Embedding C^*-algebras into $\mathcal{O}_2$

Master thesis
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Introduction

The purpose of this thesis is to prove that every separable, nuclear C^*-algebra embeds into (i.e. is isomorphic to a sub-C^*-algebra of) the Cuntz algebra $\mathcal{O}_2$. This is a slight modification of Kirchberg's Exact Embedding Theorem that every separable, exact C^*-algebra can be embedded into $\mathcal{O}_2$, see [KP].

The salient feature of nuclear C^*-algebras is that their tensor products with arbitrary C^*-algebras are particularly well-behaved, and so are *-homomorphisms which are defined on or take values in a separable, nuclear C^*-algebra. Another important property which a C^*-algebra can have is being purely infinite, which means that it contains plenty of infinite projections.

A particularly interesting nuclear and purely infinite C^*-algebra which plays a central part in this thesis is the Cuntz algebra $\mathcal{O}_2$. In addition to embedding various C^*-algebras into it we also prove that any two unital, injective *-homomorphisms $\varphi, \psi \colon A \to \mathcal{O}_2$ are approximately unitarily equivalent if A is unital, separable and exact. This leads to an important uniqueness result for $\mathcal{O}_2$.

Moreover, we show that the K_0-group of $\mathcal{O}_2$ is zero (in fact, $K_1(\mathcal{O}_2)$ is zero, too). Using Kirchberg's Embedding Theorem and the result on approximate unitary equivalence it can be shown that $\mathcal{O}_2$ can even be regarded as a tensorial zero for unital, separable, simple and nuclear algebras, i.e. $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ for every C^*-algebra A having these properties.

Kirchberg's Embedding Theorem is situated in the field of classification theory, the fundamental concept of which it is to assign to each C^*-algebra one or several objects such that two C^*-algebras are isomorphic if and only if their corresponding objects are isomorphic. Based on Kichberg's Embedding Theorem and the above $A \otimes \mathcal{O}_2$ statement, Kirchberg and Phillips independently proved a spectacular classification result for Kirchberg algebras (separable, simple, nuclear and purely infinite C^*-algebras).

The line of argument in this thesis mainly follows Chapter 6 in [R2], at some places giving modified proofs based on [KR] or personal discussions with Professor Mikael Rørdam.

The main changes happen in the following places: In the proof of the statement on approximate unitary equivalence we use recent results from [KR] to perform the step from approximate similarity via isometries to approximate unitary equivalence. In the proof of Kirchberg's Embedding Theorem we modify the first part of the embedding procedure as follows: In the original proof, a separable, nuclear C^*-algebra A is embedded into $\mathcal{K} \otimes C(\mathbb{T}) \otimes A \cong C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z}$, while we show that A embeds into $(C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A \cong C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z}$.

This thesis is organized as follows: Chapter 1 serves as a toolbox for the whole paper, providing a collection of definitions and statements both in fundamental C^*-algebra theory and in more specific areas such as purely infinite or nuclear C^*-algebras. In Chapter 2 we give the definition of the Cuntz algebras $\mathcal{O}_n$, state some of their most significant properties and present the proof that $K_0(\mathcal{O}_2)$ is zero.

Chapter 3 deals with the first steps towards the proof of Kirchberg's Embedding Theorem and concludes with the important result that any two unital, injective *-homomorphisms from a unital, separable, exact C^*-algebra into $\mathcal{O}_2$ are approximately unitarily equivalent.

In Chapter 4 we digress a bit from the main ideas of the proof of the embedding theorem to discuss limit algebras and to collect statements about them which we need later on. In particular we show that approximate unitary equivalence can be turned into exact unitary equivalence by passing to limit algebras.

Chapter 5 opens with the proof that the C^*-algebra $C(\mathbb{T})$ of continuous, complex valued functions on the unit circle embeds into $\mathcal{O}_2$, then we return to the main line of the argument and use the results from Chapter 3 to prove the first embedding results for exact C^*-algebras.

Chapter 6 is used to introduce the notion of discrete crossed products and to prove that a C^*-algebra A can be embedded into $(C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A \cong C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z}$. We apply this result in Chapter 7, where we prove Kirchberg's Embedding Theorem and present its first important consequence, namely the existence of non-zero *-homomorphisms between Kirchberg algebras.

In Appendix A we discuss a uniqueness property of $\mathcal{O}_2$ and the $A \otimes \mathcal{O}_2$-Theorem, which are also consequences of the embedding theorem. In Appendix B we just take a quick glance at Kirchberg and Phillips' Classification Theorem to give an idea of what a striking result could be proved after Kirchberg presented his famous embedding result.

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Chapter 1

Toolbox

This chapter is a collection of fundamental definitions and statements for this thesis. After some general C^*-algebra theory we first turn to equivalence relations of projections in C^*-algebras and give a short account of the definition of the K_0-group of a unital C^*-algebra. Then we introduce the concepts of purely infinite and nuclear C^*-algebras which are going to play a central part in this thesis.

1.1 C^*-algebra theory

This section starts with some basic definitions and important standard results in C^*-algebra theory, and then presents some concepts which will be needed later on in this thesis, among them matrix algebras, homotopy equivalence and special elements, like projections or positives, in C^*-algebras.

1.1.1 Preliminaries

A good reference for the material covered in this section is [Mu].

Definition 1.1.1. A *C^*-algebra* is a complex algebra A equipped with a norm $\|\cdot\|$ and an involution * such that

(i) A is complete with respect to $\|\cdot\|$;

(ii) $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$;

(iii) $\|a^*a\| = \|a\|^2$ for all $a \in A$.

A norm on A which satisfies (ii) and (iii) is called a *C^*-norm* on A. A C^*-algebra A is said to be *unital* if it has a multiplicative identity, usually denoted by 1_A, it is said to be *separable* if it contains a countable, dense subset, and it is called *simple* if it has no non-trivial, two-sided, closed ideals.

Definition 1.1.2. Let A and B be C^*-algebras. A ***-homomorphism** from A into B is a linear map $\varphi\colon A \to B$ satisfying $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = (\varphi(a))^*$ for all $a, b \in A$, i.e. φ is *multiplicative* and *self-adjoint*. If A and B are unital with units 1_A and 1_B, respectively, then a *-homomorphism φ is said to be *unital* if $\varphi(1_A) = 1_B$.

Remarks 1.1.3. Let A and B be C^*-algebras and let $\varphi\colon A \to B$ be a *-homomorphism.

(i) The image $\varphi(A)$ is a sub-C^*-algebra of B. In particular, if φ is injective, then A is isomorphic to a sub-C^*-algebra of B. We then often say that A embeds into B, in symbols $A \hookrightarrow B$, and call φ an embedding of A into B.

(ii) All *-homomorphisms are *norm-decreasing*, i.e. $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$.

(iii) A *-homomorphism is isometric, i.e. $\|\varphi(a)\| = \|a\|$ for all $a \in A$, if and only if it is injective.

Notation. For every locally compact Hausdorff space X let $C_0(X)$ denote the C^*-algebra of continuous functions $f\colon X \to \mathbb{C}$ with the following property: For each $\varepsilon > 0$ there is a compact subset K of X such that $|f(x)| \leq \varepsilon$ for all $x \in X \setminus K$. If X is compact this coincides with $C(X)$, the C^*-algebra of all continuous functions $f\colon X \to \mathbb{C}$.

Theorem 1.1.4 (Gelfand)**.** Let A be an abelian C^*-algebra. Then there exists a locally compact Hausdorff space X such that A is *-isomorphic to $C_0(X)$. If, additionally, A is unital, then A is *-isomorphic to $C(X)$ for a compact Hausdorff space X.

Remark 1.1.5. Let A be a C^*-algebra. There exists a unique unital C^*-algebra $\widetilde{A}$ that contains A as an ideal and that satisfies $\widetilde{A}/A \cong \mathbb{C}$. The C^*-algebra $\widetilde{A}$ is called the *unitization* of A and can be written as

$$\widetilde{A} = \big\{a + \alpha 1_{\widetilde{A}} \mid a \in A,\ \alpha \in \mathbb{C}\big\}.$$

Let B also be a C^*-algebra and let $\varphi\colon A \to B$ be a *-homomorphism. Then there exists a unique unital *-homomorphism $\widetilde{\varphi}\colon \widetilde{A} \to \widetilde{B}$ which extends φ, and $\widetilde{\varphi}$ is given by $\widetilde{\varphi}(a + \alpha 1_{\widetilde{A}}) = \varphi(a) + \alpha 1_{\widetilde{B}}$ for all $a \in A$ and all $\alpha \in \mathbb{C}$.

Remark 1.1.6. Let A be a separable C^*-algebra. Then $\widetilde{A}$ also is separable, because when $\{a_1, a_2, \dots\}$ is a countable, dense subset of A, then $\{a_n + \gamma 1_{\widetilde{A}} \mid n \in \mathbb{N},\ \gamma \in \mathbb{Q} + i\mathbb{Q}\}$ is a countable, dense subset of $\widetilde{A} = \{a + \alpha 1_{\widetilde{A}} \mid a \in A,\ \alpha \in \mathbb{C}\}$.

Definition 1.1.7. Let A be a C^*-algebra and let $a \in A$. If A is unital we define the *spectrum* $\sigma(a)$ of a to be the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda 1_A - a \text{ is not invertible in } A\}.$$

If A is non-unital, then a is considered to be an element of the unitization $\widetilde{A}$ of A and the spectrum $\sigma(a)$ is defined in the unital C^*-algebra $\widetilde{A}$.

Theorem 1.1.8 (Functional calculus)**.** Let A be a unital C^*-algebra and let $a \in A$ be a normal element, i.e. $a^*a = aa^*$. Let $z\colon \sigma(a) \to \mathbb{C}$ denote the inclusion map. Then there exists a unique unital, injective *-homomorphism $\varphi\colon C(\sigma(a)) \to A$ such that $\varphi(z) = a$. The image $\mathrm{Im}(\varphi)$ is the sub-C^*-algebra of A generated by a and 1_A, i.e.

$$C(\sigma(a)) \;\cong\; C^*(a, 1_A).$$

Representing C^*-algebras on Hilbert spaces

Definition 1.1.9. A *representation* of a C^*-algebra A is a pair (H, π) where H is a Hilbert space and $\pi\colon A \to B(H)$ is a *-homomorphism. A representation (H, π) is said to be *faithful* if π is injective. We often refer to π as a *-representation of A on the Hilbert space H.

Theorem 1.1.10 (Gelfand-Naimark)**.** Every C^*-algebra admits a faithful representation.

Notation. Let V be a normed linear space and let W be a subset of V. Then $[W]$ denotes the closure of the linear span of W in V.

Remark 1.1.11. Let A be a C^*-algebra and let (H, π) be a faithful representation of A. Then $K = [\pi(A)H]$ is a Hilbert space which is contained in H. For each $a \in A$ and for each $\xi \in K$ define $\pi_K(a)\xi = \pi(a)\xi$, then $\pi_K\colon A \to B(K)$ defines a faithful *-representation of A such that $[\pi_K(A)K] = K$, i.e. π_K is *non-degenerate.*

The following lemma will be needed in some norm estimates in Chapter 3, and at this stage it serves as an example of how the properties of a C^*-norm and the Gelfand-Naimark Theorem can be used.

Lemma 1.1.12. Let A be a C^*-algebra and let $a, b \in A$ with $a^*b = 0 = ab^*$. Then

$$\|a + b\| = \max\{\|a\|, \|b\|\}.$$

Proof. We first show that $\|a + b\| \leq \max\{\|a\|, \|b\|\}$. Put $x = a + b$ and notice that

$$x^*x = a^*a + a^*b + b^*a + b^*b = a^*a + b^*b$$

and that $(x^*x)^n = (a^*a)^n + (b^*b)^n$ for some $n \in \mathbb{N}$ implies that

$$\begin{aligned}(x^*x)^{n+1} &= \left((a^*a)^n + (b^*b)^n\right)(a^*a + b^*b) \\ &= (a^*a)^{n+1} + (a^*a)^n\, b^*b + (b^*b)^n\, a^*a + (b^*b)^{n+1} = (a^*a)^{n+1} + (b^*b)^{n+1}\,.\end{aligned}$$

It follows by induction that $(x^*x)^n = (a^*a)^n + (b^*b)^n$ for all $n \in \mathbb{N}$, and thus

$$\|(x^*x)^n\| \leq \|(a^*a)^n\| + \|(b^*b)^n\| \leq \|a\|^{2n} + \|b\|^{2n} \qquad \text{for all } n \in \mathbb{N},$$

which implies

$$\|x\|^{2^m} = \|(x^*x)^{2^{m-1}}\| \leq \|a\|^{2^m} + \|b\|^{2^m} \leq 2\left(\max\{\|a\|, \|b\|\}\right)^{2^m} \qquad \text{for all } m \in \mathbb{N}_0.$$

Thus,

$$\|x\| \leq 2^{\frac{1}{2^m}} \max\{\|a\|, \|b\|\} \qquad \text{for all } m \in \mathbb{N}_0,$$

and hence $\|a + b\| = \|x\| \leq \max\{\|a\|, \|b\|\}$. To show the other inequality choose a faithful *-representation $\pi\colon A \to B(H)$ for some Hilbert space H. Let $T = \pi(a)$ and $S = \pi(b)$, then $T^*S = 0 = ST^*$ and therefore

$$\langle T\xi, S\eta\rangle = \langle \xi, T^*S\eta\rangle = 0 \qquad \text{for all } \xi, \eta \in H,$$

i.e. $T(H) \perp S(H)$. This implies (see, for instance, Theorem 12.2 in [Ru]) that

$$\|T\xi\| \leq \|(T + S)\xi\| \qquad \text{for all } \xi \in H,$$

and therefore $\|T\| \leq \|T + S\|$. It follows in the same way that $\|S\| \leq \|T + S\|$ and hence $\|T + S\| \geq \max\{\|T\|, \|S\|\}$. As π is isometric this implies that $\|a + b\| \geq \max\{\|a\|, \|b\|\}$, as required. $\square$

Multiplier algebras and essential ideals

The following facts about multiplier algebras and essential ideals can all be looked up in [Mu] (see pages 38–39 for multiplier algebras and Theorem 3.1.8 for essential ideals).

Proposition/Definition 1.1.13. Let A be a C^*-algebra. A pair (L, R) of bounded, linear maps on A is called a *double centralizer* for A if

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad R(a)b = aL(b) \qquad \text{for all } a, b \in A.$$

The set of all double centralizers for A can be equipped with algebraic operations and a norm which make it a unital C^*-algebra, which is called the *multiplier algebra* of A, denoted by $\mathcal{M}(A)$.

Remark 1.1.14. Let A be a C^*-algebra. Then A is unital if and only if $A = \mathcal{M}(A)$.

Definition 1.1.15. Let A be a C^*-algebra. A closed, two-sided ideal I of A is said to be an *essential ideal in* A, in symbols $I \overset{\text{ess}}{\lhd} A$, if

$$aI = 0 \quad \Longrightarrow \quad a = 0 \qquad \text{for all } a \in A.$$

Remark 1.1.16. Each C^*-algebra A is contained as an essential ideal in its multiplier algebra $\mathcal{M}(A)$, and $\mathcal{M}(A)$ is the largest unital C^*-algebra which contains A as an essential ideal. The second statement is a consequence of the following statement, which is part of Theorem 3.1.8 in [Mu]:

Lemma 1.1.17. Let I be an essential ideal in a C^*-algebra A. Then there is a unique injective *-homomorphism $\varphi\colon A \to \mathcal{M}(I)$ extending the inclusion $I \hookrightarrow \mathcal{M}(I)$. If A is a unital C^*-algebra, then φ is a unital *-homomorphism.

We use this lemma to prove the following equivalence for essential ideals:

Lemma 1.1.18. Let A be a C^*-algebra and let I be an ideal in A. The following statements are equivalent:

(i) The ideal I is essential in A;

(ii) There exists a faithful representation $\pi\colon A \to B(H)$ for some Hilbert space H such that $[\pi(I)H] = H$.

Proof. (i)$\Rightarrow$(ii): Suppose that I is essential in A and let $\pi_0\colon I \to B(H)$ be a *-representation of I on some Hilbert space H. By Remark 1.1.11 we can assume that π_0 is non-degenerate, i.e. $[\pi_0(I)H] = H$. By Corollary 3.12.5 in [Pe1], π_0 extends to a faithful *-representation $\pi_1\colon \mathcal{M}(I) \to B(H)$. As I is essential in A, Lemma 1.1.17 yields an injective *-homomorphism $\iota\colon A \to \mathcal{M}(I)$ extending the inclusion of I into $\mathcal{M}(I)$. Set $\pi = \pi_1 \circ \iota\colon A \to B(H)$, then π is a faithful *-representation of A and

$$[\pi(I)H] = [\pi_1(I)H] = [\pi_0(I)H] = H.$$

(ii)$\Rightarrow$(i): Let $\pi\colon A \to B(H)$ be a faithful *-representation of A such that $[\pi(I)H] = H$. Let $a \in A$ be such that $ax = 0$ for all $x \in I$. Then $\pi(ax) = \pi(a) \circ \pi(x) = 0$ for all $x \in I$, i.e. $\pi(I)H \subseteq \ker(\pi(a))$. Since $\ker(\pi(a))$ is a closed linear subspace of H we can conclude that $H = [\pi(I)H] = \ker(\pi(a))$, i.e. $\pi(a) = 0$. As π is faithful, this implies that $a = 0$. $\square$

Matrix algebras

Definition 1.1.19. Let A be a C^*-algebra and let $n \in \mathbb{N}$. We denote by $M_n(A)$ the set of all $n \times n$ matrices $(a_{ij})_{i,j}$ with entries in A.

Proposition 1.1.20. Let A be a C^*-algebra and let $n \in \mathbb{N}$. Equip $M_n(A)$ with entry-wise vector space operations, matrix multiplication and the involution given by $(a_{ij})^*_{i,j} = (a^*_{ji})_{i,j}$ for each $(a_{ij})_{i,j}$ in $M_n(A)$. Then $M_n(A)$ is a complex, involutive algebra and there is a unique norm on $M_n(A)$ making it a C^*-algebra. For each element $a = (a_{ij})_{i,j}$ in $M_n(A)$ this norm satisfies

$$\max_{1 \leq i,j \leq n} \|a_{ij}\| \leq \|a\| \leq \sum_{i,j=1}^{n} \|a_{ij}\|.$$

Definition 1.1.21. Let $n \in \mathbb{N}$. The system of *standard matrix units* of $M_n(\mathbb{C})$ is defined to be $\{e_{ij} \mid i,j \in \mathbb{N}_{\leq n}\}$ where $e_{ij} \in M_n(\mathbb{C})$ is the matrix whose (i,j)th entry entry is one and whose other entries are zero for all $i,j \in \mathbb{N}_{\leq n}$.

Remark 1.1.22. Let $n \in \mathbb{N}$ and let $\{e_{ij} \mid i,j \in \mathbb{N}_{\leq n}\}$ be the system of standard matrix units in $M_n(\mathbb{C})$. Then

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases} \qquad \text{for all } i,j,k,l \in \mathbb{N}_{\leq n},$$

and $e^*_{ij} = e_{ji}$ for all $i,j \in \mathbb{N}_{\leq n}$. The system of standard matrix units is a basis for $M_n(\mathbb{C})$.

Lemma 1.1.23. Let A be a C^*-algebra. If A is unital and simple, then so is $M_n(A)$ for all $n \in \mathbb{N}$.

Proposition/Definition 1.1.24. Let A and B be C^*-algebras, let $\varphi\colon A \to B$ be a *-homomorphism and let $n \in \mathbb{N}$. The map

$$\varphi^{(n)}\colon M_n(A) \to M_n(B),\ (a_{ij})_{i,j} \mapsto (\varphi(a_{ij}))_{i,j}$$

obtained by applying φ entry-wise is a *-homomorphism and is called the nth *inflation* of φ.

1.1.2 Homotopy equivalence of C^*-algebras

Definition 1.1.25. Let A and B be C^*-algebras and let $\varphi, \psi\colon A \to B$ be *-homomorphisms. A map $\Phi\colon [0,1] \times A \to B,\ (t,a) \mapsto \Phi_t(a)$ is called a *homotopy* between φ and ψ if the following hold:

(i) The map $\Phi_t\colon A \to B,\ a \mapsto \Phi_t(a)$ is a *-homomorphism for each $t \in [0,1]$,

(ii) The map $[0,1] \to B,\ t \mapsto \Phi_t(a)$ is continuous for each $a \in A$,

(iii) $\Phi_0 = \varphi$ and $\Phi_1 = \psi$.

If such a homotopy exists, then φ and ψ are called *homotopic*, in symbols $\varphi \sim_h \psi$. The C^*-algebras A and B are said to be *homotopy equivalent*, denoted by $A \sim_h B$, if there exist *-homomorphisms $f\colon A \to B$ and $g\colon B \to A$ such that $g \circ f \sim_h \mathrm{id}_A$ and $f \circ g \sim_h \mathrm{id}_B$.

Remark 1.1.26. Homotopy equivalence of C^*-algebras is an equivalence relation.

Statement (ii) in Definition 1.1.25 can be referred to by saying that the family $(\Phi_t)_{t\in[0,1]}$ is *point-wise continuous.* The following statement will be helpful when we have to check point-wise continuity:

Lemma 1.1.27. Let A and B be C^*-algebras, let $\mathcal{T}$ be a non-empty topological space and let $(\varphi_t)_{t\in\mathcal{T}}$ be a family of *-homomorphisms from A to B. Then the set

$$D = \{a \in A \mid \mathcal{T} \to B,\ t \mapsto \varphi_t(a) \ \text{ is continuous}\}$$

on which $(\varphi_t)_{t\in\mathcal{T}}$ is point-wise continuous is a sub-C^*-algebra of A.

Proof. Let $a, b \in D$ and let $\lambda \in \mathbb{C}$. Then the maps

$$\mathcal{T} \to B,\ t \mapsto \varphi_t(\lambda a + b) = \lambda\varphi_t(a) + \varphi_t(b),$$

$$\mathcal{T} \to B,\ t \mapsto \varphi_t(ab) = \varphi_t(a)\varphi_t(b)$$

and

$$\mathcal{T} \to B,\ t \mapsto \varphi_t(a^*) = \varphi_t(a)^*$$

are continuous as the maps $t \mapsto \varphi_t(a)$ and $t \mapsto \varphi_t(b)$ are, and hence $\lambda a + b \in D,\ ab \in D$ and $a^* \in D$. Let now $a \in \overline{D}$, let $t \in \mathcal{T}$ and let $\varepsilon > 0$. Choose $d \in D$ with $\|a - d\| < \varepsilon/3$ and let U be a neighbourhood of t such that $\|\varphi_t(d) - \varphi_s(d)\| < \varepsilon/3$ for all $s \in U$. Then

$$\|\varphi_t(a) - \varphi_s(a)\| \leq \|\varphi_t(a) - \varphi_t(d)\| + \|\varphi_t(d) - \varphi_s(d)\| + \|\varphi_s(d) - \varphi_s(a)\| < \varepsilon$$

for all $s \in U$, i.e. $t \mapsto \varphi_t(a)$ is continuous and therefore $a \in D$. $\square$

Before we turn to two examples of homotopy equivalence we give the definition of the cone and the suspension of a C^*-algebra.

Definition 1.1.28. Let A be a C^*-algebra. The *cone* CA of A is defined by

$$CA = \{f \in C([0,1], A) \mid f(0) = 0\},$$

and the *suspension* SA of A is defined to be

$$SA = \{f \in C([0,1], A) \mid f(0) = f(1) = 0\}.$$

Remark 1.1.29. Let A be a C^*-algebra. Then $\varphi\colon C_0((0,1), A) \to SA$, given by

$$\varphi(f)(t) = \begin{cases} \ \ 0\ , & t \in \{0,1\}, \\ f(t), & t \in (0,1), \end{cases}$$

is a *-isomorphism (with inverse given by restriction). Let now $\lambda\colon \mathbb{R} \to (0,1)$ be a homeomorphism. Then

$$\psi\colon C_0((0,1), A) \to C_0(\mathbb{R}, A),\ f \mapsto f \circ \lambda$$

also is a *-isomorphism. Altogether,

$$SA \ \cong\ C_0((0,1), A) \ \cong\ C_0(\mathbb{R}, A).$$

Remark 1.1.30. Notice that $C([0,1])$ is separable by the Stone-Weierstraß Theorem (the polynomials with complex rational coefficients form a countable, dense subset). By Remark 1.1.29, $C_0(\mathbb{R})$ is isomorphic to the sub-C^*-algebra $S\mathbb{C}$ of $C([0,1])$ and hence also separable.

Examples 1.1.31. (i) For every C^*-algebra A the cone CA is homotopy equivalent to the zero C^*-algebra.

(ii) Let A and B be C^*-algebras and let $\widetilde{A}$ and $\widetilde{B}$ denote their unitizations. If A and B are homotopy equivalent, then so are $\widetilde{A}$ and $\widetilde{B}$.

Proof. (i): Let A be a C^*-algebra. Considering the zero maps $f\colon CA \to 0$ and $g\colon 0 \to CA$ it is clear that $g \circ f = 0$ and $f \circ g = 0$, where 0 denotes both the zero C^*-algebra and the appropriate zero maps. Hence all we have to show is that id_{CA} is homotopic to the zero map on CA. For each $t \in [0,1]$ define $\Phi_t\colon CA \to CA$ by $\Phi_t(f)(s) = f(st)$ for all $f \in CA$ and for all $s \in [0,1]$. Then each Φ_t is a *-homomorphism, the map $t \mapsto \Phi_t(f)$ is continuous for each $f \in CA$, and $\Phi_0 = 0$ and $\Phi_1 = \mathrm{id}_{CA}$.

(ii): Suppose that $A \sim_h B$ and choose *-homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $\psi \circ \varphi \sim_h \mathrm{id}_A$ and $\varphi \circ \psi \sim_h \mathrm{id}_B$. Recall that the unitizations can be written as

$$\widetilde{A} = \big\{a + \alpha 1_{\widetilde{A}} \mid a \in A,\ \alpha \in \mathbb{C}\big\}, \qquad \widetilde{B} = \big\{b + \beta 1_{\widetilde{B}} \mid b \in B,\ \beta \in \mathbb{C}\big\},$$

and consider the unique unital extensions $\tilde{\varphi}\colon \widetilde{A} \to \widetilde{B}$ and $\tilde{\psi}\colon \widetilde{B} \to \widetilde{A}$ of φ and ψ, given by $\tilde{\varphi}(a + \alpha 1_{\widetilde{A}}) = \varphi(a) + \alpha 1_{\widetilde{B}}$ and $\tilde{\psi}(b + \beta 1_{\widetilde{B}}) = \psi(b) + \beta 1_{\widetilde{A}}$. We show that $\tilde{\psi} \circ \tilde{\varphi} \sim_h \mathrm{id}_{\widetilde{A}}$. Let Φ be a homotopy between $\psi \circ \varphi$ and id_A, and define

$$\tilde{\Phi}\colon [0,1] \times \widetilde{A} \to \widetilde{A},\ (t, \tilde{a}) \mapsto \tilde{\Phi}_t(\tilde{a}),$$

where each $\tilde{\Phi}_t$ denotes the unique unital *-homomorphism extending Φ_t to $\widetilde{A}$. For each $\tilde{a} = a + \alpha 1_{\widetilde{A}} \in \widetilde{A}$ and for all $t, s \in [0,1]$ we can calculate

$$\begin{aligned}\|\tilde{\Phi}_t(\tilde{a}) - \tilde{\Phi}_s(\tilde{a})\|_{\widetilde{A}} &= \|\Phi_t(a) + \alpha 1_{\widetilde{A}} - \Phi_s(a) - \alpha 1_{\widetilde{A}}\|_{\widetilde{A}} = \|\Phi_t(a) - \Phi_s(a)\|_{\widetilde{A}} \\ &= \|\Phi_t(a) - \Phi_s(a)\|_A\end{aligned}$$

which shows that $t \mapsto \tilde{\Phi}_t(\tilde{a})$ is continuous for each $\tilde{a} \in \widetilde{A}$ as $t \mapsto \Phi_t(a)$ is continuous for each $a \in A$. It follows from $\Phi_0 = \psi \circ \varphi$ and by uniqueness of the unital extensions that $\tilde{\Phi}_0 = \widetilde{\psi \circ \varphi} = \tilde{\psi} \circ \tilde{\varphi}$, and in the same way $\Phi_1 = \mathrm{id}_A$ implies that $\tilde{\Phi}_1 = \widetilde{\mathrm{id}_A} = \mathrm{id}_{\widetilde{A}}$. Thus, $\tilde{\psi} \circ \tilde{\varphi} \sim_h \mathrm{id}_{\widetilde{A}}$. It can be shown analogously that $\tilde{\varphi} \circ \tilde{\psi} \sim_h \mathrm{id}_{\widetilde{B}}$, and altogether it follows that $\widetilde{A} \sim_h \widetilde{B}$. $\square$

1.1.3 Special elements in C^*-algebras

Definition 1.1.32. Let A be a C^*-algebra. An element $a \in A$ is said to be *self-adjoint* if $a^* = a$. An element $p \in A$ is called a *projection* if $p^* = p = p^2$, and an element $v \in A$ for which v^*v is a projection is said to be a *partial isometry* .

Suppose now that A is unital. An element $s \in A$ with $s^*s = 1_A$ is said to be an *isometry*, and an element $u \in A$ with $u^*u = uu^* = 1_A$ is called a *unitary*.

Remarks 1.1.33. Let A be a C^*-algebra.

(i) Non-zero projections and non-zero partial isometries in A (and hence also isometries and unitaries if A is unital) have norm one.

(ii) Suppose that A is unital and let s be an isometry in A. Then

$$\|sa\| = \|as^*\| = \|a\| \qquad \text{for all } a \in A.$$

Proof. (i): Let $0 \neq p$ be a projection in A. Then $\|p\| = \|p^*p\| = \|p\|^2$, and as $\|p\| \neq 0$ by assumption, it follows that $\|p\| = 1$. Let $0 \neq v$ be a partial isometry in A. Then v^*v is a non-zero projection in A and hence $\|v\|^2 = \|v^*v\| = 1$.

(ii): Let $a \in A$. Then

$$\|sa\|^2 = \|a^*s^*sa\| = \|a^*a\| = \|a\|^2,$$

i.e. $\|sa\| = \|a\|$, and hence also $\|as^*\| = \|sa^*\| = \|a^*\| = \|a\|$, or apply the same argument as for $\|as\|$. $\square$

These statements will be used without further comment in many estimates throughout this thesis. Proofs of the following statements about positive elements and projections can all be found in [Mu].

Definition 1.1.34. Let A be a C^*-algebra. An element $a \in A$ is said to be *positive* if it is self-adjoint and $\sigma(a) \subseteq \mathbb{R}_{\geq 0}$, we then write $a \geq 0$. The set of positive elements in A is denoted by A^+. For two self-adjoint elements $a, b \in A$ we write $a \leq b$ if $b - a \geq 0$.

Proposition 1.1.35. Let A be a C^*-algebra and let $a \in A^+$. Then there exists a unique element $b \in A^+$ such that $b^2 = a$. The element b is invertible if and only if so is a.

Notation. The element b in Proposition 1.1.35 is called the *positive square root* of a and is denoted by $a^{1/2}$.

Proposition 1.1.36. An element a in a C^*-algebra A is positive if and only if there is $x \in A$ with $x^*x = a$.

Proposition 1.1.37. Let A be a C^*-algebra. Every projection in A is positive. Let now $p, q \in A$ be projections. Then the following statements are equivalent:

(i) $q \leq p$;

(ii) $pq = q$;

(iii) $qp = q$.

The following statement shows that elements which behave approximately like a projection, isometry or unitary, are in fact close to a projection, an isometry, or a unitary, respectively. We prove only statement (ii) on isometries here as the other proofs are very similar. Further comments on these results can be found in Exercises 2.7 and 2.8 in [R1].

Lemma 1.1.38. Let A be a C^*-algebra and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

(i) for any q in A satisfying $\|q - q^*\| \leq \delta$ and $\|q^2 - q\| \leq \delta$ there is a projection p in A with $\|q - p\| \leq \varepsilon$.

(ii) if A is unital, then for any t in A with $\|t^*t - 1_A\| \leq \delta$ there exists an isometry s in A such that $\|t - s\| \leq \varepsilon$.

(iii) if A is unital, then for any v in A satisfying $\|v^*v - 1_A\| \leq \delta$ and $\|vv^* - 1_A\| \leq \delta$ there exists a unitary element u in A with $\|v - u\| \leq \varepsilon$.

Proof of (ii): Suppose that A is unital, choose $0 < \delta < 1$ such that $\delta(1+\delta)/(1-\delta) \leq \varepsilon$ and let $t \in A$ with $\|t^*t - 1_A\| \leq \delta$. Then $\|t^*t - 1_A\| < 1$, and therefore the positive element t^*t is invertible in A and has a positive, invertible square root in A. Put $s = t(t^*t)^{-1/2}$. Then

$$s^*s = (t^*t)^{-1/2}t^*t(t^*t)^{-1/2} = 1_A,$$

i.e. s is an isometry in A. To show that $\|t - s\| \leq \varepsilon$ we compute estimates for $\|t\|$ and for $\|1_A - (t^*t)^{-1/2}\|$, using the continuous function calculus for t^*t. The spectrum $\sigma(t^*t)$ is contained in the interval $[1-\delta, 1+\delta]$ because $\|t^*t - 1_A\| \leq \delta$ (see, for instance, Lemma 2.2.3 in [R1]). As t^*t is self-adjoint, this yields $\|t\|^2 = \|t^*t\| \leq 1 + \delta$, and hence $\|t\| \leq \sqrt{1+\delta} \leq 1 + \delta$. By the functional calculus we obtain

$$\|1_A - (t^*t)^{-1/2}\| \leq \sup\{|1 - \lambda^{-1/2}| \mid \lambda \in \sigma(t^*t)\} \leq \frac{\delta}{1-\delta},$$

where the last inequality follows by a short calculation from $\sigma(t^*t) \subseteq [1-\delta, 1+\delta]$. Altogether, we can now show that

$$\|t - s\| = \|t - t(t^*t)^{-1/2}\| \leq \|t\|\|1_A - (t^*t)^{-1/2}\| \leq \frac{\delta(1+\delta)}{1-\delta} \leq \varepsilon,$$

which completes the proof. □

1.2 Relations of projections and K_0

For proofs and details about the material in this section the reader may consult Chapters 1–4 in [R1].

1.2.1 Relations of projections and properly infinite projections

Definition 1.2.1. Let A be a C^*-algebra. Two projections p and q in A are said to be *(Murray-von Neumann) equivalent*, in symbols $p \sim q$, if there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$.

Remarks 1.2.2. (i) Let A be a C^*-algebra and let v be a partial isometry in A, i.e. v^*v is a projection. Then $v = vv^*v$, hence vv^* also is a projection, and with $p = v^*v$ and $q = vv^*$ the following holds:

$$v = qv = vp = qvp.$$

(ii) The relation $\sim$ defines an equivalence relation on the set of projections in A.

Definition 1.2.3. Let A be a C^*-algebra. A projection p in A is said to be *infinite* if it is equivalent to a proper subprojection of itself, i.e. if there is a projection q in A such that $p \sim q < p$, where $q < p$ means $q \leq p$ and $q \neq p$. A projection which is not infinite is said to be *finite*.

If A is unital then A is said to be *infinite* if 1_A is an infinite projection, otherwise A is called *finite*.

The next definition gives a stronger notion of infinity:

Definition 1.2.4. Let A be a C^*-algebra. A non-zero projection p in A is said to be *properly infinite* if there are two projections e, f in A such that $ef = 0$ (i.e. e and f are mutually orthogonal), $e \leq p,\ f \leq p$ and $e \sim p \sim f$.

A unital C^*-algebra A is said to be *properly infinite* if 1_A is a properly infinite projection.

Remark 1.2.5. Let A be a unital, properly infinite C^*-algebra. Then A contains a sequence $(t_n)_{n\in\mathbb{N}}$ of isometries with orthogonal range projections, i.e. with $t_i t_i^* \perp t_j t_j^*$ for all $i \neq j \in \mathbb{N}$. See Exercise 4.6 in [R1] for an idea of the proof.

Notation. Let $\mathcal{P}(A)$ denote the set of projections in a C^*-algebra A. For each $n \in \mathbb{N}$ let $\mathcal{P}_n(A) = \mathcal{P}(M_n(A))$, and let $\mathcal{P}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n(A)$.

In what follows we define a generalized Murray-von Neumann equivalence and introduce a binary operation on $\mathcal{P}_\infty(A)$. In Section 1.2.2 these concepts will be used to define the K_0-group of a unital C^*-algebra, and now they will, combined with one more relation on $\mathcal{P}_\infty(A)$, lead to some useful statements about properly infinite projections.

Definition 1.2.6. Let A be a C^*-algebra and define a relation on $\mathcal{P}_\infty(A)$ as follows: Let $m, n \in \mathbb{N}$, let $p \in \mathcal{P}_n(A)$ and let $q \in \mathcal{P}_m(A)$. Then $p \sim_0 q$ if there exists an element v in the $m \times n$ matrices $M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$.

Definition 1.2.7. Let A be a C^*-algebra, let p and q be projections in $\mathcal{P}_\infty(A)$ and define

$$p \oplus q = \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

such that $p \oplus q$ is contained in $\mathcal{P}_{n+m}(A)$ when $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ for some $m, n \in \mathbb{N}$.

Another relation on $\mathcal{P}_\infty(A)$ can be defined as follows:

Definition 1.2.8. Let A be a C^*-algebra, let $m, n \in \mathbb{N}$, let $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$. We write $q \precsim p$ if there is a projection $p_0 \in \mathcal{P}_n(A)$ such that $q \sim_0 p_0 \leq q$.

Remark 1.2.9. It is not true that $p \precsim q \precsim p$ implies $p \sim_0 q$, see the note to Exercise 4.7 in [R1].

Definition 1.2.10. An element a in a C^*-algebra A is said to be *full* if it is not contained in any proper, closed, two-sided ideal in A. A projection $p \in \mathcal{P}_n(A) \subseteq \mathcal{P}_\infty(A)$ is called *full* if p is a full element in $M_n(A)$.

Lemma 1.2.11. Let A be a C^*-algebra.

(i) A non-zero projection $p \in \mathcal{P}_\infty(A)$ is properly infinite if and only if $p \oplus p \precsim p$.

(ii) Let p, q be projections in A and suppose that $p \precsim q \precsim p$ and that p is properly infinite. Then q also is properly infinite.

(iii) If p is a properly infinite, full projection in $\mathcal{P}_\infty(A)$, then $q \precsim p$ for every projection q in $\mathcal{P}_\infty(A)$.

An idea of the proofs is given in Exercises 4.7, 4.8 and 4.9 in [R1].

1.2.2 The functor K_0 for unital C^*-algebras

This section gives a very short introduction to the construction of $K_0(A)$ for a unital C^*-algebra A. This construction also works for non-unital C^*-algebras, but the resulting functor lacks certain exactness properties (see [R1] for details). Therefore, K_0 is defined in a different way for non-unital C^*-algebras which we will not present here. In Chapter 2 it will be shown how the K_0-group of the (unital) Cuntz algebra $\mathcal{O}_2$ can be determined explicitly.

