]$, since $[\Omega(E, C)_v] = m[A]$. Thus, if μ is a non-zero measure we must have $\mu(A) > 0$. On the other hand, since $m \cdot [A] = n \cdot [A]$ we also have

$$m \cdot \mu(A) = n \cdot \mu(A),$$

and so we conclude that $\mu(A) = \infty$. It will follow from Proposition 5.3.6 that $\Omega(E, C)$ is a Cantor space. □

In the following we shall extend the above results to the case of a global action. Suppose that $\theta\colon G \curvearrowright X$ is a partial action on a set X and that $\beta\colon G \curvearrowright Y$ is a globalization, i.e. θ is the restriction of β to X and $Y = \bigcup_{s \in G} \beta_s(X)$. Consider an algebra $\mathbb{D}$ of admissible subsets of Y, such that β is an admissible action and X is an admissible subset. Then we define an algebra of admissible subsets of X by

$$\mathbb{D}|_X = \{A \cap X \mid A \in \mathbb{D}\}.$$

Note that θ becomes admissible, so we can consider the relationship between the associated type semigroups $S(X, G, \mathbb{D}|_X)$ and $S(Y, G, \mathbb{D})$. In general, the former embeds into the latter:

Proposition 4.1.15. *The canonical homomorphism $S(X,G,\mathbb{D}|_X) \to S(Y,G,\mathbb{D})$ given by*

$$\left[\bigcup_{i=1}^{n} A_i \times \{i\}\right]_X \mapsto \left[\bigcup_{i=1}^{n} A_i \times \{i\}\right]_Y$$

is injective.

Proof. The map is clearly well-defined since θ is the restriction of β, and any admissible subset of X is admissible in Y as well. For injectivity assume that

$$\left[\bigcup_{i=1}^{n} A_i \times \{i\}\right]_Y = \left[\bigcup_{j=1}^{m} B_j \times \{j\}\right]_Y.$$

By definition there is $l \in \mathbb{Z}_+$, $C_k \in \mathbb{D}$, $s_k \in G$ and $n_k, m_k \in \mathbb{N}$ for all $k = 1, \dots, l$ such that

$$\bigcup_{i=1}^{n} A_i \times \{i\} = \bigsqcup_{k=1}^{l} C_k \times \{n_k\} \quad \text{and} \quad \bigcup_{j=1}^{m} B_j \times \{j\} = \bigsqcup_{k=1}^{l} \beta_{s_k}(C_k) \times \{m_k\}.$$

Since $A_i \subset X$ for each i, we must have $C_k \subset X$ for all k, hence $C_k \in \mathbb{D}|_X$. Likewise we have

$$\beta_{s_k}(C_k) \subset B_{m_k} \cap \beta_{s_k}(X) \subset X \cap \beta_{s_k}(X) = X_{s_k},$$

hence $C_k \subset X_{s_k^{-1}}$. It follows that

$$\bigcup_{j=1}^{m} B_j \times \{j\} = \bigsqcup_{k=1}^{l} \theta_{s_k}(C_k) \times \{m_k\},$$

and so

$$\left[\bigcup_{i=1}^{n} A_i \times \{i\}\right]_X = \left[\bigcup_{j=1}^{m} B_j \times \{j\}\right]_X$$

as required. □

With very mild additional assumptions we also gain surjectivity in the case $\mathbb{D} = \mathbb{K}$.

Proposition 4.1.16. *Assume that Y is a Hausdorff space, and X is a compact-open subset. Then $\mathbb{K}(Y)|_X = \mathbb{K}(X)$, and the canonical map $S(X,G,\mathbb{K}(X)) \to S(Y,G,\mathbb{K}(Y))$ is an isomorphism.*

Proof. Since Y is assumed to be Hausdorff, the intersection of two compact subspaces is compact, and therefore $\mathbb{K}(X) = \mathbb{K}(Y)|_X$. We have already seen that the canonical homomorphism is injective, so we must only check surjectivity. Now, the classes $[B]_Y$ for $B \in \mathbb{K}(Y)$ generate $S(Y,G,\mathbb{K}(Y))$, so it suffices to prove that there are $A_1, \dots, A_n \in \mathbb{K}(X)$ such that

$$\sum_{i=1}^{n} [A_i]_Y = [B]_Y.$$

By assumption $Y = \bigcup_{s \in G} \beta_s(X)$, so there are $s_1, \dots, s_n \in G$ such that $B \subset \bigcup_{i=1}^{n} \beta_{s_i}(X)$ due to compactness of B. Now define admissible subsets

$$\begin{aligned}
B_1 &= B \cap \beta_{s_1}(X) \\
B_2 &= (B \cap \beta_{s_2}(X)) \setminus B_1 \\
B_3 &= (B \cap \beta_{s_3}(X)) \setminus (B_1 \cup B_2) \\
&\vdots \\
B_n &= (B \cap \beta_{s_n}(X)) \setminus (B_1 \cup \ldots \cup B_{n-1})
\end{aligned}$$

of Y such that $B = \bigsqcup_{i=1}^{n} B_i$. By construction $B_i \subset \beta_{s_i}(X)$, hence $A_i = \beta_{s_i^{-1}}(B_i)$ defines an admissible subset of X. Moreover, the canonical homomorphism maps $[A_i]_X$ to $[B_i]_Y$, so it maps $\sum_{i=1}^{n}[A_i]_X$ to $[B]_Y$. $\square$

Now we only need a minor ingredient in order to obtain our main result on the type semigroup for global actions.

Lemma 4.1.17. *If $\theta\colon G \curvearrowright X$ is a global action on a locally compact Hausdorff space, then it extends to a global action $\theta^\bullet\colon G \curvearrowright X^\bullet$ on the one-point compactification $X^\bullet$ of X. Moreover, there is a canonical order embedding*

$$S(X, G, \mathbb{K}(X)) \to S(X^\bullet, G, \mathbb{K}(X^\bullet)).$$

Proof. Write $X^\bullet = X \cup \{\bullet\}$ and extend each θ_s by $\theta_s^\bullet(\bullet) = \bullet$. Then θ_s is a homeomorphism of $X^\bullet$, and clearly the assignment $s \mapsto \theta_s^\bullet$ is an action. Arguing as in the proof of Proposition 4.1.15, there is a canonical injective homomorphism

$$S(X, G, \mathbb{K}(X)) \to S(X^\bullet, G, \mathbb{K}(X^\bullet)),$$

and it is straightforward to verify that it is an order embedding. $\square$

Theorem 4.1.18. *Let M denote any finitely generated, conical abelian monoid. Then there exists a global action $\mathbb{F} \curvearrowright X$ of a finitely generated free group on a zero-dimensional, metrizable compact space along with an order embedding $M \to S(X, \mathbb{F}, \mathbb{K}(X))$.*

Proof. From Theorem 4.1.13 there is a finite bipartite separated graph (E, C) and a unitary embedding $M \to S(\Omega(E, C), \mathbb{F}, \mathbb{K})$. Then $\Omega(E, C)^{\mathrm{e}}$ is a zero-dimensional locally compact Hausdorff space due to Remark 1.3.5 and Proposition 1.3.7, and by Proposition 4.1.16 there is an isomorphism

$$S(\Omega(E, C), \mathbb{F}, \mathbb{K}) \to S(\Omega(E, C)^{\mathrm{e}}, \mathbb{F}, \mathbb{K}).$$

Finally $S(\Omega(E, C)^{\mathrm{e}}, \mathbb{F}, \mathbb{K})$ order embeds into $S((\Omega(E, C)^{\mathrm{e}})^\bullet, \mathbb{F}, \mathbb{K})$ by Lemma 4.1.17, and

$$X = (\Omega(E, C)^{\mathrm{e}})^\bullet$$

is a zero-dimensional compact Hausdorff space. Such a space is always metrizable due to the Urysohn Metrization Lemma. $\square$

We can almost extend Corollary 4.1.14 to the case of a global action:

Corollary 4.1.19. *There exists a global action $\theta\colon \mathbb{F} \curvearrowright Z$ of a finitely generated free group on a Cantor space Z and a non-paradoxical (w.r.t. $\mathbb{K}$) clopen subset $A \subset Z$, such that $\mu(A) = \infty$ for every finitely additive θ-invariant measure $\mu\colon \mathbb{K} \to [0, \infty]$ satisfying $\mu(A) > 0$.*

Proof. Let $2 \le m < n$ and consider the graph $(E, C) := (E(m, n), C(m, n))$ as defined in Corollary 4.1.14. Then the statement holds for the action

$$(\theta^{\mathrm{e}})^\bullet\colon \mathbb{F} \curvearrowright (\Omega(E, C)^{\mathrm{e}})^\bullet$$

with $A = \Omega(E, C)_u$ precisely as in the proof of Corollary 4.1.9. Note that $\Omega(E, C)^{\mathrm{e}}$ contains no isolated points, since this is the case for $\Omega(E, C)$, and hence neither does $(\Omega(E, C)^{\mathrm{e}})^\bullet$. $\square$

Note that the above proof establishes a definite negative answer to Question 4.1.6. Before we close this chapter we also extend Theorem 4.1.11 to the global setting.

Corollary 4.1.20. *Denote by* (E, C) *a finite bipartite separated graph. Then the map*

$$\Phi\colon S(\Omega(E,C)^{\mathrm{e}}, \mathbb{F}, \mathbb{K}) \to \mathcal{V}(C_{c,K}(\Omega(E,C)^{\mathrm{e}}) \rtimes_{(\theta^{\mathrm{e}})^*} \mathbb{F})$$

is an isomorphism.

Proof. Write $X = \Omega(E,C)$ and $Y = \Omega(E,C)^{\mathrm{e}}$. We observed in Proposition 1.3.8 that the inclusion

$$C_K(X) \rtimes_{\theta^*} \mathbb{F} \hookrightarrow C_{c,K}(Y) \rtimes_{(\theta^{\mathrm{e}})^*} \mathbb{F}$$

maps onto a full corner of the codomain, hence the induced homomorphism of $\mathcal{V}$-monoids is an isomorphism. Noting that the diagram

$$\begin{tikzcd}
S(X, \mathbb{F}, \mathbb{K}(X)) \arrow[r, "\Phi_X"] \arrow[d] & \mathcal{V}(C_K(X) \rtimes_{\theta^*} \mathbb{F}) \arrow[d, "\mathcal{V}(\iota)"] \\
S(Y, \mathbb{F}, \mathbb{K}(Y)) \arrow[r, "\Phi_Y"'] & \mathcal{V}(C_{c,K}(Y) \rtimes_{(\theta^{\mathrm{e}})^*} \mathbb{F})
\end{tikzcd}$$

commutes, that the left map is an isomorphism by Theorem 4.1.16 and the upper map is an isomorphism by Proposition 4.1.11, it follows that Φ_Y is an isomorphism as well. $\square$

Chapter 5

Descriptions of $\mathbb{F} \curvearrowright \Omega(E, C)$

In this chapter, we will provide descriptions of the canonical partial (E,C)-action in terms of data from the graph (E,C). This will allow us to characterize the graphs, for which the action is topologically free. We shall continue to work under the assumption that (E,C) is a finite bipartite graph.

5.1 $\Omega(E,C)$ as a subspace of $2^{\mathbb{F}}$

As we have seen, $\mathcal{O}(E,C)$ is the universal C^*-algebra for semi-saturated partial representations satisfying the relations (PI1)-(PI4), see Proposition 2.2.9 and Corollary 2.3.7. These relations are expressed solely by the final projections of the partial representation, hence they can be translated into relations $\mathcal{R}$ on $C(X_{\mathbb{F}})$ using the machinery of Section 1.5. For this, we need the following minor lemma:

Lemma 5.1.1. *Assume that G has a length function $|\cdot|$, let $\sigma\colon G \to \mathcal{A}$ denote a partial representation and write $\varepsilon(s) = \sigma(s)\sigma(s)^*$. Then σ is semi-saturated if and only if $\varepsilon(st) \leq \varepsilon(s)$ for every $s, t \in G$ such that $|st| = |s| + |t|$.*

Proof. If $|st| = |s| + |t|$, then by the identity $\|xx^*\| = \|x\|^2$ and Proposition 1.2.12

$$\|\sigma(s)\sigma(t) - \sigma(st)\|^2 = \|\varepsilon(st) - \sigma(s)\varepsilon(t)\sigma(s^{-1})\| = \|\varepsilon(st) - \varepsilon(st)\varepsilon(s)\|,$$

hence σ is semi-saturated if and only if $\varepsilon(st) = \varepsilon(st)\varepsilon(s)$ for all such s, t. But this exactly means that $\varepsilon(st) \leq \varepsilon(s)$. $\square$

Apply the notation of Section 1.5. With the above lemma in mind, we can translate the defining relations of $\mathcal{O}(E,C)$ into the following set of relations on $C(X_{\mathbb{F}})$:

(F1) $g^1_{e,f} = 1_e 1_f - \delta_{e,f} 1_e$ for all $e, f \in X \in C$.

(F2) $g^2_{e,f} = 1_{e^{-1}} - 1_{f^{-1}}$ for all $e, f \in E^1$ such that $s(e) = s(f)$.

(F3) $g^3_{X,Y} = \sum_{e \in X} 1_e - \sum_{e \in Y} 1_e$ for all $X, Y \in C_v$ and $v \in E^{0,0}$.

(F4) $g^4 = -1 + \sum_{v \in E^{0,0}} \sum_{e \in X_v} 1_e + \sum_{v \in E^{0,1}} 1_{e_v^{-1}}$ for a choice $X_v \in C_v$ for each $v \in E^{0,0}$, and a choice $e_v \in s^{-1}(v)$ for $v \in E^{0,1}$.

(SS) $g^5_{s,t} = 1_s 1_{st} - 1_{st}$ for all $s, t \in \mathbb{F}$ such that $|st| = |s| + |t|$.

Denote by $\mathcal{R}$ the above set of functions and write $\Omega = \Omega_{\mathcal{R}}$. Then by Proposition 1.5.6 and Proposition 2.3.7, we have an isomorphism

$$\Psi\colon C(\Omega) \rtimes_{\alpha^u} \mathbb{F} \to C(\Omega(E,C)) \rtimes_\alpha \mathbb{F}$$

satisfying $\Psi(\tau(s)) = \sigma(s)$ for all $s \in \mathbb{F}$, with τ and σ being the canonical partial representations. Since $C(\Omega)$ is generated by the $\varepsilon_\tau(s)$'s and $C(\Omega(E,C))$ is generated by the $\varepsilon_\sigma(s)$'s, Ψ restricts to an isomorphism $C(\Omega) \to C(\Omega(E,C))$. Moreover

$$\Psi(\alpha_s^u(g)\delta_1) = \Psi(\tau(s)\cdot(g\delta_1)\cdot\tau(s)) = \sigma(s)\Psi(g\delta_1)\sigma(s)^* = \alpha_s(\Psi(g\delta_1))$$

for all $s \in \mathbb{F}$, so Ψ is an equivalence of the partial actions.

For the rest of this chapter, we will simply regard the partial action $\mathbb{F} \curvearrowright \Omega(E,C)$ as the one defined just above. In the following we shall give an explicit description of the points of $\Omega(E,C)$ in this picture. We will need the following definitions.

Definition 5.1.2. A subset $\omega \subset \mathbb{F}$ is called *convex* if

$$|t^{-1}s| = |t^{-1}r| + |r^{-1}s|$$

for $s,t \in \omega$ implies $r \in \omega$. Intuitively, this means that ω contains the shortest path in the Cayley graph of $\mathbb{F}$ connecting any two points $s,t \in \omega$.

Definition 5.1.3. For $\omega \subset \mathbb{F}$ and $s \in \omega$, the *local configuration* ω_s of ω at s is the set of elements $t \in E^1 \cup (E^1)^{-1}$, such that $st \in \omega$.