Proposition/Definition 1.2.12. Let A be a C^*-algebra. Then $\sim_0$ defines an equivalence relation on $\mathcal{P}_\infty(A)$, and the set of equivalence classes is denoted by $\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0$. For each $p \in \mathcal{P}_\infty(A)$ let $[p]_\mathcal{D}$ denote the equivalence class containing p. One can define an addition on $\mathcal{D}(A)$ by

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D} \qquad \text{for all } p, q \in \mathcal{P}_\infty(A),$$

and this makes $(\mathcal{D}(A), +)$ an abelian semigroup.

Remark 1.2.13. To every abelian semigroup S on can assign an abelian group $G(S)$ via the Grothendieck construction. There exists an additive map $\gamma_S : S \to G(S)$ such that

$$G(S) = \{\gamma_S(g) - \gamma_S(h) \mid g, h \in S\},$$

the so-called *Grothendieck map*. For details the reader is referred to Chapter 3 in [R1].

Definition 1.2.14. Let A be a unital C^*-algebra. Then $K_0(A)$ is defined to be the abelian group which is assigned to $\mathcal{D}(A)$ via the Grothendieck construction. Let $\gamma : \mathcal{D}(A) \to K_0(A)$ be the Grothendieck map and define

$$[\,\cdot\,]_0 : \mathcal{P}_\infty(A) \to K_0(A),\ p \mapsto [p]_0 = \gamma([p]_\mathcal{D}).$$

The following proposition gives the standard picture of $K_0(A)$ in the unital case.

Proposition 1.2.15. Let A be a unital C^*-algebra. Then

$$\begin{aligned} K_0(A) &= \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(A)\} \\ &= \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}. \end{aligned}$$

The following proposition states that to every *-homomorphism φ between unital C^*-algebras A and B one can assign a group homomorphism $K_0(\varphi) : K_0(A) \to K_0(B)$. The results in Proposition 1.2.17 and Proposition 1.2.18 show that K_0 defines a homotopy invariant functor from the category of unital C^*-algebras into the category of abelian groups.

Proposition 1.2.16. Let A and B be unital C^*-algebras. Every *-homomorphism $\varphi : A \to B$ induces a unique group homomorphism $K_0(\varphi) : K_0(A) \to K_0(B)$ that satisfies

$$K_0(\varphi)([p]_0) = [\varphi^{(n)}(p)]_0$$

for $p \in \mathcal{P}_n(A)$, where $\varphi^{(n)}$ denotes the nth inflation of φ.

Proposition 1.2.17. (i) For every unital C^*-algebra A, $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$.

(ii) Let A, B and C be unital C^*-algebras and let $\varphi\colon A \to B$ and $\psi\colon B \to C$ be *-homomorphisms. Then $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.

Proposition 1.2.18. Let A and B be unital C^*-algebras.

(i) If $\varphi, \psi\colon A \to B$ are homotopic *-homomorphisms, then $K_0(\varphi) = K_0(\psi)$.

(ii) If A and B are homotopy equivalent, then $K_0(A)$ is isomorphic to $K_0(B)$.

The following proposition gives a very useful picture of $K_0(A)$ in case that A contains a properly infinite, full projection. Its proof is given in Theorem 1.4 in [Cu2]. Notice that this holds both in the unital and in the non-unital case.

Proposition 1.2.19. Let A be a C^*-algebra.

(i) If A contains a properly infinite, full projection, then

$$K_0(A) = \big\{[p]_0 \;\big|\; p \in \mathcal{P}(A) \quad \text{is properly infinite and full}\big\}.$$

(ii) If p and q are properly infinite, full projections in A, then $[p]_0 = [q]_0$ if and only if $p \sim q$.

1.3 Purely infinite C^*-algebras

The definition of purely infinite C^*-algebras can be given in many equivalent formulations, which are given in Proposition 1.3.10 below. Before we state these equivalences we introduce the notions of hereditary sub-C^*-algebras and of C^*-algebras having real rank zero.

Definition 1.3.1. A sub-C^*-algebra B of a C^*-algebra A is said to be *hereditary* if $a \leq b$ implies $a \in B$ for all positive elements $a, b \in A$ with $b \in B$.

Remark 1.3.2. Let B be a hereditary sub-C^*-algebra of a C^*-algebra A. Then any hereditary sub-C^*-algebra of B also is a hereditary sub-C^*-algebra of A.

Proof. Recall that if C is any sub-C^*-algebra of A, then an element $x \in C$ is positive in C if and only if it is positive in A. Let D be a hereditary sub-C^*-algebra of B, and let a, d be positive elements in A with $a \leq d$ and $d \in D$. As $D \subseteq B$ and B is a hereditary sub-C^*-algebra of A, we get $a \in B$. Since D is a hereditary sub-C^*-algebra of B, this gives $a \in D$, as required. □

The following two lemmas give an idea of what hereditary sub-C^*-algebras may look like. Their proofs can be found in [Mu], Theorem 3.2.2, Corollary 3.2.4 and Theorem 3.2.5.

Lemma 1.3.3. A sub-C^*-algebra B of a C^*-algebra A is hereditary if and only if $bab' \in B$ for all $b, b' \in B$ and for all $a \in A$.

Lemma 1.3.4. Let A be a C^*-algebra. If a is a positive element in A, then $\overline{aAa}$ is a hereditary sub-C^*-algebra of A. If B is a separable hereditary sub-C^*-algebra of A, then there exists a positive element $a \in A$ such that $B = \overline{aAa}$.

Corollary 1.3.5. Let p be a projection in a C^*-algebra A. Then pAp is a hereditary sub-C^*-algebra of A which is unital with unit p.

Proof. For all $a \in A$ we have $ppap = pap = papp$, i.e. p is a unit in pAp. To prove that pAp is a C^*-algebra it suffices by the preceding lemma to show that pAp is closed. Notice that $a \in pAp$ if and only if $a = pap$ for every $a \in A$. Let $a \in \overline{pAp}$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in pAp with $a_n \to a$ as $n \to \infty$. Then $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} pa_np = pap$, and hence $a \in pAp$. □

A proof of this which only uses the definition of hereditary sub-C^*-algebras can be found in [Mu], Example 3.2.1. The following remark shows that, if we need a properly infinite projection in a hereditary sub-C^*-algebra of a C^*-algebra A, it suffices to find a properly infinite projection in A which is contained in the hereditary sub-C^*-algebra.

Remark 1.3.6. Let A be a C^*-algebra, let B be a non-zero, hereditary sub-C^*-algebra of A and let p be a non-zero projection in B. Then p is properly infinite in B if and only if p is properly infinite in A.

Proof. The "only if" part is clear. Suppose now that p is properly infinite in A and choose mutually orthogonal projections $e, f \in A$ with $e \leq p,\ f \leq p$ and $e \sim p \sim f$ in A. As B is hereditary, it follows immediately that $e, f \in B$. Choose $v \in A$ with $v^*v = e$ and $vv^* = p$, then $v = pve$ by Remark 1.2.2(i), and hence $v \in B$ by Lemma 1.3.3. Hence, $e \sim p$ in A implies that $e \sim p$ in B, and it follows in the same way that $p \sim f$ in B. Thus, p is properly infinite in B. □

Definition 1.3.7. Let A be a C^*-algebra. An *approximate unit* for A is a net $(u_\lambda)_\lambda$ of positive elements in the closed unit ball of A such that

$$\lim_\lambda u_\lambda a = \lim_\lambda a u_\lambda = a \qquad \text{for all } a \in A.$$

It is shown in Theorem 3.1.1 in [Mu] that every C^*-algebra admits an approximate unit. The reference for the following statement about C^*-algebras with real rank zero is [BP], Theorem 2.6.

Definition 1.3.8. A unital C^*-algebra A is said to have *real rank zero* if the set of invertible, self-adjoint elements in A is dense in the set of self-adjoint elements in A. A non-unital C^*-algebra A is said to have real rank zero if its unitization $\widetilde{A}$ has real rank zero.

Theorem 1.3.9. Let A be a C^*-algebra. The following conditions are equivalent:

(i) The real rank of A is zero.

(ii) The set of self-adjoint elements in A with finite spectrum is dense in the set of self-adjoint elements in A.

(iii) Every hereditary sub-C^*-algebra of A has an approximate unit consisting of projections.

The equivalent conditions in the following proposition will be used below to define what it means for a unital and simple C^*-algebra to be purely infinite.

Proposition 1.3.10. Let A be a unital, simple C^*-algebra. The following statements are equivalent:

(i) A is not isomorphic to $\mathbb{C}$ and for all non-zero, positive elements a, b in A there exists x in A such that $b = x^*ax$.

(ii) A is not isomorphic to $\mathbb{C}$ and for each non-zero, positive element a in A there exists x in A such that $1_A = x^*ax$.

(iii) A is not isomorphic to $\mathbb{C}$ and every non-zero, hereditary sub-C^*-algebra of A contains a projection p with $p \sim 1_A$.

(iv) Every non-zero, hereditary sub-C^*-algebra of A contains a properly infinite projection.

(v) The real rank of A is zero and every non-zero projection in A is properly infinite.

Proof. It is shown in the following that (i)$\Leftrightarrow$(ii), that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(ii), and that (iv)$\Leftrightarrow$(v).

(i)$\Leftrightarrow$(ii): It is clear that (i) implies (ii). Suppose now that (ii) holds and let a, b be non-zero, positive elements in A. Then there exists $x_0 \in A$ with $1_A = x_0^*ax_0$. Put $x = x_0b^{1/2}$, then $x^*ax = b^{1/2}x_0^*ax_0b^{1/2} = b$.

(ii)$\Rightarrow$(iii): Suppose that (ii) holds and let B be a non-zero, hereditary sub-C^*-algebra of A. Let a be a non-zero, positive element in B, then a also is positive in A, and by (ii) there is $x \in A$ such that $1_A = x^*ax$. Set $v = a^{1/2}x$, then $v^*v = x^*ax = 1_A$ and $p = vv^* = a^{1/2}xx^*a^{1/2}$ is a projection in B by Lemma 1.3.3, and $p = vv^* \sim v^*v = 1_A$.

(iii)$\Rightarrow$(iv): Assume that (iii) holds. We show first that 1_A is properly infinite. As A is not isomorphic to $\mathbb{C}$ there exists a positive element d in A which is not contained in $\mathbb{C}1_A$, which implies that the spectrum $\sigma(d)$ contains more than one point. Let $\lambda \neq \mu \in \sigma(d)$. By Urysohn's Lemma there exist two continuous functions $f_\lambda, f_\mu \colon \sigma(d) \to [0, 1]$ such that $f_\lambda(\lambda) = f_\mu(\mu) = 1$ and $f_\lambda f_\mu \equiv 0$. Apply the continuous function calculus for d to define $a = f_\lambda(d)$ and $b = f_\mu(d)$. Then a and b are non-zero, positive elements in A with $ab = 0$, and $\overline{aAa}$ and $\overline{bAb}$ are non-zero, hereditary sub-C^*-algebras of A. By (iii) there exist projections $p \in \overline{aAa}$ and $q \in \overline{bAb}$ with $p \sim 1_A \sim q$. As $ab = 0$ it follows that $(axa)(byb) = 0$ for all $x, y \in A$, which implies that $pq = 0$. Since $p \leq 1_A$ and $q \leq 1_A$ anyway, this shows that 1_A is properly infinite.

Let now B be a non-zero, hereditary sub-C^*-algebra of A. By (iii), B contains a projection e with $e \sim 1_A$. Then $1_A \sim e \leq 1_A$ and in particular $1_A \precsim e \precsim 1_A$, which by Lemma 1.2.11(ii) implies that e is properly infinite as 1_A is.

(iv)$\Rightarrow$(ii): Suppose that (iv) holds and notice that this implies that A is not isomorphic to $\mathbb{C}$, because $\mathbb{C}$ does not contain any properly infinite projections.

Let a be a non-zero, positive element in A. Then $\overline{aAa}$ is a non-zero, hereditary sub-C^*-algebra of A and (iv) yields a properly infinite projection p in $\overline{aAa}$. Choose $z \in A$ with $\|aza - p\| < 1$. As A is assumed to be simple, p is full, and hence $1_A \precsim p$ by Lemma 1.2.11(iii), i.e. there exists a projection p_0 in A such that $1_A \sim p_0 \leq p$. Choose $v \in A$ with $v^*v = 1_A$ and $vv^* = p_0$. Then, as $p_0p = p_0 = pp_0$ because $p_0 \leq p$, and as $p_0v = v$ by Remark 1.2.2(i), we have

$$1_A = v^*p_0v = v^*p_0pp_0v = v^*pv. \tag{1.3.1}$$

Put $w = zav$, then

$$\|1_A - v^*aw\| = \|v^*(p - aza)v\| \leq \|p - aza\| < 1$$

by choice of z, which shows that $v^*aw = v^*a^{1/2}a^{1/2}w$ is invertible. Hence, $a^{1/2}w$ is left-invertible, and therefore $(a^{1/2}w)^*a^{1/2}w = w^*aw$ is invertible (see, for example, the proof of Lemma 5.1.2 in [R1]). Moreover, w^*aw is positive, and therefore $(w^*aw)^{1/2}$ exists and is invertible as w^*aw is. Put $x = w(w^*aw)^{-1/2}$, then

$$x^*ax = (w^*aw)^{-1/2}w^*aw(w^*aw)^{-1/2} = 1_A,$$

as desired.

(iv)⇒(v): Assume that (iv) holds. We show first that this implies that every non-zero projection in A is properly infinite. Let p be a non-zero projection in A. Then pAp is a non-zero, hereditary sub-C^*-algebra of A which, by (iv), contains a properly infinite projection q. Then $pq = q = qp$ and hence $q \leq p$, and as q is a properly infinite, full projection, we know that $p \precsim q$ by Lemma 1.2.11(iii). Combining these we have that $q \leq p \precsim q$ which by Lemma 1.2.11(ii) implies that p also is properly infinite.

We now show that (iv) implies that A has real rank zero. Let a be a self-adjoint element in A. In case that a is invertible there is nothing to prove, hence assume in the following that a is not invertible. Let $\varepsilon > 0$.

Case 1: There exists a non-zero, positive element b in A such that $ab = ba = 0$, i.e. b is orthogonal to a. By (iv) there exists a properly infinite projection p in $\overline{bAb}$. As A is simple, p is full and therefore $1_A - p \precsim p$, i.e. there exists a projection $q \in A$ with $1_A - p \sim q \leq p$. Choose $v \in A$ with $v^*v = 1_A - p$ and $vv^* = q$. Notice that $ab = ba = 0$ implies $ap = pa = 0$, and that $pq = qp = q$ as $q \leq p$. Using these and the relation $v = qv = v(1_A - p)$ we obtain that

$$\begin{aligned} &v^2 = v(1_A - p)qv = 0, \qquad vp = v(1_A - p)p = 0, \qquad vq = vpq = 0, \qquad pv = pqv = qv, \\ &aq = apq = 0 = qa, \qquad av = aqv = 0. \end{aligned} \tag{1.3.2}$$

Set $s = v + v^* + (p - q)$, then s is self-adjoint and

$$\begin{aligned} s^2 &= (v + v^* + (p - q))(v + v^* + (p - q)) = vv^* + v^*v + v^*p - v^*q + p - q \\ &= q + 1_A - p + p - q = 1_A, \end{aligned}$$

i.e. s is invertible with $s = s^{-1}$. Put $d = a + \varepsilon s$, then d is self-adjoint and $\|d - a\| = \varepsilon$, as $\|s\| = 1$. We proceed to show that d is invertible with inverse $\varepsilon^{-1}s - \varepsilon^{-2}vav^*$. Notice that $ap = 0$ and the equations in (1.3.2) imply that $as = av^*$ and $sv = v^*v = 1_A - p$ (and, taking adjoints, $sa = va$ and $v^*s = 1_A - p$) and calculate

$$\begin{aligned} d(\varepsilon^{-1}s - \varepsilon^{-2}vav^*) &= (a + \varepsilon s)(\varepsilon^{-1}s - \varepsilon^{-2}vav^*) = \varepsilon^{-1}av^* + 1_A - \varepsilon^{-1}svav^* \\ &= \varepsilon^{-1}av^* + 1_A - \varepsilon^{-1}(1_A - p)av^* = \varepsilon^{-1}av^* + 1_A - \varepsilon^{-1}av^* = 1_A \end{aligned}$$

and, similarly,

$$\begin{aligned} (\varepsilon^{-1}s - \varepsilon^{-2}vav^*)d &= (\varepsilon^{-1}s - \varepsilon^{-2}vav^*)(a + \varepsilon s) = \varepsilon^{-1}va + 1_A - \varepsilon^{-1}vav^*s \\ &= \varepsilon^{-1}va + 1_A - \varepsilon^{-1}va = 1_A. \end{aligned}$$

This shows that d is invertible.

Case 2: In this case we construct a self-adjoint element $a' \in A$ such that $\|a' - a\| \leq \varepsilon/2$ and for which there exists a non-zero positive element $b' \in A$ with $a'b' = b'a' = 0$. Then

Case 1 can be applied to find an invertible, self-adjoint element $d' \in A$ with $\|a' - d'\| \leq \varepsilon/2$, and therefore with $\|a - d'\| < \varepsilon$. To find a' and b' we apply the functional calculus for a. Define

$$f\colon \sigma(a) \to \mathbb{R},\ t \mapsto \begin{cases} t + \frac{\varepsilon}{2}, & t \leq -\frac{\varepsilon}{2}, \\ 0 \quad, & t \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}], \\ t - \frac{\varepsilon}{2}, & t \geq \frac{\varepsilon}{2} \end{cases}$$

and

$$g\colon \sigma(a) \to \mathbb{R},\ t \mapsto \begin{cases} 0 \quad, & t \leq -\frac{\varepsilon}{2}, \\ t + \frac{\varepsilon}{2}\ , & t \in [-\frac{\varepsilon}{2}, 0], \\ -t + \frac{\varepsilon}{2}, & t \in [0, \frac{\varepsilon}{2}], \\ 0 \quad, & t \geq \frac{\varepsilon}{2} \end{cases}$$

and set $a' = f(a)$ and $b' = g(a)$. As f and g are real-valued functions, a' and b' are self-adjoint. As a is assumed to be not invertible we know that $0 \in \sigma(a)$, and as $g \geq 0$ and $g(0) > 0$ it follows that b' is a non-zero positive element in A. Moreover, $a'b' = b'a' = 0$ as $fg = 0$. Let now $z\colon \sigma(a) \to \mathbb{R}$ denote the inclusion (we know that $\sigma(a) \subseteq \mathbb{R}$ as a is self-adjoint) and use that the functional calculus is isometric to see that $\|a' - a\| = \|f - z\| \leq \varepsilon/2$, as required.

(v)$\Rightarrow$(iv): Suppose that (v) holds and let B be a non-zero, hereditary sub-C^*-algebra of A. By (v) and by Theorem 1.3.9(iii) we can conclude that B has an approximate unit consisting of projections, in particular this means that there exists a non-zero projection p in B. By (v), p is properly infinite. $\square$

Remark 1.3.11. Conditions (i),(iv) and (v) are also equivalent if A is non-unital, see [LZ].

Definition 1.3.12. A (unital) simple C^*-algebra A is said to be *purely infinite* if it satisfies one — and hence any — of the equivalent conditions (i), (iv), (v) (and (ii),(iii) if A is unital) in Proposition 1.3.10.

The definition of non-simple, purely infinite C^*-algebras and some important properties of purely infinite C^*-algebras are given in [R2], Chapter 4.1.

Lemma 1.3.13. Let A be a simple, purely infinite C^*-algebra.

(i) Every non-zero, hereditary sub-C^*-algebra B of A is again purely infinite.

(ii) For each $n \in \mathbb{N}$ the matrix algebra $M_n(A)$ is simple and purely infinite.

Proof. (i): We show condition (iv) in Proposition 1.3.10. Let B be a non-zero, hereditary sub-C^*-algebra of A. Then B is simple because A is, see Theorem 3.2.8 in [Mu]. By Remark 1.3.2 every non-zero, hereditary sub-C^*-algebra of B also is a non-zero, hereditary sub-C^*-algebra of A and hence contains a properly infinite projection.

(ii): We show this in the unital case only, the general case is contained in the proof of Proposition 4.1.8 in [R2]. Let $n \in \mathbb{N}$. As A is unital and simple so is $M_n(A)$ by Lemma 1.1.23. We show that $M_n(A)$ satisfies condition (ii) in Proposition 1.3.10. Let $a = (a_{ij})_{i,j}$ be a non-zero, positive element in $M_n(A)$. Then there exists a non-zero element $z = (z_{ij})_{i,j} \in M_n(A)$ such that $a = z^*z$, i.e. $a_{ii} = \sum_{k=1}^{n} z_{ki}^* z_{ki}$ for each $i \in \mathbb{N}_{\leq n}$. Therefore, each a_{ii} is positive, and there exists $i \in \mathbb{N}_{\leq n}$ such that a_{ii} is non-zero (otherwise z and hence a were forced to be zero). Since A is purely infinite there exists an element $y \in A$ such that $y^* a_{ii} y = 1_A$. Define $t = (t_{kl})_{k,l} \in M_n(A)$ by

$$t_{kl} = \begin{cases} y, & k = i \text{ and } l = 1, \\ 0, & \text{else} \end{cases}$$

and let $p = (p_{kl})_{k,l} \in M_n(A)$ denote the projection given by

$$p_{kl} = \begin{cases} 1_A, & k = l = 1, \\ 0\ , & \text{else.} \end{cases}$$

Then t is the matrix having y as the first entry of its ith row and zeros elsewhere, and p has 1_A in its upper left corner and zeros elsewhere. Matrix multiplication now shows that $t^* a t = p$. As 1_A is a properly infinite projection in A, so is p in $M_n(A)$, because if q and r are mutually orthogonal projections in A with $q \sim 1_A \sim r$, then $q^{(n)}$, $r^{(n)} \in M_n(A)$ given by

$$q^{(n)}_{kl} = \begin{cases} q, & k = l = 1, \\ 0, & \text{else,} \end{cases} \qquad \text{and} \qquad r^{(n)}_{kl} = \begin{cases} r, & k = l = 1, \\ 0, & \text{else,} \end{cases}$$

are mutually orthogonal projections with $q^{(n)} \leq p$, $r^{(n)} \leq p$ and $q^{(n)} \sim p \sim r^{(n)}$. As $M_n(A)$ is simple, p is a properly infinite, full projection in $M_n(A)$, and therefore $1_{M_n(A)} \precsim p$. As shown in the paragraph before equation (1.3.1) this entails that there is an element $v \in M_n(A)$ such that $v^* p v = 1_{M_n(A)}$. Now set $x = tv$ and check that

$$x^* a x = v^* t^* a t v = v^* p v = 1_{M_n(A)},$$

as required. □

1.4 Completely bounded/positive maps

A good reference for the theory of operator systems and completely bounded/positive maps is [Pa], and most of the statements given here are taken from this book. As an example for unital, completely positive maps we consider maps like $A \to A$, $a \mapsto s^* a s$ for an isometry s in a unital C^*-algebra A, which we shall often encounter in this thesis. We conclude this section with some extension and approximation results.

Definition 1.4.1. A closed linear subspace E of a unital C^*-algebra A is said to be an *operator system* if $1_A \in E$ and if E is self-adjoint, i.e. $E = E^* = \{x^* \mid x \in E\}$.

For each $n \in \mathbb{N}$ we regard $M_n(E)$ as a subspace of $M_n(A)$ and equip it with the norm inherited from the norm on the matrix algebra $M_n(A)$.

Definition 1.4.2. Let A be a unital C^*-algebra and let E be an operator system in A. An element x of E is said to be *positive* if it is positive in A.

Definition 1.4.3. Let E and F be operator systems. A linear map $\varrho\colon E \to F$ is said to be *positive* if it maps any positive element in E to a positive element in F, and ϱ is said to be *contractive*, or a *contraction*, if $\|\varrho\| \leq 1$.

Remark 1.4.4. Positive, linear maps are self-adjoint.

Definition 1.4.5. Let E and F be operator systems. Let $\varrho\colon E \to F$ be a linear map and let for each $n \in \mathbb{N}$ the nth inflation $\varrho^{(n)}$ of ϱ be defined as in the case of *-homomorphisms, see Definition 1.1.24. Then ϱ is said to be *completely bounded* if $\sup_{n\in\mathbb{N}}\|\varrho^{(n)}\|$ is finite, and it is said to be *completely contractive*, respectively *completely positive* if $\varrho^{(n)}$ is contractive, respectively positive for each $n \in \mathbb{N}$.

Remark 1.4.6. The linear space of completely bounded maps between two operator systems E and F can be normed with the *completely bounded norm*, given by

$$\|\varrho\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}}\|\varrho^{(n)}\|$$

for every completely bounded map $\varrho\colon E \to F$.

The following remark is a combination of Propositions 2.11 and 3.6 in [Pa].

Remark 1.4.7. Let E and F be operator systems and let $\varrho\colon E \to F$ be a linear map.

(i) If ϱ is completely positive, then ϱ is completely bounded and $\|\varrho\|_{\mathrm{cb}} = \|\varrho\| = \|\varrho(1_E)\|$.

(ii) If ϱ is unital, then ϱ is completely positive if and only if $\|\varrho\|_{\mathrm{cb}} = 1$. In particular, unital, completely positive maps have norm equal to one, which will be used in many estimates later on.

Examples 1.4.8. (i) All *-homomorphisms are completely positive.

(ii) Let A be a unital C^*-algebra and let s be an isometry in A. Then $V\colon A \to A,\ a \mapsto s^*as$ is a unital, completely positive map.

Proof. (i) Let A and B be C^*-algebras, let $\varphi\colon A \to B$ be a *-homomorphism, let $n \in \mathbb{N}$ and let $a \in M_n(A)$ be a positive element. Then there exists $x \in M_n(A)$ such that $a = x^*x$. Since the inflation $\varphi^{(n)}$ also is a *-homomorphism, we have $\varphi^{(n)}(a) = \varphi^{(n)}(x^*x) = \varphi^{(n)}(x)^*\varphi^{(n)}(x)$ which shows that $\varphi^{(n)}(a)$ is positive in $M_n(B)$.

(ii) It is easy to check that V is linear, and V is unital because s is an isometry. For every $n \in \mathbb{N}$ let $s_n = \mathrm{diag}(s, \ldots, s) \in M_n(A)$ and note that $V^{(n)}(a) = s_n^*as_n$ for every $a \in M_n(A)$. Let now $n \in \mathbb{N}$ and let a be a positive element in $M_n(A)$. Then

$$V^{(n)}(a) = s_n^*a^{1/2}a^{1/2}s_n = \big(a^{1/2}s_n\big)^*a^{1/2}s_n$$

is positive in $M_n(A)$. □

A proof of the following extension result for completely bounded maps is given in Theorem 8.2 in [Pa].

Theorem 1.4.9 (Wittstock's Extension Theorem)**.** Let A be a unital C^*-algebra and let E be an operator system in A. Let H be a Hilbert space and let $\eta\colon E \to B(H)$ be a unital, completely bounded map. Then η extends to a unital, completely bounded map $\overline{\eta}\colon A \to B(H)$ with $\|\overline{\eta}\|_{\mathrm{cb}} = \|\eta\|_{\mathrm{cb}}$.

The next result shows that a self-adjoint, unital, completely bounded map η is close to a unital, completely positive map if its completely bounded norm is close to one (we know that $\|\eta\|_{\mathrm{cb}} \geq 1$ as η is assumed to be unital). A proof can be found in Proposition 1.19 in [Wa].

Proposition 1.4.10. Let A be a unital C^*-algebra, let H be a Hilbert space and let η be a self-adjoint, unital, completely bounded map from A into $B(H)$. Then there is a unital, completely positive map $\varrho\colon A \to B(H)$ such that $\|\varrho - \eta\|_{\text{cb}} \leq \|\eta\|_{\text{cb}} - 1$.

Combining the two preceding results we can prove the following statement, which is Lemma 1.6 in [KP]:

Lemma 1.4.11. Let A be a unital C^*-algebra and let E be an operator system in A. Let H be a Hilbert space and let $\eta\colon E \to B(H)$ be a self-adjoint, unital, completely bounded map. Then there is a unital, completely positive map $\varrho\colon A \to B(H)$ with $\|\varrho|_E - \eta\|_{\text{cb}} \leq \|\eta\|_{\text{cb}} - 1$.

Proof. By Wittstock's Extension Theorem there exists a unital, completely bounded map

$$\tilde{\eta}\colon A \to B(H)$$

such that $\tilde{\eta}|_E = \eta$ and $\|\tilde{\eta}\|_{\text{cb}} = \|\eta\|_{\text{cb}}$. Define

$$\overline{\eta}\colon A \to B(H),\ a \mapsto \frac{1}{2}\left(\tilde{\eta}(a) + \tilde{\eta}(a^*)^*\right).$$

For each $x \in E$ we then have

$$\overline{\eta}(x) = \frac{1}{2}\left(\tilde{\eta}(x) + \tilde{\eta}(x^*)^*\right) = \frac{1}{2}\left(\eta(x) + \eta(x^*)^*\right) = \eta(x)$$

as $\tilde{\eta}|_E = \eta$ and as η is self-adjoint, i.e. $\overline{\eta}|_E = \eta$. For the completely bounded norm we can estimate

$$\|\eta\|_{\text{cb}} \leq \|\overline{\eta}\|_{\text{cb}} \leq \frac{1}{2}\left(\|\tilde{\eta}\|_{\text{cb}} + \|\tilde{\eta}\|_{\text{cb}}\right) = \|\tilde{\eta}\|_{\text{cb}} = \|\eta\|_{\text{cb}},$$

i.e. $\|\overline{\eta}\|_{\text{cb}} = \|\eta\|_{\text{cb}}$. Moreover,

$$\overline{\eta}(a^*) = \frac{1}{2}\left(\tilde{\eta}(a^*) + \tilde{\eta}(a)^*\right) = \frac{1}{2}\left(\tilde{\eta}(a) + \tilde{\eta}(a^*)^*\right)^* = \overline{\eta}(a)^*.$$

Altogether we have shown that $\overline{\eta}$ is a self-adjoint, unital, completely bounded map which extends η to A. Proposition 1.4.10 now yields a unital, completely positive map $\varrho\colon A \to B(H)$ such that

$$\|\varrho - \overline{\eta}\|_{\text{cb}} \leq \|\overline{\eta}\|_{\text{cb}} - 1 = \|\eta\|_{\text{cb}} - 1,$$

which implies that

$$\|\varrho|_E - \eta\|_{\text{cb}} \leq \|\eta\|_{\text{cb}} - 1,$$

as required. □

1.5 Nuclearity and exactness

After stating some frequently used results on minimal tensor products we turn to the definition of nuclear C^*-algebras and state some of their permanence properties. We then discuss nuclear maps and show that a completely positive contraction being defined on or taking values in a separable, nuclear C^*-algebra is automatically nuclear. In Sections 1.5.3 and 1.5.4 we define exactness, present Kirchberg's result that every separable, exact C^*-algebra admits a nuclear embedding into $B(H)$, and use this to prove that the identity map on a unital, separable, exact C^*-algebra can be locally factorized through matrix algebras via unital, completely bounded maps.

1.5.1 Tensor products

In the following definitions and throughout this thesis tensor products play a very important part. As their definition requires quite a bit of work it is not given here, but there are several textbooks where all the work is done. For a very first start into the theory of algebraic tensor products the reader may consult [Gr], the theory of tensor products of Hilbert spaces and of C^*-algebras is dealt with in [Mu]. Except for Appendix A.2 we only consider minimal tensor products in this thesis.

Notation. For two C^*-algebras A and B let $A \otimes_{\min} B$ denote their minimal tensor product.

The following result is a consequence of Theorem 6.3.3 in [Mu] and the definition of the minimal tensor product.

Lemma 1.5.1. Let A and B be C^*-algebras with representations (H_A, π_A) and (H_B, π_B), respectively. Then there exists a unique *-homomorphism $\pi\colon A\otimes_{\min} B \to B(H_A \otimes H_B)$, where $H_A \otimes H_B$ is the Hilbert space tensor product, such that

$$\pi(a \otimes b) = \pi_A(a) \otimes \pi_B(b) \qquad \text{for all } a \in A,\ b \in B.$$

If π_A and π_B are injective, then so is π.

The next result on *-homomorphisms between minimal tensor products is also taken from [Mu], where it is Theorem 6.5.1.

Lemma 1.5.2. Let A, B, A', B' be C^*-algebras and let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be *-homomorphisms. Then there is a unique *-homomorphism $\pi\colon A \otimes_{\min} B \to A' \otimes_{\min} B'$ satisfying

$$\pi(a \otimes b) = \varphi(a) \otimes \psi(b) \qquad \text{for all } a \in A,\ b \in B.$$

Moreover, if φ and ψ are injective, then so is π.

Notation. We denote the *-homomorphism π in Lemma 1.5.2 by $\varphi \otimes \psi$.

Remark 1.5.3. Let A and B be C^*-algebras and let p be a non-zero projection in B. Then the map

$$\iota\colon A \to A \otimes_{\min} B,\ a \mapsto a \otimes p$$

is linear as $\otimes$ is bilinear, and it is self-adjoint and multiplicative because $p = p^* = p^2$. Hence, ι is a *-homomorphism. Moreover, $\|a \otimes p\| = \|a\|\|p\| = \|a\|$ for all $a \in A$, which shows that ι is an embedding of A into $A \otimes_{\min} B$.

Remark 1.5.4. Let A and B be separable C^*-algebras. Then $A\otimes_{\min} B$ is separable, because if Q_A and Q_B are countable, dense subsets of A and B, respectively, then the set

$$\Big\{ \sum_{i=1}^{n} a_i \otimes b_i \ \Big|\ n \in \mathbb{N},\ a_i \in Q_A,\ b_i \in Q_B \Big\}$$

is a countable, dense subset of $A \otimes_{\min} B$.

The following result is Corollary 4.21 in [Ta].

Lemma 1.5.5. If two C^*-algebras A and B are simple then so is $A \otimes_{\min} B$.

1.5.2 Nuclear C^*-algebras and nuclear maps

Definition 1.5.6. A C^*-algebra A is said to be *nuclear* if for any C^*-algebra B there is a unique C^*-norm on the algebraic tensor product of A and B.

Notation. Let A and B be C^*-algebras. If A or B is nuclear, the completion of their algebraic tensor product with respect to the unique C^*-norm is denoted by $A \otimes B$.

Nuclear C^*-algebras have a number of permanence properties, i.e. nuclearity is preserved under many natural operations. Some of them are stated below, and more examples can be found in Proposition 2.1.2 in [R2], where all save the second statement below are taken from. A reference for statement (ii) is Theorem 6.3.9 in [Mu].

Proposition 1.5.7. (i) Abelian C^*-algebras are nuclear.

(ii) Finite dimensional C^*-algebras are nuclear.

(iii) If two of the C^*-algebras in a short exact sequence $0 \to I \to A \to B \to 0$ (see Definition 1.5.15 below) are nuclear, then so is the third.

(iv) If A and B are nuclear C^*-algebras, then so is $A \otimes B$.

(v) If A is a nuclear C^*-algebra and α is a *-automorphism on A, then the crossed product $A \rtimes_\alpha \mathbb{Z}$ (see Definition 6.2.1) is nuclear.

Remarks 1.5.8. (i) The C^*-algebra of complex matrices $M_n(\mathbb{C})$ is nuclear for each $n \in \mathbb{N}$ as it is finite dimensional. For any C^*-algebra A the map $\psi\colon M_n(\mathbb{C}) \otimes A \to M_n(A)$, given by

$$\psi\left((\lambda_{ij})_{i,j} \otimes a\right) = (\lambda_{ij} a)_{i,j},$$

is a *-isomorphism, see for instance Example 6.3.1 in [Mu]. In particular, if A is nuclear, then so is $M_n(A)$.

(ii) Let A and B be C^*-algebras and let $\varphi\colon A \to B$ be a *-homomorphism. For each $n \in \mathbb{N}$, the inflation $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ of φ corresponds to the map

$$\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi\colon M_n(\mathbb{C}) \otimes A \longrightarrow M_n(\mathbb{C}) \otimes B,$$

in the sense that the diagram

$$\begin{tikzcd} M_n(A) \arrow[r, "\varphi^{(n)}"] & M_n(B) \\ M_n(\mathbb{C}) \otimes A \arrow[u, "\psi_A"] \arrow[r, "\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi"'] & M_n(\mathbb{C}) \otimes B \arrow[u, "\psi_B"'] \end{tikzcd}$$

commutes, where ψ_A and ψ_B denote the *-isomorphisms corresponding to ψ in (i).

Definition 1.5.9. Let A and B be C^*-algebras and let $\varrho\colon A \to B$ be a completely positive contraction. Then ϱ is said to be *nuclear* if for each finite subset F of A and for each $\varepsilon > 0$

there exist a natural number n and completely positive contractions $\sigma\colon A \to M_n(\mathbb{C})$ and $\eta\colon M_n(\mathbb{C}) \to B$ such that

$$\|\varrho(a) - (\eta \circ \sigma)(a)\| \leq \varepsilon \qquad \text{for all } a \in F,$$

i.e. the diagram

$$\begin{tikzcd} A \arrow[rr, "\varrho"] \arrow[dr, dashed, "\sigma"'] & & B \\ & M_n(\mathbb{C}) \arrow[ur, dashed, "\eta"'] & \end{tikzcd}$$

commutes within ε on F.

Remark 1.5.10. In case that A and B are unital C^*-algebras and ϱ is a unital, completely positive map, the maps σ and η can be chosen to be unital, completely positive maps (an idea of how to do this is given in the proof of Proposition 4.3 in [EH]).

Remark 1.5.11. Let A, B and C be C^*-algebras and let $\varrho_1\colon A \to B$ and $\varrho_2\colon B \to C$ be completely positive contractions. If ϱ_1 or ϱ_2 is nuclear, then so is the composition $\varrho_2 \circ \varrho_1$.

Proof. Suppose first that ϱ_1 is nuclear. Let F be a finite subset of A and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ and completely positive contractions $\sigma\colon A \to M_n(\mathbb{C})$ and $\eta'\colon M_n(\mathbb{C}) \to B$ such that

$$\|\varrho_1(a) - (\eta' \circ \sigma)(a)\| \leq \varepsilon \qquad \text{for all } a \in F.$$

Put $\eta = \varrho_2 \circ \eta'\colon M_n(\mathbb{C}) \to C$, then η is a completely positive contraction as ϱ_2 and η' are, and

$$\|(\varrho_2 \circ \varrho_1)(a) - (\eta \circ \sigma)(a)\| \leq \|\varrho_2\| \|\varrho_1(a) - (\eta' \circ \sigma)(a)\| \leq \varepsilon$$

for all $a \in F$, which shows that $\varrho_2 \circ \varrho_1$ is nuclear.

Assume now that ϱ_2 is nuclear. Let F be a finite subset of A and let $\varepsilon > 0$. Then $\varrho_1(F)$ is a finite subset of B, and we can choose $n \in \mathbb{N}$ and completely positive contractions $\sigma'\colon B \to M_n(\mathbb{C})$ and $\eta\colon M_n(\mathbb{C}) \to C$ such that

$$\|\varrho_2(b) - (\eta \circ \sigma')(b)\| \leq \varepsilon \qquad \text{for all } b \in \varrho_1(F).$$

Put $\sigma = \sigma' \circ \varrho_1\colon A \to M_n(\mathbb{C})$, then σ is a completely positive contraction as σ' and ϱ_1 are, and

$$\|(\varrho_2 \circ \varrho_1)(a) - (\eta \circ \sigma)(a)\| = \|(\varrho_2(\varrho_1(a)) - (\eta \circ \sigma')(\varrho_1(a))\| \leq \varepsilon$$

for all $a \in F$, as required. $\square$

The following theorem, which is part of Theorem 3.1 in [CE2], sheds some light on the relation between nuclear C^*-algebras and nuclear maps.

Theorem 1.5.12 (Choi-Effros)**.** Let A be a separable C^*-algebra. The following conditions are equivalent:

(i) A is nuclear;

(ii) the identity map id_A is nuclear.

We shall frequently use the following corollary to this theorem:

Corollary 1.5.13. Let A and B be C^*-algebras and let $\varrho\colon A \to B$ be a completely positive contraction. If A or B is a separable and nuclear C^*-algebra, then ϱ is a nuclear map.

Proof. Assume first that A is separable and nuclear. Then id_A is nuclear by Choi-Effros' Theorem, and hence so is $\varrho = \varrho \circ \mathrm{id}_A$ by Remark 1.5.11.

If B is separable and nuclear, then id_B is a nuclear map and thus so is $\varrho = \mathrm{id}_B \circ \varrho$. $\square$

The following theorem states an important lifting property of nuclear maps which plays a central part in the proof of Kirchberg's Embedding Theorem. The original reference is Theorem 3.10 in [CE1]; a proof is also given in Chapter 6.3 of [Wa].

Theorem 1.5.14 (Choi-Effros' Lifting Theorem)**.** Let A be a unital C^*-algebra, let I be an ideal in A, let $\pi\colon A \to A/I$ denote the quotient map, let E be a separable operator system and let $\varrho\colon E \to A/I$ be a nuclear, unital, completely positive map. Then there is a unital, completely positive map $\lambda\colon E \to A$ which lifts ϱ, i.e. which satisfies $\pi \circ \lambda = \varrho$, i.e. the following diagram commtes:

$$\begin{tikzcd} & A \arrow[d, "\pi"] \\ E \arrow[ur, dashed, "\lambda"] \arrow[r, "\varrho"'] & A/I \end{tikzcd}$$

1.5.3 Exact C^*-algebras

Definition 1.5.15. Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of C^*-algebras and let $(\varphi_n)_{n\in\mathbb{Z}}$ be a sequence of *-homomorphisms $\varphi_n\colon A_n \to A_{n+1}$ for all n in $\mathbb{Z}$. Then the sequence

$$\cdots \longrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} A_{n+2} \longrightarrow \cdots$$

is said to be *exact* if $\mathrm{Im}(\varphi_n) = \mathrm{Ker}(\varphi_{n+1})$ for all $n \in \mathbb{Z}$. An exact sequence of the form

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0$$

is called *short exact.*

Definition 1.5.16. A C^*-algebra A is said to be *exact* if for every short exact sequence

$$0 \longrightarrow I \xrightarrow{\varphi} D \xrightarrow{\psi} B \longrightarrow 0$$

the induced sequence

$$0 \longrightarrow A \otimes_{\min} I \xrightarrow{\mathrm{id}_A \otimes \varphi} A \otimes_{\min} D \xrightarrow{\mathrm{id}_A \otimes \psi} A \otimes_{\min} B \longrightarrow 0$$

is also exact. In other words, A is exact if the functor $A \otimes_{\min} (\cdot)$ is exact (see Chapter 3.2 in [R1] for the definition of exact functors).

Remark 1.5.17. Every nuclear C^*-algebra is exact (see, for instance, in Theorem 6.5.2 in [Mu] for a proof).

Similarly to nuclear C^*-algebras, exact C^*-algebras satisfy a number of permanence properties. The following statements are taken from Proposition 7.1 in [Ki1].

Proposition 1.5.18. (i) Every sub-C^*-algebra of an exact C^*-algebra is again exact.

(ii) If A is an exact C^*-algebra and I is an ideal in A, then A/I is exact.

(iii) If A and B are exact C^*-algebras, then so is $A \otimes_{\min} B$.

(iv) If A is an exact C^*-algebra and α is a *-automorphism on A, then $A \rtimes_\alpha \mathbb{Z}$ is exact (see Definition 6.2.1).

(v) Let A be a C^*-algebra, let I be an ideal in A, let $\pi\colon A \to A/I$ denote the quotient map and suppose that there exists a *-homomorphism $\lambda\colon A/I \to A$ which lifts π, i.e. which satisfies $\pi \circ \lambda = \mathrm{id}_{A/I}$. In other words, suppose that the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] & A \arrow[r, shift left, "\pi"] & A/I \arrow[l, shift left, "\lambda"] \arrow[r] & 0 \end{tikzcd}$$

is split exact. Then A is exact if I and A/I are.

Remark 1.5.19. Notice the differences between the permanence properties of exact C^*-algebras and those of nuclear C^*-algebras: Sub-C^*-algebras of nuclear C^*-algebras need not be nuclear, and extensions of exact C^*-algebras by exact C^*-algebras need not be exact (the existence of the lift λ in statement (v) is crucial). Counterexamples can be found in [Wa].

1.5.4 Exactness and nuclear embeddability

A proof of the following theorem is given in [Wa], Proposition 7.2 and Theorem 7.3.

Theorem 1.5.20 (Kirchberg)**.** Let A be a separable C^*-algebra. Then A is exact if and only if there exists a nuclear, injective *-homomorphism $\iota\colon A \to B(H)$ for some Hilbert space H, i.e. A admits a nuclear embedding into $B(H)$.

Remark 1.5.21. If A in Theorem 1.5.20 is unital, then the Hilbert space H and the nuclear embedding $\iota\colon A \to B(H)$ can be chosen such that ι is unital.

Proof. Suppose that A is unital, and use Theorem 1.5.20 to find a Hilbert space H and a nuclear embedding $\iota\colon A \to B(H)$. Put $P = \iota(1_A)$ and set $H' = P(H)$. As P is a projection, H' is a Hilbert space. For each $a \in A$ and for each $x \in H$ we have that $\iota(a)x = (P \circ \iota(a))x \in H'$, and hence we can define a *-homomorphism

$$\pi\colon \iota(A) \to B(H'),\ \iota(a) \mapsto \iota(a)|_{H'}.$$

We use this to define

$$\iota' = \pi \circ \iota\colon A \to B(H'),\ a \mapsto \iota(a)|_{H'}.$$

Then ι' is a unital *-homomorphism from A into $B(H')$, which is injective because $\iota(a)|_{H'} = 0$ if and only if $\iota(a) = 0$ for all $a \in A$ as $\iota(a)x = \iota(a)Px \in \iota(a)(H')$ for all $x \in H$. Moreover, we know from Remark 1.5.11 that ι' is nuclear as ι is, and thus ι' is a unital, nuclear embedding of A into $B(H')$. $\square$

We shall later on need a corollary of Theorem 1.5.20 which states that the identity map on any finite dimensional operator system E in a unital, separable, exact C^*-algebra A can be factorized through a complex matrix algebra, where the involved maps are unital and completely bounded with completely bounded norm arbitrarily close to one. In the proof of that corollary we shall have to perturb the unital, completely positive maps which arise from the unital, nuclear embedding of A into $B(H)$, so we first prove a lemma which helps us to control the completely bounded norms in case of such a perturbation (see Lemma 1.5 in [KP]).

Lemma 1.5.22. Let A be a unital C^*-algebra, let E be a finite dimensional operator system in A, let $m = \dim E$ and let $\{e_1, \dots, e_m\}$ be a basis of E, let $\{\xi_1, \dots, \xi_m\}$ be the standard basis of $\mathbb{C}^m$, and let $Q\colon E \to \mathbb{C}^m$ be defined by $Q(e_i) = \xi_i$ for each $i \in \mathbb{N}_{\leq m}$. Then for any subset $\{a_1, \dots, a_m\}$ of A, the map $W\colon E \to \operatorname{span}\{a_1, \dots, a_m\}$ defined by $W(e_i) = a_i$ for each $i \in \mathbb{N}_{\leq m}$ is completely bounded and

$$\|W\|_{\mathrm{cb}} \leq 1 + m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|.$$

Moreover, if $m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\| < 1$, then W is bijective, W^{-1} is completely bounded and

$$\|W^{-1}\|_{\mathrm{cb}} \leq \Big(1 - m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|\Big)^{-1}.$$

Proof. Let $\{a_1, \dots, a_m\}$ be a subset of A and let W be defined as above. Define a map $R\colon \mathbb{C}^m \to A$ by $R(\xi_i) = a_i - e_i$ for each $i \in \mathbb{N}_{\leq m}$. Equip $\mathbb{C}^m$ with the maximum norm and compute

$$\|R\| = \sup\Big\{\|\sum_{i=1}^{m} \lambda_i(a_i - e_i)\| \;\Big|\; (\lambda_i)_{i=1}^m \in \mathbb{C}^m \text{ with } \max_{1\leq i\leq m} |\lambda_i| \leq 1\Big\} \leq \sum_{i=1}^{m} \|a_i - e_i\|.$$

As $R \circ Q$ is a linear map on the m-dimensional operator system E, Lemma 2.3 in [EH] yields that $R \circ Q$ is completely bounded and

$$\|R \circ Q\|_{\mathrm{cb}} \leq m\|R \circ Q\| \leq m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|. \tag{1.5.1}$$

For each $i \in \mathbb{N}_{\leq m}$ we have

$$W(e_i) = a_i = e_i + (R \circ Q)(e_i)$$

which implies that $W = \mathrm{id}_E + R \circ Q$. Therefore, W is completely bounded and

$$\|W\|_{\mathrm{cb}} \overset{(1.5.1)}{\leq} 1 + m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|,$$

as desired. Assume now that $m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\| < 1$. For every $n \in \mathbb{N}$ and for every $x \in M_n(E)$ we can use (1.5.1) to estimate

$$\begin{aligned}
\|W^{(n)}(x)\| &= \|\mathrm{id}_E^{(n)}(x) + (R \circ Q)^{(n)}(x)\| \\
&\geq \|\mathrm{id}_{M_n(E)}(x)\| - \|(R \circ Q)^{(n)}(x)\| \\
&\geq \|x\| \left(1 - \|R \circ Q\|_{\mathrm{cb}}\right) \\
&\geq \|x\| \Big(1 - m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|\Big),
\end{aligned} \tag{1.5.2}$$

which implies that $\|W(x)\| > 0$ for all $x \in E \setminus \{0\}$, i.e. W is injective. It is clear from the definition of W that W is also surjective and hence bijective, so we can deduce from (1.5.2) that

$$\|(W^{-1})^{(n)}\| = \|\big(W^{(n)}\big)^{-1}\| \leq \Big(1 - m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|\Big)^{-1} \qquad \text{for all } n \in \mathbb{N},$$

and thus

$$\|W^{-1}\|_{\mathrm{cb}} \leq \Big(1 - m\|Q\| \sum_{i=1}^{m} \|a_i - e_i\|\Big)^{-1}.$$

□

We now turn to the corollary to Theorem 1.5.20:

Corollary 1.5.23. Let A be a unital, separable, exact C^*-algebra, let E be a finite dimensional operator system in A and let $\varepsilon > 0$. Then there exist a natural number n, a unital, completely positive map $\sigma\colon E \to M_n(\mathbb{C})$, and a self-adjoint, unital, completely bounded map $\eta\colon \sigma(E) \to E$ such that $\eta \circ \sigma = \mathrm{id}_E$ and $\|\eta\|_{\mathrm{cb}} \leq 1 + \varepsilon$.

Proof. By Remark 1.5.21 there exists a Hilbert space H such that there is a unital, nuclear, injective *-homomorphism $\iota\colon A \to B(H)$. Let $m = \dim E$ and let $\{e_1, \ldots, e_m\}$ be a basis of E with $e_1 = 1_A$, then $\{\iota(e_1) = \mathrm{id}_H, \ldots, \iota(e_m)\}$ is a basis of the operator system $\iota(E)$ in $B(H)$ as ι is unital and injective. Let $\{\xi_1, \ldots, \xi_m\}$ denote the standard basis in $\mathbb{C}^m$ and define a map $Q\colon \iota(E) \to \mathbb{C}^m$ by $Q(\iota(e_i)) = \xi_i$ for each $i \in \mathbb{N}_{\leq m}$. Choose $0 < \delta < 1$ such that $(1-\delta)^{-1} \leq 1 + \varepsilon$, and find by nuclearity of ι a natural number n and unital, completely positive maps $\sigma\colon A \to M_n(\mathbb{C})$ and $\eta'\colon M_n(\mathbb{C}) \to B(H)$ such that

$$\|\iota(e_i) - (\eta' \circ \sigma)(e_i)\| \leq \frac{\delta}{m^2\|Q\|} \qquad \text{for all } i \in \mathbb{N}_{\leq m}. \tag{1.5.3}$$

Set $a_i = (\eta' \circ \sigma)(e_i)$ for each $i \in \mathbb{N}_{\leq m}$ and define a map $W\colon \iota(E) \to \mathrm{span}\{a_1, \ldots, a_m\}$ by $W(\iota(e_i)) = a_i$ for each $i \in \mathbb{N}_{\leq m}$. Notice that $W(\mathrm{id}_H) = W(\iota(e_1)) = a_1 = \mathrm{id}_H$, i.e. W is unital. Now

$$m\|Q\| \sum_{i=1}^{m} \|\iota(e_i) - a_i\| \leq \delta < 1$$

by (1.5.3), and therefore Lemma 1.5.22 yields that W is bijective and that W^{-1} is completely bounded with

$$\|W^{-1}\|_{\mathrm{cb}} \leq \Big(1 - m\|Q\| \sum_{i=1}^{m} \|\iota(e_i) - a_i\|\Big)^{-1} \leq (1-\delta)^{-1} \leq 1 + \varepsilon.$$

Let $\iota^{-1}\colon \iota(A) \to A$ denote the left-inverse of the injective *-homomorphism ι and set

$$\eta = \iota^{-1} \circ W^{-1} \circ \eta'|_{\sigma(E)}\colon \sigma(E) \to E.$$

Then η is unital and completely bounded with

$$\|\eta\|_{\mathrm{cb}} = \|\iota^{-1} \circ W^{-1} \circ \eta'|_{\sigma(E)}\|_{\mathrm{cb}} \leq \|W^{-1}\|_{\mathrm{cb}} \leq 1 + \varepsilon,$$

and

$$(\eta \circ \sigma)(e_i) = \big(\iota^{-1} \circ W^{-1} \circ \eta' \circ \sigma\big)(e_i) = \big(\iota^{-1} \circ W^{-1}\big)(a_i) = \iota^{-1}(\iota(e_i)) = e_i$$

for each $i \in \mathbb{N}_{\leq m}$, and thus $\eta \circ \sigma|_E = \mathrm{id}_E$. To show that η is self-adjoint let $x \in \sigma(E)$ and choose $e \in E$ with $\sigma(e) = x$. Being completely positive, σ is self-adjoint and hence

$$\eta(x^*) = \eta\left(\sigma(e)^*\right) = \eta(\sigma(e^*)) = e^* = (\eta(\sigma(e)))^* = \eta(x)^*,$$

as required. $\square$

Chapter 2

The Cuntz algebras $\mathcal{O}_n$

The Cuntz algebras, introduced by Joachim Cuntz in [Cu1], are an important example of unital, separable, simple and purely infinite C^*-algebras. We start here with their definition and some important properties, then prove in Section 2.2 that $K_0(\mathcal{O}_2) = 0$ and conclude that $p\mathcal{O}_2 p \cong \mathcal{O}_2$ for every non-zero projection in $\mathcal{O}_2$. In Section 2.3 we show that $M_n(\mathbb{C}) \hookrightarrow \mathcal{O}_n \hookrightarrow \mathcal{O}_2$ for every $n \in \mathbb{N}_{\geq 2}$ which will be used in Chapter 5 to embed quasidiagonal C^*-algebras into $\mathcal{O}_2$.

2.1 Definition and important properties

Unless otherwise stated, Cuntz' paper [Cu1] is the reference for the statements in this section. An introduction to Cuntz algebras is also given in Chapter 4.2 in [R2].

Proposition/Definition 2.1.1. Let H be a separable, infinite dimensional Hilbert space. For each $n \in \mathbb{N}_{\geq 2}$ there exist n isometries $s_1, \dots, s_n$ in $B(H)$ satisfying

$$\sum_{i=1}^{n} s_i s_i^* = \mathrm{id}_H \,. \tag{2.1.1}$$

The *Cuntz algebra* $\mathcal{O}_n$ is defined to be the sub-C^*-algebra of $B(H)$ generated by $s_1, \dots, s_n$. Equation (2.1.1) is called the *Cuntz relation.* Let $n \in \mathbb{N}_{\geq 2}$ and suppose that $s_1, \dots, s_n$ are isometries satisfying the Cuntz relation. Then

$$s_i^* s_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad \text{for all } i, j \in \mathbb{N}_{\leq n}. \tag{2.1.2}$$

The *Cuntz algebra* $\mathcal{O}_\infty$ is defined to be the C^*-algebra generated by an infinite sequence $(s_i)_{i \in \mathbb{N}}$ of isometries with (2.1.2) for all $i, j \in \mathbb{N}$.

The following theorem shows that the Cuntz algebras are well defined in the sense that they do not depend on the choice of the generating isometries.

Theorem 2.1.2 (The universal property of the Cuntz algebras)**.** Let A be a unital C^*-algebra, let $n \in \mathbb{N}_{\geq 2}$ and let $s_1, \dots, s_n$ be isometries generating $\mathcal{O}_n$. Suppose that A contains n isometries $t_1, \dots, t_n$ which satisfy the Cuntz relation, i.e. $\sum_{i=1}^n t_i t_i^* = 1_A$. Then there exists a unique, unital *-homomorphism $\varphi\colon \mathcal{O}_n \to A$ satisfying $\varphi(s_i) = t_i$ for all $i \in \mathbb{N}_{\leq n}$.

Theorem 2.1.3. For each $n \in \mathbb{N}_{\geq 2}$ and for $n = \infty$ the Cuntz algebra $\mathcal{O}_n$ is unital, separable, simple, nuclear and purely infinite.

Remark 2.1.4. With the notation from Theorem 2.1.2, the *-homomorphism φ is automatically injective as $\mathcal{O}_n$ is simple, i.e. $\mathcal{O}_n$ is isomorphic to the sub-C^*-algebra of A which is generated by $t_1, \ldots, t_n$.

The following important result is proved in Theorem 5.2.1 in [R2]. It is often referred to by saying that $\mathcal{O}_2$ is *self-absorbing*.

Theorem 2.1.5 (Elliot's $\mathcal{O}_2 \otimes \mathcal{O}_2$ Theorem)**.** The C^*-algebras $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes \mathcal{O}_2$ are isomorphic.

2.2 Computing $K_0(\mathcal{O}_2)$

In this section we prove that $K_0(\mathcal{O}_2) = 0$. This is the consequence of a number of statements on the Cuntz algebras $\mathcal{O}_n$ which are presented below. The original proof was given in [Cu2]. The first result we use here holds for general unital C^*-algebras:

Lemma 2.2.1. Let A be a unital C^*-algebra and let s be an isometry in A. Define

$$\mu \colon A \to A, \ a \mapsto sas^*.$$

Then μ is a *-endomorphism on A, and $K_0(\mu) = \mathrm{id}_{K_0(A)}$.

Proof. It is easy to check that the map μ is a *-endomorphism on A. For every $n \in \mathbb{N}$ put $s_n = \mathrm{diag}(s, \ldots, s) \in M_n(A)$ and note that the nth inflation of μ is then given by

$$\mu^{(n)} \colon M_n(A) \to M_n(A), \ a \mapsto s_n a s_n^*.$$

Let now $p \in \mathcal{P}_\infty(A)$ and take $n \in \mathbb{N}$ such that $p \in \mathcal{P}_n(A)$. Set $v = s_n p$ and check that $v^*v = ps_n^* s_n p = p$ and $vv^* = s_n p s_n^* = \mu^{(n)}(p)$. This shows that $p \sim \mu^{(n)}(p)$, hence $[p]_0 = [\mu^{(n)}(p)]_0$ and

$$K_0(\mu)([p]_0) = [\mu^{(n)}(p)]_0 = [p]_0,$$

which shows that $K_0(\mu) = \mathrm{id}_{K_0(A)}$. □

Lemma 2.2.2. Let $n \in \mathbb{N}_{\geq 2}$ and let $s_1, \ldots, s_n$ be isometries generating $\mathcal{O}_n$.

(i) For every unitary $u \in \mathcal{O}_n$ there is a unique unital *-endomorphism φ_u on $\mathcal{O}_n$ such that $\varphi_u(s_i) = us_i$ for all $i \in \mathbb{N}_{\leq n}$. Moreover, $u = \sum_{i=1}^n \varphi_u(s_i)s_i^*$.

(ii) Let φ be a unital *-endomorphism on $\mathcal{O}_n$. Then $\varphi = \varphi_u$ for $u = \sum_{i=1}^n \varphi(s_i)s_i^*$.

Proof. (i): Let u be a unitary in $\mathcal{O}_n$. Then $s_i^* u^* u s_i = 1_{\mathcal{O}_n}$ for all $i \in \mathbb{N}_{\leq n}$ and

$$\sum_{i=1}^n us_i s_i^* u^* = u\Big(\sum_{i=1}^n s_i s_i^*\Big)u^* = uu^* = 1_{\mathcal{O}_n},$$

i.e. $\{us_1, \ldots, us_n\}$ is a set of isometries in $\mathcal{O}_n$ satisfying the Cuntz relation. By the universal property of the Cuntz algebras there is a unique, unital *-homomorphism $\varphi_u \colon \mathcal{O}_n \to \mathcal{O}_n$ satisfying $\varphi_u(s_i) = us_i$ for all $i \in \mathbb{N}_{\leq n}$. Besides,

$$\sum_{i=1}^n \varphi_u(s_i)s_i^* = \sum_{i=1}^n us_i s_i^* = u.$$

(ii): Check first that u as given above is a unitary:

$$u^*u = \sum_{i,j=1}^{n} s_i\varphi(s_i^*s_j)s_j^* = \sum_{i=1}^{n} s_is_i^* = 1_{\mathcal{O}_n},$$

$$uu^* = \sum_{i,j=1}^{n} \varphi(s_i)s_i^*s_j\varphi(s_j^*) = \varphi\Big(\sum_{i=1}^{n} s_is_i^*\Big) = \varphi(1_{\mathcal{O}_n}) = 1_{\mathcal{O}_n}.$$

By (i), there is a unique unital *-endomorphism φ_u with $\varphi_u(s_i) = us_i$ for all $i \in \mathbb{N}_{\leq n}$, and as

$$\varphi(s_j) = \sum_{i=1}^{n} \varphi(s_i)s_i^*s_j = us_j = \varphi_u(s_j) \qquad \text{for all } j \in \mathbb{N}_{\leq n}$$

it follows that $\varphi = \varphi_u$. □

Lemma 2.2.3. Let $n \in \mathbb{N}_{\geq 2}$, let $s_1, \ldots, s_n$ be isometries which generate $\mathcal{O}_n$ and define

$$\lambda\colon \mathcal{O}_n \to \mathcal{O}_n,\ x \mapsto \sum_{i=1}^{n} s_ixs_i^*.$$

Then λ is a unital *-endomorphism on $\mathcal{O}_n$ and the following hold:

(i) $K_0(\lambda) = n\,\mathrm{id}_{K_0(\mathcal{O}_n)}$;

(ii) $K_0(\lambda) = \mathrm{id}_{K_0(\mathcal{O}_n)}$.

Proof. It is easy to check that λ is unital, linear and self-adjoint. To show that λ is multiplicative let $x, y \in \mathcal{O}_n$ and calculate

$$\lambda(x)\lambda(y) = \sum_{i,j=1}^{n} s_ixs_i^*s_jys_j^* = \sum_{i=1}^{n} s_ixys_i^* = \lambda(xy).$$

Hence, λ is a unital *-endomorphism on $\mathcal{O}_n$.

(i): For each $i \in \mathbb{N}_{\leq n}$ let $\mu_i\colon \mathcal{O}_n \to \mathcal{O}_n,\ x \mapsto s_ixs_i^*$. Then μ_i is a *-endomorphism on $\mathcal{O}_n$ and $K_0(\mu_i) = \mathrm{id}_{K_0(\mathcal{O}_n)}$ for all $i \in \mathbb{N}_{\leq n}$ by Lemma 2.2.1, and $\lambda = \sum_{i=1}^{n} \mu_i$. Let $k \in \mathbb{N}$ and let $p \in \mathcal{P}_k(\mathcal{O}_n)$. Note that for all $i \neq j \in \mathbb{N}_{\leq n}$ the elements $\mu_i^{(k)}(p)$ and $\mu_j^{(k)}(p)$ are mutually orthogonal projections, and hence

$$[\mu_i^{(k)}(p) + \mu_j^{(k)}(p)]_0 = [\mu_i^{(k)}(p)]_0 + [\mu_j^{(k)}(p)]_0$$

by Proposition 3.1.7(iv) in [R1]. Thus,

$$\begin{aligned} K_0(\lambda)([p]_0) = [\lambda^{(k)}(p)]_0 &= \Big[\sum_{i=1}^{n} \mu_i^{(k)}(p)\Big]_0 = \sum_{i=1}^{n}[\mu_i^{(k)}(p)]_0 \\ &= \sum_{i=1}^{n} K_0(\mu_i)([p]_0) = \sum_{i=1}^{n}[p]_0 = n[p]_0, \end{aligned}$$

which shows that $K_0(\lambda) = n\,\mathrm{id}_{K_0(\mathcal{O}_n)}$.

(ii): We will show that $\lambda \sim_h \mathrm{id}_{\mathcal{O}_n}$, because then the statement follows from Propositions 1.2.18(i) and 1.2.17(i). Let $u = \sum_{i=1}^n \lambda(s_i)s_i^*$ be the unitary with $\lambda = \varphi_u$. Then

$$u^* = \sum_{i=1}^n s_i\lambda(s_i^*) = \sum_{i,j=1}^n s_i s_j s_i^* s_j^* = \sum_{i,j=1}^n \lambda(s_j)s_j^* = u,$$

i.e. u is self-adjoint and hence the spectrum $\sigma(u)$ is contained in $\mathbb{R}$. Consequently, $\sigma(u)$ is not equal to the unit circle in $\mathbb{C}$, and hence Lemma 2.1.3(ii) in [R1] yields a continuous map $v\colon [0,1] \to \mathcal{O}_n,\ t \mapsto u_t$ such that each u_t is a unitary and such that $u_0 = u$ and $u_1 = 1_{\mathcal{O}_n}$. Use these unitaries to define

$$\Phi\colon [0,1] \times \mathcal{O}_n \to \mathcal{O}_n,\ (t,x) \mapsto \varphi_{u_t}(x).$$

Then each $\Phi_t = \varphi_{u_t}$ is a *-endomorphism on $\mathcal{O}_n$, and the map $t \mapsto \varphi_{u_t}(s_i) = u_t s_i$ is continuous for every $i \in \mathbb{N}_{\leq n}$ as v is continuous. By Lemma 1.1.27 this implies that $t \mapsto \varphi_{u_t}(x)$ is continuous for every $x \in \mathcal{O}_n$. Finally, $\Phi_0 = \varphi_{u_0} = \varphi_u = \lambda$ and $\Phi_1 = \varphi_{u_1} = \varphi_{1_{\mathcal{O}_n}} = \mathrm{id}_{\mathcal{O}_n}$, and therefore $\lambda \sim_h \mathrm{id}_{\mathcal{O}_n}$ which implies $K_0(\lambda) = K_0(\mathrm{id}_{\mathcal{O}_n}) = \mathrm{id}_{K_0(\mathcal{O}_n)}$ by the propositions mentioned above. $\square$

Corollary 2.2.4. Let $n \in \mathbb{N}$ and let $g \in K_0(\mathcal{O}_n)$. Then $(n-1)g = 0$.

Proof. By Lemma 2.2.3 we have that

$$ng = n\,\mathrm{id}_{K_0(\mathcal{O}_n)}(g) = K_0(\lambda)(g) = \mathrm{id}_{K_0(\mathcal{O}_n)}(g) = g,$$

and thus $(n-1)g = 0$. $\square$

Corollary 2.2.5. $K_0(\mathcal{O}_2) = 0$.

Proof. For every $g \in K_0(\mathcal{O}_2)$ we have $g = (2-1)g = 0$; thus $K_0(\mathcal{O}_2) = 0$. $\square$

The following statement, which is Theorem 3.7 in [Cu2], describes the K_0-groups for all $\mathcal{O}_n$ with $n \in \mathbb{N}_{\geq 2}$.

Theorem 2.2.6. For every $n \in \mathbb{N}_{\geq 2}$, the K_0-group of $\mathcal{O}_n$ is isomorphic to $\mathbb{Z}/(n-1)\mathbb{Z}$.

The following consequence of Corollary 2.2.5 will be used to prove Kirchberg's Embedding Theorem in the unital case.

Corollary 2.2.7. For each non-zero projection p in $\mathcal{O}_2$ the corner $p\mathcal{O}_2p$ is isomorphic to $\mathcal{O}_2$.

Proof. Let p be a non-zero projection in $\mathcal{O}_2$. As $\mathcal{O}_2$ is simple and purely infinite, p is properly infinite and full. Since $K_0(\mathcal{O}_2) = 0$, we have that $[p]_0 = [1_{\mathcal{O}_2}]_0$, and as $1_{\mathcal{O}_2}$ also is properly infinite and full Proposition 1.2.19(ii) yields that $p \sim 1_{\mathcal{O}_2}$. Use this to choose an isometry $s \in \mathcal{O}_2$ with $ss^* = p$. Then $sxs^* = psxs^*p$ for all $x \in \mathcal{O}_2$, and we can define

$$\varphi\colon \mathcal{O}_2 \to p\mathcal{O}_2p,\ x \mapsto sxs^*,$$

which is a unital, isometric and hence injective *-homomorphism. To see that φ is also surjective let $pyp \in p\mathcal{O}_2p$ and check that

$$\varphi(s^*ys) = ss^*yss^* = pyp.$$

Thus, $p\mathcal{O}_2p \cong \mathcal{O}_2$. $\square$

2.3 Embedding $M_n(\mathbb{C})$ into $\mathcal{O}_2$

The following two lemmas show that, for each $n \in \mathbb{N}_{\geq 2}$, the complex matrix algebra $M_n(\mathbb{C})$ embeds unitally into $\mathcal{O}_n$ which can be unitally embedded into $\mathcal{O}_2$. Consequently, each $M_n(\mathbb{C})$ can be unitally embedded into $\mathcal{O}_2$ (if $n = 1$ this simply means that $\mathbb{C}$ embeds into $\mathcal{O}_2$).

Lemma 2.3.1. For each $n \in \mathbb{N}_{\geq 2}$, the complex matrix algebra $M_n(\mathbb{C})$ embeds unitally into the Cuntz algebra $\mathcal{O}_n$.

Proof. Let $n \in \mathbb{N}_{\geq 2}$, let $\{e_{ij} \mid i, j \in \mathbb{N}_{\leq n}\}$ be the system of standard matrix units for $M_n(\mathbb{C})$ as in Definition 1.1.21 and let $\{s_i \mid i \in \mathbb{N}_{\leq n}\}$ be a system of isometries generating $\mathcal{O}_n$. Define $\varphi\colon M_n(\mathbb{C}) \to \mathcal{O}_n$ by $\varphi(e_{ij}) = s_i s_j^*$ for all $i, j \in \mathbb{N}_{\leq n}$ and extend linearly. We show that φ is a unital *-homomorphism. Let $a = \sum_{i,j=1}^n a_{ij} e_{ij}$ and $b = \sum_{i,j=1}^n b_{ij} e_{ij}$ be elements in $M_n(\mathbb{C})$, compute their product

$$ab = \sum_{i,j,k,l}^{n} a_{ij} b_{kl} e_{ij} e_{kl} = \sum_{i,k,l=1}^{n} a_{ik} b_{kl} e_{il}$$

and check that

$$\varphi(ab) = \sum_{i,k,l=1}^{n} a_{ik} b_{kl} s_i s_l^* = \sum_{i,j,k,l=1}^{n} a_{ij} b_{kl} s_i s_j^* s_k s_l^* = \varphi(a)\varphi(b),$$

i.e. φ is multiplicative. Moreover,

$$\varphi(a^*) = \varphi\Big(\sum_{i,j=1}^{n} a_{ij}^* e_{ji}\Big) = \varphi\Big(\sum_{i,j=1}^{n} a_{ji}^* e_{ij}\Big) = \sum_{i,j=1}^{n} a_{ji}^* s_i s_j^* = \sum_{i,j=1}^{n} a_{ij}^* s_j s_i^* = \varphi(a)^*$$

and

$$\varphi(1_{M_n(\mathbb{C})}) = \varphi\Big(\sum_{i=1}^{n} e_{ii}\Big) = \sum_{i=1}^{n} s_i s_i^* = 1_{\mathcal{O}_n},$$

Thus, φ is a unital *-homomorphism, which is automatically injective as $M_n(\mathbb{C})$ is simple. $\square$

Lemma 2.3.2. For all $n \in \mathbb{N}_{\geq 2}$, the Cuntz algebra $\mathcal{O}_n$ embeds unitally into $\mathcal{O}_2$.

Proof. We show that $\mathcal{O}_2$ contains, for each $n \in \mathbb{N}_{\geq 2}$, a set of isometries $\{t_i \mid i \in \mathbb{N}_{\leq n}\}$ which satisfy the $\mathcal{O}_n$-relation. By the universal property of the Cuntz algebras it then follows that there is a unital, injective *-homomorphism $\varphi\colon \mathcal{O}_n \to \mathcal{O}_2$ given by $\varphi(s_i) = t_i$ for all $i \in \mathbb{N}_{\leq n}$, where $\{s_i \mid i \in \mathbb{N}_{\leq n}\}$ is a set of generating isometries in $\mathcal{O}_n$.

Let s_1 and s_2 be two isometries generating $\mathcal{O}_2$. We show that, for each $n \in \mathbb{N}_{\geq 2}$, the set $\{t_i \mid i \in \mathbb{N}_{\leq n}\}$ where

$$t_i = \begin{cases} s_2^{i-1} s_1 & \text{if} \quad i < n \\ s_2^{n-1} & \text{if} \quad i = n \end{cases} \tag{2.3.1}$$

is a set of isometries in $\mathcal{O}_2$ which satisfy the $\mathcal{O}_n$-relation. Being a product of isometries in $\mathcal{O}_2$ each such t_i is an isometry in $\mathcal{O}_2$. If $n = 2$ then $\{t_1, t_2\} = \{s_1, s_2\}$ and there is nothing to prove. Suppose that the $\mathcal{O}_n$-relation is satisfied by $\{t_i \mid i \in \mathbb{N}_{\leq n}\}$ defined as in (2.3.1) for some $n \in \mathbb{N}_{\geq 2}$ and consider the set of isometries $\{\tilde{t}_i \mid i \in \mathbb{N}_{\leq n+1}\}$ with $\tilde{t}_i = s_2^{i-1} s_1$ if $i < n+1$ and $\tilde{t}_{n+1} = s_2^n$. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} \tilde{t}_i \tilde{t}_i^* &= \sum_{i=1}^{n} s_2^{i-1} s_1 s_1^* (s_2^{i-1})^* + s_2^n (s_2^n)^* \\
&= \sum_{i=1}^{n-1} s_2^{i-1} s_1 s_1^* (s_2^{i-1})^* + s_2^{n-1} s_1 s_1^* (s_2^{n-1})^* + s_2^{n-1} s_2 s_2^* (s_2^{n-1})^* \\
&= \sum_{i=1}^{n-1} s_2^{i-1} s_1 s_1^* (s_2^{i-1})^* + s_2^{n-1} (s_2^{n-1})^* \\
&= \sum_{i=1}^{n} t_i t_i^* = 1_{\mathcal{O}_2},
\end{aligned}$$

which completes the proof. $\square$

Chapter 3

Obtaining approximate unitary equivalence

Before discussing the purpose of this chapter we give the definition of (approximate) unitary equivalence and the weaker notion of approximate similarity via isometries.

Definition 3.0.3. Let A and B be C^*-algebras, B unital, and let $\varrho, \eta \colon A \to B$ be completely positive maps. If there is a unitary u in B such that $u\eta(a)u^* = \varrho(a)$ for all $a \in A$, then ϱ and η are said to be *unitarily equivalent*, in symbols $\varrho \sim_u \eta$.

If for every $\varepsilon > 0$ and for every finite subset F of A there is a unitary u in B with $\|u\eta(a)u^* - \varrho(a)\| \leq \varepsilon$ for all $a \in F$, then ϱ and η are said to be *approximately unitarily equivalent*, denoted by $\varrho \approx_u \eta$.

Definition 3.0.4. Let A and B be C^*-algebras, B unital, and let $\varrho, \eta \colon A \to B$ be completely positive maps. We say that ϱ is *approximately similar via isometries* to η if for every $\varepsilon > 0$ and for every finite subset F of A there is an isometry s in B such that $\|s^*\eta(a)s - \varrho(a)\| \leq \varepsilon$ for all $a \in F$.

The main purpose of this chapter is to find conditions on which two unital, completely positive maps or *-homomorphisms ϱ and η between unital C^*-algebras A and B are approximately unitarily equivalent. As it turns out that obtaining approximate similarity via isometries is an important step towards approximate unitary equivalence, the first two sections of this chapter deal with this weaker property.

In particular, the main statement of Section 3.1 is that a nuclear, unital, completely positive map on a unital, simple, purely infinite C^*-algebra A is approximately similar via isometries to the identity map on A.

In Section 3.2 we will show the following: If ϱ_1 and ϱ_2 are unital, completely positive maps defined on a finite dimensional operator system in a unital, separable, exact C^*-algebra A, then there exists, on certain assumptions, a nuclear map η such that $\|\eta \circ \varrho_1 - \varrho_2\|$ can be made arbitrarily small. Combining this with the result of Section 3.1 we will be able to establish approximate similarity via isometries between ϱ_1 and ϱ_2 if these maps are unital, injective *-homomorphisms from a unital, separable, exact C^*-algebra into a unital, separable, simple, nuclear and purely infinite C^*-algebra.

In Section 3.3 it will be shown how to get from approximate similarity via isometries to approximate unitary equivalence.

As a first application of these results it will be proved in Section 3.4 that any two unital, injective *-homomorphisms from a unital, separable, exact C^*-algebra A into $\mathcal{O}_2$ are approximately unitarily equivalent.

3.1 Approximating nuclear, unital, completely positive maps via isometries

In Proposition 3.1.3 and Lemma 3.1.4 below, which are important ingredients to the proof of Proposition 3.1.5 on approximate similarity of a nuclear map to the identity, we need the notion of *states* on a C^*-algebra:

Definition 3.1.1. Let A be a unital C^*-algebra. A linear functional $\omega\colon A \to \mathbb{C}$ is said to be a *state (on A)* if ω is positive and unital. A state ω on A is said to be *pure* if for every positive linear functional ϱ on A with $\varrho \leq \omega$ (i.e. $\omega - \varrho$ is positive) there is a number $t \in [0, 1]$ such that $\varrho = t\omega$.