Lemma 5.1.4. $\omega \in \mathcal{P}(\mathbb{F})$ *is convex if and only if it satisfies (SS).*

Proof. Let $r,s \in \mathbb{F}$ and assume that $|sr| = |s| + |r|$. Then for $t \in \omega$ we have

$$\begin{aligned} g^5_{s,r}(t^{-1}\omega) &= [sr \in t^{-1}\omega][s \in t^{-1}\omega] - [sr \in t^{-1}\omega] = [tsr \in \omega]([ts \in \omega] - 1) \\ &= 1 - [tsr \in \omega](1 - [ts \in \omega] - 1) = [tsr \in \omega \Rightarrow ts \in \omega] - 1, \end{aligned}$$

hence ω satisfies (SS), if and only if $tsr \in \omega$ implies $ts \in \omega$ for all $t \in \omega$ and all s,r with $|sr| = |s| + |r|$. We claim that this is equivalent to being convex. Assuming first that ω is convex, take $t \in \omega$ and $s,r \in \mathbb{F}$ such that $|sr| = |s| + |r|$ and $tsr \in \omega$. Now put $r' = ts$, $s' = tsr$ and $t' = t$. Then $s',t' \in \omega$ and

$$|t'^{-1}s'| = |sr| = |s| + |r| = |t'^{-1}r'| + |r'^{-1}s'|,$$

hence $ts = r' \in \omega$ as required. For the converse implication, assume that $s,t \in \omega$ and $|t^{-1}s| = |t^{-1}r| + |r^{-1}s|$ for some $r \in \mathbb{F}$. Then put $r' = r^{-1}s$, $s' = t^{-1}r$ and $t' = t$ and note that

$$|s'r'| = |t^{-1}s| = |t^{-1}r| + |r^{-1}s| = |s'| + |r'|.$$

Furthermore, $t's'r' = s \in \omega$ and $t' \in \omega$, hence $r = t's' \in \omega$. $\square$

Now we have the following description of the configurations.

Proposition 5.1.5. *An element $\omega \in \mathcal{P}(\mathbb{F})$ belongs to $\Omega(E, C)$ if and only if*

(a) $1 \in \omega$.

(b) ω *is convex.*

(c) For every $s \in \omega$, there is some $v \in E^0$ such that one of the following holds:

(c1) If $v \in E^{0,1}$, then $\omega_s = s^{-1}(v)^{-1}$.

(c2) If $v \in E^{0,0}$, there is a unique element $e_X \in X$ for each $X \in C_v$, such that

$$\omega_s = \{e_X \mid X \in C_v\}.$$

Proof. In view of Lemma 5.1.4, it suffices to prove that $\omega \in \mathcal{P}(\mathbb{F})$ satisfies (c) if and only if it satisfies (F1)-(F4). Due to Proposition 1.5.4, (F1)-(F4) amounts precisely to the following for each $s \in \omega$:

- $|X \cap \omega_s| \leq 1$ for all $X \in C$.
- $s^{-1}(v)^{-1} \cap \omega_s = s^{-1}(v)^{-1}$ or $s^{-1}(v)^{-1} \cap \omega_s = \emptyset$ for all $v \in E^{0,1}$.
- $|X \cap \omega_s| = |Y \cap \omega_s|$ for all $v \in E^{0,0}$ and $X, Y \in C_v$.
- For some choice $X_v \in C_v$ with $v \in E^{0,0}$ and $e_v \in s^{-1}(v)$ with $v \in E^{0,1}$, we have

$$\left|\omega_s \cap \left(\{e_v^{-1} \mid v \in E^{0,1}\} \cup \bigsqcup_{v \in E^{0,0}} X_v\right)\right| = 1.$$

For instance, $\omega \in \mathcal{P}(\mathbb{F})$ satisfies (F1) if and only if

$$[se \in \omega] \cdot [sf \in \omega] = 1_e(s^{-1}\omega)1_f(s^{-1}\omega) = \delta_{e,f}1_e(s^{-1}\omega) = \delta_{e,f} \cdot [se \in \omega]$$

for all $X \in C$, $e, f \in X$ and $s \in \omega$, which holds if and only if $|X \cap \omega_s| \leq 1$ for all $s \in \omega$. Clearly the above conditions correspond to (c). □

Remark 5.1.6. Note that $\omega \in \Omega(E, C)_v$ with $v \in E^{0,0}$, if and only if ω satisfies (c2) at $s = 1$, and $\omega \in \Omega(E, C)_v$ with $v \in E^{0,1}$, if and only if ω satisfies (c1) at $s = 1$.

5.2 E-functions

While the results of the previous section simplifies the description of the partial action greatly, it does involve a rather intricate presentation of the configurations. In this section, we will give a description of the configurations of type (c1) at 1 in terms of so-called E-functions that provide a better understanding of the zig-zag nature. As we have disjoint unions

$$\Omega(E,C)_v = \bigsqcup_{e \in X} \Omega(E,C)_e$$

for all $v \in E^{0,0}$ and $X \in C_v$ along with homeomorphisms $\theta_e \colon \Omega(E,C)_{s(e)} \to \Omega(E,C)_e$ for all $e \in E^1$, this provides a description of the entire space. We shall work under the following assumption in order to avoid all sorts of trivialities.

Assumption 5.2.1. (E,C) is a finite bipartite separated graph satisfying $|C_v| \geq 2$ for each $v \in E^{0,0}$.

This is justified by the following observation.

Proposition 5.2.2. *If (E,C) is a finite bipartite graph, then there is a finite bipartite graph $(\tilde{E},\tilde{C})$ with $|\tilde{C}_v| \geq 2$ for all $v \in \tilde{E}^{0,0}$ and a number $n \geq 0$ such that*

- *$L_K(E,C)$ is Morita equivalent to $L_K(\tilde{E},\tilde{C}) \oplus K^n$,*
- *$L_K^{ab}(E,C)$ is Morita equivalent to $L_K^{ab}(\tilde{E},\tilde{C}) \oplus K^n$,*
- *$C^*(E,C)$ is Morita equivalent to $C^*(\tilde{E},\tilde{C}) \oplus \mathbb{C}^n$,*
- *$\mathcal{O}(E,C)$ is Morita equivalent to $\mathcal{O}(\tilde{E},\tilde{C}) \oplus \mathbb{C}^n$.*

Proof. Let $V = \{v \in E^0 \mid |C_v| = 1\}$ and define a complete subobject of (E,C) with

$$\bar{E}^0 = E^0 \setminus V \quad \text{and} \quad \bar{E}^1 = E^1 \setminus r^{-1}(V).$$

Then we obtain the requested graph as a complete subobject of $(\bar{E},\bar{C})$ by setting

$$\tilde{E}^{0,0} = \bar{E}^{0,0}, \quad \tilde{E}^{0,1} = \bar{s}(\bar{E}^1) \quad \text{and} \quad \tilde{E}^1 = \bar{E}^1.$$

Note that $q = \sum_{v \in \bar{E}^0} v$ defines a full projection in $L_K(E,C)$. Indeed, if $I \lhd L_K(E,C)$ contains q, then $e = eq \in I$ for all $e \in E^1$ and thus $v = \sum_{e \in X} ee^* \in I$ for all $v \in V$ and $X \in C_v$. Finally we note that

$$qL_K(E,C)q = L_K(\bar{E},\bar{C}) = L_K(\tilde{E},\tilde{C}) \oplus \langle v \mid v \in \bar{E}^0 \setminus s(\bar{E}^1)\rangle \cong L_K(\tilde{E},\tilde{C}) \oplus K^n$$

with $n = |\bar{E}^0| - |s(\bar{E}^1)|$. Denoting the J-ideals of $(\bar{E},\bar{C})$ and $(\tilde{E},\tilde{C})$ by $\bar{J}$ and $\tilde{J}$, we clearly have $\bar{J} = \tilde{J}$ since $\bar{E}^1 = \tilde{E}^1$, and the deleted vertices of $\bar{E}^0$ are isolated. It is also straightforward to verify that $\bar{J} = qJq$, hence

$$\underline{q}L_K^{\mathrm{ab}}(E,C)\underline{q} = L_K^{\mathrm{ab}}(\bar{E},\bar{C}) = L_K^{\mathrm{ab}}(\tilde{E},\tilde{C}) \oplus \langle v \mid v \in \bar{E}^0 \setminus s(\bar{E}^1)\rangle \cong L_K^{\mathrm{ab}}(\tilde{E},\tilde{C}) \oplus K^n$$

as desired, and $\underline{q}$ is full in $L_K^{\mathrm{ab}}(E,C)$. The very same arguments apply to $C^*(E,C)$ and $\mathcal{O}(E,C)$. $\square$

We shall give the rather technical definition of an E-function just below, but first we should understand the motivation for this concept. Assume that $\omega \in \Omega(E,C)_v$ for some $v \in E^{0,1}$, and regard ω as a subset of the Cayley graph of $\mathbb{F}$. Recall that $\omega_1 = s^{-1}(v)^{-1}$ by Remark 5.1.6, and since the local configuration $\omega_{e^{-1}}$ contains e for all $e \in s^{-1}(v)$, it must be of type (c2). Thus $\omega_{e^{-1}}$ involves a choice $e_X \in X$ for each $[e] \neq X \in C_v$, and different choices would have determined another configuration. Now, the local configuration $\omega_{e^{-1}e_X}$ contains e_X^{-1}, hence it must be of type (c1), i.e. $\omega_{e^{-1}e_X} = s^{-1}(s(e_X))$. Continuing this way, we can build up ω by making "the right choices", and the point of E-functions is precisely to formalize this description of the configurations in terms of choices.

For $X \in C_v$ we define $Z_X = \prod_{Y \in C_v, Y \neq X} Y$, and for a particular $Y' \in C_v$ such that $Y' \neq X$, we denote by $\pi_{Y'}$ the projection $Z_X \to Y'$.

Definition 5.2.3. A *partial E-function* for (E,C) is a finite sequence

$$(\Omega_1, g_1), (\Omega_2, g_2), \ldots, (\Omega_n, g_n),$$

where each g_i is a function $\Omega_i \to \bigcup_{X \in C} Z_X$ satisfying

(a) $\Omega_1 = s^{-1}(v) \subset \mathbb{F}$ for some $v \in E^{0,1}$, and $g_1(e) \in Z_{[e]}$ for all $e \in s^{-1}(v)$.

(b) For each $i \geq 2$, $\Omega_i \subset \mathbb{F}$ is the set of elements

$$e_{2i-1}e_{2i-2}^{-1}e_{2i-3}\cdots e_3 e_2^{-1} e_1,$$

such that

- $e_{2i-3}\cdots e_3 e_2^{-1} e_1 \in \Omega_{i-1}$,
- assuming $g_{i-1}(e_{2i-3}\cdots e_3 e_2^{-1} e_1) \in Z_X$ with $X \in C_v$, there is some $X \neq Y \in C_v$ such that
$$e_{2i-2} = (\pi_Y \circ g_{i-1})(e_{2i-3}\cdot e_3 e_2^{-1} e_1),$$
- $s(e_{2i-1}) = s(e_{2i-2})$ and $e_{2i-1} \neq e_{2i-2}$.

(c) The functions g_i satisfy $g_i(e_{2i-1}e_{2i-2}^{-1}e_{2i-3}\cdots e_3 e_2^{-1} e_1) \in Z_{[e_{2i-1}]}$.

Finally, an *E-function* is an infinite sequence $\{(\Omega_i, g_i)\}_{i \geq 1}$ such that

$$(\Omega_1, g_1), (\Omega_2, g_2), \ldots, (\Omega_n, g_n)$$

is a partial E-function for each $n \geq 1$. Note that a partial E-function can always be extended to a proper E-function.

The following proposition will hopefully shed light upon the meaning of this definition.

Proposition 5.2.4. *Let $v \in E^{0,1}$. Then the points of $\Omega(E,C)_v$ are in one-to-one correspondence with the set of E-functions $\{(\Omega_i, g_i)\}_{i \geq 1}$ such that $\Omega_1 = s^{-1}(v)$. If $\omega \in \Omega(E,C)_v$ corresponds to $\{(\Omega_i, g_i)\}_{i \geq 1}$, then the partial E-function*

$$(\Omega_1, g_1), \ldots, (\Omega_n, g_n)$$

corresponds to a clopen neighbourhood $\mathfrak{U}_n$ of ω, and if V is any open neighbourhood of ω, then $\mathfrak{U}_n \subset V$ for some $n \in \mathbb{N}$. Moreover, if $\omega' \in \Omega(E,C)_v$ corresponds to $\{(\Omega'_i, g'_i)\}_{i \geq 1}$, then $\omega' \in \mathfrak{U}_n$ if and only if $\Omega_i = \Omega'_i$ and $g_i = g'_i$ for all $1 \leq i \leq n$.

Proof. Given a point $\omega \in \Omega(E,C)_v$, ω satisfies (c1) at 1 by Remark 5.1.6. Hence, for any $e \in s^{-1}(v)$ we have $e^{-1} \in \omega$, and ω satisfies (c2) at e^{-1}. Therefore there is a unique element $g_1(e) \in Z_{[e]}$ such that $e^{-1}\pi_X(g_1(e)) \in \omega$ for all $X \in C_{r(e)}$ with $X \neq [e]$, and in particular (Ω_1, g_1) is a partial E-function. Now assume that we have extended this to a partial E-function $(\Omega_1, g_1), \ldots, (\Omega_n, g_n)$ such that

$$e_1^{-1}e_2e_3^{-1}\cdots e_{2n-2}e_{2n-1}^{-1}\pi_X(g_n(e_{2n-1}e_{2n-2}^{-1}\cdots e_3e_2^{-1}e_1)) \in \omega$$

for each $t := e_{2n-1}e_{2n-2}^{-1}\cdots e_3e_2^{-1}e_1 \in \Omega_n$ and $X \in C_{r(e_{2n-1})}$ with $X \neq [e_{2n-1}]$. Then the local configuration of ω at $t^{-1}\pi_X(g_n(t))$ contains $\pi_X(g_n(t))^{-1}$, hence it satisfies $(c1)$ and

$$\omega_{t^{-1}\pi_X(g_n(t))} = s^{-1}(s(e_{2n}))^{-1}.$$

Defining Ω_{n+1} to be the set of elements $e_{2n+1}e_{2n}^{-1}t$ such that

- $t = e_{2n-1}e_{2n-2}^{-1}\cdots e_2^{-1}e_1 \in \Omega_n$,
- $e_{2n} = \pi_X(g_n(t))$ for some $X \in C_{r(e_{2n-1})}$ with $X \neq [e_{2n-1}]$,
- $e_{2n+1} \in s^{-1}(s(e_{2n})) \setminus \{e_{2n}\}$,

it therefore satisfies the conditions of Definition 5.2.3. Finally, the local configuration of ω at $r = t^{-1}e_{2n}e_{2n+1}^{-1}$ satisfies (c2), hence there is a unique element $g_{n+1}(r) \in Z_{[e_{2n+1}]}$ such that $r^{-1}\pi_X(g_{n+1}(r)) \in \omega$ for all $X \in C_{r(e_{2n+1})}$ with $X \neq [e_{2n+1}]$. We conclude that

$$(\Omega_1, g_1), \ldots, (\Omega_{n+1}, g_{n+1})$$

is a partial E-function, hence ω determines a unique E-function.