Remark 3.1.2. Let A be a C^*-algebra and let $\omega\colon A \to \mathbb{C}$ be a positive linear functional. Then the map

$$\sigma\colon A \times A \to \mathbb{C},\ (a, b) \mapsto \omega(b^*a)$$

is a positive, sesquilinear form, and hence the Cauchy-Schwarz inequality

$$|\sigma(a, b)|^2 \leq \sigma(a, a)\sigma(b, b)$$

holds for all $a, b \in A$. For ω this yields

$$|\omega(ab)|^2 = |\sigma(b, a^*)|^2 \leq \sigma(b, b)\sigma(a^*, a^*) = \omega(b^*b)\omega(aa^*)$$

for all $a, b \in A$.

Proposition 3.1.3. Let A be a unital, simple, purely infinite C^*-algebra and let ω be a state on A. For each finite subset F of A and for each $\varepsilon > 0$ there is a non-zero projection p in A such that

$$\|pap - \omega(a)p\| \leq \varepsilon \qquad \text{for all } a \in F.$$

Proof. We first consider the case that ω is a pure state. Define $L = \{x \in A \mid \omega(x^*x) = 0\}$, the so-called left kernel of ω. Since ω is bounded, L is closed, and by Remark 3.1.2 we have

$$|\omega\left((ax)^*ax\right)|^2 \leq \omega(x^*x)\omega\left((ax)^*a\left((ax)^*a\right)^*\right) = 0 \qquad \text{for all } a \in A,\ x \in L,$$

i.e. L is a closed left ideal in A. By Theorem 3.2.1 in [Mu] this implies that the set $N = L \cap L^*$ is a hereditary sub-C^*-algebra of A. We proceed to show that N is non-zero. As ω is assumed to be pure, Proposition 3.13.6 in [Pe1] yields that $\ker(\omega) = L + L^*$. Notice that, for each $a \in A$, we have $\omega(a - \omega(a)1_A) = 0$, i.e. $a - \omega(a)1_A \in \ker(\omega)$. Being purely infinite, A is not isomorphic to $\mathbb{C}$ and thus contains an element a such that $a - \omega(a)1_A \neq 0$. This shows that $\ker(\omega)$ and hence L are non-zero. For every non-zero element $x \in L$ we have $0 \neq x^*x \in L \cap L^* = N$ because L is a left ideal and x^*x is self-adjoint. Thus, N is a non-zero, hereditary sub-C^*-algebra of A.

Since, being purely infinite, A has real rank zero, Theorem 1.3.9 yields the existence of an approximate unit $(q_\lambda)_{\lambda\in\Lambda}$ for N such that each q_λ is a projection. As $N \subseteq L$ we have

$\omega(q_\lambda) = \omega(q_\lambda^* q_\lambda) = 0$ for each $\lambda \in \Lambda$. Thus, setting $p_\lambda = 1_A - q_\lambda$ for each $\lambda \in \Lambda$, we have $\omega(p_\lambda) = \omega(1_A) - \omega(q_\lambda) = \omega(1_A) = 1$, i.e. each p_λ is a non-zero projection in A. As shown above, $x \in L$ implies $x^*x \in N$ and thus

$$\|xp_\lambda\|^2 = \|p_\lambda x^* x p_\lambda\| \leq \|x^*x(1 - q_\lambda)\| \to 0 \qquad \text{for all } x \in L$$

as $(q_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for N. We have already seen that $a - \omega(a)1_A$ is contained in $\ker(\omega) = L + L^*$ for all $a \in A$, and thus we can for every $a \in A$ choose $x, y \in L$ such that $a - \omega(a)1_A = x + y^*$ to obtain

$$\begin{aligned}\|p_\lambda a p_\lambda - \omega(a)p_\lambda\| &= \|p_\lambda \left(a - \omega(a)1_A\right) p_\lambda\| = \|p_\lambda(x + y^*)p_\lambda\| \leq \|xp_\lambda\| + \|p_\lambda y^*\| \\ &= \|xp_\lambda\| + \|yp_\lambda\| \to 0. \end{aligned} \tag{3.1.1}$$

Let now F be a finite subset of A and let $\varepsilon > 0$. For each $a \in F$ we can by (3.1.1) choose $\lambda_a \in \Lambda$ such that $\|p_\lambda a p_\lambda - \omega(a)p_\lambda\| \leq \varepsilon$ for all $\lambda \geq \lambda_a$ in Λ. Let $\lambda \in \Lambda$ be a majorant of $\{\lambda_a \mid a \in F\}$ and put $p = p_\lambda$, then

$$\|pap - \omega(a)p\| \leq \varepsilon \qquad \text{for all } a \in F,$$

which proves the proposition in the case of pure states.

In the general case we use the fact that A, being simple and purely infinite, satisfies the assumptions of Lemma 11.2.4 in [Di], which then entails that the set of pure states on A is weak*-dense in the state space of A. Let F be a finite subset of A, let $\varepsilon > 0$ and choose a pure state τ on A such that $|\tau(a) - \omega(a)| \leq \varepsilon/2$ for all $a \in F$. As shown above, there is a net $(p_\lambda)_{\lambda\in\Lambda}$ of non-zero projections in A such that $\|p_\lambda a p_\lambda - \tau(a)p_\lambda\| \to 0$ for all $a \in A$. Choose $\lambda \in \Lambda$ such that $\|p_\lambda a p_\lambda - \tau(a)p_\lambda\| \leq \varepsilon/2$ for all $a \in F$, set $p = p_\lambda$ and conclude that

$$\|pap - \omega(a)p\| \leq \|pap - \tau(a)p\| + \|\tau(a)p - \omega(a)p\| \leq \varepsilon \qquad \text{for all } a \in F,$$

as desired. □

The following lemma shows that every unital, completely positive map from a complex matrix algebra $M_n(\mathbb{C})$ into a unital, properly infinite C^*-algebra A can, via an isometry, be related to a *-homomorphism from $M_n(\mathbb{C})$ into A.

Lemma 3.1.4. Let A be a unital, properly infinite C^*-algebra, let n be a natural number and let $\varrho\colon M_n(\mathbb{C}) \to A$ be a unital, completely positive map. Then there exist a *-homomorphism $\varphi\colon M_n(\mathbb{C}) \to A$ and an isometry t in A such that

$$\varrho(x) = t^*\varphi(x)t \qquad \text{for all } x \in M_n(\mathbb{C}).$$

Proof. Let $\{e_{ij} \mid i, j \in N_{\leq n}\}$ denote the system of standard matrix units in $M_n(\mathbb{C})$. Keep in mind that

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases} \qquad \text{for all } i, j, k, l \in \mathbb{N}_{\leq n}.$$

and $e_{ij}^* = e_{ji}$ for all $i, j \in \mathbb{N}_{\leq n}$. Most of the work in this proof will be needed to show the following two statements:

(i) There is a partial isometry $v \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A$ such that

$$v^* \left(x \otimes 1_{M_n(\mathbb{C})} \otimes 1_A\right) v = e_{11} \otimes e_{11} \otimes \varrho(x) \qquad \text{for all } x \in M_n(\mathbb{C});$$

(ii) There are a *-homomorphism $\psi\colon M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A \to A$ and an isometry $s \in A$ such that
$$\psi\left(e_{11} \otimes e_{11} \otimes a\right) = sas^* \qquad \text{for all } a \in A.$$

If (i) and (ii) are true we can define
$$\varphi\colon M_n(\mathbb{C}) \to A,\ x \mapsto \psi(x \otimes 1_{M_n(\mathbb{C})} \otimes 1_A).$$

Then φ is a *-homomorphism because both ψ and the map defined by $x \mapsto x \otimes 1_{M_n(\mathbb{C})} \otimes 1_A$ are *-homomorphisms (the latter as $1_{M_n(\mathbb{C})} \otimes 1_A$ is a projection in $M_n(\mathbb{C}) \otimes A$, see Remark 1.5.3). The elements v and s satisfy
$$v^*v = v^*\left(1_{M_n(\mathbb{C})} \otimes 1_{M_n(\mathbb{C})} \otimes 1_A\right) v \overset{\text{(i)}}{=} e_{11} \otimes e_{11} \otimes \varrho(1_{M_n(\mathbb{C})}) = e_{11} \otimes e_{11} \otimes 1_A \tag{3.1.2}$$
and
$$ss^* = s1_As^* \overset{\text{(ii)}}{=} \psi\left(e_{11} \otimes e_{11} \otimes 1_A\right) = \psi(v^*v). \tag{3.1.3}$$
Set $t = \psi(v)s$ and use (3.1.3) to see that $t^*t = s^*\psi(v^*v)s = s^*ss^*s = 1_A$, i.e. t is an isometry. Altogether, φ and t satisfy
$$t^*\varphi(x)t = s^*\psi\left(v^*\left(x \otimes 1_{M_n(\mathbb{C})} \otimes 1_A\right)v\right)s \overset{\text{(i)}}{=} s^*\psi\left(e_{11} \otimes e_{11} \otimes \varrho(x)\right)s \overset{\text{(ii)}}{=} s^*s\varrho(x)s^*s = \varrho(x)$$
for all $x \in M_n(\mathbb{C})$, as desired. For the proof of (i) define
$$y = \sum_{i,j=1}^n e_{ij} \otimes \varrho(e_{ij}), \tag{3.1.4}$$
then $y \in M_n(\mathbb{C}) \otimes A$ corresponds to the $n \times n$-matrix with entries in A for which the (i,j)th entry is $\varrho(e_{ij})$. Write y as
$$y = \left(\mathrm{id}_{M_n(\mathbb{C})} \otimes \varrho\right)\Big(\sum_{i,j=1}^n e_{ij} \otimes e_{ij}\Big).$$

Calculate
$$\begin{aligned}\Big(\frac{1}{n}\sum_{i,j=1}^n e_{ij} \otimes e_{ij}\Big)^2 &= \frac{1}{n^2}\sum_{i,j,k,l=1}^n e_{ij}e_{kl} \otimes e_{ij}e_{kl} \\ &= \frac{1}{n^2}\sum_{i,k,l=1}^n e_{ik}e_{kl} \otimes e_{ik}e_{kl} = \frac{1}{n}\sum_{i,l=1}^n e_{il} \otimes e_{il}\end{aligned}$$
to see that $1/n\sum_{i,j=1}^n e_{ij} \otimes e_{ij}$ is a projection and hence a positive element, which implies that $\sum_{i,j=1}^n e_{ij} \otimes e_{ij}$ is also positive. As $\mathrm{id}_{M_n(\mathbb{C})} \otimes \varrho$ is a positive map because ϱ is completely positive, this shows that y is positive and hence has a positive square root $y^{1/2}$ in $M_n(\mathbb{C}) \otimes A$. Choose elements $a_{ij} \in A$ such that $y^{1/2} = \sum_{i,j=1}^n e_{ij} \otimes a_{ij}$ and compute
$$y = \Big(\sum_{i,j=1}^n e_{ij} \otimes a_{ij}\Big)^2 = \sum_{i,j,k,l=1}^n e_{ij}e_{kl} \otimes a_{ij}a_{kl} = \sum_{i,k,l=1}^n e_{il} \otimes a_{ik}a_{kl} = \sum_{i,l=1}^n e_{il} \otimes \sum_{k=1}^n a_{ik}a_{kl}.$$

Since the matrix units $\{e_{ij} \mid i,j \in \mathbb{N}_{\le n}\}$ are linearly independent, comparing the last equation with (3.1.4) yields that

$$\varrho(e_{ij}) = \sum_{k=1}^{n} a_{ik}a_{kj} \qquad \text{for all } i,j \in \mathbb{N}_{\le n}. \tag{3.1.5}$$

Set $v = \sum_{i,j=1}^{n} e_{i1} \otimes e_{j1} \otimes a_{ji}$. Then v is an element in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A$ and corresponds to the following $n^2 \times n^2$-matrix with entries in A: The first column lists the columns of $(a_{ij})_{i,j}$, the other entries are zero. Notice that, as $y^{1/2}$ is self-adjoint, we have $a_{ji}^* = a_{ij}$ for all $i,j \in \mathbb{N}_{\le n}$ and thus

$$\begin{aligned}
&v^* \left(e_{ij} \otimes 1_{M_n(\mathbb{C})} \otimes 1_A\right) v \\
&= \Big(\sum_{k,l=1}^{n} e_{1k} \otimes e_{1l} \otimes a_{kl}\Big) \left(e_{ij} \otimes 1_{M_n(\mathbb{C})} \otimes 1_A\right) \Big(\sum_{m,r=1}^{n} e_{m1} \otimes e_{r1} \otimes a_{rm}\Big) \\
&= \sum_{k,l,m,r=1}^{n} e_{1k}e_{ij}e_{m1} \otimes e_{1l}e_{r1} \otimes a_{kl}a_{rm} = \sum_{l=1}^{n} e_{11} \otimes e_{11} \otimes a_{il}a_{lj} \\
&\overset{(3.1.5)}{=} e_{11} \otimes e_{11} \otimes \varrho(e_{ij})
\end{aligned}$$

for all $i,j \in \mathbb{N}_{\le n}$. As each $x \in M_n(\mathbb{C})$ can be written as $\sum_{i,j=1}^{n} x_{ij}e_{ij}$ this extends to

$$v^* \left(x \otimes 1_{M_n(\mathbb{C})} \otimes 1_A\right) v = e_{11} \otimes e_{11} \otimes \varrho(x) \qquad \text{for all } x \in M_n(\mathbb{C});$$

and as seen in (3.1.2) this implies that v is a partial isometry. This proves (i).

We now turn to the proof of (ii). For technical reasons we want to identify the set $\{e_{ij} \otimes e_{kl} \mid i,j,k,l \in \mathbb{N}_{\le n^2}\} \subseteq M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ with the system of standard matrix units in $M_{n^2}(\mathbb{C}) \cong M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Formally this can be done in the following way: For all $i,j,k,l \in \mathbb{N}_{\le n^2}$ define two numbers $r = (i-1)n+k$ and $s = (j-1)n+l$ and set $g_{rs} = e_{ij} \otimes e_{kl}$. A calculation shows that then $\{g_{rs} \mid r,s \in \mathbb{N}_{\le n^2}\} = \{e_{ij} \otimes e_{kl} \mid i,j,k,l \in \mathbb{N}_{\le n}\}$, and

$$g_{rs}g_{uv} = \begin{cases} g_{rv} & \text{if } \ s = u \\ 0 & \text{if } \ s \neq u \end{cases} \qquad \text{for all } r,s,u,v \in \mathbb{N}_{\le n^2}$$

and $g_{rs}^* = g_{sr}$ for all $r,s \in \mathbb{N}_{\le n^2}$. Since A is properly infinite we can by Remark 1.2.5 choose a set of isometries $\{t_i \mid i \in \mathbb{N}_{\le n^2}\}$ in A with mutually orthogonal range projections, i.e. with $t_it_i^* \perp t_jt_j^*$ whenever $i \neq j$. Define $\psi\colon M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A \to A$ by $\psi(g_{ij} \otimes a) = t_iat_j^*$ for all $i,j \le n^2$ and for all $a \in A$. Let $a,b \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A$ and choose elements $a_{ij}, b_{ij} \in A,\ i,j \in \mathbb{N}_{\le n^2}$, such that

$$a = \sum_{i,j=1}^{n^2} g_{ij} \otimes a_{ij}, \qquad b = \sum_{i,j=1}^{n^2} g_{ij} \otimes b_{ij}.$$

Extending the definition of ψ linearly gives $\psi(a) = \sum_{i,j=1}^{n^2} t_ia_{ij}t_j^*$, correspondingly for b. To see that ψ is a *-homomorphism use

$$a^* = \sum_{i,j=1}^{n^2} g_{ij} \otimes a_{ji}^*, \qquad\qquad ab = \sum_{i,l=1}^{n^2} g_{il} \otimes \sum_{k=1}^{n^2} a_{ik}b_{kl},$$

to compute

$$\psi(a^*) = \sum_{i,j=1}^{n^2} t_i a_{ji}^* t_j^* = \sum_{i,j=1}^{n^2} t_j a_{ij}^* t_i^* = \Big(\sum_{i,j=1}^{n^2} t_i a_{ij} t_j^* \Big)^* = \psi(a)^*$$

and

$$\psi(ab) = \psi\Big(\sum_{i,l=1}^{n^2} g_{il} \otimes \sum_{k=1}^{n^2} a_{ik} b_{kl} \Big) = \sum_{i,l=1}^{n^2} t_i \big(\sum_{k=1}^{n^2} a_{ik} b_{kl} \big) t_l^* = \sum_{i,j,k,l=1}^{n^2} t_i a_{ij} t_j^* t_k b_{kl} t_l^* = \psi(a)\psi(b).$$

By definition of ψ and g_{11} we have

$$\psi(e_{11} \otimes e_{11} \otimes a) = \psi(g_{11} \otimes a) = t_1 a t_1^* \qquad \text{for all } a \in A,$$

and thus (ii) holds with $s = t_1$. This completes the proof. □

We are now able to show that each nuclear, unital, completely positive map on a unital, simple, purely infinite C^*-algebra A is approximately similar via isometries to id_A. In the proof we shall need the fact that an element which is almost an isometry is close to an isometry, which was proved in Lemma 1.1.38.

Proposition 3.1.5. Let A be a unital, simple, purely infinite C^*-algebra and let $\varrho\colon A \to A$ be a nuclear, unital, completely positive map. Then for each finite subset F of A and for each $\varepsilon > 0$ there exists an isometry s in A such that

$$\|s^* a s - \varrho(a)\| \le \varepsilon \qquad \text{for all } a \in F.$$

Proof. Let F be a finite subset of A and let $\varepsilon > 0$. Assume that $1_A \in F$. By nuclearity of ϱ choose a natural number n and unital, completely positive maps $\sigma\colon A \to M_n(\mathbb{C})$ and $\eta\colon M_n(\mathbb{C}) \to A$ such that

$$\|\varrho(a) - (\eta \circ \sigma)(a)\| \le \frac{\varepsilon}{2} \qquad \text{for all } a \in F. \tag{3.1.6}$$

Being purely infinite, A is properly infinite, and so Lemma 3.1.4 yields a (possibly non-unital) *-homomorphism $\varphi\colon M_n(\mathbb{C}) \to A$ and an isometry $t \in A$ with

$$\eta(x) = t^* \varphi(x) t \qquad \text{for all } x \in M_n(\mathbb{C}). \tag{3.1.7}$$

We will show the following:

($\star$) There exists an element w in A such that $w^* w = \varphi(1_{M_n(\mathbb{C})})$ and

$$\|w^* a w - (\varphi \circ \sigma)(a)\| \le \frac{\varepsilon}{2} \qquad \text{for all } a \in F. \tag{3.1.8}$$

Then set $s = wt$, check that $s^* s = t^* w^* w t = t^* \varphi(1_{M_n(\mathbb{C})}) t = \eta(1_{M_n(\mathbb{C})}) = 1_A$ to see that s is an isometry in A and use (3.1.7), (3.1.8) and (3.1.6) to estimate

$$\|s^* a s - \varrho(a)\| \le \|t^* w^* a w t - t^*(\varphi \circ \sigma)(a) t\| + \|(\eta \circ \sigma)(a) - \varrho(a)\| \le \varepsilon \qquad \text{for all } a \in F.$$

We now proceed to prove ($\star$). Put $M = \max_{a \in F} \|a\|$ and choose $0 < \delta < \min\{1, \varepsilon/4\}$ such that Lemma 1.1.38 holds with $\varepsilon/(12M)$, i.e. any element that behaves "δ-almost" like an isometry is $\varepsilon/(12M)$-close to an isometry. We start with the construction of an element $u \in A$ which satisfies

$$\|u^*au - (\varphi \circ \sigma)(a)\| \leq \delta \qquad \text{for all } a \in F, \tag{3.1.9}$$

then we perturb u slightly to obtain w. Let $\{e_{ij} \mid i,j \in \mathbb{N}_{\leq n}\}$ denote the standard matrix units in $M_n(\mathbb{C})$ and let $\{\xi_i \mid i \in \mathbb{N}_{\leq n}\}$ be the standard basis for $\mathbb{C}^n$. Define a map

$$\omega\colon M_n(\mathbb{C}) \otimes A \to \mathbb{C}, \ \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} \mapsto \frac{1}{n} \sum_{i,j=1}^{n} \langle \sigma(a_{ij})\xi_j, \xi_i \rangle.$$

Since σ is a unital, completely positive map, ω is a unital, positive linear functional (see Theorem 6.1 and the paragraph preceding it in [Pa]), i.e. ω is a state. We shall use later on that

$$\sigma(a) = (\sigma(a)_{ij})_{i,j} = \sum_{i,j=1}^{n} \sigma(a)_{ij} e_{ij} = \sum_{i,j=1}^{n} \langle \sigma(a)\xi_j, \xi_i \rangle e_{ij} = n \sum_{i,j=1}^{n} \omega(e_{ij} \otimes a) e_{ij} \tag{3.1.10}$$

for all $a \in A$. As A is unital, simple and purely infinite, so is $M_n(\mathbb{C}) \otimes A$ by Remarks 1.3.13(ii) and 1.5.8(i), and thus Proposition 3.1.3 yields a non-zero projection $p \in M_n(\mathbb{C}) \otimes A$ with

$$\|p\,(e_{kl} \otimes a)\,p - \omega\,(e_{kl} \otimes a)\,p\| \leq \frac{\delta}{n^3} \tag{3.1.11}$$

for all $k,l \in \mathbb{N}_{\leq n}$ and for all $a \in F$. As $M_n(\mathbb{C}) \otimes A$ is simple and purely infinite, p is a full and properly infinite projection. By Lemma 1.2.11(iii) this entails that $e_{11} \otimes \varphi(e_{11}) \precsim p$ for the projection $e_{11} \otimes \varphi(e_{11})$ in $M_n(\mathbb{C}) \otimes A$, i.e. there exists a partial isometry $v \in M_n(\mathbb{C}) \otimes A$ with $v^*v = e_{11} \otimes \varphi(e_{11})$ and $vv^* \leq p$. As usual, we can choose elements $v_{ij} \in A$ with

$$v = \sum_{i,j=1}^{n} e_{ij} \otimes v_{ij},$$

but we will show that it is here possible to choose elements $v_i \in A$, $i \in \mathbb{N}_{\leq n}$ such that

$$v = \sum_{i=1}^{n} e_{i1} \otimes v_i. \tag{3.1.12}$$

It follows from

$$e_{11} \otimes \varphi(e_{11}) = v^*v = \sum_{i,j,k,l=1}^{n} e_{ji}e_{kl} \otimes v_{ij}^* v_{kl} = \sum_{j,l=1}^{n} e_{jl} \otimes \sum_{k=1}^{n} v_{kj}^* v_{kl}$$

that $v_{kl}^* v_{kl} = 0$ if $l \neq 1$ and thus $v_{kl} = 0$ if $l \neq 1$. Putting $v_i = v_{i1}$ for each $i \in \mathbb{N}_{\leq n}$ gives (3.1.12). As $vv^* \leq p$ we know that $vv^*p = vv^* = pvv^*$ and, combining this with the usual relations from Remark 1.2.2(i), we obtain that $v = vv^*v = pvv^*v = pv$. We use this to establish the following two equalities, which we shall need below: For all $k,l \in \mathbb{N}_{\leq n}$ and for all $a \in A$ we have

$$\begin{aligned} e_{11} \otimes v_k^* a v_l &= \sum_{i,j=1}^{n} e_{1i}e_{kl}e_{j1} \otimes v_i^* a v_j = \Big(\sum_{i=1}^{n} e_{1i} \otimes v_i^*\Big)\,(e_{kl} \otimes a)\,\Big(\sum_{j=1}^{n} e_{j1} \otimes v_j\Big) \\ &= v^*\,(e_{kl} \otimes a)\,v = v^*p\,(e_{kl} \otimes a)\,pv \end{aligned} \tag{3.1.13}$$

and

$$
\begin{aligned}
e_{11} \otimes \omega(e_{kl} \otimes a)\varphi(e_{11}) &= \omega(e_{kl} \otimes a)(e_{11} \otimes \varphi(e_{11})) = \omega(e_{kl} \otimes a)v^*v \\
&= \omega(e_{kl} \otimes a)v^*pv = v^*\omega(e_{kl} \otimes a)pv. \qquad (3.1.14)
\end{aligned}
$$

Combining equations (3.1.13) and (3.1.14) and using (3.1.11) we can estimate

$$
\begin{aligned}
\|v_k^* a v_l - \omega(e_{kl} \otimes a)\varphi(e_{11})\| &= \|e_{11} \otimes v_k^* a v_l - e_{11} \otimes \omega(e_{kl} \otimes a)\varphi(e_{11})\| \\
&= \|v^* p(e_{kl} \otimes a)pv - v^*\omega(e_{kl} \otimes a)pv\| \\
&\leq \|p(e_{kl} \otimes a)p - \omega(e_{kl} \otimes a)p\| \leq \frac{\delta}{n^3} \qquad (3.1.15)
\end{aligned}
$$

for all $k, l \in \mathbb{N}_{\leq n}$ and for all $a \in F$. Put

$$
u = \sqrt{n} \sum_{k=1}^{n} v_k \varphi(e_{1k}),
$$

calculate

$$
u^* a u = n \sum_{k,l=1}^{n} \varphi(e_{k1}) v_k^* a v_l \varphi(e_{1l}) \qquad \text{for all } a \in A,
$$

and estimate, using (3.1.15) and that $\sigma(a)$ can be written as in (3.1.10), that

$$
\begin{aligned}
\|u^* a u - (\varphi \circ \sigma)(a)\| &= \|n \sum_{k,l=1}^{n} \varphi(e_{k1}) v_k^* a v_l \varphi(e_{1l}) - n \sum_{k,l=1}^{n} \omega(e_{kl} \otimes a)\varphi(e_{kl})\| \\
&\leq n \sum_{k,l=1}^{n} \|\varphi(e_{k1})\big(v_k^* a v_l - \omega(e_{kl} \otimes a)\varphi(e_{11})\big)\varphi(e_{1l})\| \\
&\leq n \sum_{k,l=1}^{n} \|v_k^* a v_l - \omega(e_{kl} \otimes a)\varphi(e_{11})\| \\
&\leq n \sum_{k,l=1}^{n} \frac{\delta}{n^3} = \delta
\end{aligned}
$$

for all $a \in F$, which proves (3.1.9). To obtain w write $q = \varphi(1_{M_n(\mathbb{C})})$, notice that q is a non-zero projection in A and consider the sub-C^*-algebra qAq. Using $\varphi(x) = q\varphi(x)q$ for all $x \in M_n(\mathbb{C})$ we get

$$
\|qu^* a uq - (\varphi \circ \sigma)(a)\| = \|qu^* a uq - q(\varphi \circ \sigma)(a)q\| \leq \|u^* a u - (\varphi \circ \sigma)(a)\| \leq \delta
$$

for all $a \in F$. Hence, the element uq has the same approximation property as u, so we can assume that $u = uq$. Then $u^*u = qu^*uq$ is contained in the unital C^*-algebra qAq and satisfies

$$
\|u^*u - 1_{qAq}\| = \|u^*u - q\| = \|u^*u - (\varphi \circ \sigma)(1_A)\| \leq \delta,
$$

because we assumed that $1_A \in F$. Now, Lemma 1.1.38 provides us with an isometry $w \in qAq$, i.e. with $w^*w = q$, satisfying $\|w - u\| \leq \varepsilon/(12M)$. To complete the proof it is only left to check that w has the desired approximation property. Notice that $\|w\| = 1$ and $\|u\| \le 1+\delta$ and calculate

$$\begin{aligned}
\|w^*aw - (\varphi\circ\sigma)(a)\| &\le \|w^*aw - u^*aw\| + \|u^*aw - u^*au\| + \|u^*au - (\varphi\circ\sigma)(a)\| \\
&\le \|w^* - u^*\|\|aw\| + \|u^*a\|\|w-u\| + \delta \\
&\le \|w-u\|M + \|w-u\|(1+\delta)M + \delta \\
&\le 3\|w-u\|M + \delta \le \frac{3M\varepsilon}{12M} + \delta \le \frac{\varepsilon}{2}
\end{aligned}$$

for all $a \in F$, using that $\delta < \min\{1, \varepsilon/4\}$. This shows ($\star$) and completes the proof. $\square$

3.2 Approximate similarity via isometries

Proposition 3.2.1. Let A be a unital, separable, exact C^*-algebra, let E be a finite dimensional operator system in A and let $\varepsilon > 0$. Then there exists a natural number n, depending on A, E and ε, such that if B_1 and B_2 are unital C^*-algebras and $\varrho_1\colon E \to B_1$ and $\varrho_2\colon E \to B_2$ are unital, completely positive maps satisfying

(i) ϱ_1 is injective;

(ii) $\|\mathrm{id}_n \otimes \varrho_1^{-1}\| \le 1+\varepsilon/2$, where $\varrho_1^{-1}\colon \varrho_1(E) \to E$;

(iii) ϱ_2 is nuclear;

then there is a unital, completely positive map $\eta\colon B_1 \to B_2$ such that $\|\eta\circ\varrho_1 - \varrho_2\| \le \varepsilon$, i.e. such that the diagram

$$\begin{tikzcd}
 & E \arrow[dl, "\varrho_1"'] \arrow[dr, "\varrho_2"] & \\
B_1 \arrow[rr, dashed, "\eta"'] & & B_2
\end{tikzcd}$$

commutes within ε on the unit sphere of E.

If now $B_1 = B_2 = B$ is a unital, simple and purely infinite C^*-algebra and η is nuclear and hence by Proposition 3.1.5 approximately similar via isometries to id_B, then Proposition 3.2.1 yields that ϱ_2 is approximately similar via isometries to ϱ_1. We will pursue this further in Corollary 3.2.3 below.

Proof. The basic idea of the proof is to construct the following diagram

$$\begin{tikzcd}
 & E \arrow[r, "\mathrm{id}_E"] \arrow[dl, "\varrho_1"'] \arrow[d, dashed, "\sigma_1"'] & E \arrow[d, dashed, "\sigma_2"] \arrow[dr, "\varrho_2"] & \\
B_1 \arrow[r, dashed, "\tau_1"'] & M_n(\mathbb{C}) \arrow[r, dashed, "\tau_2"'] \arrow[ur, dotted, "\eta_1"] & M_r(\mathbb{C}) \arrow[r, dashed, "\eta_2"'] & B_2
\end{tikzcd}$$

such that $\eta = \eta_2\circ\tau_2\circ\tau_1$ is a unital, completely positive map satisfying $\|\eta\circ\varrho_1 - \varrho_2\| \le \varepsilon$. To this end it will be shown that one can construct maps (as indicated in the diagram by dashed or dotted arrows) such that the diagram is almost commutative on the unit spheres of the involved spaces. The dashed arrows stand for unital, completely positive maps and the dotted one indicates a unital, completely bounded map defined on $\sigma_1(E)$. We start with the construction of the second triangle:

Since A is a unital, separable, exact C^*-algebra and E is a finite dimensional operator system in A, Corollary 1.5.23 yields a natural number n, a unital, completely positive map $\sigma_1\colon E \to M_n(\mathbb{C})$, and a self-adjoint, unital, completely bounded map $\eta_1\colon \sigma_1(E) \to E$ such that $\eta_1 \circ \sigma_1 = \mathrm{id}_E$ and

$$\|\eta_1\|_{\mathrm{cb}} \leq 1 + \varepsilon/4. \tag{3.2.1}$$

We continue with the fourth triangle: As the unit sphere of any finite dimensional normed space is compact there exists an $\varepsilon/16$-dense finite subset F of the unit sphere of E. By nuclearity of ϱ_2 there are $r \in \mathbb{N}$ and unital, completely positive maps $\sigma_2\colon E \to M_r(\mathbb{C})$ and $\eta_2\colon M_r(\mathbb{C}) \to B_2$ such that $\|(\varrho_2 - \eta_2 \circ \sigma_2)(a)\| \leq \varepsilon/8$ for all $a \in F$. For any x in the unit sphere of E choose $a \in F$ with $\|x - a\| \leq \varepsilon/16$ and calculate

$$\|(\varrho_2 - \eta_2 \circ \sigma_2)(x)\| \leq (\|\varrho_2\| + \|\eta_2 \circ \sigma_2\|)\,\|x - a\| + \|(\varrho_2 - \eta_2 \circ \sigma_2)(a)\| \leq 2\frac{\varepsilon}{16} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

Hence,

$$\|\varrho_2 - \eta_2 \circ \sigma_2\| \leq \varepsilon/4. \tag{3.2.2}$$

We now turn to the third triangle. Notice that $\sigma_1(E)$, being the image of the finite dimensional operator system E under the unital and self-adjoint map σ_1, also is a finite dimensional (hence closed) operator system in the unital C^*-algebra $M_n(\mathbb{C})$. Notice that $\sigma_2 \circ \eta_1\colon \sigma_1(E) \to M_r(\mathbb{C})$ is a self-adjoint, unital, completely bounded map taking values in $M_r(\mathbb{C}) \cong B(\mathbb{C}^r)$, and hence Lemma 1.4.11 yields a unital, completely positive map $\tau_2\colon M_n(\mathbb{C}) \to M_r(\mathbb{C})$ with

$$\|\tau_2|_{\sigma_1(E)} - \sigma_2 \circ \eta_1\|_{\mathrm{cb}} \leq \|\sigma_2 \circ \eta_1\|_{\mathrm{cb}} - 1 \leq \|\eta_1\|_{\mathrm{cb}} - 1 \overset{(3.2.1)}{\leq} \frac{\varepsilon}{4}. \tag{3.2.3}$$

Similarly, we construct the map τ_1 in the first triangle: As above it follows that $\varrho_1(E)$ is an operator system. Consider the map $\sigma_1 \circ \varrho_1^{-1}\colon \varrho_1(E) \to M_n(\mathbb{C})$. Proposition 8.11 in [Pa] tells us that the map $\sigma_1 \circ \varrho_1^{-1}$, being a linear map from an operator system into the complex matrix algebra $M_n(\mathbb{C})$, is completely bounded with

$$\begin{aligned}\|\sigma_1 \circ \varrho_1^{-1}\|_{\mathrm{cb}} &= \|\mathrm{id}_n \otimes (\sigma_1 \circ \varrho_1^{-1})\| = \|(\mathrm{id}_n \otimes \sigma_1) \circ \big(\mathrm{id}_n \otimes \varrho_1^{-1}\big)\| \\ &\leq \|\mathrm{id}_n \otimes \sigma_1\|\|\mathrm{id}_n \otimes \varrho_1^{-1}\| \leq 1 + \frac{\varepsilon}{2}.\end{aligned}$$

Moreover, $\sigma_1 \circ \varrho_1^{-1}$ is unital and self-adjoint, so we can apply Lemma 1.4.11 once more to obtain a unital, completely positive map $\tau_1\colon B_1 \to M_n(\mathbb{C})$ with

$$\|\tau_1|_{\varrho_1(E)} - \sigma_1 \circ \varrho_1^{-1}\|_{\mathrm{cb}} \leq \|\sigma_1 \circ \varrho_1^{-1}\|_{\mathrm{cb}} - 1 \leq \frac{\varepsilon}{2}. \tag{3.2.4}$$

Now put $\eta = \eta_2 \circ \tau_2 \circ \tau_1\colon B_1 \to B_2$. Then η is a unital, completely positive map and applying

(3.2.4), (3.2.3), $\eta_1 \circ \sigma_1 = \mathrm{id}_E$ and (3.2.2) we get

$$\begin{aligned}\|\eta \circ \varrho_1 - \varrho_2\| &\leq \|\eta_2 \circ \tau_2 \circ \tau_1 \circ \varrho_1 - \eta_2 \circ \tau_2 \circ \sigma_1 \circ \varrho_1^{-1} \circ \varrho_1\| + \|\eta_2 \circ \tau_2 \circ \sigma_1 - \varrho_2\| \\ &\leq \|\tau_1|_{\varrho_1(E)} - \sigma_1 \circ \varrho_1^{-1}\| + \|\eta_2 \circ \tau_2 \circ \sigma_1 - \eta_2 \circ \sigma_2 \circ \eta_1 \circ \sigma_1\| \\ &\qquad\qquad\qquad\qquad + \|\eta_2 \circ \sigma_2 \circ \eta_1 \circ \sigma_1 - \varrho_2\| \\ &\leq \frac{\varepsilon}{2} + \|\tau_2|_{\sigma_1(E)} - \sigma_2 \circ \eta_1\| + \|\eta_2 \circ \sigma_2 - \varrho_2\| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,\end{aligned}$$

as required. □

Definition 3.2.2. A *Kirchberg algebra* is a C^*-algebra which is separable, simple, nuclear and purely infinite.

Corollary 3.2.3. Let A be a unital, separable, exact C^*-algebra, let B_1 and B_2 be unital C^*-algebras and let $\varphi_1 \colon A \to B_1$ and $\varphi_2 \colon A \to B_2$ be unital *-homomorphisms with φ_1 injective and φ_2 nuclear.

(i) There exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ of unital, completely positive maps from B_1 to B_2 such that $\|(\eta_n \circ \varphi_1)(a) - \varphi_2(a)\| \to 0$ as $n \to \infty$ for each $a \in A$.

(ii) If $B_1 = B_2 = B$ is a unital Kirchberg algebra, then there is a sequence $(s_n)_{n\in\mathbb{N}}$ of isometries in B such that $\|s_n^*\varphi_1(a)s_n - \varphi_2(a)\| \to 0$ as $n \to \infty$ for each $a \in A$.

Proof. (i): By separability of A there exists an increasing sequence of finite dimensional operator systems $(E_n)_{n\in\mathbb{N}}$ in A such that $\overline{\bigcup_{n\in\mathbb{N}} E_n} = A$. Then, for each $n \in \mathbb{N}$, we have that $\varphi_1|_{E_n}$ is injective and $\varphi_2|_{E_n}$ is nuclear. Moreover, $\varphi_1^{-1} \colon \varphi_1(A) \to A$ is a *-homomorphism, and hence $\|\mathrm{id}_k \otimes \varphi_1^{-1}\| \leq 1$ for all $k \in \mathbb{N}$, and, consequently, $\|\mathrm{id}_k \otimes (\varphi_1|_{E_n})^{-1}\| \leq 1$ for all $k, n \in \mathbb{N}$. We can hence apply Proposition 3.2.1 to obtain for each $n \in \mathbb{N}$ a unital, completely positive map $\eta_n \colon B_1 \to B_2$ with $\|\eta_n \circ \varphi_1|_{E_n} - \varphi_2|_{E_n}\| \leq 1/n$, which implies that

$$\|(\eta_n \circ \varphi_1)(b) - \varphi_2(b)\| \leq \frac{\|b\|}{n} \qquad \text{for all } n \in \mathbb{N},\ b \in E_n. \tag{3.2.5}$$

As $(E_n)_{n\in\mathbb{N}}$ is increasing, there exists for every $b \in \bigcup_{n\in\mathbb{N}} E_n$ a number $n_0 \in \mathbb{N}$ such that $b \in E_n$ for all $n \in \mathbb{N}_{\geq n_0}$, and hence (3.2.5) implies $\|(\eta_n \circ \varphi_1)(b) - \varphi_2(b)\| \to 0$ as $n \to \infty$. As $\overline{\bigcup_{n\in\mathbb{N}} E_n} = A$ and as $\eta_n \circ \varphi_1 - \varphi_2$ is bounded, this implies $\|(\eta_n \circ \varphi_1)(a) - \varphi_2(a)\| \to 0$ as $n \to \infty$ for each $a \in A$.