Conversely, given an E-function $\{(\Omega_i, g_i)\}_{i\geq 1}$ such that $\Omega_1 = s^{-1}(v)$, we can define an element $\omega \in \Omega(E,C)_v$ by

$$\omega = \{1\} \cup \bigsqcup_{i\geq 1} \Omega_i^{-1} \cup \{t^{-1}\pi_X(g_i(t)) \mid t = e_{2i-1}e_{2i-2}^{-1}\cdots e_2^{-1}e_1 \in \Omega_i, [e_{2i-1}] \neq X \in C_{r(e_{2i-1})}\}.$$

Surely these two constructions are mutual inverses. Now if ω corresponds to $\{(\Omega_i, g_i)\}_{i\geq 1}$, to every partial E-function $(\Omega_1, g_1), \ldots, (\Omega_n, g_n)$ we associate the set

$$\mathfrak{U}_n = \bigcap_{\substack{i=1,\ldots,n \\ t=e_{2i-1}\cdots e_1\in\Omega_i}} \bigcap_{[e_{2i-1}]\neq X\in C_{r(e_{2i-1})}} \Omega(E,C)_{t^{-1}\pi_X(g_i(t))}.$$

Surely $\mathfrak{U}_n$ is a clopen neighbourhood of ω. If V is any open neighbourhood of ω, then by definition of the topology on $\Omega(E,C)$, there are finitely many $s_1, \ldots, s_k \in \omega$ such that

$$\omega \in \bigcap_{i=1}^{k} \Omega(E,C)_{s_i} \subset V.$$

But clearly $\mathfrak{U}_n \subset \bigcap_{i=1}^k \Omega(E,C)_{s_i}$ for sufficiently large n, hence $\omega \in \mathfrak{U}_n \subset V$. If $\omega' \in \Omega(E,C)_v$ corresponds to $\{(\Omega_i', g_i')\}_{i\geq 1}$, then surely $\omega' \in \mathfrak{U}_n$ if and only if $\Omega_i = \Omega_i'$ and $g_i = g_i'$ for $i = 1, \ldots, n$. $\square$

Finally we give an explicit description of the action on E-functions.

Lemma 5.2.5. *Given a reduced word $s \in \mathbb{F}$, we have $Dom(\theta_s) = \emptyset$ unless*

$$s = f_n^{-1} e_n f_{n-1}^{-1} e_{n-1} \cdots f_1^{-1} e_1 \in \mathbb{F},$$

with $r(e_i) = r(f_i)$, $[e_i] \neq [f_i]$ for all $i = 1, \dots, n$ and $s(f_i) = s(e_{i+1})$ for all $i = 1, \dots, n-1$. In that case, regard θ_s as a map of E-functions. Then the domain of θ_s consists of the E-functions $\mathfrak{g} = \{(\Omega_i, g_i)\}_{i \geq 1}$ such that $\Omega_1 = s^{-1}(s(e_1))$ and

$$f_i = \pi_{[f_i]}(g_i(e_i f_{i-1}^{-1} e_{i-1} \cdots f_1^{-1} e_1))$$

for all $i = 1, \dots, n$. Likewise, the range of θ_s consists of the E-functions $\mathfrak{g}' = \{(\Omega_i', g_i')\}_{i \geq 1}$ such that $\Omega_1' = s^{-1}(s(f_n))$ and

$$e_i = \pi_{[e_i]}(g_i'(f_i e_{i+1}^{-1} f_{i+1} \cdots e_n^{-1} f_n))$$

for all $i = 1, \dots, n$. For $\mathfrak{g} \in Dom(\theta_s)$, write $\mathfrak{g}' = \theta_s(\mathfrak{g})$. If $x_{2j-1} x_{2j-2}^{-1} \cdots x_3 x_2^{-1} x_1 \in \Omega_j$ and $x_1 \neq e_1$, then

$$g_{n+j}'(x_{2j-1} x_{2m-2}^{-1} \cdots x_3 x_2^{-1} x_1 e_1^{-1} f_1 \cdots e_n^{-1} f_n) = g_n(x_{2j-1} x_{2j-2}^{-1} \cdots x_3 x_2^{-1} x_1).$$

Proof. The claims concerning the domain and range of θ_s follow directly from Proposition 5.1.5 and Proposition 5.2.4. Consider the other claims and write

$$\omega \in \Omega(E, C)_{s(e_1)} \quad \text{and} \quad \omega' \in \Omega(E, C)_{s(f_n)}$$

for the elements corresponding to $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Also write

$$t = x_{2j-1} x_{2j-2}^{-1} \cdots x_3 x_2^{-1} x_1.$$

From $t^{-1} \in \omega$ and $\omega' = s\omega$ we get $st^{-1} \in \omega'$, and since $e_1 \neq x_1$ we must have $ts^{-1} \in \Omega_{n+j}'$. We deduce that

$$\begin{aligned} \{\pi_X(g_j(t)) \mid X \in C_{r(x_{2j-1})}, X \neq [x_{2j-1}]\} &= \omega_{t^{-1}} = \omega'_{st^{-1}} \\ &= \{\pi_X(g_{n+j}(ts^{-1}) \mid X \in C_{r(x_{2j-1})}, X \neq [x_{2j-1}]\}, \end{aligned}$$

hence $g_j(t) = g_{n+j}(ts^{-1})$. $\square$

5.3 Topologically free actions

In the following we shall give a characterization of the graphs (E, C) (still satisfying Assumption 5.2.1) for which the canonical partial action is topologically free (see Definition 1.3.1). As we noted in Section 1.4, this is of particular importance when investigating the reduced crossed products. We will be working with paths in the so-called double $\widehat{E}$ of E, defined by $\widehat{E}^0 = E^0$ and $\widehat{E}^1 = E^1 \cup (E^1)^*$ with extended range and source maps such that $r(e^*) = s(e)$ and $s(e^*) = r(e)$. First, we need a couple of definitions.

Definition 5.3.1. A path γ in $\widehat{E}$ is called *admissible*, if for any subpath e^*f we have $[e] \neq [f]$, and for any subpath ef^* we have $e \neq f$, with $e, f \in E^1$. A *closed path* in (E, C) is a non-trivial admissible path $\gamma = e_{2n}^* e_{2n-1} \cdots e_3 e_2^* e_1$ in $\widehat{E}$, such that $s(e_1) = s(e_{2n})$, and a *cycle* in (E, C) is a closed path γ, such that $e_1 \neq e_{2n}$ and $s(e_{2i-1}) \neq s(e_{2j})$ for all $1 \leq i \leq j < n$. An *entry* based at $w \in E^{0,1}$ is a non-trivial admissible path $f_{2m-1} f_{2m-2}^* \cdots f_3 f_2^* f_1$, such that $s(f_1) = w$ and there is some $X \in C_{r(f_{2m-1})}$ with $X \neq [f_{2m-1}]$ and $|X| \geq 2$. Finally, given a closed path γ in (E, C) with notation as above, an *entry* of γ is an entry based at $s(e_1)$.

Definition 5.3.2. The graph (E, C) is said to *satisfy condition (L)*, if any cycle in (E, C) has an entry.

Before we can examine topological freeness of the canonical partial action, we need a couple of purely graph-theoretic lemmas.

Lemma 5.3.3. *Denote by $\gamma = e_{2n}^* e_{2n-1} \cdots e_3 e_2^* e_1$ a closed path in (E, C), and assume that for some $1 \leq i < n$ there exists an admissible path*

$$\nu = f_{2m-1} f_{2m-2}^* \cdots f_3 f_2^* f_1,$$

such that $s(f_1) = s(e_{2i})$ and $|X| \geq 2$ for some $X \in C_{r(f_{2m-1})}$ with $X \neq [f_{2m-1}]$. Then γ has an entry.

Proof. If $f_1 \neq e_{2i}$, then surely $\nu e_{2i}^* e_{2i-1} \cdots e_3 e_2^* e_1$ is an admissible path and thus an entry of γ, so we shall assume that $f_1 = e_{2i}$. We will divide the problem into a number of cases. If $m \geq i + 1$ and

$$f_1 = e_{2i}, f_2 = e_{2i-1}, \ldots, f_{2i} = e_1,$$

then $f_{2m-1} f_{2m-2}^* \cdots f_{2i+2}^* f_{2i+1}$ is an entry for γ. If $m \leq i$ and

$$f_1 = e_{2i}, f_2 = e_{2i-1}, \ldots, f_{2m-1} = e_{2i-2m+2},$$

then there is $X \in C_{r(f_{2m-1})} = C_{r(e_{2i-2m+2})} = C_{r(e_{2i-2m+1})}$ such that $|X| \geq 2$ and $e_{2i-2m+2} \notin X$, hence $e_{2i-2m+2} e_{2i-2m+3}^* \cdots e_{2n-1}^* e_{2n}$ is an entry for γ. Also, if

$$f_1 = e_{2i}, f_2 = e_{2i-1}, \ldots, f_{2k} = e_{2i-2k+1}$$

for some $k < i$ but $f_{2k+1} \neq e_{2i-2k}$, then

$$f_{2m-1} f_{2m-2}^* \cdots f_{2k+2}^* f_{2k+1} e_{2i-2k}^* e_{2i-2k-1} \cdots e_3 e_2^* e_1$$

is an entry for γ. Finally, we consider the case where

$$f_1 = e_{2i}, f_2 = e_{2i-1}, \ldots, f_{2k+1} = e_{2i-2k}$$

for some $k < i$, but $f_{2k+2} \neq e_{2i-2k-1}$. Write $\nu = \nu' f_{2k+2}^* f_{2k+1} \cdots f_3 f_2^* f_1$ with ν' a non-trivial admissible path. If $[f_{2k+2}] = [e_{2i-2k-1}]$, then $|[e_{2i-2k-1}]| \geq 2$ since $e_{2i-2k-1} \neq f_{2k+2}$ by assumption, hence in that case $e_{2i-2k} e_{2i-2k+1}^* \cdots e_{2n-1}^* e_{2n}$ is an entry for γ. If, on the other hand, $[f_{2k+2}] \neq [e_{2i-2k-1}]$, then

$$\nu' f_{2k+2}^* e_{2i-2k-1} e_{2i-2k-2}^* \cdots e_3 e_2^* e_1$$

is an entry for γ. This finishes the proof. □

Lemma 5.3.4. *Assume that (E, C) satisfies condition (L). Then every closed path in (E, C) has an entry.*

Proof. Given a closed path $\gamma = e_{2n}^* e_{2n-1} \cdots e_2^* e_1$, we can assume that $s(e_{2i-1}) \neq s(e_{2j})$ for all $1 \leq i \leq j < n$. Indeed, if there are $1 \leq i \leq j < n$ such that $s(e_{2i-1}) = s(e_{2j})$, then choosing (i, j) with $j - i$ minimal, we have $s(e_{2k-1}) \neq s(e_{2l})$ for all $i \leq k \leq l < j$. We have thus obtained a closed path $e_{2j}^* e_{2j-1} \cdots e_{2i}^* e_{2i-1}$ with the required property, and by Lemma 5.3.3, it suffices to prove that this path has an entry.

Now if $e_1 \neq e_{2n}$, then γ is a cycle and therefore has an entry by assumption. Assuming $e_1 = e_{2n}$, we must have $n \geq 2$ and we claim that $e_2 \neq e_{2n-1}$. For $n = 2$ this is trivial, and if $n \geq 3$, then

$$s(e_2) = s(e_3) \neq s(e_{2(n-1)}) = s(e_{2n-1}),$$

so in particular $e_2 \neq e_{2n-1}$. We have $r(e_{2n-1}) = r(e_{2n}) = r(e_1) = r(e_2)$, and we shall divide the problem into the two cases $[e_{2n-1}] = [e_2]$ and $[e_{2n-1}] \neq [e_2]$. Assuming the former, we must have $|[e_2]| \geq 2$, and so e_1 is an entry for γ. If, on the other hand, $[e_{2n-1}] \neq [e_2]$, then

$$e_2^* e_{2n-1} e_{2n-2}^* \cdots e_5 e_4^* e_3$$

is a cycle, hence it has an entry by condition (L). It follows from Lemma 5.3.3 that γ has an entry as well. □

Theorem 5.3.5. *Suppose that (E, C) satisfies Assumption 5.2.1. Then the canonical partial action is topologically free if and only if (E, C) satisfies condition (L).*

Proof. First, assume that (E, C) does not satisfy condition (L) and denote by

$$\gamma = e_{2n}^* e_{2n-1} \cdots e_3 e_2^* e_1$$

a cycle in (E, C) with no entries. Also write $s = e_{2n}^{-1} e_{2n-1} \cdots e_3 e_2^{-1} e_1 \in \mathbb{F}$. Then one can easily see, proceeding inductively, that there is a unique E-function $\{(\Omega_i, g_i)\}_{i \geq 1}$ such that $\Omega_1 = s^{-1}(s(e_1))$. Denote by ω the point in $\Omega(E, C)_{s(e_1)}$ corresponding to the unique E-function. Then $\Omega(E, C)_{s^{-1}}$ and $\Omega(E, C)_s$ are both non-empty subsets of $\Omega(E, C)_{s(e_1)}$ since γ is a cycle, hence they both equal $\{\omega\}$. In particular $\{\omega\}$ is a s-invariant subset with non-empty interior, so the action is not topologically free.

Now assume that (E, C) does in fact satisfy condition (L). Given any $s \in \mathbb{F}$, we shall prove that the set of fixed points of θ_s has empty interior. From Proposition 5.1.5 it follows that that we need only consider words $s \in \mathbb{F}$, where the letters oscillate between belonging to E^1 and $(E^1)^{-1}$. We may also assume that the first and last letter in s do not both belong to E^1 or $(E^1)^{-1}$, for then θ_s will have no fixed points at all. Indeed, if ω is a fixed point of θ_s with s of this form, then both $s \in \omega$ and $s^{-1} \in \omega$, so by convexity of ω, the local configuration of ω

at 1 will contain both an element of E^1 and $(E^1)^{-1}$, which is a contradiction. First, we shall consider a non-trivial reduced word of the form

$$s = f_n^{-1} e_n \cdots f_1^{-1} e_1,$$

and we may assume that $\mathrm{Dom}(\theta_s) \neq \emptyset$, for else the claim is trivial. From Lemma 5.2.5 it follows that $r(e_i) = r(f_i)$ with $[e_i] \neq [f_i]$ for all $i = 1, \ldots, n$ and $s(f_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$, i.e.

$$\gamma = f_n^* e_n \cdots f_1^* e_1$$

is an admissible path. Surely we can also assume that there actually is some fixed point ω of θ_s, and we denote by $\mathfrak{g} = \{(\Omega_i, g_i)\}_{i \geq 1}$ the corresponding E-function with $\Omega_1 = s^{-1}(s(e_1))$. On the other hand, since $\theta_s(\omega) = \omega$ it follows from Lemma 5.2.5 that $\Omega_1 = s^{-1}(s(f_n))$, so $s(e_1) = s(f_n)$. We conclude that γ is a closed path in (E, C).

In the following we shall suppose that $e_1 \neq f_n$ and let V denote an open neighbourhood of ω. Then by Proposition 5.2.4 there is an clopen neighbourhood $\mathfrak{U}_k$ of ω corresponding to each partial E-function $(\Omega_1, g_1), \ldots, (\Omega_k, g_k)$, and choosing $k \geq n$ to be sufficiently large we will have $\mathfrak{U}_k \subset V$. We shall construct a point $\omega' \in \mathfrak{U}_k$ such that $\theta_s(\omega') \neq \omega'$. It follows from Lemma 5.3.4 that γ has an entry

$$\eta = x_{2j-1} x_{2j-2}^* \cdots x_3 x_2^* x_1.$$

Being an entry, η is an admissible path with $s(x_1) = s(e_1)$, having $[x_{2j-1}] \neq X \in C_{r(x_{2j-1})}$ with $|X| \geq 2$. Assuming that j is minimal with this property, we must have

$$t := x_{2j-1} x_{2j-2}^{-1} \cdots x_3 x_2^{-1} x_1 \in \Omega_j.$$

Indeed $x_1 \in s^{-1}(s(e_1)) = \Omega_1$, and assuming that $x_{2i-1} x_{2i-2}^{-1} \cdots x_3 x_2^{-1} x_1 \in \Omega_i$ for some $i < j$ we should prove the identity

$$x_{2i} = \pi_{[x_{2i}]}(g_i(x_{2i-1} x_{2i-2} \cdots x_3 x_2^{-1} x_1)).$$

From η being admissible we have $r(x_{2i-1}) = r(x_{2i})$ but $[x_{2i-1}] \neq [x_{2i}]$, and by minimality of j it follows that $|[x_{2i}]| = 1$. Thus x_{2i} is the only possible choice in $[x_{2i}]$, and the identity holds. Picking a positive integer m with $mn + j > k$, we shall divide the argument into two cases.