(ii): If $B_1 = B_2 = B$ is a unital Kirchberg algebra then each of the maps η_n in (i) is nuclear by Corollary 1.5.13. As each E_n is finite dimensional we can for each $n \in \mathbb{N}$ choose a $1/n$-dense finite subset F_n of the unit sphere of E_n. Then Proposition 3.1.5 yields for each $n \in \mathbb{N}$ an isometry $s_n \in B$ such that

$$\|s_n^*\varphi_1(a)s_n - \eta_n(\varphi_1(a))\| \leq \frac{1}{n} \qquad \text{for all } a \in F_n.$$

For each $n \in \mathbb{N}$ and for each b in the unit sphere of E_n choose $a \in F_n$ with $\|a - b\| \leq 1/n$ to obtain

$$\begin{aligned}&\|s_n^*\varphi_1(b)s_n - (\eta_n \circ \varphi_1)(b)\| \\ &\leq \|s_n^*\varphi_1(b)s_n - s_n^*\varphi_1(a)s_n\| + \|s_n^*\varphi_1(a)s_n - (\eta_n \circ \varphi_1)(a)\| + \|(\eta_n \circ \varphi_1)(a) - (\eta_n \circ \varphi_1)(b)\| \\ &\leq \|b - a\| + \frac{1}{n} + \|a - b\| \leq \frac{3}{n}.\end{aligned}$$

This implies that

$$\|s_n^*\varphi_1(b)s_n - (\eta_n \circ \varphi_1)(b)\| \leq \frac{3\|b\|}{n} \qquad \text{for all } n \in \mathbb{N},\ b \in E_n,$$

and combining this with (3.2.5) gives

$$\|s_n^*\varphi_1(b)s_n - \varphi_2(b)\| \leq \|s_n^*\varphi_1(b)s_n - (\eta_n \circ \varphi_1)(b)\| + \|(\eta_n \circ \varphi_1)(b) - \varphi_2(b)\| \leq \frac{4\|b\|}{n}$$

for each $n \in \mathbb{N}$ and for each $b \in E_n$. As in (i) we can conclude that $\|s_n^*\varphi_1(a)s_n - \varphi_2(a)\| \to 0$ as $n \to \infty$ for all all $a \in A$. $\square$

3.3 From approximate similarity via isometries to unitary equivalence

The results in this section are based on Lemmas 2.4 and 2.5 in [KP]. We start with some technical results and estimations which shall be useful later on, when we construct unitaries out of isometries to obtain approximate unitary equivalence.

Lemma 3.3.1. Let A be a unital C^*-algebra, let s be an isometry in A and let $V\colon A \to A$, $a \mapsto s^*as$ be the corresponding unital, completely positive map. Then

$$\|[a, ss^*]\| = \max\left\{\|V(a^*a) - V(a^*)V(a)\|^{1/2},\ \|V(aa^*) - V(a)V(a^*)\|^{1/2}\right\}$$

for all $a \in A$.

That the map V as defined in the lemma is indeed unital and completely positive was proved in Example 1.4.8(ii).

Proof. Set $p = ss^*$, then p is a projection. Note that $x = (1_A - p)ap$ and $y = -pa(1_A - p)$ satisfy $x^*y = 0 = xy^*$, so that we can apply Lemma 1.1.12 to get

$$\|[a, ss^*]\| = \|ap - pa\| = \|(1_A - p)ap - pa(1_A - p)\| = \max\left\{\|(1_A - p)ap\|,\ \|pa(1_A - p\|\right\}.$$

The expressions on the right hand side can be modified using

$$\|(1_A - p)ap\|^2 = \|pa^*(1_A - p)ap\| = \|s^*a^*(1_A - ss^*)as\| = \|V(a^*a) - V(a^*)V(a)\|$$

and, similarly,

$$\|pa(1_A - p)\|^2 = \|pa(1_A - p)a^*p\| = \|s^*a(1_A - ss^*)a^*s\| = \|V(aa^*) - V(a)V(a^*)\|,$$

and therefore

$$\|[a, ss^*]\| = \max\left\{\|V(a^*a) - V(a^*)V(a)\|^{1/2},\ \|V(aa^*) - V(a)V(a^*)\|^{1/2}\right\}.$$

$\square$

Remark 3.3.2. Let A, s and V be as in Lemma 3.3.1. Then ss^* commutes with all $a \in A$ if and only if V is multiplicative. The "if"-part is an immediate consequence of Lemma 3.3.1. To see the converse, assume that ss^* commutes with all $a \in A$ and let $a, b \in A$. Then

$$V(ab) = s^*abs = s^*abss^*s = s^*ass^*bs = V(a)V(b).$$

Definition 3.3.3. Let A be a unital C^*-algebra and let s, t be isometries in A that *satisfy the $\mathcal{O}_2$-relation,* i.e. $ss^* + tt^* = 1_A$. For all $a, b \in A$ define the *Cuntz sum* of a and b with respect to s and t by $a \oplus_{s,t} b = sas^* + tbt^*$.

Lemma 3.3.4. Let A be a unital C^*-algebra, let s be an isometry in A and let $V : A \to A$, $a \mapsto s^*as$ be the corresponding unital, completely positive map. Suppose that v_1 and v_2 are isometries in A which satisfy the $\mathcal{O}_2$-relation. Then $w_1 = (1_A - ss^*) + sv_1s^*$ and $w_2 = sv_2$ are also isometries satisfying the $\mathcal{O}_2$-relation, and

$$\|a \oplus_{w_1,w_2} V(a) - a\| \leq \|[v_1, V(a)]\| + \|[v_2, V(a)]\| + 2\|[a, ss^*]\|$$

for all $a \in A$.

Proof. Put $p = ss^*$. Then p is a projection, $(1_A - p)s = s - ss^*s = 0 = s^*(1_A - p)$ and therefore

$$w_1^*w_1 = ((1_A - p) + sv_1^*s^*)((1_A - p) + sv_1s^*) = (1_A - p) + sv_1^*s^*sv_1s^* = 1_A - p + p = 1_A.$$

This shows that w_1 is an isometry, and w_2 is an isometry being a product of isometries. To show that the $\mathcal{O}_2$-relation holds for w_1 and w_2 we use that it is satisfied by v_1 and v_2:

$$\begin{aligned} w_1w_1^* + w_2w_2^* &= ((1_A - p) + sv_1s^*)((1_A - p) + sv_1^*s^*) + sv_2v_2^*s^* \\ &= 1_A - p + sv_1v_1^*s^* + sv_2v_2^*s^* \\ &= 1_A - p + p = 1_A. \end{aligned}$$

Now the Cuntz sum with respect to w_1 and w_2 is well-defined, and we can calculate for any $a \in A$:

$$\begin{aligned} &\|a \oplus_{w_1,w_2} V(a) - a\| = \|w_1aw_1^* + w_2V(a)w_2^* - a\| \\ &= \|((1_A - p) + sv_1s^*)\,a\,((1_A - p) + sv_1^*s^*) + sv_2V(a)v_2^*s^* - a\| \\ &= \|(1_A - p)a(1_A - p) + (1_A - p)asv_1^*s^* + sv_1s^*a(1_A - p) + sv_1V(a)v_1^*s^* + sv_2V(a)v_2^*s^* - a\| \\ &= \|(1_A - p)a(1_A - p) + (1_A - p)asv_1^*s^* + sv_1s^*a(1_A - p) + s\,(V(a) \oplus_{v_1,v_2} V(a))\,s^* \\ &\qquad\qquad - sV(a)s^* + pap - a\| \\ &\leq \|(1_A - p)a(1_A - p) + pap - a\| + \|(1_A - p)asv_1^*s^* + sv_1s^*a(1_A - p)\| \\ &\qquad\qquad + \|V(a) \oplus_{v_1,v_2} V(a) - V(a)\| \end{aligned} \tag{3.3.1}$$

Check that $(1_A - p - p)(1_A - p - p) = 1_A$ to see that $1_A - p - p$ is a unitary and use this to transform the first term in (3.3.1) as follows:

$$\begin{aligned} \|(1_A - p)a(1_A - p) + pap - a\| &= \|((1_A - p)a(1_A - p) + pap - a)\,(1_A - p - p)\| \\ &= \|(1_A - p)a(1_A - p) - a(1_A - p) - pap + ap\| \\ &= \|ap - pa\| = \|[a, ss^*]\|. \end{aligned}$$

To deal with the second term use $ps = s$ to see that

$$\|(1_A - p)asv_1^*s^*\| = \|apsv_1^*s^* - pasv_1^*s^*\| = \|[a, p]\,sv_1^*s^*\| \leq \|[a, ss^*]\|$$

and, similarly,

$$\|sv_1s^*a(1_A - p)\| = \|sv_1s^*pa - sv_1s^*ap\| = \|sv_1s^*[p, a]\| \leq \|[a, ss^*]\|.$$

Notice that with $x = (1_A - p)asv_1^*s^*$ and $y = sv_1s^*a(1_A - p)$ we have $x^*y = 0 = xy^*$ (because $(1_A - p)s = 0$), and hence we can apply Lemma 1.1.12 to give

$$\|(1_A - p)asv_1^*s^* + sv_1s^*a(1_A - p)\| = \max\{\|(1_A - p)asv_1^*s^*\|, \|sv_1s^*a(1_A - p)\|\} \le \|[a, ss^*]\|.$$

For the third term in (3.3.1) we get

$$\begin{aligned} &\|V(a) \oplus_{v_1,v_2} V(a) - V(a)\| \\ &\le \|v_1V(a)v_1^* - V(a)v_1v_1^*\| + \|v_2V(a)v_2^* - V(a)v_2v_2^*\| + \|V(a)(v_1v_1^* + v_2v_2^*) - V(a)\| \\ &= \|[v_1, V(a)]\| + \|[v_2, V(a)]\|. \end{aligned}$$

This completes the proof. □

Definition 3.3.5. A C^*-algebra A is said to be *$\mathcal{O}_2$-absorbing* if $A \otimes \mathcal{O}_2 \cong A$.

In the proof of Lemma 3.3.10 below we shall need the existence of an asymptotically central sequence of *-homomorphisms from $\mathcal{O}_2$ into a unital, $\mathcal{O}_2$-absorbing C^*-algebra A. We start with the definition of asymptotically central sequences and then turn to the result we need.

Definition 3.3.6. Let A and B be C^*-algebras. A bounded sequence $(x_n)_{n\in\mathbb{N}}$ in B is said to be *asymptotically central* if $\lim_{n\to\infty}\|[x_n, b]\| = 0$ for all b in B. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ of *-homomorphisms from A into B is called *asymptotically central* if $(\varphi_n(a))_{n\in\mathbb{N}}$ is an asymptotically central sequence in B for every $a \in A$.

The following remark will be helpful when we have to prove that a given sequence is asymptotically central.

Remark 3.3.7. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a C^*-algebra A. Then the set

$$D = \{a \in A \mid \lim_{n\to\infty}\|[x_n, a]\| = 0\}$$

is a closed linear subspace of A.

Proof. Let $a, b \in D$ and let $\lambda \in \mathbb{C}$. Then

$$\|x_n(\lambda a + b) - (\lambda a + b)x_n\| \le |\lambda|\|x_na - ax_n\| + \|x_nb - bx_n\| \to 0$$

as $n \to \infty$, i.e. D is a linear space. To see that D is closed let $a \in \overline{D}$ and let $\varepsilon > 0$. Let $M \in \mathbb{R}_{\ge 0}$ be an upper bound of $\{\|x_n\| \mid n \in \mathbb{N}\}$ and choose $d \in D$ such that $\|d - a\| < \varepsilon/(3M)$. Next, find $N \in \mathbb{N}$ such that $\|[x_n, d]\| < \varepsilon/3$ for all $n \in \mathbb{N}_{\ge N}$, then

$$\begin{aligned} \|[x_n, a]\| = \|x_na - ax_n\| &\le \|x_na - x_nd\| + \|x_nd - dx_n\| + \|dx_n - ax_n\| \\ &< M\|a - d\| + \frac{\varepsilon}{3} + \|d - a\|M < \varepsilon \end{aligned}$$

for all $n \in \mathbb{N}_{\ge N}$, and hence $a \in D$. □

The result we need here is a corollary of the following statement, which is proved in [R2], Lemma 5.2.3.

Lemma 3.3.8. There exists an asymptotically central sequence of unital *-endomorphisms of $\mathcal{O}_2$.

Corollary 3.3.9. Let A be a unital, $\mathcal{O}_2$-absorbing C^*-algebra. Then there exists an asymptotically central sequence of unital *-homomorphisms from $\mathcal{O}_2$ into A.

Proof. Lemma 3.3.8 provides us with an asymptotically central sequence $(\varrho_n)_{n\in\mathbb{N}}$ of unital *-endomorphisms of $\mathcal{O}_2$. For each $n \in \mathbb{N}$ define $\psi_n \colon \mathcal{O}_2 \to A \otimes \mathcal{O}_2,\ x \mapsto 1_A \otimes \varrho_n(x)$. Then each ψ_n is a unital *-homomorphism and for each $x \in \mathcal{O}_2$ and for each elementary tensor $a \otimes y \in A \otimes \mathcal{O}_2$ we have

$$\begin{aligned}\lim_{n\to\infty} \|[\psi_n(x), a\otimes y]\| &= \lim_{n\to\infty} \|(1_A \otimes \varrho_n(x))\,(a\otimes y) - (a\otimes y)\,(1_A\otimes \varrho_n(x))\| \\ &= \lim_{n\to\infty} \|a \otimes \big(\varrho_n(x)y - y\varrho_n(x)\big)\| \\ &= \|a\| \lim_{n\to\infty} \|[\varrho_n(x), y]\| = 0. \end{aligned} \tag{3.3.2}$$

Since the sequence $(\psi_n(x))_{n\in\mathbb{N}}$ is bounded by $\|x\|$ for each $x \in \mathcal{O}_2$, it follows from Remark 3.3.7 and from (3.3.2) that $(\psi_n(x))_{n\in\mathbb{N}}$ is asymptotically central for each $x \in \mathcal{O}_2$, i.e. $(\psi_n)_{n\in\mathbb{N}}$ is an asymptotically central sequence of unital *-homomorphisms from $\mathcal{O}_2$ into $A \otimes \mathcal{O}_2$. As A is assumed to be $\mathcal{O}_2$-absorbing we can find a *-isomorphism $\lambda\colon A\otimes\mathcal{O}_2 \to A$ and use this to define $\varphi_n = \lambda \circ \psi_n$ for each $n \in \mathbb{N}$. Then each φ_n is a unital *-homomorphism from $\mathcal{O}_2$ into A, and it is easily seen that $(\varphi_n)_{n\in\mathbb{N}}$ is asymptotically central: Let $x \in \mathcal{O}_2$, let $a \in A$ and let $z = \lambda^{-1}(a) \in A \otimes \mathcal{O}_2$. Then, for all $n \in \mathbb{N}$:

$$\|\varphi_n(x)a - a\varphi_n(x)\| = \|\lambda\big(\psi_n(x)z - z\psi_n(x)\big)\| = \|\psi_n(x)z - z\psi_n(x)\|,$$

and as $(\psi_n)_{n\in\mathbb{N}}$ is asymptotically central so is $(\varphi_n)_{n\in\mathbb{N}}$. □

The following lemma is the first step on the way from approximate similarity via isometries to approximate unitary equivalence: It shows how, for an isometry s in a unital, $\mathcal{O}_2$-absorbing C^*-algebra A, the unital, completely positive map corresponding to s can, via a unitary, be related to the identity map on A.

Lemma 3.3.10. Let A be a unital, $\mathcal{O}_2$-absorbing C^*-algebra, let s and t be isometries in A and let $V\colon A \to A,\ a \mapsto s^*as$ and $W\colon A\to A,\ a\mapsto t^*at$ be the corresponding unital, completely positive maps. Then for every finite subset F of A and for every $\varepsilon > 0$ there is a unitary u in A with

$$\|uV(a)u^* - a\| \le 5\kappa + \varepsilon \qquad \text{for all } a \in F,$$

where

$$\kappa = \max_{a\in F\cup F^*} \Big\{ \|V(a^*a) - V(a^*)V(a)\|^{1/2},\ \|WV(a^*a) - WV(a^*)WV(a)\|^{1/2},\ \|WV(a) - a\| \Big\}.$$

Before we turn to the proof we have a look at the special case where $\kappa = 0$ for all subsets F of A. By Lemma 3.3.1 and Remark 3.3.2 we then have that V is multiplicative and that W is multiplicative on $V(A)$. Moreover, W is a left inverse of V. The conclusion of Lemma 3.3.10 is in this case that V is approximately unitarily equivalent to id_A.

Let now F be a finite subset of A, let $\varepsilon > 0$, and let u be a unitary in A such that $\|uV(a)u^* - a\| \le \varepsilon$ for all $a \in F$. Then we also have that $\|V(a) - u^*au\| \le \varepsilon$ for all $a \in F$ which reveals another way of looking at the statement of the lemma: If s and t are isometries in A such that the corresponding unital, completely positive maps V and W are multiplicative and W is a left inverse of V, then s and t can be used to construct unitary elements u such that V is approximated by the *-automorphisms on A given by $a \mapsto u^*au$.

In this way Lemma 3.3.10 may remind one of the Cantor-Bernstein Theorem, which states that if $f\colon M \to N$ and $g\colon N \to M$ are injective maps between sets M and N, then there exists a bijection between M and N. A slightly different version of Lemma 3.3.10 is given as Lemma 6.3.7 in [R2], where the proof uses methods similar to those in the proof of the Cantor-Bernstein Theorem.

Proof of Lemma 3.3.10. By Corollary 3.3.9 there exists an asymptotically central sequence $(\varphi_n)_{n\in\mathbb{N}}$ of unital *-homomorphisms from $\mathcal{O}_2$ into A. Let F be a finite subset of A and let $\varepsilon > 0$. Let r_1 and r_2 be isometries in $\mathcal{O}_2$ satisfying the $\mathcal{O}_2$-relation. As $(\varphi_n)_{n\in\mathbb{N}}$ is asymptotically central we can choose $N \in \mathbb{N}$ such that

$$\begin{aligned} \|[\varphi_N(r_1), V(a)]\| + \|[\varphi_N(r_2), V(a)]\| &\leq \frac{\varepsilon}{2}, \\ \|[\varphi_N(r_1), WV(a)]\| + \|[\varphi_N(r_2), WV(a)]\| &\leq \frac{\varepsilon}{2} \end{aligned} \tag{3.3.3}$$

for all $a \in F$. Put $v_1 = \varphi_N(r_1)$ and $v_2 = \varphi_N(r_2)$. Since φ_N is a unital *-homomorphism, v_1 and v_2 are isometries in A and satisfy the $\mathcal{O}_2$-relation. As in Lemma 3.3.4 define

$$\begin{aligned} s_1 &= (1_A - ss^*) + sv_1s^*, & s_2 &= sv_2, \\ t_1 &= (1_A - tt^*) + tv_1t^*, & t_2 &= tv_2, \end{aligned}$$

to obtain two pairs of isometries (s_1, s_2) and (t_1, t_2) in A satisfying the $\mathcal{O}_2$-relation. Define $u = s_2t_1^* + s_1t_2^*$ and calculate

$$\begin{aligned} u^*u &= (t_1s_2^* + t_2s_1^*)(s_2t_1^* + s_1t_2^*) = t_1t_1^* + t_1s_2^*s_1t_2^* + t_2s_1^*s_2t_1^* + t_2t_2^* = 1_A, \\ uu^* &= (s_2t_1^* + s_1t_2^*)(t_1s_2^* + t_2s_1^*) = s_2s_2^* + s_2t_1^*t_2s_1^* + s_1t_2^*t_1s_2^* + s_1s_1^* = 1_A \end{aligned}$$

to see that u is a unitary in A. We show that, for all $a, b \in A$, adjoining with u turns the Cuntz sum of a and b with respect to t_1, t_2 into the Cuntz sum of b and a with respect to s_1, s_2:

$$\begin{aligned} u\,(a \oplus_{t_1,t_2} b)\,u^* &= (s_2t_1^* + s_1t_2^*)(t_1at_1^* + t_2bt_2^*)(t_1s_2^* + t_2s_1^*) = (s_2at_1^* + s_1bt_2^*)(t_1s_2^* + t_2s_1^*) \\ &= s_2as_2^* + s_1bs_1^* = b \oplus_{s_1,s_2} a. \end{aligned}$$

Thus we can calculate

$$\begin{aligned} &\|uV(a)u^* - a\| \\ &= \|uV(a)u^* - u\,(V(a) \oplus_{t_1,t_2} WV(a))\,u^* + WV(a) \oplus_{s_1,s_2} V(a) - a\| \\ &\leq \|V(a) - V(a) \oplus_{t_1,t_2} WV(a)\| + \|s_1WV(a)s_1^* - s_1as_1^*\| + \|s_2V(a)s_2^* + s_1as_1^* - a\| \\ &\leq \|V(a) \oplus_{t_1,t_2} WV(a) - V(a)\| + \|WV(a) - a\| + \|a \oplus_{s_1,s_2} V(a) - a\| \end{aligned} \tag{3.3.4}$$

for all $a \in A$. We have that $\|WV(a) - a\| \leq \kappa$ for all $a \in F$ by definition of κ. We proceed to obtain an estimation for the first term. By Lemma 3.3.4 and by (3.3.3) we know that

$$\begin{aligned} \|V(a) \oplus_{t_1,t_2} WV(a) - V(a)\| &\leq \|[v_1, WV(a)]\| + \|[v_2, WV(a)]\| + 2\|[V(a), tt^*]\| \\ &\leq 2\|[V(a), tt^*]\| + \frac{\varepsilon}{2} \end{aligned} \tag{3.3.5}$$

for all $a \in F$. We show next that $\|[V(a), tt^*]\| \leq \kappa$ for all $a \in F$. To achieve this, note that $1_A - tt^*$ is a projection and hence positive, which implies that

$$\begin{aligned} W\left(V(a^*)V(a)\right) - WV(a^*)WV(a) &= t^*s^*a^*ss^*ast - t^*s^*a^*stt^*s^*ast \\ &= t^*s^*a^*s(1_A - tt^*)s^*ast \\ &= (s^*ast)^*(1_A - tt^*)s^*ast \geq 0 \end{aligned}$$

for all $a \in A$. In the same way we obtain

$$WV(a^*a) - W\left(V(a^*)V(a)\right) = t^*s^*a^*ast - t^*s^*a^*ss^*ast = (ast)^*(1 - ss^*)ast \geq 0,$$

and combining the last two equations gives

$$0 \leq W\left(V(a^*)V(a)\right) - WV(a^*)WV(a) \leq WV(a^*a) - WV(a^*)WV(a) \qquad \text{for all } a \in A.$$

Combining this with Lemma 3.3.1 yields

$$\begin{aligned} &\|[V(a), tt^*]\| \\ &= \max\left\{\|W\left(V(a^*)V(a)\right) - WV(a^*)WV(a)\|^{1/2},\ \|W\left(V(a)V(a^*)\right) - WV(a)WV(a^*)\|^{1/2}\right\} \\ &\leq \max\left\{\|WV(a^*a) - WV(a^*)WV(a)\|^{1/2},\ \|WV(aa^*) - WV(a)WV(a^*)\|^{1/2}\right\} \leq \kappa \end{aligned}$$

for all $a \in F$ and hence, together with (3.3.5), we have

$$\|V(a) \oplus_{t_1,t_2} WV(a) - V(a)\| \leq 2\kappa + \frac{\varepsilon}{2} \qquad \text{for all } a \in F.$$

For the remaining term in (3.3.4) notice first that, by Lemma 3.3.1 and by definition of κ, we have

$$\|[a, ss^*]\| = \max\left\{\|V(a^*a) - V(a^*)V(a)\|^{1/2},\ \|V(aa^*) - V(a)V(a^*)\|^{1/2}\right\} \leq \kappa$$

for all $a \in F$ and thus, by Lemma 3.3.4 and by (3.3.3)

$$\|a \oplus_{s_1,s_2} V(a) - a\| \leq \|[v_1, V(a)]\| + \|[v_2, V(a)]\| + 2\|[a, ss^*]\| \leq 2\kappa + \frac{\varepsilon}{2}$$

for all $a \in F$. Altogether we have shown that

$$\|uV(a)u^* - a\| \leq 5\kappa + \varepsilon \qquad \text{for all } a \in F,$$

which completes the proof. □

In the following lemma it is shown that if two unital, completely positive maps ϱ and η are, on finite sets, almost multiplicative and similar via two isometries s and t, then the constant κ from the previous lemma can be made arbitrarily small.

Lemma 3.3.11. Let A and B be unital C^*-algebras, let s and t be isometries in B and let $V, W\colon B \to B$ be the corresponding unital, completely positive maps. Let $\varrho, \eta\colon A \to B$ be unital, completely positive maps, let F be a finite subset of A and let $\delta > 0$. Put

$$\kappa = \max_{b \in \varrho(F \cup F^*)} \left\{\|V(b^*b) - V(b^*)V(b)\|^{1/2},\ \|WV(b^*b) - WV(b^*)WV(b)\|^{1/2},\ \|WV(b) - b\|\right\}$$

and $M = \max_{a \in F}\|a\|$. If

(i) $\|\varrho(a^*a) - \varrho(a^*)\varrho(a)\| \le \delta$ and $\|\eta(a^*a) - \eta(a^*)\eta(a)\| \le \delta$ for all $a \in F \cup F^*$, and

(ii) $\|V\varrho(a) - \eta(a)\| \le \delta$ and $\|W\eta(a) - \varrho(a)\| \le \delta$ for all $a \in F \cup F^* \cup \{x^*x \mid x \in F \cup F^*\}$,

then

$$\kappa \le \max\left\{2\delta,\ 2\sqrt{(1+M)\delta}\right\}.$$

Note that this lemma is actually true for arbitrary unital, completely positive maps V and W on B.

Proof. Let $b \in \varrho(F \cup F^*)$ and choose $a \in F \cup F^*$ with $\varrho(a) = b$. Then

$$\begin{aligned}
\|V(b^*b) - V(b^*)V(b)\| &= \|V\left(\varrho(a^*)\varrho(a)\right) - V\varrho(a^*)V\varrho(a)\| \\
&= \|V\big(\varrho(a^*)\varrho(a) - \varrho(a^*a)\big)\| + \|V\varrho(a^*a) - \eta(a^*a)\| \\
&\quad\ + \|\eta(a^*a) - \eta(a^*)\eta(a)\| + \|\eta(a^*)\eta(a) - V\varrho(a^*)V\varrho(a)\| \\
&\le \delta + \delta + \delta + \|\eta(a^*)\|\|\eta(a) - V\varrho(a)\| + \|\eta(a^*) - V\varrho(a^*)\|\|V\varrho(a)\| \\
&\le (3+2M)\delta
\end{aligned}$$

and

$$\|WV(b) - b\| = \|WV\varrho(a) - \varrho(a)\| \le \|W\big(V\varrho(a) - \eta(a)\big)\| + \|W\eta(a) - \varrho(a)\| \le 2\delta.$$

The last calculation can also be done with a^*a or a^* instead of a to yield

$$\|WV\varrho(a^*a) - \varrho(a^*a)\| \le 2\delta, \qquad \|WV\varrho(a^*) - \varrho(a^*)\| \le 2\delta.$$

This helps us in the estimation of the remaining term:

$$\begin{aligned}
&\|WV(b^*b) - WV(b^*)WV(b)\| \\
&= \|WV\left(\varrho(a^*)\varrho(a)\right) - WV\varrho(a^*)WV\varrho(a)\| \\
&\le \|WV\big(\varrho(a^*)\varrho(a) - \varrho(a^*a)\big)\| + \|WV\varrho(a^*a) - WV\varrho(a^*)WV\varrho(a)\| \\
&\le \delta + \|WV\varrho(a^*a) - \varrho(a^*a)\| + \|\varrho(a^*a) - \varrho(a^*)\varrho(a)\| + \|\varrho(a^*)\varrho(a) - WV\varrho(a^*)WV\varrho(a)\| \\
&\le \delta + 2\delta + \delta + \|\varrho(a^*)\|\|\varrho(a) - WV\varrho(a)\| + \|\varrho(a^*) - WV\varrho(a^*)\|\|WV\varrho(a)\| \\
&\le \delta + 2\delta + \delta + 4\delta M = 4(1+M)\delta.
\end{aligned}$$

Altogether we have obtained now that

$$\kappa \le \max\left\{\sqrt{(3+2M)\delta},\ 2\delta,\ 2\sqrt{(1+M)\delta}\right\} = \max\left\{2\delta,\ 2\sqrt{(1+M)\delta}\right\}.$$

□

Combining the results of Lemmas 3.3.10 and 3.3.11 we can now obtain approximate unitary equivalence of two almost multiplicative, unital, completely positive maps which are mutually approximately similar via isometries and which take values in an $\mathcal{O}_2$-absorbing C^*-algebra. The following lemma is designed to be used in the proof of Lemma 5.2.1. A less technical version in the case where ϱ and η are unital *-homomorphisms is given in Lemma 3.4.1 in the beginning of the following section.

Lemma 3.3.12. Let A and B be unital C^*-algebras such that B is $\mathcal{O}_2$-absorbing, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all unital, completely positive maps $\varrho, \eta \colon A \to B$ and for all finite subsets F of the unit sphere of A the following holds: If

(i) $\|\varrho(a^*a) - \varrho(a^*)\varrho(a)\| \le \delta$ and $\|\eta(a^*a) - \eta(a^*)\eta(a)\| \le \delta$ for all $a \in F \cup F^*$, and

(ii) there exist isometries $s, t \in B$ with $\|s^*\varrho(a)s - \eta(a)\| \le \delta$ and $\|t^*\eta(a)t - \varrho(a)\| \le \delta$ for all $a \in F \cup F^* \cup \{x^*x \mid x \in F \cup F^*\}$,

then there is a unitary $u \in B$ such that

$$\|u\eta(a)u^* - \varrho(a)\| \le \varepsilon \qquad \text{for all } a \in F.$$

Proof. Choose $0 < \delta < 1$ with $\delta + 10\sqrt{2\delta} \le \varepsilon/2$ and let F be a finite subset of the unit sphere of A. Let $\varrho, \eta \colon A \to B$ be unital, completely positive maps satisfying (i), and suppose that there are isometries $s, t \in B$ satisfying (ii). Then Lemma 3.3.10 yields a unitary $u \in B$ with

$$\|us^*\varrho(a)su^* - \varrho(a)\| \le 5\kappa + \frac{\varepsilon}{2} \qquad \text{for all } a \in F, \tag{3.3.6}$$

where

$$\kappa = \max_{b \in \varrho(F \cup F^*)} \left\{ \|V(b^*b) - V(b^*)V(b)\|^{1/2},\ \|WV(b^*b) - WV(b^*)WV(b)\|^{1/2},\ \|WV(b) - b\| \right\}.$$

By Lemma 3.3.11,

$$\kappa \le \max\left\{2\delta,\ 2\sqrt{(1+M)\delta}\right\},$$

where $M = \max_{a \in F}\|a\|$. As F is a subset of the unit sphere of A, we have $M = 1$ and hence $\kappa \le \max\{2\delta, 2\sqrt{2\delta}\} \le 2\sqrt{2\delta}$ since we assumed $\delta < 1$. Therefore, $\delta + 5\kappa \le \delta + 10\sqrt{2\delta} \le \varepsilon/2$ and hence, using (3.3.6),

$$\begin{aligned} \|u\eta(a)u^* - \varrho(a)\| &\le \|u\eta(a)u^* - us^*\varrho(a)su^*\| + \|us^*\varrho(a)su^* - \varrho(a)\| \\ &\le \|\eta(a) - s^*\varrho(a)s\| + 5\kappa + \frac{\varepsilon}{2} \\ &\le \delta + 5\kappa + \frac{\varepsilon}{2} \le \varepsilon \end{aligned}$$

for all $a \in F$. □

3.4 Unitary equivalence of unital, injective *-homomorphisms into $\mathcal{O}_2$

This section starts with a simplified version of Lemma 3.3.12, and then deals with the important result that any two unital, injective *-homomorphisms from a unital, separable, exact C^*-algebra into $\mathcal{O}_2$ are approximately unitarily equivalent.

Lemma 3.4.1. Let A and B be unital C^*-algebras such that B is $\mathcal{O}_2$-absorbing and let $\varphi, \psi \colon A \to B$ be unital *-homomorphisms. Let $\varepsilon > 0$ and let F be a finite subset of A. Then there exists $\delta > 0$ such that if there are isometries s, t in B with

$$\|s^*\varphi(a)s - \psi(a)\| \le \delta, \qquad \|t^*\psi(a)t - \varphi(a)\| \le \delta \tag{3.4.1}$$

for all $a \in F \cup F^* \cup \{x^*x \mid x \in F \cup F^*\}$, then there is a unitary u in B such that

$$\|u\psi(a)u^* - \varphi(a)\| \le \varepsilon \qquad \text{for all } a \in F.$$

Proof. Put $M = \max_{a\in F}\|a\|$ and choose $\delta > 0$ with $\max\{2\delta,\ 2\sqrt{(1+M)\delta}\} \le \varepsilon/15$. Assume that there are isometries s, t in B satisfying (3.4.1), and let $V\colon B \to B,\ b \mapsto s^*bs$ and $W\colon B \to B,\ b \mapsto t^*bt$ be the corresponding unital, completely positive maps on B. By Lemma 3.3.10 there is a unitary u in B such that

$$\|us^*\varphi(a)su^* - \varphi(a)\| \le 5\kappa + \frac{\varepsilon}{3} \qquad \text{for all } a \in F,$$

where

$$\kappa = \max_{b\in\varphi(F\cup F^*)} \left\{\|V(b^*b) - V(b^*)V(b)\|^{1/2},\ \|WV(b^*b) - WV(b^*)WV(b)\|^{1/2},\ \|WV(b) - b\|\right\}.$$

By Lemma 3.3.11,

$$\kappa \le \max\left\{2\delta,\ 2\sqrt{(1+M)\delta}\right\} \le \frac{\varepsilon}{15},$$

and therefore

$$\begin{aligned}\|u\psi(a)u^* - \varphi(a)\| &\le \|u\psi(a)u^* - us^*\varphi(a)su^*\| + \|us^*\varphi(a)su^* - \varphi(a)\| \\ &\le \delta + 5\kappa + \frac{\varepsilon}{3} \le \varepsilon.\end{aligned}$$

for all $a \in F$. □

Theorem 3.4.2. Let A be a unital, separable, exact C^*-algebra.

(i) Let B be a unital, separable, simple and nuclear C^*-algebra. Then any two unital, injective *-homomorphisms $\varphi, \psi\colon A \to B \otimes \mathcal{O}_2$ are approximately unitarily equivalent.

(ii) Any two unital, injective *-homomorphisms $\varphi, \psi\colon A \to \mathcal{O}_2$ are approximately unitarily equivalent.

Proof. (i): Let $\varphi, \psi\colon A \to B \otimes \mathcal{O}_2$ be unital, injective *-homomorphisms. We want to apply Corollary 3.2.3(ii) to show that φ and ψ are mutually approximately similar via isometries. First of all we have to check that $B \otimes \mathcal{O}_2$ is a unital Kirchberg algebra: We know that $B \otimes \mathcal{O}_2$ is unital with unit $1_B \otimes 1_{\mathcal{O}_2}$, separable as a tensor product of separable C^*-algebras (Remark 1.5.4), nuclear as the tensor product of nuclear C^*-algebras (Proposition 1.5.7(iv)), and simple as the minimal tensor product of two simple C^*-algebras (Lemma 1.5.5). To get that $B \otimes \mathcal{O}_2$ is purely infinite we consider two cases: At first assume that B is of type I, i.e. B is isomorphic either to some $M_n(\mathbb{C})$ or to $\mathcal{K}$, the compact operators on an infinite dimensional, separable Hilbert space. As B is assumed to be unital and $\mathcal{K}$ is not, we have $B \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, and therefore $B \otimes \mathcal{O}_2 \cong M_n(\mathbb{C}) \otimes \mathcal{O}_2 \cong M_n(\mathcal{O}_2)$, which is purely infinite as $\mathcal{O}_2$ is by Lemma 1.3.13(ii). If B is not of type I, then $B \otimes \mathcal{O}_2$ is purely infinite by Theorem 4.1.10 in [R2]. Altogether this gives that $B \otimes \mathcal{O}_2$ is a unital Kirchberg algebra. Moreover,

$$(B \otimes \mathcal{O}_2) \otimes \mathcal{O}_2 \ \cong\ B \otimes (\mathcal{O}_2 \otimes \mathcal{O}_2) \ \cong\ B \otimes \mathcal{O}_2,$$

i.e. $B \otimes \mathcal{O}_2$ is $\mathcal{O}_2$-absorbing as $\mathcal{O}_2$ is self-absorbing.

As $B \otimes \mathcal{O}_2$ is a nuclear C^*-algebra, φ and ψ are nuclear *-homomorphisms by Corollary 1.5.13, and hence we can apply Corollary 3.2.3(ii) to find two sequences of isometries $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in $B \otimes \mathcal{O}_2$ with

$$\lim_{n\to\infty}\|s_n^*\varphi(a)s_n - \psi(a)\| = 0, \qquad \lim_{n\to\infty}\|t_n^*\psi(a)t_n - \varphi(a)\| = 0 \qquad \text{for all } a \in A. \tag{3.4.2}$$

Let $\varepsilon > 0$, let F be a finite subset of A and choose $\delta > 0$ as in Lemma 3.4.1. By (3.4.2) there exists $N \in \mathbb{N}$ such that

$$\|s_N^* \varphi(a) s_N - \psi(a)\| \leq \delta, \qquad \|t_N^* \psi(a) t_N - \varphi(a)\| \leq \delta$$

for all $a \in F \cup F^* \cup \{x^*x \mid x \in F \cup F^*\}$, and hence Lemma 3.4.1 yields a unitary $u \in B \otimes \mathcal{O}_2$ with

$$\|u\psi(a)u^* - \varphi(a)\| \leq \varepsilon \qquad \text{for all } a \in F.$$

This proves that φ and ψ are approximately unitarily equivalent.

(ii): Let $\varphi, \psi \colon A \to \mathcal{O}_2$ be unital, injective *-homomorphisms. We consider two possibilities of proving that φ and ψ are approximately unitarily equivalent. The first is to apply part (i): Let $\lambda \colon \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be a *-isomorphism and consider the unital, injective *-homomorphisms $\widetilde{\varphi} = \lambda \circ \varphi$ and $\widetilde{\psi} = \lambda \circ \psi$ from A into $\mathcal{O}_2 \otimes \mathcal{O}_2$. By (i), $\widetilde{\varphi}$ and $\widetilde{\psi}$ are approximately unitarily equivalent. Let now $\varepsilon > 0$, let F be a finite subset of A and choose a unitary $\tilde{u} \in \mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\|\tilde{u}\widetilde{\psi}(a)\tilde{u}^* - \widetilde{\varphi}(a)\| \leq \varepsilon \qquad \text{for all } a \in F.$$

Set $u = \lambda^{-1}(\tilde{u})$, then u is a unitary in $\mathcal{O}_2$ and

$$\|u\psi(a)u^* - \varphi(a)\| = \|\lambda^{-1}\big(\tilde{u}\widetilde{\psi}(a)\tilde{u}^* - \widetilde{\varphi}(a)\big)\| = \|\tilde{u}\widetilde{\psi}(a)\tilde{u}^* - \widetilde{\varphi}(a)\| \leq \varepsilon \qquad \text{for all } a \in F,$$

which shows that φ and ψ are approximately unitarily equivalent.

Alternatively we can apply Corollary 3.2.3(ii) and Lemma 3.4.1 directly as in (i): Note that both φ and ψ are nuclear as $\mathcal{O}_2$ is. As $\mathcal{O}_2$ is a unital Kirchberg algebra we can apply Corollary 3.2.3(ii) to find two sequences of isometries $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ in $\mathcal{O}_2$ with

$$\lim_{n\to\infty} \|s_n^* \varphi(a) s_n - \psi(a)\| = 0, \qquad \lim_{n\to\infty} \|t_n^* \psi(a) t_n - \varphi(a)\| = 0 \qquad \text{for all } a \in A.$$

As in the proof of (i) we can now apply Lemma 3.4.1 to prove that φ and ψ are approximately unitarily equivalent. □

Part (ii) of the preceding theorem will play an important part in the proof of Kirchberg's Embedding Theorem. In Appendix A we will show how both theorems can be combined to obtain a uniqueness statement for $\mathcal{O}_2$, which shows that $\mathcal{O}_2$ is basically the only unital, separable, exact C^*-algebra satisfying these theorems.

Chapter 4

Limit algebras

Limit algebras are, roughly speaking, C^*-algebras consisting of equivalence classes of sequences in a given C^*-algebra. They can be used to turn properties which are only approximately satisfied in the original algebra into exact statements. An example of this occurs in Lemma 4.5.2 in Section 4.5, where approximate unitary equivalence will be turned into exact unitary equivalence by passing to limit algebras. Sections 4.1–4.4 are used to present the concepts of products of C^*-algebras, filters and limit algebras, and Section 4.6 gives some results on limit algebras of minimal tensor products and matrix algebras.

4.1 Products and sums of C^*-algebras

In the definition of limit algebras below we introduce the C^*-algebra $\ell^\infty(A)$ of bounded sequences in a C^*-algebra A which is a special case of a product of C^*-algebras. As more general products will appear in Section 5.3 we use this opportunity to give the general definition and to state some facts, the proofs of which are either given here or can be looked up in [R1] or [Mu].

Definition 4.1.1. Let I be a non-empty set and let $(A_i)_{i\in I}$ be a family of C^*-algebras. The *product* $\prod_{i\in I} A_i$ is defined to be the set of all functions $a\colon I \to \bigcup_{i\in I} A_i$ with $a(i) \in A_i$ for all $i \in I$ and with

$$\sup\{\|a(i)\|_{A_i} \mid i \in I\} < \infty.$$

We usually write a_i instead of $a(i)$ for each $i \in I$ and denote an element of $\prod_{i\in I} A_i$ by $(a_i)_{i\in I}$.

Remark 4.1.2. The product $\prod_{i\in I} A_i$ is a C^*-algebra with coordinate-wise addition, scalar multiplication, multiplication and involution, and with norm given by

$$\|a\| = \sup\{\|a(i)\|_{A_i} \mid i \in I\}.$$

Definition 4.1.3. Let I and $(A_i)_{i\in I}$ be as before. The *sum* $\sum_{i\in I} A_i$ is defined to be the closure of the set

$$\Big\{a \in \prod_{i\in I} A_i \;\Big|\; a(i) \neq 0 \text{ for only finitely many } i \in I\Big\}.$$

Remark 4.1.4. The sum $\sum_{i\in I} A_i$ is a closed, two-sided ideal in $\prod_{i\in I} A_i$, and hence the quotient
$$\prod_{i\in I} A_i / \sum_{i\in I} A_i$$
is a C^*-algebra. Let
$$\pi\colon \prod_{i\in I} A_i \to \prod_{i\in I} A_i / \sum_{i\in I} A_i$$
denote the quotient map.

The following is Lemma 6.1.3 in [R1].

Lemma 4.1.5. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C^*-algebras and let $a = (a_n)_{n\in\mathbb{N}}$ be an element in $\prod_{n\in\mathbb{N}} A_n$. Then
$$\|\pi(a)\| = \limsup_{n\to\infty} \|a_n\|.$$
In particular, a belongs to $\sum_{n\in\mathbb{N}} A_n$ if and only if $\lim_{n\to\infty}\|a_n\| = 0$.

We now examine matrix algebras of products and vice versa, and use this result to describe completely positive maps taking values in a product C^*-algebra.

Lemma 4.1.6. Let I be a non-empty set, let $(A_i)_{i\in I}$ be a family of C^*-algebras and let $n \in \mathbb{N}$. Then the map
$$\varphi\colon \prod_{i\in I} M_n(A_i) \to M_n\Big(\prod_{i\in I} A_i\Big),\ \Big((a^i_{kl})_{k,l}\Big)_{i\in I} \mapsto \Big((a^i_{kl})_{i\in I}\Big)_{k,l}$$
is a *-isomorphism.

Proof. It is easily seen that φ is linear, self-adjoint and bijective. Showing multiplicativity also comes down to doing matrix multiplication and shifting brackets: Let $\big((a^i_{kl})_{k,l}\big)_{i\in I}$ and $\big((b^i_{kl})_{k,l}\big)_{i\in I}$ be elements in $\prod_{i\in I} M_n(A_i)$. Then
$$\begin{aligned}\varphi\Big(\big((a^i_{kl})_{k,l}\big)_{i\in I}\big((b^i_{kl})_{k,l}\big)_{i\in I}\Big) &= \varphi\bigg(\Big(\Big(\sum_{m=1}^n a^i_{km}b^i_{ml}\Big)_{k,l}\Big)_{i\in I}\bigg) = \bigg(\Big(\sum_{m=1}^n a^i_{km}b^i_{ml}\Big)_{i\in I}\bigg)_{k,l}\\ &= \big((a^i_{kl})_{i\in I}\big)_{k,l}\big((b^i_{kl})_{i\in I}\big)_{k,l} = \varphi\Big(\big((a^i_{kl})_{k,l}\big)_{i\in I}\Big)\varphi\Big(\big((b^i_{kl})_{k,l}\big)_{i\in I}\Big).\end{aligned}$$
Thus, φ is a *-isomorphism. □

Remark 4.1.7. Let I be a non-empty set, let $(A_i)_{i\in I}$ be a family of C^*-algebras and let $a = (a_i)_{i\in I} \in \prod_{i\in I} A_i$. Then a is positive in $\prod_{i\in I} A_i$ if and only if a_i is positive in A_i for all $i \in I$. To see this recall that a is positive if and only if there exists $x \in \prod_{i\in I} A_i$ with $x^*x = a$. This is the case if and only if for each $i \in I$ there exists $x_i \in A_i$ such that $x_i^*x_i = a_i$.

Lemma 4.1.8. Let A be a C^*-algebra, let I be a non-empty set, let $(B_i)_{i\in I}$ be a family of C^*-algebras and let
$$\varrho\colon A \to \prod_{i\in I} B_i$$
be a linear map. For each i in I let $\varrho_i\colon A \to B_i$ be the ith component map of ϱ, i.e. $\varrho(a) = (\varrho_i(a))_{i\in I}$ for all a in A. Then ϱ is completely positive if and only if ϱ_i is completely positive for each i in I.

Proof. Let $n \in \mathbb{N}$ and let $\varrho^{(n)}$ denote the nth inflation of ϱ, accordingly define $\varrho_i^{(n)}$ for each $i \in I$. By Lemma 4.1.6 there is a *-isomorphism

$$\varphi\colon \prod_{i\in I} M_n(B_i) \to M_n\Big(\prod_{i\in I} B_i\Big).$$

For each $a = (a_{kl})_{k,l} \in M_n(A)$ we can compute

$$\varrho^{(n)}(a) = (\varrho(a_{kl}))_{k,l} = \big((\varrho_i(a_{kl}))_{i\in I}\big)_{k,l} = \varphi\left(\big((\varrho_i(a_{kl}))_{k,l}\big)_{i\in I}\right) = \varphi\left(\Big(\varrho_i^{(n)}(a)\Big)_{i\in I}\right),$$

and as both φ and φ^{-1} are positive, this implies by Remark 4.1.7 that $\varrho^{(n)}(a)$ is positive if and only if so is $\varrho_i^{(n)}(a)$ for each $i \in I$. $\square$

4.2 The limit algebra $(A)_\infty$

This first example of limit algebras is constructed as the quotient of the bounded sequences on a C^*-algebra by the ideal of sequences which are convergent to zero.

Notation. For every C^*-algebra A the product $\prod_{n\in\mathbb{N}} A$ is simply the C^*-algebra of bounded sequences on A, which is denoted by $\ell^\infty(A)$. The sum $\sum_{n\in\mathbb{N}} A$ equals the subset of all sequences $(a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ that converge to zero, i.e. $\lim_{n\to\infty}\|a_n\| = 0$, and is denoted by $c_0(A)$.

Definition 4.2.1. Let A be a C^*-algebra. The quotient $\ell^\infty(A)/c_0(A)$ is denoted by $(A)_\infty$. Let $\pi_\infty\colon \ell^\infty(A) \to (A)_\infty$ be the quotient map, define the diagonal embedding $\delta_A\colon A \to \ell^\infty(A)$, $a \mapsto (a, a, a, \dots)$ and put $\iota_A = \pi_\infty \circ \delta_A\colon A \to (A)_\infty$.

Remark 4.2.2. The map ι_A embeds A into $(A)_\infty$ because $\pi_\infty(a, a, a, \dots) \neq \pi_\infty(b, b, b, \dots)$ for all $a \neq b \in A$.

4.3 Filters

To generalize the idea of limit algebras given in Section 4.2 one needs a more general notion of convergence, which can be achieved by introducing the concept of filters and convergence along filters.

Definition 4.3.1. Let ω be a non-empty subset of the power set of $\mathbb{N}$. If

(i) $\emptyset \notin \omega$,

(ii) ω is closed under finite intersections, i.e. $X \cap Y \in \omega$ for all $X, Y \in \omega$,

(iii) ω is upwards directed, i.e. if $X \in \omega$ and $Y \subseteq \mathbb{N}$ with $X \subseteq Y$, then $Y \in \omega$,

then ω is called a *filter* on $\mathbb{N}$. A filter ω on $\mathbb{N}$ is said to be

- an *ultrafilter* if it is not properly contained in any other filter on $\mathbb{N}$;
- *free* if $\bigcap_{X\in\omega} X = \emptyset$.

We first present two standard examples and then turn to a lemma with some important facts on ultrafilters and free filters.

Examples 4.3.2. (i) For each $n \in \mathbb{N}$ define $\omega_n = \{X \subseteq \mathbb{N} \mid n \in X\}$, then ω_n is an ultrafilter on $\mathbb{N}$, but it is not free (see Lemma 4.3.3(ii),(iii)).

(ii) Define $\omega_0 = \{X \subseteq \mathbb{N} \mid \mathbb{N} \setminus X \text{ is finite}\}$, the filter of cofinite subsets of $\mathbb{N}$. This is a free filter on $\mathbb{N}$, but it is not an ultrafilter (see Lemma 4.3.3(ii)).

Lemma 4.3.3. (i) Every filter on $\mathbb{N}$ is contained in an ultrafilter on $\mathbb{N}$.

(ii) A filter ω on $\mathbb{N}$ is an ultrafilter if and only if either $X \in \omega$ or $\mathbb{N} \setminus X \in \omega$ for each $X \subseteq \mathbb{N}$.

(iii) An ultrafilter ω on $\mathbb{N}$ is free if and only if $\omega \neq \omega_n$ for all $n \in \mathbb{N}$.

(iv) If ω is a free filter on $\mathbb{N}$ then each $X \in \omega$ is an infinite set and ω contains infinitely many sets.

(v) Each free ultrafilter ω on $\mathbb{N}$ contains the free filter ω_0 of all cofinite subsets of $\mathbb{N}$.

Proof. (i): Let ω be a filter on $\mathbb{N}$. We use Zorn's Lemma to prove that

$$\mathcal{M} = \big\{\tilde{\omega} \;\big|\; \tilde{\omega} \text{ is a filter on } \mathbb{N} \text{ and } \omega \subseteq \tilde{\omega}\big\},$$

partially ordered by inclusion, contains a maximal element. Let $\mathcal{N}$ be a totally ordered subset of $\mathcal{M}$ and set $\omega_{\mathcal{N}} = \bigcup_{\tilde{\omega} \in \mathcal{N}} \tilde{\omega}$. We show first that $\omega_{\mathcal{N}}$ is a filter. As $\emptyset \notin \tilde{\omega}$ for all $\tilde{\omega} \in \mathcal{N}$ it follows that $\emptyset \notin \omega_{\mathcal{N}}$. Let $X, Y \in \omega_{\mathcal{N}}$. As $\mathcal{N}$ is totally ordered there is $\tilde{\omega} \in \mathcal{N}$ with $X \in \tilde{\omega}$ and $Y \in \tilde{\omega}$. Then $X \cap Y \in \tilde{\omega}$ and hence $X \cap Y \in \omega_{\mathcal{N}}$. Let now $Z \in \omega_{\mathcal{N}}$ and let $K \subseteq \mathbb{N}$ with $Z \subseteq K$. Choose $\tilde{\omega} \in \mathcal{N}$ with $Z \in \tilde{\omega}$, then $K \in \tilde{\omega}$ and therefore $K \in \omega_{\mathcal{N}}$. This shows that $\omega_{\mathcal{N}}$ is a filter, and as $\omega \subseteq \tilde{\omega} \subseteq \omega_{\mathcal{N}}$ for all $\tilde{\omega} \in \mathcal{N}$ it follows that $\omega_{\mathcal{N}}$ is contained in $\mathcal{M}$ and is a majorant of $\mathcal{N}$. Hence there exists a maximal filter containing ω.

(ii): Let ω be a filter on $\mathbb{N}$. Suppose first that ω is an ultrafilter, and assume that there is $X \subseteq \mathbb{N}$ such that $X \notin \omega$ and $\mathbb{N} \setminus X \notin \omega$. Define

$$\tilde{\omega} = \big\{Y \subseteq \mathbb{N} \mid \exists A \in \omega :\ A \cap X \subseteq Y\big\}$$

and show that $\tilde{\omega}$ is a filter on $\mathbb{N}$ with $\omega \subset \tilde{\omega}$, which contradicts the assumption that ω is an ultrafilter. As $\mathbb{N} \setminus X \notin \omega$ we know that $B \not\subseteq \mathbb{N} \setminus X$ and hence $B \cap X \neq \emptyset$ for all $B \in \omega$. Thus, $\emptyset \notin \tilde{\omega}$. Let $Y, Z \in \tilde{\omega}$ and choose $A, B \in \omega$ such that $A \cap X \subseteq Y$ and $B \cap X \subseteq Z$, then $A \cap B \in \omega$ and $(A \cap B) \cap X \subseteq Y \cap Z$, i.e. $Y \cap Z \in \tilde{\omega}$. Let now $K \subseteq \mathbb{N}$ with $Y \subseteq K$. Then $A \cap X \subseteq Y \subseteq K$ and hence $K \in \tilde{\omega}$. Thus, $\tilde{\omega}$ is a filter on $\mathbb{N}$. For all $A \in \omega$ we have that $A \cap X \subseteq A$ and $A \cap X \subseteq X$. The first implies that $\omega \subseteq \tilde{\omega}$, and the second implies that $X \in \tilde{\omega}$, and hence $\omega \subset \tilde{\omega}$.

Conversely, assume that ω is not an ultrafilter and show that there is $X \subseteq \mathbb{N}$ such that $X \notin \omega$ and $\mathbb{N} \setminus X \notin \omega$. By (i) there exists an ultrafilter $\tilde{\omega}$ with $\omega \subset \tilde{\omega}$. Choose $X \in \tilde{\omega} \setminus \omega$, then clearly $X \notin \omega$. But as $\tilde{\omega}$ is a filter, we have that $\mathbb{N} \setminus X \notin \tilde{\omega}$, because otherwise $\emptyset = X \cap \mathbb{N} \setminus X \in \tilde{\omega}$, and consequently $\mathbb{N} \setminus X \notin \omega$.

(iii): Let ω be an ultrafilter on $\mathbb{N}$. If $\omega = \omega_n$ for some $n \in \mathbb{N}$, then $\bigcap_{X \in \omega} = \{n\} \neq \emptyset$, i.e. ω is not free. Conversely, suppose that ω is not free, and show that there is $n \in \mathbb{N}$ with

$\omega = \omega_n$. Since ω is an ultrafilter, statement (ii) implies that for all $m, n \in \bigcap_{X\in\omega} X$ we have $\{m\}, \{n\} \in \omega$, consequently also $\{m\} \cap \{n\} \in \omega$ and thus $\{m\} \cap \{n\} \neq \emptyset$, and hence $m = n$. We can thus choose $n \in \mathbb{N}$ with $\bigcap_{X\in\omega} X = \{n\}$. This implies that $\omega_n \subseteq \omega$ because for every $X \in \omega_n$ we know that $X \in \omega$, because otherwise (ii) implied $\mathbb{N} \setminus X \in \omega$ and hence $n \notin \bigcap_{Y\in\omega} Y$. Thus $\omega_n \subseteq \omega$, and as ω_n is an ultrafilter this implies $\omega_n = \omega$.

(iv): Let ω be a free filter on $\mathbb{N}$ and let $X \in \omega$. As $\bigcap_{Y\in\omega} Y = \emptyset$ there is for every $n \in X$ a set $Y_n \in \omega$ with $n \notin Y_n$, i.e. $X \cap \bigcap_{n\in X} Y_n = \emptyset$, but on the other hand, $X \cap \bigcap_{n\in E} Y_n \neq \emptyset$ for every finite subset E of X. Thus, X is infinite. Now, $\{Y_n \mid n \in X\}$ contains infinitely many sets, and $\{Y_n \mid n \in X\} \subseteq \omega$.

(v): Let ω be a free ultrafilter and let X be a finite subset of $\mathbb{N}$. By (iv) this implies that $X \notin \omega$, and hence $\mathbb{N} \setminus X \in \omega$ by (ii). Thus, $\omega_0 \subseteq \omega$. $\square$

Convergence along filters is defined as follows:

Definition 4.3.4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Hausdorff space $\mathcal{T}$, let x_0 be a point in $\mathcal{T}$ and let ω be a filter on $\mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is said to *converge to* x_0 *along* ω, in symbols

$$\lim_{\omega} x_n = x_0,$$

if for every neighbourhood U of x_0 there exists $X \in \omega$ such that $x_n \in U$ for all $n \in X$.

Lemma 4.3.5. Let $\mathcal{T}$, $(x_n)_{n\in\mathbb{N}}$ and ω be as in Definition 4.3.4.

(i) If $\omega = \omega_m$ for some $m \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}}$ converges to x_m.

(ii) Assume that ω is an ultrafilter and that $(x_n)_{n\in\mathbb{N}}$ is a sequence in a compact subset K of $\mathcal{T}$. Then $(x_n)_{n\in\mathbb{N}}$ converges along ω.

(iii) Assume that $\omega = \omega_0$ is the free filter of cofinite subsets of $\mathbb{N}$ and let $x_0 \in \mathcal{T}$. Then $\lim_{\omega_0} x_n = x_0$ if and only if $\lim_{n\to\infty} x_n = x_0$.

(iv) Assume there is $x_0 \in \mathcal{T}$ with $\lim_{n\to\infty} x_n = x_0$. Then $\lim_{\omega} x_n = x_0$ for every free ultrafilter ω on $\mathbb{N}$.

Proof. (i): Let $m \in \mathbb{N}$ and suppose that $\omega = \omega_m$. Then $\{m\} \in \omega$ and we can simply choose $X = \{m\}$ for every neighbourhood U of x_m.

(ii): For each $X \subseteq \mathbb{N}$ set $T_X = \overline{\{x_n \mid n \in X\}}$. Then T_X is a non-empty, compact subset of K whenever X is a non-empty subset of $\mathbb{N}$. We show first that $\bigcap_{X\in\omega} T_X \neq \emptyset$. To this end let F be a finite subset of ω. Then $Y = \bigcap_{X\in F} X \neq \emptyset$ as ω is a filter, and therefore $T_Y \neq \emptyset$. Moreover, if $x \in T_Y$, then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ in Y such that $x_{n_k} \to x$ as $k \to \infty$. By definition of Y this entails that $(n_k)_{k\in\mathbb{N}}$ is a sequence in each $X \in F$, and hence $x \in T_X$ for each $X \in F$. Thus,

$$\emptyset \neq T_Y \subseteq \bigcap_{X\in F} T_X,$$

and by the finite intersection property of compact sets it follows that $\bigcap_{X\in\omega} T_X \neq \emptyset$.

Choose $x \in \bigcap_{X\in\omega} T_X$ and let U be a neighbourhood of x. Let $A = \{n \in \mathbb{N} \mid x_n \in U\}$ and $B = \mathbb{N} \setminus A$. Then $x \notin T_B$, which implies that $B \notin \omega$. As ω is an ultrafilter, this implies that

$A \in \omega$ and therefore $x = \lim_\omega x_n$.

(iii) Let U be a neighbourhood of x_0 and suppose that $\lim_{\omega_0} x_n = x_0$. Then there is a finite subset F of $\mathbb{N}$ such that for each $n \in X = \mathbb{N} \setminus F$ we have $x_n \in U$. Then $x_n \in U$ for all $n \geq \max F$ and hence $\lim_{n\to\infty} x_n = x_0$. Conversely, if $\lim_{n\to\infty} x_n = x_0$, choose $N \in \mathbb{N}$ with $x_n \in U$ for all $n \geq N$ and put $X = \mathbb{N} \setminus \mathbb{N}_{\leq N}$, then $X \in \omega_0$ and $x_n \in U$ for all $n \in X$.

(iv) This follows from statement (iii) and Lemma 4.3.3(v). □

Definition 4.3.6. Let ω be a filter on $\mathbb{N}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define

$$\limsup_\omega x_n = \inf_{X\in\omega} \sup_{n\in X} x_n.$$

Remark 4.3.7. If ω in Definition 4.3.6 is a free filter on $\mathbb{N}$, then

$$\limsup_\omega x_n \leq \limsup_{n\to\infty} x_n,$$

because for each $k \in \mathbb{N}$ there is $X \in \omega$ with $X \subseteq \mathbb{N}_{\geq k}$: Let $k \in \mathbb{N}$. As $\bigcap_{X\in\omega} X = \emptyset$ there exists for each $n \in \mathbb{N}$ a set $X_n \in \omega$ with $n \notin X_n$. Then $X = \bigcap_{n<k} X_n \in \omega$ and $X \subseteq \mathbb{N}_{\geq k}$.

4.4 The ultrapower C^*-algebra $(A)_\omega$

We now perform a construction similar to the one in Section 4.2 to obtain the so-called ultrapower C^*-algebra $(A)_\omega$ of a given C^*-algebra A, which defines a more general class of limit algebras.

Remark 4.4.1. Let A be a C^*-algebra and let ω be a filter on $\mathbb{N}$. Denote by $c_\omega(A)$ the subset of all sequences $(a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ that converge to zero along ω, i.e. $\lim_\omega \|a_n\| = 0$. Then $c_\omega(A)$ is a closed, two-sided ideal in A.

Definition 4.4.2. Let A be a C^*-algebra and let ω be a filter on $\mathbb{N}$. The quotient C^*-algebra $\ell^\infty(A)/c_\omega(A)$, denoted by $(A)_\omega$, is called the *ultrapower* of A with respect to the filter ω. Let $\pi_\omega \colon \ell^\infty(A) \to (A)_\omega$ denote the quotient map and set $\iota_A = \pi_\omega \circ \delta_A \colon A \to (A)_\omega$, where δ_A is the diagonal embedding of A into $\ell^\infty(A)$ as defined in Definition 4.2.1.

Remarks 4.4.3. Let A be a C^*-algebra and let ω be a filter on $\mathbb{N}$.

(i) It follows as in Remark 4.2.2 that ι_A embeds A into $(A)_\omega$.

(ii) If ω_0 is the filter of cofinite sets in $\mathbb{N}$, then Lemma 4.3.5(iii) implies that $c_{\omega_0}(A) = c_0(A)$ and therefore $(A)_{\omega_0} = (A)_\infty$.

The following lemma establishes, for every sequence $a \in \ell^\infty(A)$, a very useful relation between the norm of its limit (or lim sup) and the norm of its image in the ultrapower algebra.

Lemma 4.4.4. Let A be a C^*-algebra and let ω be a filter on $\mathbb{N}$.

(i) For each $a = (a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ one has $\|\pi_\omega(a)\| = \limsup_\omega \|a_n\|$; and if ω is an ultrafilter, then $\|\pi_\omega(a)\| = \lim_\omega \|a_n\|$.

(ii) Let $k \in \mathbb{N}$ and consider for each $j \in \mathbb{N}_{\leq k}$ an element $a^{(j)} = (a_n^{(j)})_{n\in\mathbb{N}}$ in $l^\infty(A)$. Let $\varepsilon > 0$ and assume that $\|\pi_\omega(a^{(j)})\| < \varepsilon$ for all $j \in \mathbb{N}_{\leq k}$. Then there exists $X \in \omega$ such that $\|a_n^{(j)}\| < \varepsilon$ for all $n \in X$ and for all $j \in \mathbb{N}_{\leq k}$.

Proof. (i): Consider two sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ with $a - b \in c_\omega(A)$, i.e. $\pi_\omega(a) = \pi_\omega(b)$ in $(A)_\omega$. Then, by definition of $c_\omega(A)$, $\lim_\omega \|a_n - b_n\| = 0$ and therefore $\limsup_\omega \|a_n\| = \limsup_\omega \|b_n\|$. Thereby the map

$$\|\cdot\|_\omega \colon (A)_\omega \to \mathbb{R}_{\geq 0},\ \pi_\omega(a) \mapsto \limsup_\omega \|a_n\|$$

is well-defined, and it is easily shown that $\|\cdot\|_\omega$ defines a C^*-norm on $(A)_\omega$. Besides, the quotient C^*-algebra $(A)_\omega$ is canonically equipped with the complete C^*-norm defined by

$$\|\pi_\omega(a)\| = \inf\{\|a + x\| \mid x \in c_\omega(A)\}$$

for each $a \in \ell^\infty(A)$, which implies that $(A)_\omega$ admits only one C^*-norm (see Remark 6.3.3 in [Mu]). Hence,

$$\|\pi_\omega(a)\| = \limsup_\omega \|a_n\| \qquad \text{for all } a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A).$$

Assume now that ω is an ultrafilter and let $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$. Then $(\|a_n\|)_{n\in\mathbb{N}}$ is a sequence in the compact interval $[0, \sup_{n\in\mathbb{N}} \|a_n\|]$ and hence convergent by Lemma 4.3.5(ii). Combining this with the first statement gives

$$\|\pi_\omega(a)\| = \limsup_\omega \|a_n\| = \lim_\omega \|a_n\|.$$

(ii): By (i), the assumption that $\|\pi_\omega(a^{(j)})\| < \varepsilon$ implies

$$\limsup_\omega \|a_n^{(j)}\| < \varepsilon \qquad \text{for all } j \in \mathbb{N}_{\leq k},$$

and hence there exists for each $j \in \mathbb{N}_{\leq k}$ a set $X_j \in \omega$ such that $\sup_{n\in X_j} \|a_n^{(j)}\| < \varepsilon$. Put $X = \bigcap_{j\leq k} X_j$, then $X \in \omega$ as a finite intersection, and $\|a_n^{(j)}\| < \varepsilon$ for all $n \in X$ and for all $j \in \mathbb{N}_{\leq k}$. □

The following lemma extracts a detail from the proof of Kirchberg's Embedding Theorem.

Lemma 4.4.5. Let A and B be C^*-algebras and let $\iota \colon A \to B$ be an embedding of A into B. Then $(A)_\omega$ embeds into $(B)_\omega$ for every filter ω on $\mathbb{N}$. If A, B and ι are unital, then $(A)_\omega$ also embeds unitally into $(B)_\omega$.

Proof. Let ω be a filter on $\mathbb{N}$. Let $\pi_\omega^B \colon \ell^\infty(B) \to (B)_\omega$ denote the quotient map and define

$$\iota_\infty \colon \ell^\infty(A) \to \ell^\infty(B),\ (a_n)_{n\in\mathbb{N}} \mapsto (\iota(a_n))_{n\in\mathbb{N}},$$

then $\pi_\omega^B \circ \iota_\infty$ is a *-homomorphism from $\ell^\infty(A)$ into $(B)_\omega$. As ι is isometric and by Lemma 4.4.4 we have

$$\limsup_\omega \|a_n\| = \limsup_\omega \|\iota(a_n)\| = \|\left(\pi_\omega^B \circ \iota_\infty\right)(a)\| \qquad \text{for all } a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$$

and hence

$$\ker\left(\pi_\omega^B \circ \iota_\infty\right) = c_\omega(A).$$

The first isomorphism theorem now yields an injective *-homomorphism $\bar{\iota} \colon (A)_\omega \to (B)_\omega$. If A, B and ι are unital, then so is $\bar{\iota}$ by construction. □

4.5 Lifting problems and unitary equivalence

We shall next be concerned with the lifting of projections, isometries and unitaries in $(A)_\omega$ to projections, isometries and unitaries in $\ell^\infty(A)$, i.e. for a projection $p \in (A)_\omega$ we try to find a projection $q \in \ell^\infty(A)$ with $\pi_\omega(q) = p$ and so on. In the proof of these statements we shall need the fact that an element which behaves almost like a projection, an isometry or a unitary, is close to a projection, an isometry or a unitary, respectively, see Lemma 1.1.38.

Lemma 4.5.1. Let A be a C^*-algebra and let ω be a filter on $\mathbb{N}$. Then the following hold:

(i) Each projection in $(A)_\omega$ lifts to a projection in $\ell^\infty(A)$;

(ii) If A is unital, then each isometry in $(A)_\omega$ lifts to an isometry in $\ell^\infty(A)$;

(iii) If A is unital, then each unitary in $(A)_\omega$ lifts to a unitary in $\ell^\infty(A)$.

Proof. (i): Let p be a projection in $(A)_\omega$ and choose an element $a = (a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ with $\pi_\omega(a) = p$. Then

$$\|\pi_\omega(a - a^*)\| = \|p - p^*\| = 0, \qquad \|\pi_\omega(a - a^2)\| = \|p - p^2\| = 0,$$

and therefore

$$\limsup_\omega \|a_n - a_n^*\| = 0, \qquad \limsup_\omega \|a_n - a_n^2\| = 0 \tag{4.5.1}$$

by Lemma 4.4.4. We will use this to define a sequence $q = (q_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ consisting of projections satisfying $\lim_\omega \|a_n - q_n\| = 0$ and therefore $\pi_\omega(q) = \pi_\omega(a) = p$.

For each $k \in \mathbb{N}$ choose $\delta_k > 0$ such that Lemma 1.1.38(i) holds with $\varepsilon = 1/k$, and such that $(\delta_k)_{k\in\mathbb{N}}$ converges to zero. By (4.5.1) we can for each $k \in \mathbb{N}$ find $X_k \in \omega$ such that

$$\|a_n - a_n^*\| \le \delta_k, \qquad \|a_n - a_n^2\| \le \delta_k \qquad \text{for all } n \in X_k. \tag{4.5.2}$$

Let $n \in \mathbb{N}$.
Case 1: The element a_n is itself a projection. Then set $q_n = a_n$.
Case 2: For all $k \in \mathbb{N}$ we have $\|a_n - a_n^*\| > \delta_k$ or $\|a_n - a_n^2\| > \delta_k$. Then set $q_n = 0$.
Case 3: Assume that neither Case 1 nor Case 2 holds. Then

$$k_n = \max\left\{k \in \mathbb{N} \;\middle|\; \|a_n - a_n^*\| \le \delta_k \text{ and } \|a_n - a_n^2\| \le \delta_k\right\}$$

is well-defined (the set on the right hand side is bounded as $(\delta_k)_{k\in\mathbb{N}}$ converges to zero and as a_n is not a projection), and Lemma 1.1.38(i) yields a projection q_n with $\|a_n - q_n\| \le 1/k_n$.

As q_n is a projection in A for each $n \in \mathbb{N}$, the element $q = (q_n)_{n\in\mathbb{N}}$ is a projection in $\ell^\infty(A)$. Let now $\varepsilon > 0$, choose $K \in \mathbb{N}$ with $1/K < \varepsilon$ and let $n \in X_K$. By (4.5.2) we know that $\|a_n - a_n^*\| \le \delta_K$ and $\|a_n - a_n^2\| \le \delta_K$, and hence Case 1 or Case 3 applies. In Case 1 we have $\|a_n - q_n\| = 0 < \varepsilon$, and in Case 3 we have

$$\|a_n - q_n\| \le \frac{1}{k_n} \le \frac{1}{K} < \varepsilon.$$

Thus, $\lim_\omega \|a_n - q_n\| = 0$ and hence $\pi_\omega(q) = \pi_\omega(a) = p$.

(ii): Suppose that A is unital, let s be an isometry in $(A)_\omega$ and let $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ with $\pi_\omega(a) = s$ be a lift of s. As in the proof of (i) we conclude that

$$\limsup_\omega \|a_n^* a_n - 1_A\| = \|\pi_\omega(a^*a - 1_{\ell^\infty(A)})\| = \|s^*s - 1_{(A)_\omega}\| = 0. \tag{4.5.3}$$

For each $k \in \mathbb{N}$ choose a number $\delta_k > 0$ such that Lemma 1.1.38(ii) holds for $\varepsilon = 1/k$, and such that $(\delta_k)_{k\in\mathbb{N}}$ converges to zero. For each $k \in \mathbb{N}$ use (4.5.3) to find $X_k \in \omega$ with

$$\|a_n^* a_n - 1_A\| \leq \delta_k \qquad \text{for all } n \in X_k.$$

Let $n \in \mathbb{N}$. As in (i) we consider three cases:
Case 1: The element a_n is an isometry. Set $t_n = a_n$.
Case 2: For all $k \in \mathbb{N}$ we have $\|a_n^* a_n - 1_A\| > \delta_k$. Then set $t_n = 1_A$.
Case 3: Suppose that neither Case 1 nor Case 2 holds. Then

$$k_n = \max\left\{k \in \mathbb{N} \;\middle|\; \|a_n^* a_n - 1_A\| \leq \delta_k\right\}$$

is well-defined, and Lemma 1.1.38(ii) yields an isometry t_n with $\|a_n - t_n\| \leq 1/k_n$.

Now $t = (t_n)_{n\in\mathbb{N}}$ is an isometry in $\ell^\infty(A)$, and it follows as in the proof of (i) that $\lim_\omega \|a_n - t_n\| = 0$ and hence $\pi_\omega(t) = s$.

(iii): Suppose that A is unital, let u be a unitary in $(A)_\omega$ and let $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ be a lift of u. As before conclude that

$$\limsup_\omega \|a_n^* a_n - 1_A\| = \|\pi_\omega(a^*a - 1_{\ell^\infty(A)})\| = \|u^*u - 1_{(A)_\omega}\| = 0$$

and in the same way

$$\limsup_\omega \|a_n a_n^* - 1_A\| = \|uu^* - 1_{(A)_\omega}\| = 0.$$

Using the same procedure as in (i) and (ii) one can prove the existence of a unitary $v = (v_n)_{n\in\mathbb{N}}$ in $\ell^\infty(A)$ with $\lim_\omega \|a_n - v_n\| = 0$ and hence $\pi_\omega(v) = u$. ☐

The following lemma shows how approximate unitary equivalence of *-homomorphisms can be turned into exact unitary equivalence by passing over to limit algebras.

Lemma 4.5.2. Let A and B be C^*-algebras such that B is unital, and let $\varphi, \psi\colon A \to B$ be *-homomorphisms. Let ω be a filter on $\mathbb{N}$ and let $\iota_B = \pi_\omega \circ \delta_B\colon B \to (B)_\omega$ be the canonical embedding of B into $(B)_\omega$. Then the following hold:

(i) We have $\iota_B \circ \varphi \approx_u \iota_B \circ \psi$ in $(B)_\omega$ if and only if $\varphi \approx_u \psi$ in B.

(ii) If A is separable and ω is a free filter, then $\iota_B \circ \varphi \sim_u \iota_B \circ \psi$ if and only if $\iota_B \circ \varphi \approx_u \iota_B \circ \psi$.

Proof. (i): Assume that $\iota_B \circ \varphi \approx_u \iota_B \circ \psi$ in $(B)_\omega$, let F be a finite subset of A and let $\varepsilon > 0$. By assumption there is a unitary $v \in (B)_\omega$ such that

$$\|v(\iota_B \circ \psi)(a)v^* - (\iota_B \circ \varphi)(a)\| < \varepsilon \qquad \text{for all } a \in F.$$

Lemma 4.5.1(iii) yields a unitary $u = (u_n)_{n\in\mathbb{N}}$ in $\ell^\infty(B)$ with $\pi_\omega(u) = v$, and we have that

$$\|\pi_\omega\big(u\delta_B(\psi(a))u^* - \delta_B(\varphi(a))\big)\| = \|v(\iota_B \circ \psi)(a)v^* - (\iota_B \circ \varphi)(a)\| < \varepsilon \qquad \text{for all } a \in F.$$

Now Lemma 4.4.4(ii) provides us with a set $X \in \omega$ such that

$$\|u_n\psi(a)u_n^* - \varphi(a)\| < \varepsilon \qquad \text{for all } n \in X,\ a \in F.$$

Hence, φ and ψ are approximately unitarily equivalent in B.

Conversely, suppose that $\varphi \approx_u \psi$ in B. Let $\varepsilon > 0$, let F be a finite subset of A and choose $u \in B$ with

$$\|u\psi(a)u^* - \varphi(a)\| < \varepsilon \qquad \text{for all } a \in F.$$

As ι_B is a unital, isometric *-homomorphism, $\iota_B(u)$ is a unitary in $(B)_\omega$, and

$$\|\iota_B(u)(\iota_B \circ \psi)(a)\iota_B(u)^* - (\iota_B \circ \varphi)(a)\| = \|u\psi(a)u^* - \varphi(a)\| < \varepsilon \qquad \text{for all } a \in F,$$

which shows that $\iota_B \circ \varphi$ and $\iota_B \circ \psi$ are approximately unitarily equivalent in $(B)_\omega$.

(ii): Suppose now that A is separable and that ω is free. It is clear by definition that $\iota_B \circ \varphi \sim_u \iota_B \circ \psi$ implies $\iota_B \circ \varphi \approx_u \iota_B \circ \psi$. Assume now that $\iota_B \circ \varphi \approx_u \iota_B \circ \psi$. By (i) this implies that $\varphi \approx_u \psi$ in B. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of A with $\overline{\bigcup_{n\in\mathbb{N}} F_n} = A$ and choose for each $n \in \mathbb{N}$ a unitary $u_n \in B$ with

$$\|u_n\psi(a)u_n^* - \varphi(a)\| \leq \frac{1}{n} \qquad \text{for all } a \in F_n.$$

Since $(F_n)_{n\in\mathbb{N}}$ is increasing, we can as in the proof of Corollary 3.2.3(i) conclude that

$$\lim_{n\to\infty} \|u_n\psi(a)u_n^* - \varphi(a)\| = 0 \qquad \text{for all } a \in \bigcup_{n\in\mathbb{N}} F_n,$$

and as $u_n\psi(\cdot)u_n^* - \varphi(\cdot)$ is bounded for each $n \in \mathbb{N}$ this can be extended to hold for each $a \in A$. Put $v = \pi_\omega((u_n)_{n\in\mathbb{N}})$, then v is a unitary in $(B)_\omega$ and by Lemma 4.4.4(i) and Remark 4.3.7 we obtain

$$\begin{aligned} \|v(\iota_B \circ \psi)(a)v^* - (\iota_B \circ \varphi)(a)\| &= \limsup_\omega \|u_n\psi(a)u_n^* - \varphi(a)\| \\ &\leq \limsup_{n\to\infty} \|u_n\psi(a)u_n^* - \varphi(a)\| = 0 \end{aligned}$$

for all $a \in A$, which shows that $\iota_B \circ \varphi \sim_u \iota_B \circ \psi$. $\square$

Corollary 4.5.3. Let A and B be C^*-algebras with A separable and B unital, let ω be a free filter on $\mathbb{N}$ and let $\varphi, \psi \colon A \to B$ be *-homomorphisms. Then $\varphi \approx_u \psi$ in B if and only if $\iota_B \circ \varphi \sim_u \iota_B \circ \psi$ in $(B)_\omega$.