Suppose first that $e_1 \neq x_1$. Let ω' denote the element corresponding to the E-function defined as follows: For $1 \leq l \leq mn + j - 1$ define $\Omega_l' = \Omega_l$, $g_l' = g_l$ and take $X \in C_{r(x_{2j-1})}$ with $X \neq [x_{2j-1}]$ and $|X| \geq 2$. Then there is some $x \in X$ with $x \neq \pi_X(g_j(t))$, and we define

$$\pi_X(g_{mn+j}'(ts^{-m})) = x.$$

Apart from this, we define g_{mn+j}' arbitrarily and also extend the partial E-function

$$(\Omega_1', g_1'), (\Omega_2', g_2'), \ldots, (\Omega_{mn+j}', g_{mn+j}')$$

arbitrarily to an E-function $\{(\Omega_i', g_i')\}_{i \geq 1}$. Since $k \leq mn + j - 1$ we have $(\Omega_i, g_i) = (\Omega_i', g_i')$ for all $1 \leq i \leq k$, hence $\omega' \in V$. Finally we need to check that ω' is not fixed by θ_s. We have $ts^{-m} \in \Omega_{mn+j}$ since $e_1 \neq x_1$ by assumption, and in order to reach a contradiction we suppose that ω' is actually a fixed point. Then by Lemma 5.2.5 we have

$$x = \pi_X(g_{mn+j}'(ts^{-m})) = \pi_X(g_j'(t)) = \pi_X(g_j(t)),$$

contradicting the fact that $\pi_X(g_j(t)) \neq x$.

Now assuming that $e_1 = x_1$ we have $x_1 \neq f_n$, since we are also assuming that $e_1 \neq f_n$. Proceeding as above with the element ts^{-m} replaced by ts^m, we obtain a non-fixed point $\omega' \in U$.

We shall use what we have just proved to cover the rest of the cases. First suppose that s is of the form

$$s = e_n f_{n-1}^{-1} \cdots f_1^{-1} e_1 f_0^{-1}$$

with $e_n \neq f_0$ and $n \geq 1$, and assume that V is a non-empty open set of fixed points of θ_s. From $\mathrm{Dom}(\theta_s) \neq \emptyset$ we deduce that $f_{n-1}^* e_{n-1} \cdots f_1^* e_1$ is an admissible path, $s(f_0) = s(e_1)$ with $f_0 \neq e_1$, and $s(e_n) = s(f_{n-1})$ with $e_n \neq f_{n-1}$. Moreover, since θ_s has a fixed point ω, we must have both $s \in \omega$ and $s^{-1} \in \omega$, hence $e_n, f_0 \in \omega_1$. Since we have assumed that $e_n \neq f_0$, we get $[e_n] \neq [f_0]$. Now define $t = f_0^{-1} e_n f_{n-1}^{-1} e_{n-1} \cdots f_1^{-1} e_1$ and note that surely $\theta_{f_0^{-1}}(V) \subset \mathrm{Dom}(\theta_t)$. Since

$$\theta_t(\theta_{f_0^{-1}}(\omega)) = \theta_{tf_0^{-1}}(\omega) = \theta_{f_0^{-1}s}(\omega) = \theta_{f_0^{-1}}(\omega)$$

for any $\omega \in V$, we conclude that $\theta_{z_0^{-1}}(V)$ is a non-empty open set of fixed points of θ_t, contradicting the first part of the proof.

To deal with the remaining two cases, we consider the statements

(a) the set of fixed points for θ_s has empty interior if s is of the form

$$s = e_1^{-1} e_n f_{n-1}^{-1} e_{n-1} \cdots f_1^{-1} e_1,$$

(b) the set of fixed points for θ_s has empty interior if s is of the form

$$s = f_0 f_n^{-1} e_n \cdots f_1^{-1} e_1 f_0^{-1},$$

for $n \geq 1$. We shall prove that

(a) holds for $n \Rightarrow$ (b) holds for $n \Rightarrow$ (a) holds for $n+1$,

so first we assume that (a) holds for n. Consider an element s of the form

$$s = f_0 f_n^{-1} e_n \cdots f_1^{-1} e_1 f_0^{-1}$$

and define $t = f_n^{-1} e_n \cdots f_1^{-1} e_1$. Once again, let V denote a non-empty open set of fixed points of θ_s. Then surely $\theta_{f_0^{-1}}(V) \subset \mathrm{Dom}(\theta_t)$ and

$$\theta_t(\theta_{f_0^{-1}}(\omega)) = \theta_{tf_0^{-1}s^{-1}}(\omega) = \theta_{f_0^{-1}}(\omega)$$

for all $\omega \in V$, hence $\theta_{f_0^{-1}}(V)$ is a non-empty open set of fixed points for θ_t. However, this contradicts either the first part of the proof or the inductive assumption. Now suppose that (b) holds for some $n \geq 1$. Given $s = e_1^{-1} e_{n+1} f_n^{-1} e_n \cdots f_1^{-1} e_1$, we assume that V is a non-empty set of fixed points of θ_s and define $t = e_{n+1} f_n^{-1} e_n \cdots e_2 f_1^{-1}$. Then surely $\theta_{e_1}(V) \subset \mathrm{Dom}(\theta_t)$ and

$$\theta_t(\theta_{e_1}\omega) = \theta_t(\theta_{e_1}(\theta_{s^{-1}}(\omega))) = \theta_{te_1e_1^{-1}t^{-1}e_1}(\omega) = \theta_{e_1}(\omega)$$

for all $\omega \in V$, so $\theta_{e_1}(V)$ is a non-empty open set of fixed points of θ_t. But depending on whether $e_{n+1} = f_1$ or $e_{n+1} \neq f_1$, this contradicts the induction assumption or what we have just shown above. As (a) trivially holds for $n = 1$, this finishes the proof. $\square$

Recall that a compact zero-dimensional and metrizable space is a Cantor space (of which there is only one up to homeomorphism), if it contains no isolated points. We are also able to give sufficient conditions for $\Omega(E,C)$ to be a Cantor space.

Proposition 5.3.6. *Assume that for each $w \in E^{0,1}$ there are entries*

$$e_{2n-1}e_{2n-2}^{*}\cdots e_3e_2^{*}e_1 \quad \textit{and} \quad f_{2m-1}f_{2m-2}^{*}\cdots f_3f_2^{*}f_1$$

of w with $e_1 \neq f_1$. Then $\Omega(E,C)$ is a Cantor space.

Proof. We first note that $\Omega(E,C)$ is a Cantor space, if and only if $\Omega(E,C)_w$ is a Cantor space for all $w \in E^{0,1}$ — this is a direct consequence of Definition 2.4.9. Therefore we consider some $\omega \in \Omega(E,C)_w$ with $w \in E^{0,1}$ and denote by $\{(\Omega_i, g_i)\}_{i\geq 1}$ the E-function corresponding to ω. Then given any $n \in \mathbb{N}$, we should construct another E-function $\{(\Omega_i', g_i')\}_{i\geq 1}$ with $\Omega_i = \Omega_i'$ and $g_i = g_i'$ for all $i \leq n$. We of course define the partial E-function $(\Omega_1', g_1'), \ldots, (\Omega_n', g_n')$ accordingly, and we will automatically have $\Omega_{n+1}' = \Omega_{n+1}$ as well. Now take any

$$e_{2n+1}e_{2n}^{-1}e_{2n-1}\cdots e_3e_2^{-1}e_1 \in \Omega_{n+1}'.$$

Then by assumption, there is an entry

$$f_{2m-1}f_{2m-2}^{*}\cdots f_3f_2^{*}f_1$$

at $s(e_{2n})$ with $f_1 \neq e_{2n}$, and we can assume that m is chosen minimally. Setting $\Omega_i' = \Omega_i$ and $g_i' = g_i$ for $n+1 \leq i \leq m+n-1$ as well, we have

$$f_{2m-1}f_{2m-2}^{-1}\cdots f_3f_2^{-1}f_1e_{2n}^{-1}e_{2n-1}\cdots e_3e_2^{-1}e_1 \in \Omega_{m+n}'.$$

Then there is some color $[f_{2m-1}] \neq X \in C_{r(f_{2m-1})}$ with $|X| \geq 2$, and we simply define

$$\pi_X\Big(g_{m+n}'(f_{2m-1}f_{2m-2}^{-1}\cdots f_3f_2^{-1}f_1e_{2n}^{-1}e_{2n-1}\cdots e_3e_2^{-1}e_1 \in \Omega_{m+n}')\Big)$$

to be different from

$$\pi_X\Big(g_{m+n}(f_{2m-1}f_{2m-2}^{-1}\cdots f_3f_2^{-1}f_1e_{2n}^{-1}e_{2n-1}\cdots e_3e_2^{-1}e_1 \in \Omega_{m+n}')\Big).$$

Finishing off the definition of g_{m+n}' arbitrarily and extending $(\Omega_1', g_1'), \ldots, (\Omega_{m+n}', g_{m+n}')$ arbitrarily to an E-function $\{(\Omega_i', g_i')\}_{i\geq 1}$, we have obtained an E-function with the desired properties. □

Likewise, we can give sufficient conditions for the existence of isolated points.

Proposition 5.3.7. *Assume that there are no entries based at $w \in E^{0,1}$. Then $\Omega(E,C)_w$ is a one-point space. In particular, $\Omega(E,C)$ contains an isolated point.*

Proof. Simply note that there is only one E-function $\{(\Omega_i, g_i)\}_{i\geq 1}$ with $\Omega_1 = s^{-1}(w)$. □

We continue the investigation of isolated points with a few examples in the final chapter.

Chapter 6

Examples and discussion

We close this thesis with a minor chapter of examples and discussions. The most obvious example is that of a trivially separated graph.

Example 6.1.1. If E is a trivially separated finite graph, we have $L_K(E) = L_K^{\mathrm{ab}}(E)$ and $C^*(E) = \mathcal{O}(E)$. In particular, we obtain a description of $L_K(E)$ and $C^*(E)$ as crossed products

$$L_K(E) \cong C_K(\Omega(E)) \rtimes_{\theta^*} \mathbb{F} \quad \text{and} \quad C^*(E) \cong C(\Omega(E)) \rtimes_{\theta^*} \mathbb{F}$$

with a partial action. Changing (F1)-(F4) slightly to take into account the fact that E need not be bipartite, one can obtain a description of the partial action similar to that of Proposition 5.1.5. However, it is clear that our theory does not offer a lot of new insights into classical graph algebras.

Proposition 6.1.2. *Given a separated graph (E, C) and any $X \in C$, we write E_X for the trivially separated graph given by*

$$E_X^0 = E^0 \quad \textit{and} \quad E_X^1 = X.$$

Then there are isomorphisms

$$L_K(E, C) \cong \mathop{\ast}_{E^0} L_K(E_X) \quad \textit{and} \quad C^*(E, C) \cong \mathop{\ast}_{E^0} C^*(E_X),$$

where the amalgamated free products run over all colors $X \in C$.

Proof. Note that a $*$-homomorphism $L_K(E_X) \to A$ is the same thing as a set

$$\{p_v, s_e \mid v \in E^0, e \in X\} \subset A$$

satisfying

- $p_v p_{v'} = \delta_{v,v'} p_v$ for all $v, v' \in E^0$,
- $p_{r(e)} s_e = s_e p_{s(e)} = s_e$ for all $e \in X$,
- $s_e^* s_{e'} = \delta_{e,e'} p_{s(e)}$ for all $e, e' \in X$,
- $p_v = \sum_{e \in X} s_e s_e^*$ if $X \in C_v$.

From the universal property of the amalgamated free product, a $*$-homomorphism

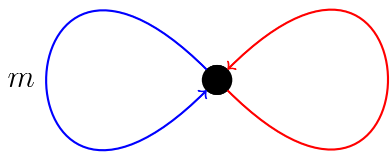
$$\mathop{\scalebox{1.5}{$*$}}_{E^0} L_K(E_X) \to A$$

is the same thing as $*$-homomorphisms $L_K(E_X) \to A$ for each $X \in C$ that agree on E^0, i.e. a set $\{p_v, s_e \mid v \in E^0, e \in E^1\} \subset A$ satisfying

- $p_v p_{v'} = \delta_{v,v'} p_v$ for all $v, v' \in E^0$,
- $p_{r(e)} s_e = s_e p_{s(e)} = s_e$ for all $e \in E^1$,
- $s_e^* s_{e'} = \delta_{e,e'} p_{s(e)}$ for all $e, e' \in X \in C$,
- $p_v = \sum_{e \in X} s_e s_e^*$ for all $X \in C_v$, $v \in E^0$.

But this is the same thing as a $*$-homomorphism $L_K(E, C) \to A$. One argues similarly for $C^*(E, C)$. $\square$

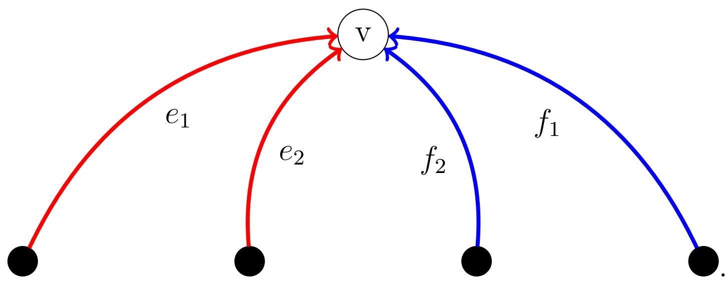
Example 6.1.3. Note that in case (E, C) only has one vertex, the amalgamation is trivial. For instance, we can consider the graph



,

for which the graph C^*-algebra is simply the universal unital free product $\mathcal{O}_m * \mathcal{O}_n$ of the Cuntz-algebras.

Example 6.1.4. Consider the bipartite graph



.

Then it is easily seen that the full corner $vL_K(E, C)v$ is generated by the set of distinct projections

$$\{p(e_1), p(e_2), p(f_1), p(f_2)\}$$

with the relation

$$p(e_1) + p(e_2) = p(f_1) + p(f_2).$$

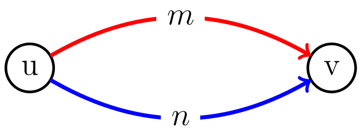
This is the universal unital free product $K^2 * K^2$. However, the corresponding full corner $\underline{v}L_K^{\mathrm{ab}}(E, C)\underline{v}$ of the abelianized Leavitt path algebra is generated by four distinct and commuting projections $\{\underline{p(e_1)}, \underline{p(e_2)}, \underline{p(f_1)}, \underline{p(f_2)}\}$ such that

$$\underline{p(e_1)} + \underline{p(e_2)} = \underline{p(f_1)} + \underline{p(f_2)},$$

and this is simply isomorphic to K^4. Passing from $L_K(E, C)$ to $L_K^{\mathrm{ab}}(E, C)$ thus gives a drastic change in complexity.

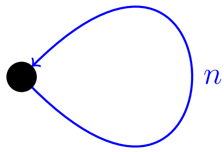
The following example, which we have already encountered, is really the motivational example of the entire theory.

Example 6.1.5. Let $1 \leq m < n$ and consider the graph $(E, C) := (E(m,n), C(m,n))$ of Corollary 4.1.14



.

Note that if $m = 1$, then $(E, C) = (\tilde{F}, \tilde{D})$ as in Proposition 3.2.2, where F is the trivially separated graph



,

hence the algebras associated to $(E(1,n), C(1,n))$ are simply the two-by-two matrices over the Cuntz algebra $\mathcal{O}_n$ and its algebraic counterpart. The C^*-algebras

$$\mathcal{O}_{m,n} = \mathcal{O}(E, C)$$

for $m \geq 2$ were studied in detail in [4], and as one might expect, the behavior is very different from the case $m = 1$. Indeed, it is proven that $\mathcal{O}_{m,n}$ is not even exact (see [4, Theorem 7.2]), while the reduced crossed product

$$\mathcal{O}^r_{m,n} = C(\Omega(E, C)) \rtimes_{r,\theta^*} \mathbb{F}$$

is exact from Theorem 1.4.9. However, $\mathcal{O}^r_{m,n}$ is still not nuclear, because then we would in fact have $\mathcal{O}_{m,n} = \mathcal{O}^r_{m,n}$ (see [4, Theorem 6.4]). In the purely algebraic context we have

$$\mathcal{V}(L_K(E, C)) = \langle u \mid m \cdot u = n \cdot u \rangle,$$

hence $k \cdot u$ is a finite if and only if $k < m$, and $k \cdot u$ is properly infinite if and only if $k \geq m$. Since $\mathcal{V}(L_K(E, C)) \to \mathcal{V}(L^{\mathrm{ab}}_K(E, C))$ is a refinement, the same holds in $L^{\mathrm{ab}}_K(E, C)$.

The Leavitt path algebra $L_K(E, C)$ is closely related to the *classical Leavitt algebra* $L_K(m, n)$, which is the $*$-algebra generated by elements $a_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ such that the matrix $a = [a_{i,j}]$ is unitary, i.e. $aa^* = 1_m$ and $a^*a = 1_n$. In fact, we claim that there are isomorphisms

(a) $L_K(E, C) \cong M_{m+1}(L_K(m, n)) \cong M_{n+1}(L_K(m, n))$

(b) $vL_K(E, C)v \cong M_m(L_K(m, n)) \cong M_n(L_K(m, n))$

(c) $uL_K(E, C)u \cong L_K(m, n)$.

Proof. Let $e_{i,j}$ denote the (i, j)'th standard matrix unit in $M_{m+1}(K)$ and define a projection

$$p = \sum_{i=1}^{m} 1 \otimes e_{i,i}$$

in $M_{m+1}(L_K(m, n))$. Then we can define a $*$-homomorphism

$$\psi \colon L_K(E, C) \to M_{m+1}(L_K(m, n))$$

by

- $\psi(u) = 1 \otimes e_{m+1,m+1}$ and $\psi(v) = p$,
- $\psi(e_i) = 1 \otimes e_{i,m+1}$ for all $1 \leq i \leq m$,
- $\psi(f_j) = \sum_{l=1}^{m} a_{l,j} \otimes e_{l,m+1}$ for all $1 \leq j \leq n$.

Indeed it is straightforward to verify that ψ respects the defining relations of $L_K(E, C)$. In order to build an inverse of ψ, we define

$$\varphi_1 \colon L_K(m, n) \to L_K(E, C) \quad \text{and} \quad \varphi_2 \colon M_{m+1}(K) \to L_K(E, C)$$

by

$$\varphi_1(a_{i,j}) = e_i^* f_j + \sum_{l=1}^{m} e_l e_i^* f_j e_l^*$$

and

- $\varphi_2(e_{i,j}) = e_i e_j^*$ for $1 \leq i, j \leq m$,
- $\varphi_2(e_{i,m+1}) = e_i$ for $1 \leq i \leq m$,
- $\varphi_2(e_{m+1,m+1}) = u$.

It is straightforward to check that the $\varphi_2(e_{i,j})$'s form a set of matrix units, hence φ_2 is well defined. For φ_1 to be well-defined, we need the matrix $[\varphi_1(a_{i,j})]$ to be unitary, and we will simply verify that $[\varphi_1(a_{i,j})][\varphi_1(a_{i,j})]^* = 1_m$, as the other identity can be checked in a similar fashion. For this we simply observe that the (i, j)'th entry of the above product equals

$$\sum_{k=1}^{n} \varphi_1(a_{i,k})\varphi_1(a_{j,k})^* = \sum_{k=1}^{n} \left(e_i^* f_k f_k^* e_j + \sum_{l=1}^{m} e_l e_i^* f_k f_k^* e_j e_l^* \right) = \delta_{i,j} u + \delta_{i,j} v = \delta_{i,j}.$$

Finally, one checks that the images of φ_1 and φ_2 commute, hence there is a product homomorphism $\varphi = \varphi_1 \times \varphi_2$, and this is an inverse of ψ. The other isomorphism is essentially obtained by interchanging the roles of the e_i's and f_j's.

For (b) and (c), we simply note that the isomorphisms restrict to isomorphisms

$$vL_K(E, C)v \cong pM_{m+1}(L_K(m, n))p \cong M_m(L_K(m, n))$$

and

$$vL_K(E, C)v \cong pM_{n+1}(L_K(m, n))p \cong M_n(L_K(m, n)),$$

as well as

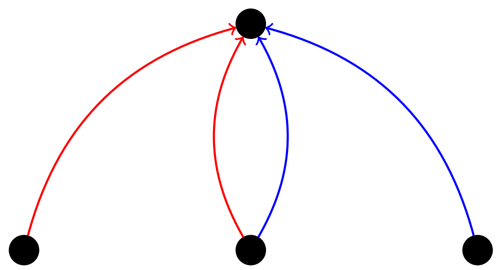
$$uL_K(E, C)u \cong (1 \otimes e_{m+1,m+1})M_{m+1}(L_K(m, n))(1 \otimes e_{m+1,m+1}) \cong L_K(m, n).$$

□

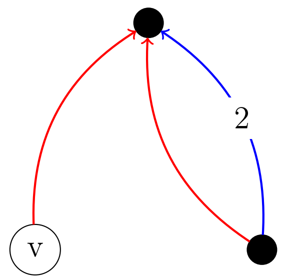
Using universality, we of course have similar results in the C^*-algebraic context. Note that $uL_K(E, C)u$ is a full corner of $L_K(E, C)$, so in particular $\mathcal{V}(L_K(m, n)) \cong \mathcal{V}(L_K(E, C))$. We conclude that $L_K(m, n)$ is a finite $*$-algebra (i.e. the identity is a finite projection), such that $M_m(L_K(m, n)) \cong M_n(L_K(m, n))$ is properly infinite (i.e. the identities in these $*$-algebras are properly infinite projections).

Now we shall consider two examples of graphs that neither satisfy the requirements of Proposition 5.3.6 nor those of Proposition 5.3.7. They show that vertices $v \in E^{0,1}$ with $|s^{-1}(v)| = 1$ may produce isolated points, but need not do so.

Example 6.1.6. Note that even though all three vertices of $E^{0,1}$ in the graph



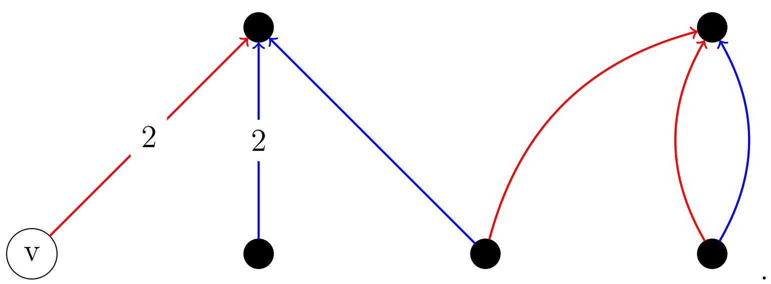
have entries, the associated space contains a lot of isolated points. On the other hand, the space associated to the graph



is indeed a Cantor space, even though $|s^{-1}(v)| = 1$.

One might think that the requirements of Proposition 5.3.6 could be relaxed to just demanding that v has an entry and $|s^{-1}(v)| \geq 2$ for every $v \in E^{0,1}$, but as the following example shows, this is not the case.

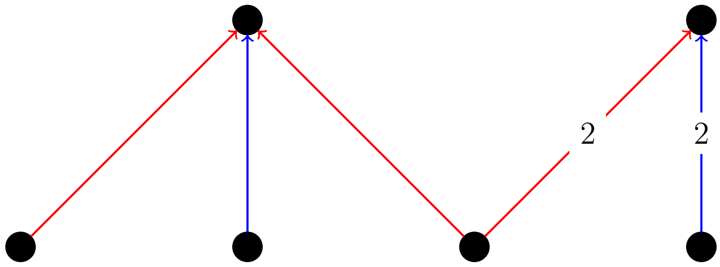
Example 6.1.7. Consider the graph



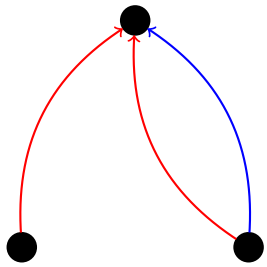
Clearly every $w \in E^{0,1}$ has an entry and satisfies $|s^{-1}(w)| = 2$. However, it is easily seen $\Omega(E, C)_v$ contains infinitely many isolated points.

Finally we shall see that the cardinality of the isolated points versus the cardinality of non-isolated points of $\Omega(E, C)$ may vary greatly, although there is at most countably infinite many isolated points.

Example 6.1.8. If $\Omega(E, C)$ contains isolated points and is infinite, usually there will be infinitely many isolated points. However, it is fairly easy to see that the space associated to the graph



is uncountable and has precisely 3 isolated points. In a completely different direction, the space associated to the graph



decomposes as $\Omega(E, C) = X \sqcup Y$, where $X \cong Y$ are both homeomorphic to the one point compactification $\mathbb{N}^{\bullet}$. In particular, the space is countably infinite and has only two non-isolated points.

We finally give a few remarks on the future of the subject.

Remark 6.1.9. It was conjectured in [2] that the canonical homomorphism

$$\mathcal{V}(L(E, C)) \to \mathcal{V}(C^*(E, C))$$

is in fact an isomorphism, and this is surely the most important open problem in the theory at the moment. If the conjecture holds, then many of the results in the purely algebraic context presented in this thesis will hold in the C^*-context as well. First of all, the canonical homomorphism $\mathcal{V}(L^{\mathrm{ab}}(E, C)) \to \mathcal{V}(\mathcal{O}(E, C))$ will be an isomorphism as well by Theorem 3.3.11 and Theorem 3.3.12, and for any finitely generated conical abelian monoid M, there will exist a graph (E, C) along with a refinement

$$M \to \mathcal{V}(\mathcal{O}(E, C)).$$

Thus the abelianized graph C^*-algebras $\mathcal{O}(E, C)$ will form a fairly accessible class of C^*-algebras, considering the description as a crossed product, with wild $\mathcal{V}$-monoids. The conjecture is known to be true for trivially separated graphs.

Appendices

As we are working quite intensively with all sorts of algebraic objects, the relevant general facts needed have been compiled in the following three appendices. Hopefully, this will allow some focus on the things that really matter in the rest of the thesis.

A.1 Abelian Monoids

Definition A.1.1. An abelian monoid is a set M with an abelian and associative binary relation, usually denoted $+$, with neutral element 0. Note that 0 is necessarily unique. A homomorphism of abelian monoids $\varphi\colon M \to N$ is simply an additive map satisfying $\varphi(0) = 0$.

Definition A.1.2. M is called *cancellative* if $a + b = a + c$ implies $b = c$, and it is called *conical* if $a + b = 0$ implies $a = b = 0$.

Definition A.1.3 (The algebraic preorder)**.** Any abelian monoid can be equipped with a relation $\leq$ defined as follows: We write $a \leq b$ if and only if there is some $c \in M$ such that $a + c = b$. All preorderings of monoids will be of this type.

Obviously, $\leq$ is reflexive and transitive, but in general it is not antisymmetric. However, if M is both conical and cancellative, then $\leq$ will be antisymmetric. Obviously any homomorphism preserves the algebraic preorder. Finally, an element satisfying $2a \leq a$ will be called *properly infinite*.

Example A.1.4 (The free abelian monoid on a set)**.** Given a set X, the *free abelian monoid* on X is the set $\mathbb{Z}_+(X)$ of formal sums $\sum_{x\in X} n_x x$, where the n_x's are non-negative integers with $n_x = 0$ for all but finitely many $x \in X$. Equipped with the obvious addition, $\mathbb{Z}_+(X)$ becomes a cancellative and conical abelian monoid.

Example A.1.5 (The monoid $\langle X \mid \mathcal{R}\rangle$)**.** Given a set X and a relation $\mathcal{R}$ on $\mathbb{Z}_+(X)$ we can form an abelian monoid $\langle X \mid \mathcal{R}\rangle$ "generated by X with relations $\mathcal{R}$" as follows: Let $\sim$ denote the smallest equivalence relation on $\mathbb{Z}_+(X)$ containing $\mathcal{R}$ such that $a_1 \sim b_1$ and $a_2 \sim b_2$ implies $a_1 + a_2 \sim b_1 + b_2$. Then the addition on $\mathbb{Z}_+(X)$ drops to an addition on the set of equivalence classes $\langle X \mid \mathcal{R}\rangle := \frac{\mathbb{Z}_+(X)}{\sim}$, hence the quotient defines a monoid. Clearly, this monoid enjoys the following universal property: Given a homomorphism $\varphi\colon \mathbb{Z}_+(X) \to N$ such that $\varphi(a) = \varphi(a')$ for all $(a, a') \in \mathcal{R}$, there is a unique homomorphism $\overline{\varphi}\colon \langle X \mid \mathcal{R}\rangle \to N$ such that $\overline{\varphi}([a]) = \varphi(a)$ for all $a \in \mathbb{Z}_+(X)$. Usually, we will omit the brackets when denoting elements of $\langle X \mid \mathcal{R}\rangle$, and shall simply write $a = a'$ whenever $a \sim a'$. Abusing the notation in an obvious way, any such relation $\sim$ is generated by a collection of relations $\mathcal{R} = \{\mathbf{r}_j\}_{j\in J}$, where each $\mathbf{r}_j$ is simply a relation of the form

$$\mathbf{r}_j\colon \qquad \sum_{x\in X} m_{x,j} x = \sum_{x\in X} n_{x,j} x.$$

Note that any abelian monoid M can be presented as a monoid of the form $\langle X \mid \mathcal{R}\rangle$, but in general one desires that both X and $\mathcal{R}$ are as small as possible.

Definition A.1.6. An abelian monoid M is called *finitely presented*, if it is isomorphic to $\langle X \mid \mathcal{R}\rangle$ for finite X and $\mathcal{R} = \{\mathbf{r}_j\}_{j\in J}$ with finite J.

Lemma A.1.7. *Every conical abelian monoid M has a presentation $\langle X \mid \{\boldsymbol{r}_j\}_{j\in J}\rangle$ with*

$$\boldsymbol{r}_j : \qquad \sum_{x\in X} m_{x,j} x = \sum_{x\in X} n_{x,j} x,$$

such that $\sum_{x\in X} m_{x,j}, \sum_{x\in X} n_{x,j} > 0$ for all $j \in J$ and $\sum_{j\in J} m_{x,j} + n_{x,j} > 0$ for all $x \in X$. Furthermore, if M is finitely presented, then this can be accomplished by a finite presentation.

Proof. As noted above, any abelian monoid has a presentation $\langle X \mid \{\mathbf{r}_j\}_{j\in J}\rangle$. Now if

$$\sum_{x\in X} m_{x,j} = 0$$

(or analogously $\sum_{x\in X} n_{x,j} = 0$) for some j, then $\mathbf{r}_j$ reduces to

$$0 = \sum_{x\in X} n_{x,j} x.$$

If $n_{x,j} > 0$ for some $x \in X$, we can conclude that $x = 0$ since our monoid is assumed to be conical. Discarding all such x's along with the entire relation $\mathbf{r}_j$, we obtain the same monoid. Doing this for every j, our monoid will satisfy the first of the above inequalities. For the other one, assume that $\sum_{j\in J} m_{x,j} + n_{x,j} = 0$ for some x. Then we may simply add the relation

$$\mathbf{r}_x : \qquad x = x.$$

Doing this for every such x, we obtain the same monoid but with a presentation satisfying both the first and second of the above inequalities. Finally, note that if our initial presentation is finite, then so is the one we have produced. $\square$

The following proposition is known as *Redei's Theorem*, but the very elegant proof was given by Peter Freyd in [10].