Proof. This follows immediately from Lemma 4.5.2. $\square$

4.6 Limit algebras of minimal tensor products and matrix algebras

The following result will be useful as a part of the embedding procedure in the proof of Kirchberg's Embedding Theorem.

Lemma 4.6.1. Let A and B be unital C^*-algebras, B nuclear, and let ω be a filter on $\mathbb{N}$. Then there exists a unital, injective *-homomorphism $\lambda\colon (A)_\omega \otimes B \to (A \otimes B)_\omega$.

Proof. Let $\pi_\omega\colon \ell^\infty(A) \to (A)_\omega$ and $\pi_\omega^\otimes\colon \ell^\infty(A \otimes B) \to (A \otimes B)_\omega$ be the respective quotient maps. Define $\varphi\colon (A)_\omega \to (A \otimes B)_\omega$ as follows: Let $x \in (A)_\omega$ and choose $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ with $x = \pi_\omega(a)$. Then $(a_n \otimes 1_B)_{n\in\mathbb{N}} \in \ell^\infty(A \otimes_{\min} B)$ and we can set

$$\varphi(x) = \pi_\omega^\otimes\big((a_n \otimes 1_B)_{n\in\mathbb{N}}\big).$$

To see that this does not depend on the choice of a consider $\tilde{a} = (\tilde{a}_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ with $x = \pi_\omega(\tilde{a})$ and note that

$$\limsup_\omega \|a_n \otimes 1_B - \tilde{a}_n \otimes 1_B\| = \limsup_\omega \|a_n - \tilde{a}_n\| = 0$$

which implies that

$$\pi_\omega^\otimes\big((\tilde{a}_n \otimes 1_B)_{n\in\mathbb{N}}\big) = \pi_\omega^\otimes\big((a_n \otimes 1_B)_{n\in\mathbb{N}}\big) = \varphi(x).$$

Hence, φ is well-defined and it is easy to check that φ is a unital *-homomorphism. Let $\iota_\otimes\colon A \otimes B \to (A \otimes B)_\omega$ be the canonical embedding and define

$$\psi\colon B \to (A \otimes B)_\omega,\ b \mapsto \iota_\otimes(1_A \otimes b),$$

then ψ is a unital *-homomorphism. Let now $x \in (A)_\omega$, choose $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ with $\pi_\omega(a) = x$ and let $b \in B$. Then

$$\begin{aligned}\varphi(x)\psi(b) &= \pi_\omega^\otimes\big((a_n \otimes 1_B)_{n\in\mathbb{N}}\big)\iota_\otimes(1_A \otimes b) = \pi_\omega^\otimes\big((a_n \otimes 1_B)_{n\in\mathbb{N}}\big)\pi_\omega^\otimes\big((1_A \otimes b)_{n\in\mathbb{N}}\big)\\ &= \pi_\omega^\otimes\big((a_n \otimes b)_{n\in\mathbb{N}}\big) = \pi_\omega^\otimes\big((1_A \otimes b)_{n\in\mathbb{N}}\big)\pi_\omega^\otimes\big((a_n \otimes 1_B)_{n\in\mathbb{N}}\big)\\ &= \psi(b)\varphi(x),\end{aligned}$$

i.e. $\varphi((A)_\omega)$ and $\psi(B)$ commute. As B is nuclear, Theorem 6.3.7 in [Mu] implies that there is a *-homomorphism

$$\lambda\colon (A)_\omega \otimes B \to (A \otimes B)_\omega$$

such that $\lambda(x \otimes b) = \varphi(x)\psi(b)$ for all $x \in (A)_\omega$ and for all $b \in B$, which implies that λ is unital. It follows from Theorem 3.3 in [Bl] that every closed, two-sided ideal in $(A)_\omega \otimes B$ is generated by elementary tensors as B is nuclear. We use this to show that $\ker(\lambda) = \{0\}$. Let $x \otimes b \in (A)_\omega \otimes B$ such that $\lambda(x \otimes b) = 0$. Choose a lift $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ of x. Then

$$0 = \lambda(x \otimes b) = \pi_\omega^\otimes\big((a_n \otimes b)_{n\in\mathbb{N}}\big)$$

which implies that

$$\limsup_\omega \|a_n \otimes b\| = \|b\| \lim_\omega \|a_n\| = 0.$$

Hence, $b = 0$ or $\lim_\omega a_n = 0$ and thus $x = 0$, i.e. $x \otimes b = 0$. Consequently, $\ker(\lambda) = \{0\}$ and λ is injective. $\square$

We conclude this chapter with the following result on matrix algebras and limit algebras and a corollary to it which will be needed in the proof of Lemma 5.2.1 in the following chapter.

Lemma 4.6.2. Let A be a C^*-algebra, let ω be a filter on $\mathbb{N}$ and let $n \in \mathbb{N}$. Then

$$M_n\left((A)_\omega\right) \;\cong\; \left(M_n(A)\right)_\omega .$$

Proof. Consider the diagram

$$\begin{tikzcd} M_n\left(\ell^\infty(A)\right) \arrow[r, "\psi"] \arrow[d, "\pi_\omega^{(n)}"'] & \ell^\infty\left(M_n(A)\right) \arrow[d, "\pi_\omega^{M_n}"] \\ M_n\left((A)_\omega\right) \arrow[r, dashed, "\varphi"'] & \left(M_n(A)\right)_\omega \end{tikzcd}$$

where $\pi_\omega^{(n)}$ is the nth inflation of the quotient map $\pi_\omega\colon \ell^\infty(A) \to (A)_\omega$, where

$$\pi_\omega^{M_n}\colon \ell^\infty\left(M_n(A)\right) \to \left(M_n(A)\right)_\omega$$

is the quotient map in case of the matrix algebras, and where ψ is the inverse of the *-isomorphism from Lemma 4.1.6. We want to construct a *-isomorphism φ as indicated by the dashed arrow.

Let $x = (x_{ij})_{i,j} \in M_n((A)_\omega)$, choose $a = \left((a_{ij}^k)_{k\in\mathbb{N}}\right)_{i,j} \in M_n(\ell^\infty(A))$ with $\pi_\omega^{(n)}(a) = x$ and set

$$\varphi(x) = \left(\pi_\omega^{M_n} \circ \psi\right)(a).$$

To check that this is independent of the choice of a let $b = \left((b_{ij}^k)_{k\in\mathbb{N}}\right)_{i,j} \in M_n(\ell^\infty(A))$ with $\pi_\omega^{(n)}(b) = x$. Then, using Proposition 1.1.20 at the inequalities marked with $(\star)$,

$$\begin{aligned} \|\left(\pi_\omega^{M_n} \circ \psi\right)(a-b)\| &= \|\pi_\omega^{M_n}\left(\left((a_{ij}^k - b_{ij}^k)_{i,j}\right)_{k\in\mathbb{N}}\right)\| = \limsup_\omega \|\left(a_{ij}^k - b_{ij}^k\right)_{i,j}\| \\ &\overset{(\star)}{\leq} \limsup_\omega \sum_{i,j=1}^n \|a_{ij}^k - b_{ij}^k\| = \sum_{i,j=1}^n \limsup_\omega \|a_{ij}^k - b_{ij}^k\| \\ &= \sum_{i,j=1}^n \|\pi_\omega\left((a_{ij}^k - b_{ij}^k)_{k\in\mathbb{N}}\right)\| \\ &\overset{(\star)}{\leq} n^2 \|\pi_\omega^{(n)}\left(\left((a_{ij}^k - b_{ij}^k)_{k\in\mathbb{N}}\right)_{i,j}\right)\| \\ &= n^2 \|\pi_\omega^{(n)}(a-b)\| = 0. \end{aligned}$$

Hence, φ is well-defined, and it is easy to check that φ is a *-homomorphism. Let now $x \in M_n((A)_\omega)$ with $\varphi(x) = 0$. Choose $a = \left((a_{ij}^k)_{k\in\mathbb{N}}\right)_{i,j} \in M_n(\ell^\infty(A))$ such that $0 = \varphi(x) = \left(\pi_\omega^{M_n} \circ \psi\right)(a)$, i.e.

$$\begin{aligned} 0 = \|\left(\pi_\omega^{M_n} \circ \psi\right)(a)\| &= \|\pi_\omega^{M_n}\left(\left((a_{ij}^k)_{i,j}\right)_{k\in\mathbb{N}}\right)\| \\ &= \limsup_\omega \|(a_{ij}^k)_{i,j}\| \overset{(\star)}{\geq} \limsup_\omega \|a_{ml}^k\| \end{aligned}$$

and hence $\left(a_{ml}^k\right)_{k\in\mathbb{N}} \in c_\omega(A)$ for all $m, l \in \mathbb{N}_{\leq n}$. Consequently,

$$x = \pi_\omega^{(n)}(a) = \left(\pi_\omega\left((a_{ij}^k)_{k\in\mathbb{N}}\right)\right)_{i,j} = 0$$

in $M_n((A)_\omega)$ and φ is injective. To see that φ is surjective let $y \in (M_n(A))_\omega$, choose a lift $a \in \ell^\infty(M_n(A))$ of y and set $x = \left(\pi_\omega^{(n)} \circ \psi^{-1}\right)(a)$. Then $\varphi(x) = \left(\pi_\omega^{M_n} \circ \psi\right)(\psi^{-1}(a)) = y$. $\square$

Corollary 4.6.3. Let A and B be C^*-algebras, let ω be a filter on $\mathbb{N}$ and let $\varrho\colon A \to \ell^\infty(B)$ be a linear map with component maps $\varrho_k\colon A \to B$ for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ and let $\pi_\omega\colon \ell^\infty(B) \to (B)_\omega$ and $\pi_\omega^{M_n}\colon \ell^\infty(M_n(B)) \to (M_n(B))_\omega$ denote the quotient maps as before, and let $\varphi\colon M_n((B)_\omega) \to (M_n(B))_\omega$ be the *-isomorphism from Lemma 4.6.2. Then

$$\varphi \circ (\pi_\omega \circ \varrho)^{(n)} = \pi_\omega^{M_n} \circ \left(\varrho_k^{(n)}\right)_{k\in\mathbb{N}},$$

i.e. the diagram

$$\begin{tikzcd} M_n(A) \arrow[r, "(\varrho_k^{(n)})_{k\in\mathbb{N}}"] \arrow[d, "(\pi_\omega\circ\varrho)^{(n)}"'] & \ell^\infty(M_n(B)) \arrow[d, "\pi_\omega^{M_n}"] \\ M_n((B)_\omega) \arrow[r, "\varphi"'] & (M_n(B))_\omega \end{tikzcd}$$

commutes. In particular, if $(\pi_\omega \circ \varrho)^{(n)}$ is injective, then so is $\pi_\omega^{M_n} \circ \left(\varrho_k^{(n)}\right)_{k\in\mathbb{N}}$.

Proof. To see this, take $a = (a_{ij})_{i,j} \in M_n(A)$. Then

$$\begin{aligned}\left(\varphi \circ (\pi_\omega \circ \varrho)^{(n)}\right)(a) &= \left(\varphi \circ \pi_\omega^{(n)}\right)\left(\left(\varrho(a_{ij})\right)_{i,j}\right) = \pi_\omega^{M_n}\left(\left(\left(\varrho_k(a_{ij})\right)_{i,j}\right)_{k\in\mathbb{N}}\right) \\ &= \pi_\omega^{M_n}\left(\left(\varrho_k^{(n)}(a)\right)_{k\in\mathbb{N}}\right) = \left(\pi_\omega^{M_n} \circ \left(\varrho_k^{(n)}\right)_{k\in\mathbb{N}}\right)(a).\end{aligned}$$

□

Chapter 5

First embeddings into $\mathcal{O}_2$

In this chapter we construct the first embeddings of exact C^*-algebras into $\mathcal{O}_2$. Section 5.1 deals with the embedding of $C(\mathbb{T})$ into $\mathcal{O}_2$. Though this result goes directly into the proof of Kirchberg's Embedding Theorem, it is quite independent of the rest of the theory developed here. In Section 5.2 we resume our main line of argumentation and use several results from Chapter 3 to show that a unital, separable, exact C^*-algebra which has a liftable, unital embedding into $(\mathcal{O}_2)_\omega$ for a free ultrafilter ω embeds unitally into $\mathcal{O}_2$. In Section 5.3 this result will be applied to embed unital, separable, exact and quasidiagonal C^*-algebras into $\mathcal{O}_2$.

5.1 Embedding $C(\mathbb{T})$ into $\mathcal{O}_2$

Lemma 5.1.1. The C^*-algebra $C(\mathbb{T})$ of continuous complex-valued functions on the unit circle embeds unitally into $\mathcal{O}_2$.

In the proof of this lemma we first obtain a topological space X such that $C(X)$ embeds into $\mathcal{O}_2$. To construct then an embedding of $C(\mathbb{T})$ into $C(X)$ we shall use the following topological statements. Moreover, we shall need the notion of maximal abelian subalgebras which is introduced below. We start with the following facts about the Cantor Set $\mathcal{C}$, which are part of Theorem VI′ in [AH].

Theorem 5.1.2. (i) Every compact, totally disconnected, metrizable topological space without isolated points is homeomorphic to the Cantor Set $\mathcal{C}$.

(ii) For every compact, metrizable topological space Y there exists a continuous, surjective map $f\colon \mathcal{C} \to Y$ from the Cantor Set $\mathcal{C}$ onto Y.

We also consider the following corollary to statement (i) of the previous theorem:

Corollary 5.1.3. Let X be a compact, metrizable topological space without isolated points. Then there exists a closed subset X_0 of X which is homeomorphic to the Cantor Set $\mathcal{C}$.

Proof. By Theorem 5.1.2(i) it suffices to construct a closed, totally disconnected subset of X which has no isolated points. To this end we inductively construct a decreasing sequence $(X_n)_{n\in\mathbb{N}}$ of subsets of X such that for every $n \in \mathbb{N}$ the following holds: The set X_n is the disjoint union of 2^n non-empty, closed balls B_j^n, $j \in \mathbb{N}_{\leq 2^n}$, with radius less than 2^{-n}.

Moreover, $B_{2j-1}^{n+1}, B_{2j}^{n+1} \subseteq B_j^n$ for all $j \in \mathbb{N}_{\leq 2^n}$. All balls are defined with respect to the metric which induces the topology on X and are supposed to have radius greater than zero.

As X has no isolated points and is metrizable and hence Hausdorff there are points $x, y \in X$ such that there are disjoint, closed balls $B_1^1, B_2^1 \subseteq X$ with radius less than $1/2$ and $x \in B_1^1$, $y \in B_2^1$. Set $X_1 = B_1^1 \cup B_2^1$. Again as X has no isolated points and is metrizable we find non-empty, disjoint, closed balls $B_1^2, B_2^2 \subseteq B_1^1$ and $B_3^2, B_4^2 \subseteq B_2^1$ with radius less than $1/4$. Set $X_2 = \bigcup_{1 \leq j \leq 4} B_j^2$. Similarly, when $X_n = \bigcup_{1 \leq j \leq 2^n} B_j^n$ has been constructed for some $n \in \mathbb{N}$, we find for each $j \in \mathbb{N}_{\leq 2^n}$ two non-empty, disjoint, closed balls $B_{2j-1}^{n+1}, B_{2j}^{n+1} \subseteq B_j^n$ with radius less than $2^{-(n+1)}$ and set $X_{n+1} = \bigcup_{1 \leq l \leq 2^{n+1}} B_l^{n+1}$. Being a finite union of non-empty, closed balls each X_n is non-empty and closed in X. Thus, $(X_n)_{n \in \mathbb{N}}$ is a decreasing sequence of non-empty, compact subsets of X, and hence

$$X_0 = \bigcap_{n \in \mathbb{N}} X_n$$

also is a non-empty, compact subset of X. We show that X_0 is totally disconnected and has no isolated points.

For the first property we show that every subset of X_0 which contains more than one point is disconnected. Let $K \subseteq X_0$ and let $x \neq y \in K$. There are $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\leq 2^n}$ such that $x \in B_k^n$ and $y \notin B_k^n$. As each B_j^n is a closed ball in X, the sets $B_j^n \cap K$ are closed in K for every $j \in \mathbb{N}_{\leq 2^n}$. On the other hand,

$$K = X_n \cap K = \bigcup_{1 \leq j \leq 2^n} \left(B_j^n \cap K\right),$$

and hence

$$B_k^n \cap K = K \setminus \bigcup_{\substack{1 \leq j \leq 2^n \\ j \neq k}} \left(B_j^n \cap K\right)$$

is open in K as the complement of a finite union of closed sets. Altogether, $B_k^n \cap K$ is a non-empty, clopen, proper subset of K, and hence K is not connected.

Let now $x \in X_0$ and show that x is not an isolated point in X_0. Let $\varepsilon > 0$, and choose $n \in \mathbb{N}$ with $2^{-n+1} < \varepsilon$ and $k \in \mathbb{N}_{\leq 2^n}$ with $x \in B_k^n$. By construction of X_0 this implies that $x \in B_{2k-1}^{n+1}$ or $x \in B_{2k}^{n+1}$, assume the first. Take $y \in B_{2k}^{n+1} \cap X_0$, then $y \neq x$, but both x and y are contained in B_k^n, and hence have distance less than $2 \cdot 2^{-n} < \varepsilon$. □

We also need the following statement on metrizability:

Lemma 5.1.4. Let (X, τ) be a compact Hausdorff space such that $C(X)$ is separable. Then X is metrizable.

Proof. Let $\{f_n \mid n \in \mathbb{N}\}$ be a countable, dense subset of $C(X)$. Define a function

$$d\colon X \times X \to \mathbb{R}_{\geq 0},\ (x, y) \mapsto \sum_{n=1}^{\infty} \frac{1}{2^n} \max\{|f_n(x) - f_n(y)|, 1\}. \tag{5.1.1}$$

It is easily checked that $d(x, x) = 0$ for all $x \in X$, that d is symmetric and satisfies the triangle inequality. To show that $d(x, y) > 0$ if $x \neq y \in X$ we need the separability of $C(X)$: Let

$x \neq y \in X$. There exists (by Urysohn's Lemma) a function $f \in C(X)$ with $f(x) \neq f(y)$. Set $\varepsilon = |f(x) - f(y)|$ and choose $n \in \mathbb{N}$ such that $\|f - f_n\| < \varepsilon/2$. Then

$$|f_n(x) - f_n(y)| \geq |f(x) - f(y)| - |f_n(x) - f(x)| - |f(y) - f_n(y)| > 0,$$

and hence $d(x, y) > 0$, as required. Thus, (5.1.1) defines a metric on X, and it is left to show that the topology induced by this metric coincides with the given topology τ on X. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X, let $x \in X$ and suppose first that $(x_k)_{k\in\mathbb{N}}$ converges to x with respect to τ. Let $\varepsilon > 0$ and choose $q \in (0, 1)$ with $\sum_{n=1}^{\infty} q^n < \varepsilon/2$. As each f_n is continuous with respect to τ we can for every $n \in \mathbb{N}$ choose a number $K_n \in \mathbb{N}$ with $|f_n(x_k) - f_n(x)| < q^n$ for all $k \in \mathbb{N}_{\geq K_n}$. Choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \varepsilon/2$ and set $K = \max\{K_n \mid n \in \mathbb{N}_{\leq N}\}$. Then, for all $k \in \mathbb{N}_{\geq K}$:

$$d(x_k, x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max\{|f_n(x) - f_n(y)|, 1\} \leq \sum_{n=1}^{N} \frac{q^n}{2^n} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \varepsilon,$$

i.e. $(x_k)_{k\in\mathbb{N}}$ also converges to x with respect to d. Conversely, assume that $(x_k)_{k\in\mathbb{N}}$ does not converge to x with respect to τ, i.e. there exists a neighbourhood U of x such that $x_k \in X \setminus U$ for infinitely many $k \in \mathbb{N}$. By Urysohn's Lemma there exists $f \in C(X)$ with $f(x) = 1$ and $f|_{X\setminus U} \equiv 0$. By separability of $C(X)$ there is $j \in \mathbb{N}$ with $\|f - f_j\| \leq 1/3$. For all $k \in \mathbb{N}$ with $x_k \in X \setminus U$ we thus have

$$\begin{aligned} d(x_k, x) &\geq \frac{1}{2^j} |f_j(x_k) - f_j(x)| \\ &\geq \frac{1}{2^j} \left(|f(x_k) - f(x)| - |f_j(x_k) - f(x_k)| - |f(x) - f_j(x)|\right) \\ &\geq \frac{1}{3 \cdot 2^j}, \end{aligned}$$

i.e. $(x_k)_{k\in\mathbb{N}}$ does not converge to x with respect to d either. □

We now turn to some statements about maximal abelian sub-C^*-algebras:

Definition 5.1.5. Let A be a C^*-algebra. An abelian sub-C^*-algebra D of A is called a *maximal abelian subalgebra*, or *masa*, if it is not properly contained in any other abelian sub-C^*-algebra of A.

Remarks 5.1.6. (i) An application of Zorn's Lemma shows that every abelian sub-C^*-algebra of a C^*-algebra A is contained in a masa of A. Therefore, every C^*-algebra contains a masa.

(ii) Let A be a C^*-algebra and let D be a masa of A. If an element a in A commutes with every element in D, then a belongs to D.

For more information about masas the reader may consult [R1]. We have now collected enough material to prove that $C(\mathbb{T})$ embeds unitally into $\mathcal{O}_2$:

Proof of Lemma 5.1.1. Let $D \subseteq \mathcal{O}_2$ be a maximal abelian sub-C^*-algebra of $\mathcal{O}_2$. As $1_{\mathcal{O}_2}$ commutes with every element of $\mathcal{O}_2$ and hence of D, D contains $1_{\mathcal{O}_2}$ by Remark 5.1.6(ii). Since D is a unital, commutative C^*-algebra, there exists a *-isomorphism $\varphi\colon C(X) \to D$

(automatically unital) for some compact Hausdorff space X. Being a sub-C^*-algebra of $\mathcal{O}_2$, D is separable, and hence so is $C(X)$. By Lemma 5.1.4 this implies that X is metrizable.

We show indirectly that X has no isolated points. Assume that there is an isolated point x_0 in X. Then the characteristic function $\chi_{\{x_0\}}$ of $\{x_0\}$ is continuous, and we can consider $p = \varphi(\chi_{\{x_0\}})$ in D. As $\chi_{\{x_0\}}^2 = \chi_{\{x_0\}} = \chi_{\{x_0\}}^*$, the element p is a projection in D. For every $d \in D$ choose $f_d \in C(X)$ such that $\varphi(f_d) = d$ and calculate

$$pdp = \varphi(\chi_{\{x_0\}} f_d)p = \varphi\left(f_d(x_0)\chi_{\{x_0\}}\right)p = f_d(x_0)\varphi(\chi_{\{x_0\}})p = f_d(x_0)p^2 = f_d(x_0)p.$$

This shows that $pDp = \mathbb{C}p$. Using this and that D is abelian we obtain for each $d \in D$ that $pd = dp = dp^2 = pdp \in \mathbb{C}p$, i.e. there is $\lambda \in \mathbb{C}$ such that $pd = dp = \lambda p$. Therefore we can calculate for any $x \in p\mathcal{O}_2 p$ and any $d \in D$ that

$$xd = xpd = \lambda xp = \lambda x = \lambda px = dpx = dx.$$

Thus, every $x \in p\mathcal{O}_2 p$ commutes with every $d \in D$, and hence $p\mathcal{O}_2 p \subseteq D$ by Remark 5.1.6(ii). But this implies $p\mathcal{O}_2 p = pDp = \mathbb{C}p$, which is a contradiction as $p\mathcal{O}_2 p$, being a hereditary sub-C^*-algebra of $\mathcal{O}_2$, is purely infinite and $\mathbb{C}p$ is finite.

Thus, X has no isolated points, and Corollary 5.1.3 yields the existence of a closed subset X_0 of X which is homeomorphic to the Cantor Set $\mathcal{C}$. By Theorem 5.1.2(ii) there exists a continuous, surjective function from $\mathcal{C}$ onto the interval $[0,1]$, and as X_0 is homeomorphic to $\mathcal{C}$, this also yields a continuous, surjective function $f_0 \colon X_0 \to [0,1]$. By Tietze's Extension Theorem (see Proposition 1.5.8 and its proof in [Pe2]), the function f_0 can be extended to a continuous, surjective function $f \colon X \to [0,1]$. This can be used to define a function

$$h \colon X \to \mathbb{T},\ x \mapsto \mathrm{e}^{2\pi \mathrm{i} f(x)},$$

which is continuous and surjective as f is. Now we can define

$$\psi \colon C(\mathbb{T}) \to C(X),\ g \mapsto g \circ h.$$

It is easy to check that ψ is a unital *-homomorphism. To see that ψ is injective, take $g \in C(\mathbb{T})$ with $\psi(g) = 0$. Since h is surjective, this implies that $g(t) = 0$ for all $t \in \mathbb{T}$, and hence $g = 0$. Thus, ψ defines a unital embedding of $C(\mathbb{T})$ into $C(X) \cong D \subseteq \mathcal{O}_2$, i.e. $C(\mathbb{T})$ embeds unitally into $\mathcal{O}_2$. □

Remark 5.1.7. Another way to obtain an embedding of $C(\mathbb{T})$ into $\mathcal{O}_2$ is the following: Let s be one of the isometries which generate $\mathcal{O}_2$ and show that $\sigma(s + s^*) = [-2, 2]$ (which requires some work). Then $C([-2,2])$ is isomorphic to $C^*(s + s^*, 1_{\mathcal{O}_2})$, the sub-$C^*$-algebra of $\mathcal{O}_2$ generated by $s + s^*$ and $1_{\mathcal{O}_2}$, and one can construct embeddings

$$C(\mathbb{T}) \hookrightarrow C([-2,2]) \hookrightarrow \mathcal{O}_2.$$

5.2 A first embedding result for exact C^*-algebras

We now return to the main argument of this thesis and show how the results from Chapter 3 on obtaining approximate similarity via isometries and then approximate unitary equivalence can be used to construct an embedding into $\mathcal{O}_2$.

Lemma 5.2.1. Let A be a unital, separable, exact C^*-algebra and let ω be a free ultrafilter on $\mathbb{N}$. Suppose that there is a unital, injective *-homomorphism $\varphi\colon A \to (\mathcal{O}_2)_\omega$ with a unital, completely positive lift $\varrho\colon A \to l^\infty(\mathcal{O}_2)$, i.e. the diagram

$$\begin{tikzcd} & \ell^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\ A \arrow[ur, "\varrho"] \arrow[r, "\varphi"'] & (\mathcal{O}_2)_\omega \end{tikzcd}$$

commutes. Then there is a unital, injective *-homomorphism from A into $\mathcal{O}_2$.

Notation. In the following proof let id_n denote the identity map on $M_n(\mathbb{C})$ for every $n \in \mathbb{N}$.

Proof. Let $(\varrho_k)_{k\in\mathbb{N}}$ be the sequence of component maps of ϱ, i.e. $\varrho(a) = (\varrho_k(a))_{k\in\mathbb{N}}$ for each $a \in A$. Then each ϱ_k is a unital, completely positive map by Lemma 4.1.8. By separability of A choose an increasing sequence of finite dimensional operator systems $(E_k)_{k\in\mathbb{N}}$ in A such that $\overline{\bigcup_{k\in\mathbb{N}} E_k} = A$. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a summable sequence in $\mathbb{R}_{>0}$ and choose another summable, decreasing sequence $(\delta_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_{>0}$ such that Lemma 3.3.12 holds with ε_k and $4\delta_k$ for each $k \in \mathbb{N}$.

We construct two increasing sequences $(k_j)_{j\in\mathbb{N}}$ and $(l_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that Proposition 3.2.1 can be applied to find sequences $(\eta_j)_{j\in\mathbb{N}}$ and $(\sigma_j)_{j\in\mathbb{N}}$ of unital, completely positive maps from $\mathcal{O}_2$ into $\mathcal{O}_2$ such that the diagrams

$$\begin{tikzcd} & E_{l_j} \arrow[dl, "\varrho_{k_j}"'] \arrow[dr, "\varrho_{k_{j+1}}"] & \\ \mathcal{O}_2 \arrow[rr, dashed, "\eta_j"'] & & \mathcal{O}_2 \end{tikzcd} \qquad \begin{tikzcd} & E_{l_{j+1}} \arrow[dl, "\varrho_{k_{j+1}}"'] \arrow[dr, "\varrho_{k_j}"] & \\ \mathcal{O}_2 \arrow[rr, dashed, "\sigma_j"'] & & \mathcal{O}_2 \end{tikzcd}$$

commute within δ_j and δ_{j+1} on the unit spheres of E_{l_j} and $E_{l_{j+1}}$, respectively. For each $n \in \mathbb{N}$ let

$$\pi_\omega^{M_n}\colon \ell^\infty(M_n(\mathcal{O}_2)) \to (M_n(\mathcal{O}_2))_\omega$$

denote the quotient map in the case of matrix algebras. As $\pi_\omega \circ \varrho = \varphi$ is an injective and hence isometric *-homomorphism so is $(\pi_\omega \circ \varrho)^{(n)}$, and hence also $\pi_\omega^{M_n} \circ \big(\varrho_k^{(n)}\big)_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$ by Corollary 4.6.3. As ω is an ultrafilter we have by Lemma 4.4.4 that

$$\lim_\omega \|(\mathrm{id}_n \otimes \varrho_k)(a)\| = \lim_\omega \|\varrho_k^{(n)}(a)\| = \big\|\big(\pi_\omega^{M_n} \circ \big(\varrho_k^{(n)}\big)_{k\in\mathbb{N}}\big)(a)\big\| = \|a\| \tag{5.2.1}$$

for all $n \in \mathbb{N}$ and for all $a \in M_n(A)$. Moreover we have

$$\begin{aligned} \pi_\omega\big((\varrho_k(ab) - \varrho_k(a)\varrho_k(b))_{k\in\mathbb{N}}\big) &= (\pi_\omega \circ \varrho)(ab) - (\pi_\omega \circ \varrho)(a)(\pi_\omega \circ \varrho)(b) \\ &= \varphi(ab) - \varphi(a)\varphi(b) = 0 \end{aligned}$$

and therefore

$$\lim_\omega \big(\varrho_k(ab) - \varrho_k(a)\varrho_k(b)\big) = 0 \qquad \text{for all } a, b \in A. \tag{5.2.2}$$

As each E_j is finite dimensional we can for each $j \in \mathbb{N}$ find a δ_j-dense finite subset F_j of the unit sphere of E_j. Choose $(F_j)_{j\in\mathbb{N}}$ to be increasing and set $\widetilde{F}_j = F_j \cup F_j^* \cup \{a^*a \mid a \in F \cup F^*\}$

for each $j \in \mathbb{N}$. Since each $\widetilde{F}_j$ is contained in the unit sphere of A and since $\overline{\bigcup_{k\in\mathbb{N}} E_k} = A$ we can find an increasing sequence $(l_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that for each $j \in \mathbb{N}$ and for each $b \in \widetilde{F}_j$ there is an element a in the unit sphere of E_{l_j} with $\|b - a\| \leq \delta_j$. For each $j \in \mathbb{N}$ choose $n_j = n(A, E_{l_j}, \delta_j)$ as in Proposition 3.2.1, put $r_j = 1 - (1 + \delta_j/2)^{-1}$ and choose an $r_j/2$-dense finite subset G_j of the unit sphere of $M_{n_j}(E_{l_j})$.

For each $j \in \mathbb{N}$ we can now perform the following construction: For each $a \in G_j$ use (5.2.1) and $(1 + \delta_j/2)^{-1} + r_j/2 < 1 = \|a\|$ to choose a set $\overline{X}_{j,a} \in \omega$ such that

$$\|(\mathrm{id}_{n_j} \otimes \varrho_m)(a)\| \geq \left(1 + \frac{\delta_j}{2}\right)^{-1} + \frac{r_j}{2} \qquad \text{for all } m \in \overline{X}_{j,a}.$$

Put $\overline{X}_j = \bigcap_{a\in G_j} \overline{X}_{j,a}$ to obtain a set $\overline{X}_j \in \omega$ with

$$\|(\mathrm{id}_{n_j} \otimes \varrho_m)(a)\| \geq \left(1 + \frac{\delta_j}{2}\right)^{-1} + \frac{r_j}{2} \qquad \text{for all } m \in \overline{X}_j,\ a \in G_j. \tag{5.2.3}$$

Using equations (5.2.2) and (5.2.1) with $n = 1$ we can in the same way find $\widetilde{X}_j$ and $\widehat{X}_j$ in ω such that

$$\|\varrho_m(ab) - \varrho_m(a)\varrho_m(b)\| \leq \delta_j \qquad \text{for all } m \in \widetilde{X}_j,\ a, b \in F_j \cup F_j^* \tag{5.2.4}$$

and

$$\|\varrho_m(a)\| \geq \|a\| - \delta_j \qquad \text{for all } m \in \widehat{X}_j,\ a \in F_j. \tag{5.2.5}$$

Now choose for each $j \in \mathbb{N}$ an element $k_j \in \overline{X}_j \cap \widetilde{X}_j \cap \widehat{X}_j$ such that $(k_j)_{j\in\mathbb{N}}$ is increasing, which is possible because $\overline{X}_j \cap \widetilde{X}_j \cap \widehat{X}_j \in \omega$ and each set in a free filter is infinite.

For any $j \in \mathbb{N}$ and for any b in the unit sphere of $M_{n_j}(E_{l_j})$ there is an element $a \in G_j$ with $\|b - a\| \leq r_j/2$. Hence, by (5.2.3):

$$\begin{aligned}\|\left(\mathrm{id}_{n_j} \otimes \varrho_{k_j}\right)(b)\| &\geq \|\left(\mathrm{id}_{n_j} \otimes \varrho_{k_j}\right)(a)\| - \|\left(\mathrm{id}_{n_j} \otimes \varrho_{k_j}\right)(b - a)\| \\ &\geq \left(1 + \frac{\delta_j}{2}\right)^{-1} + \frac{r_j}{2} - \frac{r_j}{2} \\ &= \left(1 + \frac{\delta_j}{2}\right)^{-1}.\end{aligned} \tag{5.2.6}$$

For every x in the unit sphere of E_{l_j} this implies that

$$\|\varrho_{k_j}(x)\| = \|1_{M_{n_j}(\mathbb{C})} \otimes \varrho_{k_j}(x)\| = \|\left(\mathrm{id}_{n_j} \otimes \varrho_{k_j}\right)\left(1_{M_{n_j}(\mathbb{C})} \otimes x\right)\| \geq \left(1 + \frac{\delta_j}{2}\right)^{-1} > 0.$$

Thus, $\varrho_{k_j}|_{E_{l_j}}$ is injective, i.e. $(\varrho_{k_j}|_{E_{l_j}})^{-1} \colon \varrho_{k_j}(E_{l_j}) \to E_{l_j}$ is well-defined, and it follows from (5.2.6) that

$$\|\mathrm{id}_{n_j} \otimes (\varrho_{k_j}|_{E_{l_j}})^{-1}\| \leq 1 + \frac{\delta_j}{2} \qquad \text{for each } j \in \mathbb{N}.$$

Moreover, each ϱ_{k_j} is nuclear as $\mathcal{O}_2$ is, and hence Proposition 3.2.1 yields for each $j \in \mathbb{N}$ two unital, completely positive maps $\eta_j \colon \mathcal{O}_2 \to \mathcal{O}_2$ and $\sigma_j \colon \mathcal{O}_2 \to \mathcal{O}_2$ as indicated in the two diagrams above such that

$$\|\left(\eta_j \circ \varrho_{k_j} - \varrho_{k_{j+1}}\right)\big|_{E_{l_j}}\| \leq \delta_j, \qquad \|\left(\sigma_j \circ \varrho_{k_{j+1}} - \varrho_{k_j}\right)\big|_{E_{l_{j+1}}}\| \leq \delta_{j+1}. \tag{5.2.7}$$

As $\mathcal{O}_2$ is unital, simple, purely infinite and nuclear, all η_j and σ_j are nuclear and we can apply Proposition 3.1.5 to find for each $j \in \mathbb{N}$ two isometries $s_j, t_j \in \mathcal{O}_2$ such that

$$\|s_j^* \varrho_{k_j}(a) s_j - \left(\eta_j \circ \varrho_{k_j}\right)(a)\| \leq \delta_j, \qquad \|t_j^* \varrho_{k_{j+1}}(a) t_j - \left(\sigma_j \circ \varrho_{k_{j+1}}\right)(a)\| \leq \delta_j \tag{5.2.8}$$

for all $a \in \widetilde{F}_j$. For any $j \in \mathbb{N}$ and for any $a \in \widetilde{F}_j$ choose an element b in the unit sphere of E_{l_j} with $\|b - a\| \leq \delta_j$ and combine (5.2.7) and (5.2.8) to obtain

$$\begin{aligned}
&\|s_j^* \varrho_{k_j}(a) s_j - \varrho_{k_{j+1}}(a)\| \\
&\leq \|s_j^* \varrho_{k_j}(a) s_j - \left(\eta_j \circ \varrho_{k_j}\right)(a)\| + \|\left(\eta_j \circ \varrho_{k_j}\right)(a) - \left(\eta_j \circ \varrho_{k_j}\right)(b)\| + \|\left(\eta_j \circ \varrho_{k_j}\right)(b) - \varrho_{k_{j+1}}(b)\| \\
&\qquad + \|\varrho_{k_{j+1}}(b) - \varrho_{k_{j+1}}(a)\| \\
&\leq \delta_j + \|a - b\| + \delta_j + \|b - a\| \\
&\leq 4\delta_j,
\end{aligned}$$

$$\begin{aligned}
&\|t_j^* \varrho_{k_{j+1}}(a) t_j - \varrho_{k_j}(a)\| \\
&\leq \|t_j^* \varrho_{k_{j+1}}(a) t_j - \left(\sigma_j \circ \varrho_{k_{j+1}}\right)(a)\| + \|\left(\sigma_j \circ \varrho_{k_{j+1}}\right)(a) - \left(\sigma_j \circ \varrho_{k_{j+1}}\right)(b)\| \\
&\qquad + \|\left(\sigma_j \circ \varrho_{k_{j+1}}\right)(b) - \varrho_{k_j}(b)\| + \|\varrho_{k_j}(b) - \varrho_{k_j}(a)\| \\
&\leq \delta_j + \|a - b\| + \delta_j + \|a - b\| \\
&\leq 4\delta_j
\end{aligned}$$

for all $a \in \widetilde{F}_j$. By (5.2.4) we also have

$$\|\varrho_{k_j}(a^* a) - \varrho_{k_j}(a^*) \varrho_{k_j}(a)\| \leq \delta_j, \qquad \|\varrho_{k_{j+1}}(a^* a) - \varrho_{k_{j+1}}(a^*) \varrho_{k_{j+1}}(a)\| \leq \delta_{j+1} \leq \delta_j$$

for all $a \in F_j \cup F_j^*$, and so Lemma 3.3.12 provides us with a sequence of unitaries $(u_j)_{j \in \mathbb{N}}$ in $\mathcal{O}_2$ such that

$$\|u_j \varrho_{k_{j+1}}(a) u_j^* - \varrho_{k_j}(a)\| \leq \varepsilon_j \qquad \text{for all } j \in \mathbb{N},\ a \in F_j.$$

Since each F_j is δ_j-dense in the unit sphere of E_j we can extend this as follows: For each $j \in \mathbb{N}$ and for each b in the unit sphere of E_j choose $a \in F_j$ with $\|b - a\| \leq \delta_j$ and compute

$$\begin{aligned}
&\|u_j \varrho_{k_{j+1}}(b) u_j^* - \varrho_{k_j}(b)\| \\
&\leq \|u_j \varrho_{k_{j+1}}(b) u_j^* - u_j \varrho_{k_{j+1}}(a) u_j^*\| + \|u_j \varrho_{k_{j+1}}(a) u_j^* - \varrho_{k_j}(a)\| + \|\varrho_{k_j}(a) - \varrho_{k_j}(b)\| \\
&\leq 2\delta_j + \varepsilon_j \overset{\text{def}}{=} \alpha_j.
\end{aligned} \tag{5.2.9}$$

The sequence $(\alpha_j)_{j \in \mathbb{N}}$ is summable as $(\delta_j)_{j \in \mathbb{N}}$ and $(\varepsilon_j)_{j \in \mathbb{N}}$ are. We want to define

$$\psi(a) = \lim_{j \to \infty} \operatorname{Ad}(u_1 \cdots u_j)\, \varrho_{k_{j+1}}(a) \qquad \text{for all } a \in A, \tag{5.2.10}$$

where

$$\operatorname{Ad}(u) \colon \mathcal{O}_2 \to \mathcal{O}_2,\ x \mapsto u x u^*$$

for each unitary $u \in \mathcal{O}_2$ (notice that $\operatorname{Ad}(u)$ is a *-automorphism on $\mathcal{O}_2$ for each unitary $u \in \mathcal{O}_2$ and that any product of unitaries is a unitary). We proceed to show that the limit in (5.2.10) exists and that ψ then defines a unital, injective *-homomorphism from A into $\mathcal{O}_2$.