Proposition A.1.8. *Every finitely generated abelian monoid is finitely presented.*

Proof. Assume in order to reach a contradiction that M is not finitely presented. Then there is a free abelian monoid F on finitely many generators and a chain of abelian monoids M_i with non-injective, surjective homomorphisms

$$F \to M_1 \to M_2 \to M_3 \to \dots .$$

Denoting by $\mathbb{Z}(N)$ the monoid ring on N, this gives a chain of non-injective, surjective ring homomorphisms

$$\mathbb{Z}[F] \to \mathbb{Z}[M_1] \to \mathbb{Z}[M_2] \to \mathbb{Z}[M_3] \to \dots ,$$

so the kernels I_i of the compositions $\mathbb{Z}[F] \to Z[M_i]$ form a strictly increasing chain of ideals

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots \subsetneq \mathbb{Z}[F].$$

Recall that a ring is called *Noetherian*, if it has no infinite strictly increasing chains of ideals. But $\mathbb{Z}[F]$ is a commutative polynomial ring over $\mathbb{Z}$, and since $\mathbb{Z}$ is clearly Noetherian, so is $\mathbb{Z}[F]$ by Hilbert's Basis Theorem. We have thus reached a contradiction. $\square$

Corollary A.1.9. *Every finitely generated, conical abelian monoid has a finite presentation* $\langle X \mid \{\boldsymbol{r}_j\}_{j\in J}\rangle$ *with*

$$\boldsymbol{r}_j\colon \quad \sum_{x\in X} m_{x,j}x = \sum_{x\in X} n_{x,j}x,$$

such that $\sum_{x\in X} m_{x,j}, \sum_{x\in X} n_{x,j} > 0$ *for all* $j \in J$ *and* $\sum_{j\in J} m_{x,j} + n_{x,j} > 0$ *for all* $x \in X$.

Proof. This is immediate by Lemma A.1.7 and Proposition A.1.8. □

Unitary embeddings

Now we shall prove a few lemmas that allow us to construct so-called unitary embeddings. This will be very handy in Chapter 3.

Definition A.1.10. A homomorphism of abelian monoids $\varphi\colon M \to N$ is called a *unitary embedding* if the following holds:

(a) φ is injective.

(b) $\varphi(M)$ is *cofinal* in N: For any $b \in N$ there is some $a \in A$ such that $b \leq \varphi(a)$.

(c) Whenever $\varphi(a) + b \in \varphi(M)$ we have $b \in \varphi(M)$.

Note that a unitary embedding is an order embedding.

Lemma A.1.11. *If* $\{\varphi_{i,j}\colon M_i \to M_j \mid i,j \in I, i \leq j\}$ *is a directed system of unitary embeddings, then the limit homomorphisms* $M_i \to \varinjlim M_j$ *are unitary embeddings as well.*

Proof. Abusing the notation in the usual way, recall that the direct limit of a directed system of modules may be constructed as $M := \frac{\bigoplus_{i\in I} M_i}{\sim}$ where $\sim$ is the equivalence relation generated by $a \sim \varphi_{i,j}(a)$ for all $a \in M_i$, and $i,j \in I$ with $i \leq j$, and the limit maps $\lambda_i\colon M_i \to M$ are simply the inclusions into the sum followed by the quotient map associated to the equivalence relation. Thus injectivity of the limit maps follows as usual by injectivity of the $\varphi_{i,j}$'s. To see that each $\lambda_i(M_i)$ is cofinal in M, pick $b \in M$. Then $b = \lambda_j(a')$ for some $j \in I$ and $a' \in M_j$. Picking $k \geq i,j$ we define $b' = \varphi_{j,k}(a')$. Then since $\varphi_{i,k}(M_i)$ is cofinal in M_k, there is some $a \in M_i$ such that $b' \leq \varphi_{i,k}(a)$. We conclude that

$$b = \lambda_j(a') = \lambda_k(\varphi_{j,k}(a')) = \lambda_k(b') \leq \lambda_k(\varphi_{i,k}(a)) = \lambda_i(a).$$

It remains only to check the third condition, so take $a, a' \in M_i$ and assume that

$$\lambda_i(a) + b = \lambda_i(a')$$

for some $b \in M$. Then there is some $i \leq j$ and $b' \in M_j$ such that $\lambda_j(b') = b$, hence

$$\lambda_j(\varphi_{i,j}(a)) + \lambda_j(b') = \lambda_j(\varphi_{i,j}(a')).$$

By injectivity of λ_j, we infer that $\varphi_{i,j}(a)+b' = \varphi_{i,j}(a')$, so from $\varphi_{i,j}$ being a unitary embedding we deduce that $b' \in \varphi_{i,j}(M_i)$. In particular

$$b = \lambda_j(b') \in \lambda_j(\varphi_{i,j}(M_i)) = \lambda_i(M_i).$$

□

The following lemma may come across as slightly odd, but it will be a crucial ingredient in the proof of Lemma A.1.13

Lemma A.1.12. *Given finite sets $X_1, \ldots, X_k$ and a function $\mu\colon \bigsqcup_{i=1}^k X_i \to \mathbb{Z}$ such that*

$$\sum_{i=1}^{k} \mu(x_i) \geq 0$$

for all $(x_1, \ldots, x_k) \in X_1 \times \ldots \times X_k$. Then there is a function $\nu\colon \bigsqcup_{i=1}^k X_k \to \mathbb{Z}_+$ such that

$$\sum_{i=1}^{k} \nu(x_i) = \sum_{i=1}^{k} \mu(x_i)$$

for all $(x_1, \ldots, x_k) \in X_1 \times \ldots \times X_k$.

Proof. We argue by induction over k. In case $k = 1$ the claim is vacuously satisfied, so let $k \geq 2$ by arbitrary and assume it holds for $k-1$. Clearly, there is some j such that $\sum_{i \neq j} \mu(x_i) \geq 0$ for all $(x_1, \ldots, \widehat{x_j}, \ldots, x_k) \in X_1 \times \ldots \widehat{X_j} \times \ldots \times X_k$ — we shall assume without loss of generality that $j = k$. By the induction hypothesis there is some $\mu'\colon \bigsqcup_{i=1}^{k-1} X_i \to \mathbb{Z}_+$ such that

$$\sum_{i=1}^{k-1} \mu'(x_i) = \sum_{i=1}^{k-1} \mu(x_i)$$

for all $(x_1, \ldots, x_{k-1}) \in X_1 \times \ldots \times X_{k-1}$. If $\mu(x_k) \geq 0$ for all $x_k \in X_k$, then $\mu' \sqcup \mu|_{X_k}$ satisfies the required properties, so we shall assume that $\mu'(x_k) < 0$ for some $x_k \in X_k$. Fix $\overline{x}_k \in X_k$ such that $\mu(\overline{x}_k) \leq \mu(x_k)$ for all $x_k \in X_k$ and take $(\overline{x}_1, \ldots, \overline{x}_{k-1}) \in X_1 \times \ldots \times X_{k-1}$ such that

$$\sum_{i=1}^{k-1} \mu'(\overline{x}_i) \leq \sum_{i=1}^{k-1} \mu'(x_i)$$

for all $(x_1, \ldots, x_{k-1}) \in X_1 \times \ldots \times X_{k-1}$. We have

$$\left(\sum_{i=1}^{k-1} \mu'(\overline{x}_i)\right) + \mu(\overline{x}_k) = \sum_{i=1}^{k} \mu(\overline{x}_i) \geq 0,$$

allowing us to define ν inductively in the following way: For any $x_1 \in X_1$, set

$$\nu(x_1) = \begin{cases} \mu'(x_1) + \mu(\overline{x}_k) & \text{if} \quad -\mu(\overline{x}_k) < \mu'(x_1) \\ \mu'(x_1) - \mu'(\overline{x}_1) & \text{if} \quad -\mu(\overline{x}_k) \geq \mu'(x_1) \end{cases}$$

and assuming that $\nu(x)$ has been defined for $x \in \bigsqcup_{i=1}^{n-1} X_i$ with $n < k$, we set

$$\nu(x_n) = \begin{cases} \mu'(x_n) + \mu(\overline{x}_k) + \sum_{i=1}^{n-1} \mu'(\overline{x}_i) & \text{if} \quad \sum_{i=1}^{n-1} \mu'(\overline{x}_i) < -\mu(\overline{x}_k) < \sum_{i=1}^{n} \mu'(\overline{x}_i) \\ \mu'(x_n) - \mu'(\overline{x}_n) & \text{if} \quad -\mu(\overline{x}_k) \geq \sum_{i=1}^{n} \mu'(\overline{x}_i) \\ \mu'(x_n) & \text{if} \quad \sum_{i=1}^{n-1} \mu'(\overline{x}_i) \geq -\mu(\overline{x}_k) \end{cases}.$$

Finally, for $x_k \in X_k$ we define $\nu(x_k) = \mu(x_k) - \mu(\overline{x}_k)$. It is clear that ν maps into $\mathbb{Z}_+$, so it remains only to check the summing up condition. By definition there is some $1 \leq n \leq k-1$ such that

$$\sum_{i=1}^{n-1} \mu'(\overline{x}_i) < -\mu(\overline{x}_k) \leq \sum_{i=1}^{n} \mu'(\overline{x}_i),$$

so for arbitrary $(x_1, \ldots, x_k)$ we have

$$\nu(x_i) = \begin{cases} \mu'(x_i) - \mu'(\overline{x}_i) & \text{if } \; i \leq n-1 \\ \mu'(x_n) + \mu(\overline{x}_k) + \sum_{j=1}^{n-1} \mu'(\overline{x}_j) & \text{if } \; i = n \\ \mu'(x_i) & \text{if } \; n < i < k \\ \mu(x_k) - \mu(\overline{x}_k) & \text{if } \; i = k \end{cases}.$$

We deduce that

$$\begin{aligned} \sum_{i=1}^{k} \nu(x_i) &= \left(\sum_{i=1}^{n-1} \mu'(x_i) - \mu'(\overline{x}_i) \right) + \mu'(x_n) + \mu(\overline{x}_k) + \left(\sum_{j=1}^{n-1} \mu'(\overline{x}_j) \right) \\ &+ \left(\sum_{i=n+1}^{k-1} \mu'(x_i) \right) + \mu(x_k) - \mu(\overline{x}_k) = \sum_{i=1}^{k} \mu(x_i). \end{aligned}$$

□

Lemma A.1.13. *Given finite sets $X_1, \ldots, X_k$, write $T = \bigsqcup_{i=1}^k X_i$, $S = \prod_{i=1}^k X_i$ and*

$$S_i = X_1 \times \ldots \times \widehat{X_i} \times \ldots \times X_k.$$

Define $M = \langle T \mid \{\boldsymbol{r}_{i,j}\} \rangle$ with

$$\boldsymbol{r}_{i,j} \colon \quad \sum_{x \in X_i} x = \sum_{x \in X_j} x$$

for all $1 \leq i, j \leq k$, and consider any function $f \colon S \to \mathbb{N}$. Then there is a well defined unitary embedding $\psi \colon M \to \mathbb{Z}_+(S)$ given by

$$\psi(x_i) = \sum_{(x_1, \ldots, \widehat{x_i}, \ldots, x_k) \in S_i} f(x_1, \ldots, x_k) \cdot (x_1, \ldots, x_k)$$

for $x_i \in X_i$.

Proof. First of all, let us see that ψ is in fact well-defined. Formally, we first define ψ on $\mathbb{Z}_+(T)$ as above. Then

$$\psi \left(\sum_{x \in X_i} x \right) = \sum_{y \in S} f(y) y$$

for any $1 \leq i \leq k$, so ψ drops to a homomorphism $M \to \mathbb{Z}_+(S)$. Note $\psi(M)$ is cofinal in $\mathbb{Z}_+(S)$ for the simple reason that $(x_1, \ldots, x_k) \leq \psi(x_i)$ for every $i = 1, \ldots, k$. Now, assume that $\psi(a) + b = \psi(a')$ for $a, a' \in M$. Writing

$$a = \sum_{i=1}^{k} \sum_{x_i \in X_i} \lambda_{x_i} x_i \quad \text{and} \quad a' = \sum_{i=1}^{k} \sum_{x_i \in X_i} \lambda'_{x_i} x_i,$$

we have

$$\psi(a) = \sum_{(x_1,\dots,x_k)\in S} \left(\sum_{i=1}^k \lambda_{x_i}\right) f(x_1,\dots,x_k)\cdot(x_1,\dots,x_k)$$
$$\psi(a') = \sum_{(x_1,\dots,x_k)\in S} \left(\sum_{i=1}^k \lambda'_{x_i}\right) f(x_1,\dots,x_k)\cdot(x_1,\dots,x_k),$$

hence $\sum_{i=1}^k \lambda_{x_i} \leq \sum_{i=1}^k \lambda'_{x_i}$ for all $(x_1,\dots,x_k)\in S$. Defining a function $\mu\colon T\to\mathbb{Z}$ by

$$\mu(x_i) = \lambda'_{x_i} - \lambda_{x_i}$$

for $x_i \in X_i$, we have $\sum_{i=1}^k \mu(x_i) \geq 0$ for all $(x_1,\dots,x_k)$. Then by Lemma A.1.12, there is a function $\nu\colon T\to\mathbb{Z}$ such that $\sum_{i=1}^k \nu(x_i) = \sum_{i=1}^k \mu(x_i)$ for all $(x_1,\dots,x_k)$. This allows us to set

$$a'' = \sum_{i=1}^k \sum_{x_i\in X_i} \nu(x_i)x_i \in M$$

and then $\psi(a)+\psi(a'') = \psi(a')$, hence $b=\psi(a'')$. Now it remains only to prove that ψ is injective. Writing $a,a'\in M$ as above, we can assume that for each $2\leq i\leq k$ there are $\overline{x}_i, x_i^1 \in X_i$ such that $\lambda_{\overline{x}_i} = \lambda_{x_i^1} = 0$, by simply imposing the relation in M. Assuming $\psi(a)=\psi(a')$ exactly means that

$$\sum_{i=1}^k \lambda_{x_i} = \sum_{i=1}^k \lambda'_{x_i}$$

for all $(x_1,\dots,x_k)$. We claim that $\lambda'_{\overline{x}_i} = \lambda_{x_i^1} = 0$ for all $2\leq i\leq k$. Indeed taking any $\overline{x}_1\in X_1$ we have

$$\lambda_{\overline{x}_1} = \sum_{i=1}^k \lambda_{\overline{x}_i} = \sum_{i=1}^k \lambda'_{\overline{x}_i} = \sum_{i=1}^k \lambda'_{\overline{x}_i} + \sum_{i=2}^k \lambda'_{x_i^1} = \lambda_{\overline{x}_1} + \sum_{i=2}^k (\lambda'_{\overline{x}_i} + \lambda_{x_i^1}).$$

It follows that

$$\lambda_{x_1} = \lambda_{x_1} + \sum_{i=2}^k \lambda_{\overline{x}_i} = \lambda'_{x_1} + \sum_{i=2}^k \lambda'_{\overline{x}_i} = \lambda'_{x_1}$$

for all $x_1\in X_1$. Similarly we deduce that $\lambda_x = \lambda'_x$ for all $x\in T$. This finishes the proof. □

Lemma A.1.14. *If*

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\mu"'] & N \arrow[d, "\overline{\mu}"] \\ P \arrow[r, "\overline{\psi}"'] & Q \end{tikzcd}$$

is a pushout in the category of abelian monoids and ψ is a unitary embedding, then $\overline{\psi}$ is a unitary embedding as well.

Proof. Given a diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ {\scriptstyle\mu}\downarrow & & \\ P & & \end{array}$$

as above, we define a relation $\to$ on $N \oplus P$ by $(n + \psi(m), p) \to (n, p + \mu(m))$ for all $m \in M, n \in N$ and $p \in P$. We note that $\to$ respects the addition on $N \oplus P$, hence so does the smallest equivalence relation $\sim$ containing $\to$. Now define $Q = \frac{N \oplus P}{\sim}$ and write $[n, p]$ for the equivalence class of (n, p). Setting $\overline{\psi}(p) = [0, p]$ and $\overline{\mu}(n) = [n, 0]$, it is easily seen that

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\overline{\mu}} \\ P & \xrightarrow[\overline{\psi}]{} & Q \end{array}$$

is a pushout diagram. By uniqueness of pushouts, it suffices to prove the claim for a pushout of this form. We start by proving that $\overline{\psi}(P)$ is cofinal in Q: Given $[n, p] \in Q$ we can take $m \in M$ such that $n \leq \psi(m)$. Then

$$[n, p] \leq [\psi(m), p] = [0, p + \mu(m)] = \overline{\psi}(p + \mu(m))$$

as required. For the other parts, we need the following claim.

Claim: Assume that $[\psi(m), p] = [n, p']$. Then there is some $m' \in M$ such that $n = \psi(m')$ and

$$\mu(m) + p = \mu(m') + p'.$$

Proof of claim: Since $\to$ is both reflexive and transitive, we either have

$$(\psi(m), p) \to (n, p') \quad \text{or} \quad (n, p') \to (\psi(m), p).$$

In the former case there is some $m'' \in M$ such that $\psi(m) = n + \psi(m'')$, hence $n = \psi(m')$ for some $m' \in M$ since ψ is a unitary embedding — by injectivity we even have $m = m' + m''$. Now we get

$$p' + \mu(m') = p + \mu(m'') + \mu(m') = p + \mu(m)$$

as promised. In the latter case $n = \psi(m) + \mu(m'')$ for some $m'' \in M$, and we set $m' = m + m''$. Then $n = \psi(m')$ and

$$p + \mu(m) = p' + \mu(m'') + \mu(m) = p' + \mu(m'),$$

finishing the proof of the claim.

Now assuming that $[0, p] = \overline{\psi}(p) = \overline{\psi}(p') = [0, p']$, the claim immediately implies $p = p'$, hence $\overline{\psi}$ in injective. Finally, assuming that

$$[n, p + p''] = \overline{\psi}(p) + [n, p''] = \overline{\psi}(p') = [0, p'],$$

there is some $m \in M$ such that $n = \psi(m)$. But then

$$[n, p''] = [\psi(m), p''] = [0, p'' + \mu(m)] = \overline{\psi}(p'' + \mu(m)) \in \overline{\psi}(P),$$

hence $\overline{\psi}$ is indeed a unitary embedding. $\square$

Refinement monoids

In the following we shall prove a multidimensional Soduko-like result on refinement properties in abelian monoids.

Definition A.1.15. M is said to be a *refinement monoid* if whenever $a+b=c+d$, there are $x,y,z,w \in M$ such that

$$a = x + y \ , \ b = z + w \ , \ c = x + z \quad \text{and} \quad d = y + w.$$

This is most easily understood visually as the ability to fill out a diagram as follows

$$\begin{array}{c|cc} + & c & d \\ \hline a & x & y \\ b & z & w \end{array}$$

Definition A.1.16. Consider an abelian monoid M and finite subsets $X_1, \ldots, X_k$ of M such that

$$\sum_{x_i \in X_i} x_i = \sum_{x_j \in X_j} x_j \quad \text{for all } 1 \leq i, j \leq k. \tag{A.1}$$

Then a *refinement* of the system of equations (A.1) is a set of elements

$$\{a(x_1, \ldots, x_k) \mid x_i \in X_i, i = 1, \ldots, k\}$$

such that

$$x_i = \sum_{j \neq i} \sum_{x_j \in X_j} a(x_1, \ldots, x_k)$$

for all $x_i \in X_i$ and $i = 1, \ldots, k$. Assuming that (and otherwise reordering such that)

$$|X_1| \leq \ldots \leq |X_k|,$$

we shall call (A.1) a $(|X_1|, \ldots, |X_k|)$-equation system and refer to it as having *dimension* k. With this terminology, a refinement monoid always has refinements of $(2,2)$-equation systems. However, as one might expect from the name, this property allows refinements of arbitrary equation systems.

Example A.1.17. Before we present the proof of the below lemma formally, it might be useful to get a visual presentation of what is going on. Let us assume that M is a refinement monoid, i.e. that we can always fill out a 2×2-diagram as above. Then we claim that we can always fill out a 2×3-diagram as well. Indeed, given a diagram

$$\begin{array}{c|ccc} + & c & d & e \\ \hline a & & & \\ b & & & \end{array}$$

we can fill out the 2×2 diagram

$$\begin{array}{c|cc} + & c & d+e \\ \hline a & x & y \\ b & z & w \end{array} .$$

In particular we have $d + e = y + w$, so we can fill out the diagram

$$\begin{array}{c|cc} + & d & e \\ \hline y & x' & y' \\ w & z' & w' \end{array}.$$

Combining these, we obtain the diagram

$$\begin{array}{c|ccc} + & c & d & e \\ \hline a & x & x' & y' \\ b & z & z' & w' \end{array},$$

thereby proving the claim. Essentially, we just apply this trick over and over again in the below proof.

Lemma A.1.18. *Assume that M is a refinement monoid. Then M has refinements of any equation system.*

Proof. We argue by induction over the dimension of the equation system as follows:

(a) If M is a refinement monoid, then it possesses refinements of all 2-dimensional equation systems.

(b) If M is an abelian monoid with refinements of all 2- and k-dimensional equation systems, then it possesses refinements of all $(k+1)$-dimensional equation systems as well.

Proof of (1): For the proof of this claim, we shall also proceed by induction, in this case over the first coordinate of the type of the system. However, then we will first need to prove that every $(2, m)$-equation system has a refinement, and (surprise, surprise) we shall do this by induction. The induction start is trivial, so assume that every $(2, m)$-equation system has a refinement, and consider a $(2, m+1)$-equation system

$$\sum_{i=1}^{2} x_1^i = \sum_{i=1}^{m+1} x_2^i.$$

We can regard this as the $(2, m)$-equation system $\sum_{i=1}^{2} x_1^i = (\sum_{i=1}^{m-1} x_2^i) + (x_2^m + x_2^{m+1})$, hence by assumption there is a set $\{a(x_1^i, x_2^j) \mid 1 \leq i \leq 2, 1 \leq j \leq m\}$ such that

$$\begin{aligned} x_1^i &= \sum_{j=1}^{m} a(x_1^i, x_2^j) \text{ for } i = 1, 2 \\ x_2^j &= \sum_{i=1}^{2} a(x_1^i, x_2^j) \text{ for } j = 1, \dots, m-1 \\ x_2^m + x_2^{m+1} &= \sum_{i=1}^{2} a(x_1^i, x_2^m). \end{aligned}$$

Note that the last equation is a $(2,2)$-equation system, hence there is a set

$$\{b(x_1^i, x_2^j) \mid i = 1,2 \text{ and } j = m, m+1\}$$

such that

$$a(x_1^i, x_2^m) = \sum_{j=m}^{m+1} b(x_1^i, x_2^j) \text{ for } i = 1,2 \quad \text{and} \quad x_2^j = \sum_{i=1}^{2} b(x_1^i, x_2^j) \text{ for } j = m, m+1.$$

Finally, define

$$c(x_1^i, x_2^j) = \begin{cases} a(x_1^i, x_2^j) & \text{if} \quad j \leq m-1 \\ b(x_1^i, x_2^j) & \text{if} \quad j = m, m+1 \end{cases}$$

and note that the set $\{c(x_1^i, x_2^j) \mid 1 \leq i \leq 2, 1 \leq j \leq m+1\}$ is a refinement of the $(2, m+1)$-equation system. This finishes the proof of the induction start — the proof of the induction step will follow the same pattern. Let $m_1 \geq 2$, assume that all (m_1, m_2)-equation systems have refinements and consider any (m_1+1, m_2)-equation system

$$\sum_{i=1}^{m_1+1} x_1^i = \sum_{i=1}^{m_2} x_2^i.$$

Regarding it as the (m_1, m_2)-equation system $\left(\sum_{i=1}^{m_1-1} x_1^i\right) + (x_1^m + x_1^{m+1}) = \sum_{i=1}^{m_2} x_2^i$, by assumption there is a set $\{a(x_1^i, x_2^j) \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2\}$ such that

$$\begin{aligned} x_1^i &= \sum_{j=1}^{m_2} a(x_1^i, x_2^j) \text{ for } i = 1, \ldots, m_1 - 1 \\ x_2^j &= \sum_{i=1}^{m_1} a(x_1^i, x_2^j) \text{ for } j = 1, \ldots, m_2 \\ x_1^{m_1} + x_1^{m_1+1} &= \sum_{j=1}^{m_2} a(x_1^{m_1}, x_2^j). \end{aligned}$$

Since the latter of the above equations is a $(2, m_2)$-equation system, there is a set

$$\{b(x_1^i, x_2^j) \mid m_1 \leq i \leq m_1+1, 1 \leq j \leq m_2\}$$

such that

$$x_1^i = \sum_{j=1}^{m_2} a(x_1^i, x_2^j) \text{ for } i = m_1, m_1+1$$

and

$$a(x_1^{m_1}, x_2^j) = \sum_{i=m_1}^{m_1+1} b(x_1^i, x_2^j) \text{ for } j = 1, \ldots, m_2.$$

Finally, we define

$$c(x_1^i, x_2^j) = \begin{cases} a(x_1^i, x_2^j) & \text{if} \quad i \leq m_1 - 1 \\ b(x_1^i, x_2^j) & \text{if} \quad i = m_1, m_1+1 \end{cases}$$

and note that $\{c(x_1^i, x_2^j) \mid 1 \leq i \leq m_1+1, 1 \leq j \leq m_2\}$ is a refinement of the (m_1+1, m_2)-equation system. This finishes the proof of the induction step.

Proof of (2): Given a $(k+1)$-dimensional equation system

$$\sum_{x_1 \in X_1} x_1 = \ldots = \sum_{x_k \in X_k} x_k = \sum_{x_{k+1} \in X_{k+1}} x_{k+1}$$

consider the k-dimensional equation system obtained by discarding the latter equality. By assumption there is a set $\{a(x_1, \ldots, x_k) \mid x_i \in X_i\}$ such that

$$x_i = \sum_{j \neq i} \sum_{x_j \in X_j} a(x_1, \ldots, x_k)$$

for all $x_i \in X_i$ and $i = 1, \ldots, k$. In particular

$$\sum_{x_k \in X_k} x_k = \sum_{x_k \in X_k} \sum_{j \neq k} \sum_{x_j \in X_j} a(x_1, \ldots, x_k) = \sum_{i=1}^{k} \sum_{x_i \in X_i} a(x_1, \ldots, x_k)$$

providing us with the 2-dimensional equation system

$$\sum_{i=1}^{k} \sum_{x_i \in X_i} a(x_1, \ldots, x_k) = \sum_{x_{k+1} \in X_{k+1}} x_{k+1}.$$

Thus there is a set $\{b(x_1, \ldots, x_{k+1}) \mid x_i \in X_i\}$ such that

$$a(x_1, \ldots, x_k) = \sum_{x_{k+1} \in X_{k+1}} b(x_1, \ldots, x_{k+1}) \text{ for } (x_1, \ldots, x_k) \in X_1 \times \ldots \times X_k$$
$$x_{k+1} = \sum_{i \neq k+1} \sum_{x_i \in X_i} b(x_1, \ldots, x_{k+1}) \text{ for } x_{k+1} \in X_{k+1}.$$

Hence for $i \leq k$ and $x_i \in X_i$ we also have

$$\begin{aligned} x_i &= \sum_{\substack{j=1 \\ j \neq i}}^{k} \sum_{x_j \in X_j} a(x_1, \ldots, x_k) = \sum_{j \neq i} \sum_{x_j \in X_j} \sum_{x_{k+1} \in X_{k+1}} b(x_1, \ldots, x_{k+1}) \\ &= \sum_{j \neq i} \sum_{x_j \in X_j} b(x_1, \ldots, x_{k+1}), \end{aligned}$$

proving that $\{b(x_1, \ldots, x_{k+1} \mid x_i \in X_i\}$ is indeed a refinement of the $(k+1)$-dimensional equation system. □

Definition A.1.19. Let M denote a conical abelian monoid. A *refinement* of M is another conical abelian monoid N together with a homomorphism $\iota\colon M \to N$ such that

(a) ι is a unitary embedding;

(b) N is a refinement monoid;

(c) Given a homomorphism $\varphi\colon M \to P$ with P a refinement monoid, there is a homomorphism $\widetilde{\varphi}\colon N \to P$ such that $\widetilde{\varphi} \circ \iota = \varphi$.

A.2 Rings, algebras and the functor $\mathcal{V}$

This appendix contains the ring-theoretic results and constructions that will be of use to us at some point. First however, we shall review some definitions to make sure that we are on the same page. Rings are in general not assumed to be unital, and homomorphisms of unital rings need not preserve the unit. If they do so, they will be referred to as *unital*.

Definition A.2.1. Let K denote an arbitrary field. A K-algebra is a ring A together with a scalar multiplication $K \times A \to A$ such that $1_K \cdot a = a$ for all $a \in A$. A $*$-algebra over an involutive field K is a K-algebra A equipped with an *involution*, that is a conjugate linear map $A \to A$, denoted $a \mapsto a^*$, of order two such that $(ab)^* = b^*a^*$ for all $a, b \in A$. Of course, a K-algebra homomorphism $\varphi\colon A \to B$ is nothing but a ring homomorphism satisfying $\varphi(ka) = k\varphi(a)$ for all $a \in A$, $k \in K$. Also, a $*$-homomorphism $\varphi\colon A \to B$ of $*$-algebras is an algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

Definition A.2.2. Given a $*$-algebra A. An element $p \in A$ is called a *projection* if

$$p = p^* = p^2,$$

and $x \in A$ is called a *partial isometry* if $xx^*x = x$. Note that for a partial isometry x, the elements $p(x) = xx^*$ and $p(x^*) = x^*x$ are projections, referred to as the *final* and the *initial* projection of x, respectively. Finally, a set of partial isometries X is called *tame*, if every element of the multiplicative semigroup U generated by $X \cup X^*$ is a partial isometry as well and the final projections $p(s), p(t)$ commute for any $s, t \in U$.

Definition A.2.3 (Morita equivalence)**.** Two rings A and B are said to be *Morita equivalent*, if there is an equivalence of the categories of left A-modules and the category of left B-modules. Morita equivalent rings share many properties, for instance they have equivalent categories of finitely generated projective left modules and have isomorphic ideal lattices. An idempotent $p \in A$ is called *full* if the corner pAp is not contained in any proper ideal of A, and in that case pAp is called a *full corner*. If a ring A is isomorphic to a full corner $pM_n(B)p$ for some n, then A and B are Morita-equivalent. In fact, for unital rings A and B, this is also a necessary condition. It is worth noting that Morita equivalence is in fact an equivalence relation.

Recall the following fact.

Lemma A.2.4. *A left A-module P is finitely generated and projective if and only if there is another left A-module Q such that $P \oplus Q \cong A^n$ for some n.*

Definition A.2.5 (The functor $\mathcal{V}$)**.** In the following we shall give a description and mention the very basic properties of the covariant functor $\mathcal{V}\colon$ **Rings** $\to$ **AbMon**. Now, one can define $\mathcal{V}$ in two naturally isomorphic ways, and both will be of use to us at some point. The first is as follows: For a ring A, let $\mathcal{V}(A)$ denote the set of isomorphism classes of finitely generated projective left A-modules. Then we can define an addition by $[P] + [Q] := [P \oplus Q]$, making $\mathcal{V}(A)$ into an abelian monoid. Functoriality is achieved by defining $\mathcal{V}(\varphi)\colon \mathcal{V}(A) \to \mathcal{V}(B)$ by $\mathcal{V}(\varphi)([P]) = [B \otimes_A P]$, where B is made into a right A-module via the ring homomorphism $\varphi\colon A \to B$. Note that $B \otimes_A P$ is a finitely generated projective B-module by Lemma A.2.4.

The other definition is as follows: Let $\mathcal{P}_\infty(A)$ denote the set of idempotents of $M_\infty(A)$, $M_\infty(A)$ being the ring of infinite matrices with entries in A of which only finitely many are non-zero. Define an equivalence relation on $\mathcal{P}_\infty(A)$ by

$$p \sim q \Leftrightarrow \text{there exist } a, b \in M_\infty(A) \text{ such that } p = ab \text{ and } q = ba.$$

Then we define $\mathcal{V}(A) = \mathcal{P}_\infty(A)/\sim$ and equip $\mathcal{V}(A)$ with an addition defined as follows: Given idempotents $p, q \in M_\infty(A)$, we may regard each of them as lying in finite matrix rings, i.e. $p \in M_m(A)$ and $q \in M_m(A)$. Then we can form the diagonal sum

$$p \oplus q = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \in M_{m+n}(A)$$

and set $[p]+[q] = [p\oplus q]$. This is easily seen to be independent of the choice of m and n. Finally, given a ring homomorphism $\varphi\colon A \to B$, we define $\mathcal{V}(\varphi)([p]) = [\varphi(p)]$, where $\varphi(p)$ is the matrix obtained by applying φ to every entry in p. We can translate between these two descriptions as follows: Given a finitely generated projective left A-module P, by possible changing P to an isomorphic module we have $P \oplus Q = A^n$ for some A-module Q. Thus the projection $P \oplus Q \to P$ is an idempotent linear map $A^n \to A^n$, so it corresponds to right multiplication by an idempotent element of $M_n(A)$. Conversely, given an idempotent $p \in M_n(A)$, the left A-module $A^n p$ is a finitely generated projective left A-module, and one can check that these maps give mutually inverse natural isomorphisms of the associated monoids. It is fairly easy to check that $\mathcal{V}$ is continuous, i.e. that $\mathcal{V}(\varinjlim_i A_i) \cong \varinjlim_i \mathcal{V}(A_i)$ for any directed system of rings. If A and B are Morita equivalent then $\mathcal{V}(A) \cong \mathcal{V}(B)$, and referring to K-theory, $K_0(A)$ is exactly the Groethendieck group of $\mathcal{V}(A)$ in case A is unital.

Definition A.2.6 (The Bergman algebra)**.** Given a unital K-algebra A and finitely generated projective left A-modules P and Q, George M. Bergman constructed in [7] a unital K-algebra B together with a unital K-algebra homomorphism $\iota\colon A \to B$ (making B into an A-module) and a left B-module isomorphism $\mu\colon B \otimes_A P \to B \otimes_A Q$, satisfying the following universal property: For any unital K-algebra homomorphism $A \to C$ and any left C-module isomorphism

$$\varphi\colon C \otimes_A P \to C \otimes_A Q,$$

there is a unique K-algebra homomorphism $\psi\colon B \to C$ (giving C a B-module structure) such that the diagram

$$\begin{tikzcd} C \otimes_B (B \otimes_A P) \arrow[r, "\mathrm{Id}_C \otimes \mu"] \arrow[d, "\cong"'] & C \otimes_B (B \otimes_A Q) \arrow[d, "\cong"] \\ C \otimes_A P \arrow[r, "\varphi"'] & C \otimes_A Q \end{tikzcd}$$

commutes. We shall refer to B as the *Bergman algebra obtained from adjoining an isomorphism between* P and Q. This construction is important to us because of the following deep result:

Theorem A.2.7. *Assume that A is a unital K-algebra with finitely generated projective left A-modules P and Q, and denote by $(B, \iota\colon A \to B)$ the Bergman algebra obtained from adjoining an isomorphism between P and Q along with the universal algebra homomorphism. Then $\mathcal{V}(\iota)$ drops to an isomorphism*

$$\frac{\mathcal{V}(A)}{[P]=[Q]} \to \mathcal{V}(B).$$

Proof. See [7, Theorem 5.2]. $\square$

We shall also need the concept of a double centralizer.

Definition A.2.8. Let A denote a K-algebra. A *double centralizer* is an ordered pair (L, R) of K-linear maps $A \to A$ such that

(a) $L(ab) = L(a)b$

(b) $R(ab) = aR(b)$

(c) $R(a)b = aL(b)$

for all $a, b \in A$. For such a pair, we shall refer to L as a left multiplier and R as a *right multiplier*. Note that for any $a \in A$, we can define a double centralizer (L_a, R_a) by

$$L_a(b) = ab \quad \text{and} \quad R_a(b) = ba.$$

Definition A.2.9. An algebra A is called

- *non-degenerate* if for any non-zero $a \in A$, there is some $b \in A$ such that $ab \neq 0$ or $ba \neq 0$.
- *idempotent* if $A^2 = A$.
- (L, R)-*associative* if
$$R' \circ L = L \circ R'$$
for all double centralizers $(L, R), (L', R')$ of A.

Proposition A.2.10. *Assume that A is non-degenerate or idempotent. Then it is (L, R)-associative as well.*

Proof. Assume first that A is non-degenerate and take double centralizers $(L, R), (L', R')$ of A along with $a \in A$. Then

$$R'(L(a))b = L(a)L'(b) = L(aL'(b)) = L(R'(a)b) = L(R'(a))b$$

and

$$bR'(L(a)) = R'(bL(a)) = R'(R(b)a) = R(b)R'(a) = bL(R'(a))$$

for any $b \in A$, hence $L \circ R'(a) = R' \circ L(a)$.

Now assume that A is idempotent and take $a \in A$. By assumption we may write $a = bc$ for $b, c \in A$, hence

$$R'(L(a)) = R'(L(bc)) = R'(L(b)c) = L(b)R'(c) = L(bR'(c)) = L(R'(bc)) = L(R'(a))$$

as desired. □

Later on we shall also need the following minor lemma.

Lemma A.2.11. *Given a ring A and elements $a_1, \dots, a_k, b_1, \dots, b_l \in A$. Then*

$$[a_1 \cdots a_k, b_1 \cdots b_l] \in \sum_{i,j} A[a_i, b_j]A.$$

Proof. We proceed by induction over $k+l$. If $k+l = 2$ the claim is trivial, so we shall assume that the claim holds for all k, l such that $k+l \leq n$ for some $n \geq 2$. Taking k, l with $k+l = n+1$ and elements $a_1, \dots, a_k, b_1, \dots, b_l \in A$, we can assume without loss of generality that $k \geq 2$. Writing $a = a_1 \cdots a_{k-1}$ and $b = b_1 \cdots b_l$ we then have

$$\begin{aligned}[a_1 \cdots a_k, b_1 \cdots b_l] &= aa_kb - baa_k = aa_kb - aba_k + aba_k - baa_k \\ &= a[a_k, b] + [a, b]a_k,\end{aligned}$$

and we can apply the inductive assumption to obtain the claim. □

A.3 C^*-algebras

Although C^*-algebras play a crucial role in this thesis, strictly speaking the reader need not have any prior knowledge of C^*-algebras. Therefore, we will provide some basic definitions and results in this minor appendix.

Definition A.3.1. A C^*-algebra $\mathcal{A}$ is a complex $*$-algebra equipped with a Banach space norm satisfying $\|ab\| \leq \|a\| \cdot \|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in \mathcal{A}$. A $*$-homomorphism of C^*-algebras is just an algebraic $*$-homomorphism, since they are automatically contractive, and a $*$-homomorphism $\mathcal{A} \to B(H)$ for some Hilbert space H is called a *representation* of $\mathcal{A}$. Finally, an ideal $\mathcal{I}$ of $\mathcal{A}$ is a two-sided closed ideal — it will then automatically be self-adjoint — and the quotient $\mathcal{A}/\mathcal{I}$ is again a C^*-algebra.

Remark A.3.2 (Commutative C^*-algebras)**.** It is a basic fact that any unital commutative C^*-algebra $\mathcal{A}$ is canonically isomorphic to the C^*-algebra $C(X)$ of continuous functions $X \to \mathbb{C}$ on a uniquely determined compact Hausdorff space X. The assignment $X \mapsto C(X)$ can be made into a contravariant functor in the obvious way, i.e. for any continuous map $\theta\colon X \to Y$, we define $\theta^*\colon C(Y) \to C(X)$ by $\theta^*(f) = f \circ \theta$. Then the correspondence $X \leftrightarrow C(X)$ is in fact a contravariant equivalence between the category of compact Hausdorff spaces and the category of unital commutative C^*-algebras. Finally, the reader should note that an ideal $\mathcal{I}$ in the C^*-algebra $C(X)$ is necessarily of the form

$$\{f \in C(X) \mid f(x) = 0 \text{ for } x \notin U\}$$

for some open subspace $U \subset X$.

Definition A.3.3 (Morita equivalence)**.** Morita equivalence can be defined for arbitrary C^*-algebras in terms of so-called *imprimitivity bimodules*, but for C^*-algebras with countable approximate identities (which include all unital and separable C^*-algebras), there is an equivalent and more accessible definition. Indeed, two such C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent* if $\mathcal{A} \otimes \mathbb{K} \cong \mathcal{B} \otimes \mathbb{K}$, where $\mathbb{K}$ denotes the C^*-algebra of compact operators on some separable Hilbert space. As for rings, Morita equivalence preserves many properties, including the K-theory and the ideal lattice, and if $p \in \mathcal{A}$ is a full projection of $\mathcal{A}$, then the full corner $p\mathcal{A}p$ is Morita-equivalent to $\mathcal{A}$.

Remark A.3.4 (The universal enveloping C^*-algebra)**.** Given a $*$-algebra A, one can define the (possibly infinite) number

$$\|a\| = \sup\{\|\pi(a)\| \mid \pi : A \to B(H) \text{ is a representation}\}$$

for $a \in A$. In case $\|a\| < \infty$ for all $a \in A$, $\|\cdot\|$ defines a semi-norm on A, hence

$$I = \{a \in A \mid \|a\| = 0\}$$

is a two-sided ideal in A. The norm induced on the quotient A/I satisfies all the C^*-axioms, however A/I need not be complete. The completion $C_u^*(A)$ of A/I is called the *universal enveloping C^*-algebra of A*. It enjoys the following universal property: For any $*$-homomorphism $\varphi\colon A \to \mathcal{B}$ into a C^*-algebra $\mathcal{B}$, there is a $*$-homomorphism $C_u^*(A) \to \mathcal{B}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & \mathcal{B} \\ C_u^*(A) \arrow[ur, dashed] & \end{tikzcd}$$

commutes. It is a general fact that any Banach $*$-algebra with an approximate unit has a universal enveloping C^*-algebra, so in particular this holds for unital Banach $*$-algebras.

Limits always exist in the category of C^*-algebras. At one point we shall need the following result on inductive limits, which can easily be generalized to arbitrary inductive sequences if one desires so.

Lemma A.3.5. *Given an inductive sequence of C^*-algebras*

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \mathcal{A}_3 \xrightarrow{\varphi_3} \ldots$$

with surjective transit maps and inductive limit $(A, \{\mu_n\})$. Then

$$\ker(\mu_1) = \overline{\bigcup_{n\geq 2} \ker(\varphi_{1,n})}$$

where $\varphi_{1,n} = \varphi_{n-1} \circ \ldots \circ \varphi_1 \colon \mathcal{A}_1 \to \mathcal{A}_n$.

Proof. Define an ideal of $\mathcal{A}_1$ by $\mathcal{I} = \overline{\bigcup_{n\geq 2} \ker(\varphi_{1,n})}$ and write $\overline{\varphi_{1,n}}$ for the induced isomorphism $\mathcal{A}/\ker(\varphi_{1,n}) \to \mathcal{A}_n$. Since all the transit maps are surjections, we can define $*$-homomorphisms

$$\pi_n \colon \mathcal{A}_n \xrightarrow{\overline{\varphi_{1,n}}^{-1}} \frac{\mathcal{A}_1}{\ker(\varphi_{1,n})} \to \frac{\mathcal{A}_1}{\mathcal{I}}$$

for all $n \geq 1$ — note that π_1 is just the quotient map. We claim that the diagram

$$\begin{tikzcd} \mathcal{A}_n \arrow[r, "\varphi_n"] \arrow[dr, "\pi_n"'] & \mathcal{A}_{n+1} \arrow[d, "\pi_{n+1}"] \\ & \mathcal{A}_1/\mathcal{I} \end{tikzcd}$$

commutes for all n. To see this, we note that both the upper and lower triangle of the diagram

$$\begin{tikzcd} \mathcal{A}_n \arrow[r, "\varphi_n"] \arrow[d, "\overline{\varphi_{1,n}}^{-1}"'] & \mathcal{A}_{n+1} \arrow[r, "\overline{\varphi_{1,n+1}}^{-1}"] & \mathcal{A}_1/\ker(\varphi_{1,n+1}) \arrow[d] \\ \mathcal{A}_1/\ker(\varphi_{1,n}) \arrow[urr] \arrow[rr] & & \mathcal{A}_1/\mathcal{I} \end{tikzcd},$$

where all the non-labeled arrows are the obvious quotient maps, commutes. In particular the outer diagram commutes, i.e. $\pi_{n+1} \circ \varphi_n = \pi_n$. Thus we obtain a $*$-homomorphism $\pi \colon \mathcal{A} \to \mathcal{A}_1/\mathcal{I}$ as in the diagram

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, "\mu_1"] \arrow[dr, "\pi_1"'] & \mathcal{A} \arrow[d, "\pi"] \\ & \mathcal{A}_1/\mathcal{I} \end{tikzcd}.$$

By definition $\mathcal{I} \subset \ker(\mu_1)$, and μ_1 clearly drops to an inverse of π, hence π is an isomorphism. In particular we have

$$\ker(\mu_1) = \ker(\pi_1) = \mathcal{I}.$$

$\square$

Bibliography

[1] Fernando Abadie. Enveloping actions and Takai duality for partial actions. *J. Funct. Anal.*, 197, 2003.

[2] Pere Ara. Purely infinite simple reduced C*-algebras of one-relator separated graphs. *J. Math. Anal. Appl.*, 393, 2012.

[3] Pere Ara and Ruy Exel. Dynamical systems associated to separated graphs, graph algebras and paradoxical decompositions. *Advances in Mathematics*, 252, 2014.

[4] Pere Ara, Ruy Exel, and Takeshi Katsura. Dynamical systems of type (m, n) and their C*-algebras. *Ergodic Theory and Dynamical Systems*, 33, 2013.

[5] Pere Ara and Ken R. Goodearl. C*-algebras of separated graphs. *J. Funct. Anal.*, 261, 2011.

[6] Pere Ara and Ken R. Goodearl. Leavitt path algebras of separated graphs. *J. reine angew.*, 669, 2012.

[7] George M. Bergman. Coproducts and some universal ring constructions. *Transactions of the American Mathematical Society*, 200, 1974.

[8] Ruy Exel and Michael Dokuchaev. Associativity of crossed products by partial actions, enveloping actions and partial representations. *Trans. Amer. Math. Soc.*, 357, 2005.

[9] Ruy Exel, Marcelo Laca, and John Quigg. Partial dynamical systems and C*-algebras generated by partial isometries. *J. Operator Theory*, 47, 2002.

[10] Peter Freyd. Redei's finiteness theorem for commutative semigroups. *Proc. Amer. Math. Soc.*, 19, 1968.

[11] David Kerr. C*-algebras and topological dynamics: Finite approximation and paradoxicality.

[12] Kevin McClanahan. K-theory for partial crossed products by discrete groups. *J. Funct. Anal.*, 130, 1995.

[13] Mikael Rørdam and Adam Sierakowski. Purely infinite C*-algebras arising from crossed products. *Ergodic Theory and Dynamical Systems*, 32, 2012.

[14] Stan Wagon. *The Banach-Tarski Paradox*. Cambridge University Press, 1985.