Consider the set D of all $a \in A$ for which the sequence $\left(\mathrm{Ad}\,(u_1 \cdots u_j)\, \varrho_{k_{j+1}}(a)\right)_{j\in\mathbb{N}}$ converges. Since $\|\mathrm{Ad}\,(u_1 \cdots u_j)\, \varrho_{k_{j+1}}\| \leq 1$ for every $j \in \mathbb{N}$ it follows easily that D is closed. Hence it suffices to show that $\bigcup_{k\in\mathbb{N}} E_k \subseteq D$ to conclude that $D = A$.

We obviously have $0 \in D$, hence consider $0 \neq b \in \bigcup_{k\in\mathbb{N}} E_k$, say $b \in E_l$ for some $l \in \mathbb{N}$. We show that $\left(\mathrm{Ad}\,(u_1 \cdots u_j)\, \varrho_{k_{j+1}}(b)\right)_{j\in\mathbb{N}}$ is a Cauchy sequence. For each $m \in \mathbb{N}_{\geq l}$ we have $b \in E_m$ and hence by (5.2.9):

$$\begin{aligned}
&\|\mathrm{Ad}\,(u_1 \cdots u_{m+1})\, \varrho_{k_{m+2}}(b) - \mathrm{Ad}\,(u_1 \cdots u_m)\, \varrho_{k_{m+1}}(b)\| \\
&= \|\mathrm{Ad}\,(u_{m+1})\, \varrho_{k_{m+2}}(b) - \varrho_{k_{m+1}}(b)\| \\
&= \|b\|\|\mathrm{Ad}\,(u_{m+1})\, \varrho_{k_{m+2}}\left(\tfrac{b}{\|b\|}\right) - \varrho_{k_{m+1}}\left(\tfrac{b}{\|b\|}\right)\| \\
&\leq \|b\|\alpha_{m+1},
\end{aligned}$$

and therefore for all $m, n \in \mathbb{N}_{\geq l}$ with $m < n$

$$\begin{aligned}
\|\mathrm{Ad}\,(u_1 \cdots u_n)\, \varrho_{k_{n+1}}(b) - \mathrm{Ad}\,(u_1 \cdots u_m)\, \varrho_{k_{m+1}}(b)\| &= \|\mathrm{Ad}\,(u_{m+1} \cdots u_n)\, \varrho_{k_{n+1}}(b) - \varrho_{k_{m+1}}(b)\| \\
&\leq \|b\| \sum_{j=m+1}^{n} \alpha_j.
\end{aligned}$$

As $(\alpha_j)_{j\in\mathbb{N}}$ is summable, this shows that $\left(\mathrm{Ad}\,(u_1 \cdots u_j)\, \varrho_{k_{j+1}}(b)\right)_{j\in\mathbb{N}}$ is a Cauchy sequence and hence convergent in $\mathcal{O}_2$. Thus, $D = A$ and we can define

$$\psi \colon A \to \mathcal{O}_2,\ a \mapsto \lim_{j\to\infty} \mathrm{Ad}\,(u_1 \cdots u_j)\, \varrho_{k_{j+1}}(a).$$

As each ϱ_k is a unital, completely positive map it is clear that ψ is linear, unital and self-adjoint. Moreover we have

$$\|\psi(a)\| = \|\lim_{j\to\infty} \mathrm{Ad}\,(u_1 \cdots u_j)\, \varrho_{k_{j+1}}(a)\| = \lim_{j\to\infty} \|\varrho_{k_{j+1}}(a)\| \leq \|a\|$$

for each $a \in A$, i.e. ψ is bounded. Recall from (5.2.4) that we have

$$\|\varrho_{k_j}(ab) - \varrho_{k_j}(a)\varrho_{k_j}(b)\| \leq \delta_j \qquad \text{for all } j \in \mathbb{N},\ a, b \in F_j \cup F_j^*.$$

Let $0 \neq a, b \in \bigcup_{k\in\mathbb{N}} E_k$ and choose $l \in \mathbb{N}$ such that $a, b \in E_l$. Write $\tilde{a} = a/\|a\|$, $\tilde{b} = b/\|b\|$ and choose for each $j \in \mathbb{N}_{\geq l}$ elements $a_j, b_j \in F_j$ with $\|\tilde{a} - a_j\| \leq \delta_j$ and $\|\tilde{b} - b_j\| \leq \delta_j$ to get

$$\begin{aligned}
\|\varrho_{k_j}(ab) - \varrho_{k_j}(a)\varrho_{k_j}(b)\| &= \|a\|\|b\|\|\varrho_{k_j}(\tilde{a}\tilde{b}) - \varrho_{k_j}(\tilde{a})\varrho_{k_j}(\tilde{b})\| \\
&\leq \|a\|\|b\| \Big(\|\varrho_{k_j}(\tilde{a}\tilde{b}) - \varrho_{k_j}(a_j b_j)\| + \|\varrho_{k_j}(a_j b_j) - \varrho_{k_j}(a_j)\varrho_{k_j}(b_j)\| \\
&\qquad + \|\varrho_{k_j}(a_j)\varrho_{k_j}(b_j) - \varrho_{k_j}(\tilde{a})\varrho_{k_j}(\tilde{b})\|\Big) \\
&\leq \|a\|\|b\|\,(2\delta_j + \delta_j + 2\delta_j) = \|a\|\|b\| 5\delta_j
\end{aligned}$$

for each $j \in \mathbb{N}_{\geq l}$ and therefore

$$\begin{aligned}
\|\psi(ab) - \psi(a)\psi(b)\| &= \lim_{j\to\infty} \|\mathrm{Ad}\,(u_1 \cdots u_j)\left(\varrho_{k_{j+1}}(ab) - \varrho_{k_{j+1}}(a)\varrho_{k_{j+1}}(b)\right)\| \\
&= \lim_{j\to\infty} \|\varrho_{k_j}(ab) - \varrho_{k_j}(a)\varrho_{k_j}(b)\| \leq \|a\|\|b\| \lim_{j\to\infty} 5\delta_j = 0.
\end{aligned}$$

By continuity, this implies that $\psi(ab) = \psi(a)\psi(b)$ for all $a, b \in A$.

We have shown now that ψ is a unital *-homomorphism from A into $\mathcal{O}_2$. For injectivity it hence suffices to show that ψ is isometric. Let $j \in \mathbb{N}$ and let b be in the unit sphere of E_j. For each $l \in \mathbb{N}_{\geq j}$ choose $a_l \in F_l$ with $\|b - a_l\| \leq \delta_l$, and use (5.2.5) to see that

$$\|\varrho_{k_l}(b)\| \geq \|\varrho_{k_l}(a_l)\| - \|\varrho_{k_l}(b - a_l)\| \geq \|a_l\| - \delta_l - \delta_l \geq \|b\| - 3\delta_l \qquad \text{for all } l \in \mathbb{N}_{\geq j}.$$

This leads to

$$\|\psi(b)\| = \lim_{l\to\infty} \|\mathrm{Ad}\,(u_1 \cdots u_l)\, \varrho_{k_{l+1}}(b)\| = \lim_{l\to\infty} \|\varrho_{k_l}(b)\| \geq \lim_{l\to\infty} (\|b\| - 3\delta_l) = \|b\|.$$

As ψ is a *-homomorphism we have that $\|\psi(b)\| \leq \|b\|$ for all $b \in A$ anyway, and it follows that $\|\psi(b)\| = \|b\|$ for any b in the unit sphere of $\bigcup_{k\in\mathbb{N}} E_k$. Therefore,

$$\|\psi(b)\| = \|b\| \|\psi\left(\tfrac{b}{\|b\|}\right)\| = \|b\| \qquad \text{for all } b \in \bigcup_{k\in\mathbb{N}} E_k,$$

which by continuity implies that ψ is isometric. This completes the proof. □

5.3 Embedding quasidiagonal C^*-algebras into $\mathcal{O}_2$

In this section it will be shown how Lemma 5.2.1 can be applied to embed quasidiagonal C^*-algebras into $\mathcal{O}_2$. We start with the definition of quasidiagonal C^*-algebras and state a theorem of Voiculescu which yields a class of non-trivial examples.

5.3.1 Quasidiagonal C^*-algebras

Definition 5.3.1. A separable C^*-algebra A is said to be *quasidiagonal* if it admits a faithful representation (H, π) on a separable Hilbert space H such that there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of finite rank projections in $B(H)$ with $P_n H \subseteq P_{n+1} H$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|P_n(x) - x\| = 0 \qquad \text{for all } x \in H,$$

i.e. $(P_n)_{n\in\mathbb{N}}$ converges to id_H in the strong operator topology, and

$$\lim_{n\to\infty} \|P_n \pi(a) - \pi(a) P_n\| = 0 \qquad \text{for all } a \in A.$$

Remarks 5.3.2. (i) A sub-C^*-algebra of a quasidiagonal C^*-algebra is again quasidiagonal: Simply restrict the *-representation π in Definition 5.3.1 to the sub-C^*-algebra and use the same sequence of projections.

(ii) Both the zero C^*-algebra and the complex numbers $\mathbb{C}$ are quasidiagonal.

The following result, which is Theorem 5 in [Vo], enables us to show that certain C^*-algebras are quasidiagonal.

Theorem 5.3.3. Let A and B be C^*-algebras such that B homotopically dominates A, i.e. there are *-homomorphisms $f\colon A \to B$ and $g\colon B \to A$ with $g \circ f \sim_h \mathrm{id}_A$. If B is quasidiagonal then so is A.

Corollary 5.3.4. Let A be a C^*-algebra. Then both the cone CA and its unitization $\widetilde{CA}$ are quasidiagonal.

Proof. As shown in Example 1.1.31 the cone CA is homotopy equivalent to the zero C^*-algebra, and its unitization $\widetilde{CA}$ is homotopy equivalent to the unitization of the zero C^*-algebra, which is $\mathbb{C}$. It now follows from Remark 5.3.2(ii) and Theorem 5.3.3 that CA and $\widetilde{CA}$ are quasidiagonal. □

5.3.2 Embedding quasidiagonal C^*-algebras

Lemma 5.3.5 below is taken from [BK], where it is contained in Proposition 3.1.3 and the remarks preceding it. Slightly modified it will help us to show that for any unital, separable, exact and quasidiagonal C^*-algebra A there is a commutative diagram as in the assumptions of Lemma 5.2.1, which then implies that A embeds unitally into $\mathcal{O}_2$.

Notation. To make this text more readable, the complex $n \times n$-matrices are denoted by M_n instead of $M_n(\mathbb{C})$ in this section.

Lemma 5.3.5. Let A be a unital, separable, quasidiagonal C^*-algebra. Then there exists a sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that there is a unital embedding

$$\varphi\colon A \to \prod_{n\in\mathbb{N}} M_{k_n} \Big/ \sum_{n\in\mathbb{N}} M_{k_n}$$

which has a unital, completely positive lift, i.e. there exists a unital, completely positive map $\varrho\colon A \to \prod_{n\in\mathbb{N}} M_{k_n}$ such that the diagram

$$\begin{tikzcd} & \prod_{n\in\mathbb{N}} M_{k_n} \arrow[d, "\pi"] \\ A \arrow[ur, "\varrho"] \arrow[r, "\varphi"'] & \prod_{n\in\mathbb{N}} M_{k_n} \big/ \sum_{n\in\mathbb{N}} M_{k_n} \end{tikzcd} \tag{5.3.1}$$

commutes, where π is the quotient map.

We show in the following how this statement can be slightly refined:

Lemma 5.3.6. Let A be a unital, separable, quasidiagonal C^*-algebra. Then there exist $(k_n)_{n\in\mathbb{N}}$, φ and ϱ as in Lemma 5.3.5 with the additional property that

$$\lim_{n\to\infty} \|\varrho_n(a)\| = \|a\| \qquad \text{for all } a \in A,$$

where $\varrho_n\colon A \to M_{k_n}$ denotes the nth component map of ϱ for every $n \in \mathbb{N}$.

Proof. Choose $(k_n)_{n\in\mathbb{N}}$, φ and ϱ as in Lemma 5.3.5 and let ϱ_n denote the nth component map of ϱ for each $n \in \mathbb{N}$. For all natural numbers n_1 and n_2 with $n_1 < n_2$ consider the map

$$\varrho_{n_1,n_2}\colon A \to \sum_{n=n_1}^{n_2} M_{k_n},\ a \mapsto (\varrho_{n_1}(a), \dots, \varrho_{n_2}(a)).$$

It follows from Lemma 4.1.8 that each ϱ_{n_1,n_2} is a unital, completely positive map. This implies that

$$\|\varrho_{n_1,n_2}(a)\| \le \|a\| \qquad \text{for all } n_1 < n_2 \in \mathbb{N},\ a \in A,$$

and thus

$$\lim_{\substack{n\to\infty\\ n>n_1}} \|\varrho_{n_1,n}(a)\| \leq \|a\| \qquad \text{for all } n_1 \in \mathbb{N},\ a \in A. \tag{5.3.2}$$

On the other hand, we can use that

$$\|\pi(b)\| = \limsup_{n\to\infty} \|b_n\| \qquad \text{for all } b = (b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} M_{k_n}$$

by Lemma 4.1.5, and that φ is isometric as it is injective to estimate

$$\begin{aligned}\lim_{\substack{n\to\infty\\ n>n_1}} \|\varrho_{n_1,n}(a)\| &= \lim_{\substack{n\to\infty\\ n>n_1}} \max_{n_1\leq j\leq n} \|\varrho_j(a)\| = \sup_{n>n_1} \|\varrho_n(a)\| \geq \inf_{k\in\mathbb{N}} \sup_{n>k} \|\varrho_n(a)\| \\ &= \limsup_{n\to\infty} \|\varrho_n(a)\| = \|(\pi\circ\varrho)(a)\| = \|\varphi(a)\| = \|a\|\end{aligned}$$

for all $n_1 \in \mathbb{N}$ and for all $a \in A$. Together with (5.3.2) this shows that

$$\lim_{\substack{n\to\infty\\ n>n_1}} \|\varrho_{n_1,n}(a)\| = \|a\| \qquad \text{for all } n_1 \in \mathbb{N},\ a \in A.$$

In particular, this shows that for every $n_1 \in \mathbb{N}$, for every $\varepsilon > 0$, and for every finite subset F of A there exists $N \in \mathbb{N}_{>n_1}$ such that

$$\|\varrho_{n_1,N}(a)\| \geq \|a\| - \varepsilon \qquad \text{for all } a \in F. \tag{5.3.3}$$

Let $\{a_1, a_2, \dots\}$ be a countable dense subset of A. By (5.3.3) we can inductively construct a strictly increasing sequence $(n_j)_{j\in\mathbb{N}}$ such that

$$\|\varrho_{n_j,n_{j+1}}(a_i)\| \geq \|a_i\| - \frac{1}{j} \qquad \text{for all } j \in \mathbb{N},\ i \in \mathbb{N}_{\leq j},$$

which implies

$$\lim_{j\to\infty} \|\varrho_{n_j,n_{j+1}}(a_i)\| = \|a_i\| \qquad \text{for all } i \in \mathbb{N}.$$

As each $\varrho_{n_j,n_{j+1}}$ is bounded, this extends to

$$\lim_{j\to\infty} \|\varrho_{n_j,n_{j+1}}(a)\| = \|a\| \qquad \text{for all } a \in A. \tag{5.3.4}$$

For each $j \in \mathbb{N}$ put $\tilde{k}_j = \sum_{n=n_j}^{n_{j+1}} k_n$ and define

$$\iota_j\colon \sum_{n=n_j}^{n_{j+1}} M_{k_n} \longrightarrow M_{\tilde{k}_j},\ \left(a^{n_j}, a^{n_j+1}, \dots, a^{n_{j+1}}\right) \longmapsto \begin{pmatrix} a^{n_j} & 0 & \cdots & 0 \\ 0 & a^{n_j+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{n_{j+1}} \end{pmatrix},$$

where each a^{n_l} denotes a $k_{n_l} \times k_{n_l}$-matrix (with brackets erased when embedded into $M_{\tilde{k}_j}$). Then ι_j is a unital, injective and hence isometric *-homomorphism for each $j \in \mathbb{N}$, and so we can define unital, completely positive maps

$$\tilde{\varrho}_j = \iota_j \circ \varrho_{n_j,n_{j+1}}\colon A \longrightarrow M_{\tilde{k}_j}.$$

As ι_j is isometric for each $j \in \mathbb{N}$, equation (5.3.4) gives

$$\lim_{j\to\infty} \|\tilde{\varrho}_j(a)\| = \|a\| \qquad \text{for all } a \in A. \tag{5.3.5}$$

Let

$$\tilde{\pi}\colon \prod_{j\in\mathbb{N}} M_{\tilde{k}_j} \longrightarrow \prod_{j\in\mathbb{N}} M_{\tilde{k}_j} / \sum_{j\in\mathbb{N}} M_{\tilde{k}_j}$$

be the projection map and set

$$\tilde{\varrho} = (\tilde{\varrho}_j)_{j\in\mathbb{N}} : A \longrightarrow \prod_{j\in\mathbb{N}} M_{\tilde{k}_j}$$

and

$$\tilde{\varphi} = \tilde{\pi} \circ \tilde{\varrho}\colon A \longrightarrow \prod_{j\in\mathbb{N}} M_{\tilde{k}_j} / \sum_{j\in\mathbb{N}} M_{\tilde{k}_j}.$$

Then $\tilde{\varrho}$ is a unital, completely positive map as each $\tilde{\varrho}_j$ is, and thus $\tilde{\varphi}$ also is a unital, completely positive map. To prove the lemma it is left to show that $\tilde{\varphi}$ is an injective *-homomorphism. To see that $\tilde{\varphi}$ is injective it suffices to show that it is isometric, which is easily done: Using Lemma 4.1.5 we have for each $a \in A$ that

$$\|\tilde{\varphi}(a)\| = \|(\tilde{\pi} \circ \tilde{\varrho})(a)\| = \limsup_{j\to\infty} \|\tilde{\varrho}_j(a)\| \overset{(5.3.5)}{=} \|a\|.$$

For multiplicativity take $a, b \in A$ and estimate

$$\begin{aligned}
\|\tilde{\varphi}(ab) - \tilde{\varphi}(a)\tilde{\varphi}(b)\| &= \|\tilde{\pi}(\tilde{\varrho}(ab) - \tilde{\varrho}(a)\tilde{\varrho}(b))\| = \limsup_{j\to\infty} \|\tilde{\varrho}_j(ab) - \tilde{\varrho}_j(a)\tilde{\varrho}_j(b)\| \\
&= \limsup_{j\to\infty} \max_{n_j \le n \le n_{j+1}} \|\varrho_n(ab) - \varrho_n(a)\varrho_n(b)\| \\
&= \inf_{k\in\mathbb{N}} \sup_{j\ge k} \max_{n_j \le n \le n_{j+1}} \|\varrho_n(ab) - \varrho_n(a)\varrho_n(b)\| \\
&= \inf_{k\in\mathbb{N}} \sup_{j\ge n_k} \|\varrho_j(ab) - \varrho_j(a)\varrho_j(b)\| \\
&\le \inf_{k\in\mathbb{N}} \sup_{j\ge k} \|\varrho_j(ab) - \varrho_j(a)\varrho_j(b)\| \\
&= \limsup_{j\to\infty} \|\varrho_j(ab) - \varrho_j(a)\varrho_j(b)\| \\
&= \|\pi(\varrho(ab) - \varrho(a)\varrho(b)\| = \|\varphi(ab) - \varphi(a)\varphi(b)\| = 0,
\end{aligned}$$

which shows that $\tilde{\varphi}$ is multiplicative. Hence, $\big(\tilde{k}_j\big)_{j\in\mathbb{N}}$, $\tilde{\varphi}$ and $\tilde{\varrho}$ satisfy the statement of the lemma. □

Using these results we can now verify the assumptions of Lemma 5.2.1 for every unital, separable, exact and quasidiagonal C^*-algebra.

Lemma 5.3.7. Let A be a unital, separable, exact, quasidiagonal C^*-algebra. Then A embeds unitally into $\mathcal{O}_2$.

Proof. As A is unital, separable and quasidiagonal, Lemma 5.3.6 provides us with a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers and a commutative diagram as in (5.3.1), where φ is a unital, injective *-homomorphism and ϱ is a unital, completely positive lift of φ such that the sequence of component maps $(\varrho_n)_{n\in\mathbb{N}}$ of ϱ satisfies

$$\lim_{n\to\infty}\|\varrho_n(a)\| = \|a\| \qquad \text{for all } a \in A.$$

Let ω be a free ultrafilter on $\mathbb{N}$, then

$$\lim_{\omega}\|\varrho_n(a)\| = \lim_{n\to\infty}\|\varrho_n(a)\| = \|a\| \qquad \text{for all } a \in A \tag{5.3.6}$$

by Lemma 4.3.5(iv). As shown in section 2.3, there is for each $n \in \mathbb{N}$ a unital, injective *-homomorphism $\iota_n\colon M_{k_n} \to \mathcal{O}_2$. We use these to define

$$\iota\colon \prod_{n\in\mathbb{N}} M_{k_n} \to \ell^\infty(\mathcal{O}_2),\ \big((a_{ij}^n)\big)_{n\in\mathbb{N}} \mapsto \big(\iota_n(a_{ij}^n)\big)_{n\in\mathbb{N}},$$

where each (a_{ij}^n) stands for a $k_n \times k_n$-matrix. Then ι is a unital, injective *-homomorphism as each ι_n is. In the following we show that there is a unital *-homomorphism

$$\bar{\iota}\colon \prod_{n\in\mathbb{N}} M_{k_n}\Big/\sum_{n\in\mathbb{N}} M_{k_n} \to (\mathcal{O}_2)_\omega$$

such that the diagram

$$\begin{tikzcd}
 & \prod_{n\in\mathbb{N}} M_{k_n} \arrow[r, "\iota"] \arrow[d, "\pi"] & \ell^\infty(\mathcal{O}_2) \arrow[d, "\pi_\omega"] \\
A \arrow[ur, "\varrho"] \arrow[r, "\varphi"'] & \prod_{n\in\mathbb{N}} M_{k_n}/\sum_{n\in\mathbb{N}} M_{k_n} \arrow[r, dashed, "\bar{\iota}"'] & (\mathcal{O}_2)_\omega
\end{tikzcd} \tag{5.3.7}$$

commutes. By the homomorphism theorem it suffices to show that $\sum_{n\in\mathbb{N}} M_{k_n} \subseteq \ker(\pi_\omega \circ \iota)$. Let thus $a = (a_n)_{n\in\mathbb{N}} \in \sum_{n\in\mathbb{N}} M_{k_n}$. By Lemma 4.1.5 this implies that $\lim_{n\to\infty}\|a_n\| = 0$, and as ι_n is injective and thus isometric for each $n \in \mathbb{N}$, and as ω is a free ultrafilter, we can conclude that

$$\|(\pi_\omega \circ \iota)(a)\| = \lim_{\omega}\|\iota_n(a_n)\| = \lim_{n\to\infty}\|a_n\| = 0,$$

and thus $a \in \ker(\pi_\omega \circ \iota)$. Hence there exists a *-homomorphism

$$\bar{\iota}\colon \prod_{n\in\mathbb{N}} M_{k_n}\Big/\sum_{n\in\mathbb{N}} M_{k_n} \to (\mathcal{O}_2)_\omega$$

that makes the diagram (5.3.7) commutative, and as both ι and π_ω are unital, so is $\bar{\iota}$.

Using Lemma 4.4.4 and that each ι_n is isometric we can show for each $a \in A$ that

$$\|(\bar{\iota} \circ \varphi)(a)\| = \|(\pi_\omega \circ \iota \circ \varrho)(a)\| = \lim_{\omega}\|(\iota_n \circ \varrho_n)(a)\| = \lim_{\omega}\|\varrho_n(a)\| \overset{(5.3.6)}{=} \|a\|,$$

i.e. $\bar{\iota} \circ \varphi$ is isometric and hence injective. Altogether, we have obtained a unital, injective *-homomorphism $\bar{\iota} \circ \varphi\colon A \to (\mathcal{O}_2)_\omega$ with a unital, completely positive lift $\iota \circ \varrho\colon A \to \ell^\infty(\mathcal{O}_2)$, and as A is assumed to be exact it follows from Lemma 5.2.1 that there exists a unital, injective *-homomorphism from A into $\mathcal{O}_2$. $\square$

Chapter 6

Discrete crossed products

In the proof of Kirchberg's Embedding Theorem we will need some results about crossed products by the integers. Those results will be proved here, but the underlying facts about the definition of crossed products and some first results are stated here without proof. For a more detailed treatment of this material the reader may consult [Da], for instance.

Sections 6.1 and 6.2 mainly deal with the definition of crossed products. In Section 6.3 we show how, in some cases, an injective *-homomorphism between C^*-algebras A and B can be extended to an injective *-homomorphism from $A \rtimes_\alpha \mathbb{Z}$ into B. This result will be applied both in the proof of the embedding theorem and in Section 6.4 where we deal with crossed products of minimal tensor products. In Section 6.5 we then prove the existence of a non-zero projection in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$, where τ is the left-shift on $C_0(\mathbb{R})$, and use this to embed a C^*-algebra A into $(C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A$.

6.1 Crossed products by countable, discrete groups

Though the statements involved in the proof of Kirchberg's Embedding Theorem only deal with crossed products by the integers, it seems to make more sense to describe the construction of a crossed product in the case of a general countable, discrete group.

Definition 6.1.1. A *C^*-dynamical system* is a triple (A, G, α) consisting of a C^*-algebra A, a countable, discrete group G and a group homomorphism α from G into the group of *-automorphisms on A, $\mathrm{Aut}(A)$. The group homomorphism α is often called a *(group) action* of G on A, and we denote $\alpha(g)$ by α_g for all $g \in G$.

Definition 6.1.2. Let (A, G, α) be a C^*-dynamical system and let H be a Hilbert space. A pair of maps (π_A, π_G) is called a *covariant representation* of (A, G, α) if the following hold: The map π_A is a *-representation of A on H, the map π_G is a group homomorphism from G into the unitary group $U(H)$ in $B(H)$, and π_A and π_G satisfy the following condition:

$$\pi_G(g)\pi_A(a)\pi_G(g)^* = \pi_A(\alpha_g(a)) \qquad \text{for all } a \in A,\ g \in G.$$

Remark 6.1.3. Each C^*-dynamical system has a covariant representation, see [Da] for its construction.

Proposition 6.1.4. Let (A, G, α) be a C^*-dynamical system and define

$$AG = \Big\{ \sum_{g \in G} a_g g \mid a_g \in A \text{ and } a_g = 0 \text{ for all but finitely many } g \in G \Big\}.$$

By establishing the formal rules

$$gag^{-1} = \alpha_g(a) \qquad \text{for all } a \in A,\ g \in G$$

for multiplication and

$$(ag)^* = \alpha_g^{-1}(a^*)g^{-1} = g^{-1}a^*gg^{-1} = g^{-1}a^* \qquad \text{for all } a \in A,\ g \in G$$

for involution the set AG is turned into a *-algebra.

Remark 6.1.5. To see how multiplication and involution work for general elements of AG, let $a = \sum_{g\in G} a_g g$ and $b = \sum_{g\in G} b_g g$ be elements of AG and compute

$$ab = \sum_{g\in G}\sum_{h\in G} a_g g b_h h = \sum_{g\in G}\sum_{h\in G} a_g g b_h g^{-1} g h = \sum_{g\in G}\sum_{h\in G} a_g \alpha_g(b_h) g h = \sum_{s\in G}\sum_{g\in G} a_g \alpha_g(b_{g^{-1}s}) s$$

and

$$a^* = \sum_{g\in G} g^{-1} a_g^* = \sum_{g\in G} \alpha_g^{-1}(a_g^*) g^{-1} = \sum_{g\in G} \alpha_g(a_{g^{-1}}^*) g.$$

Remark 6.1.6. Every covariant representation (π_A, π_G) of a C^*-dynamical system (A, G, α) on a Hilbert space H yields a *-representation $\pi\colon AG \to B(H)$ defined by

$$\pi\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} \pi_A(a_g)\pi_G(g) \qquad \text{for all } \sum_{g\in G} a_g g \in AG.$$

Proposition/Definition 6.1.7. Let (A, G, α) be a C^*-dynamical system. For each $a \in AG$ the supremum in

$$\|a\| = \sup\big\{\|\pi(a)\| \;\big|\; \pi\colon AG \to B(H) \text{ is a } ^*\text{-representation of } AG \text{ on } H\big\}.$$

exists in $\mathbb{R}$, and this defines a C^*-norm on AG. The *crossed product* $A \rtimes_\alpha G$ of A by G is defined to be the completion of AG with respect to this norm.

Remark 6.1.8. One can define an embedding $\iota\colon A \to A \rtimes_\alpha G,\ a \mapsto ae$ where e is the neutral element in G. With this in mind we shall treat A as a sub-C^*-algebra of $A \rtimes_\alpha G$ in what follows.

The crossed product $A \rtimes_\alpha G$ has the following universal property:

Remark 6.1.9. Let (A, G, α) be a C^*-dynamical system. If (π_A, π_G) is any covariant representation of (A, G, α), then there is a *-representation π of $A \rtimes_\alpha G$ on C^* $((\pi_A(A), \pi_G(G))$ defined by

$$\pi\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} \pi_A(a_g)\pi_G(g) \qquad \text{for all } \sum_{g\in G} a_g g \in AG.$$

The existence of such a *-representation of AG is just Remark 6.1.6, and the definition of $A \rtimes_\alpha G$ and its norm implies that every *-representation of AG can be extended to a *-representation of $A \times_\alpha G$.

Remark 6.1.10. Let (A, G, α) be a C^*-dynamical system. Recall from Remark 1.1.16 that $A \rtimes_\alpha G$ is contained as an essential ideal in its multiplier algebra $\mathcal{M}(A \rtimes_\alpha G)$. There exists a group homomorphism

$$U\colon G \to \mathcal{M}(A \rtimes_\alpha G),\ g \mapsto u_g$$

such that

$$ag = au_g \qquad \text{for all } a \in A,\ g \in G.$$

If A is unital, then so is $A \rtimes_\alpha G$ with unit $1_A e$, and U can be defined by $U(g) = 1_A g$ for all $g \in G$.

6.2 Crossed products by $\mathbb{Z}$

Definition 6.2.1. Let A be a C^*-algebra, let α be a *-automorphism on A and consider the group homomorphism $\widetilde{\alpha}\colon \mathbb{Z} \to \mathrm{Aut}(A),\ n \mapsto \alpha^n$. The crossed product $A \rtimes_{\widetilde{\alpha}} \mathbb{Z}$ is called the *crossed product of* A *by* $\mathbb{Z}$ and is denoted by $A \rtimes_\alpha \mathbb{Z}$. Similarly, we write the dynamical system $(A, \mathbb{Z}, \widetilde{\alpha})$ as $(A, \mathbb{Z}, \alpha)$ and call α an action on A.

Remark 6.2.2. Let A and α be as in Definition 6.2.1. By Remark 6.1.10 there exists a group homomorphism $U\colon \mathbb{Z} \to \mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ with $an = aU(n)$ for all $n \in \mathbb{Z}$. Set $u = U(1)$, then u is a unitary in $\mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ with

$$an = aU(n) = au^n \qquad \text{for all } a \in A,\ n \in \mathbb{Z}.$$

It is very common to write the elements of $A\mathbb{Z}$ in the form $\sum_{n\in\mathbb{Z}} a_n u^n$ instead of $\sum_{n\in\mathbb{Z}} a_n n$, and we shall do so as well. In this notation, the formal rules for multiplication and involution become

$$uau^* = \alpha(a), \qquad (au)^* = u^*a \qquad \text{for all } a \in A.$$

We refer to u as the *unitary element implementing the action* α.

We now use the universal property of the crossed product to deduce a statement about crossed products of isomorphic C^*-algebras:

Lemma 6.2.3. Let A and B be isomorphic C^*-algebras with a *-isomorphism $\varphi\colon A \to B$, let α be a *-automorphism on A and let β be a *-automorphism on B. If

$$\beta \circ \varphi = \varphi \circ \alpha, \tag{6.2.1}$$

i.e. if the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

commutes, then

$$A \rtimes_\alpha \mathbb{Z} \;\cong\; B \rtimes_\beta \mathbb{Z}.$$

Proof. Let u be a unitary in $\mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ implementing the action α and let v be a unitary in $\mathcal{M}(B \rtimes_\beta \mathbb{Z})$ implementing the action β. Consider the map $\pi\colon \mathbb{Z} \to \mathcal{M}(B \rtimes_\beta \mathbb{Z}),\ n \mapsto v^n$. Then π is a group homomorphism and the pair (φ, π) defines a covariant representation of $A \rtimes_\alpha \mathbb{Z}$ on $\mathcal{M}(B \rtimes_\beta \mathbb{Z})$, because

$$\pi(n)\varphi(a)\pi(n)^* = v^n\varphi(a)(v^*)^n = \beta^n(\varphi(a)) \overset{(6.2.1)}{=} \varphi(\alpha^n(a)) \qquad \text{for all } a \in A.$$

By the universal property (Remark 6.1.9) of $A \rtimes_\alpha \mathbb{Z}$ this gives rise to a *-homomorphism $\psi\colon A \rtimes_\alpha \mathbb{Z} \to \mathcal{M}(B \rtimes_\beta \mathbb{Z})$ defined by

$$\psi\Big(\sum_{n\in\mathbb{Z}} a_n u^n\Big) = \sum_{n\in\mathbb{Z}} \varphi(a_n)v^n \qquad \text{for all } \sum_{n\in\mathbb{Z}} a_n u^n \in A\mathbb{Z}.$$

Note that $\psi(A\mathbb{Z}) \subseteq B\mathbb{Z}$ and therefore

$$\psi(A \rtimes_\alpha \mathbb{Z}) = \psi\left(\overline{A\mathbb{Z}}\right) \subseteq \overline{\psi(A\mathbb{Z})} \subseteq \overline{B\mathbb{Z}} = B \rtimes_\beta \mathbb{Z},$$

i.e. ψ is a *-homomorphism from $A \rtimes_\alpha \mathbb{Z}$ into $B \rtimes_\beta \mathbb{Z}$. As $\beta \circ \varphi = \varphi \circ \alpha$ clearly implies $\alpha \circ \varphi^{-1} = \varphi^{-1} \circ \beta$ we can repeat this reasoning with A and B interchanged to obtain a *-homomorphism $\eta\colon B \rtimes_\beta \mathbb{Z} \to A \rtimes_\alpha \mathbb{Z}$ defined by

$$\eta\Big(\sum_{n\in\mathbb{Z}} b_n v^n\Big) = \sum_{n\in\mathbb{Z}} \varphi^{-1}(b_n)u^n \qquad \text{for all } \sum_{n\in\mathbb{Z}} b_n v^n \in B\mathbb{Z}.$$

Clearly, $\eta = \psi^{-1}$, and thus ψ is a *-isomorphism between $A \rtimes_\alpha \mathbb{Z}$ and $B \rtimes_\beta \mathbb{Z}$. $\square$

Example 6.2.4. Let A be a C^*-algebra. For each $f \in C_0(\mathbb{R})$ and for each $a \in A$ define a map in $C_0(\mathbb{R}, A)$ by

$$fa\colon \mathbb{R} \to A,\ t \mapsto f(t)a,$$

and use this to define a map

$$\varphi\colon C_0(\mathbb{R}) \otimes A \to C_0(\mathbb{R}, A)$$

by

$$\varphi(f \otimes a) = fa \qquad \text{for all } f \in C_0(\mathbb{R}),\ a \in A.$$

Then φ is a *-isomorphism, see Theorem 6.4.17 in [Mu]. Let $\tau\colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be the so-called *left-shift* on $C_0(\mathbb{R})$, defined by $\tau(f)(t) = f(t+1)$ for all $f \in C_0(\mathbb{R})$ and all $t \in \mathbb{R}$. Let τ_A be defined correspondingly on $C_0(\mathbb{R}, A)$. Then τ and τ_A are *-automorphisms on $C_0(\mathbb{R})$ and $C_0(\mathbb{R}, A)$, respectively, and as $C_0(\mathbb{R})$ is nuclear, $\tau \otimes \mathrm{id}_A$ is a *-automorphism on $C_0(\mathbb{R}) \otimes A$ by Lemma 1.5.2. For each elementary tensor $f \otimes a$ in $C_0(R) \otimes A$ we can calculate

$$\begin{aligned}(\tau_A \circ \varphi)(f \otimes a) &= \tau_A(fa) = f(\cdot + 1)a = \varphi(f(\cdot + 1) \otimes a) \\ &= \varphi(\tau(f) \otimes a) = (\varphi \circ (\tau \otimes \mathrm{id}_A))(f \otimes a),\end{aligned}$$

and as both $\tau_A \circ \varphi$ and $\varphi \circ (\tau \otimes \mathrm{id}_A)$ are *-homomorphisms it follows by linearity and continuity that $\tau_A \circ \varphi = \varphi \circ (\tau \otimes \mathrm{id}_A)$. Thus, by Lemma 6.2.3,

$$(C_0(\mathbb{R}) \otimes A) \rtimes_{\tau \otimes \mathrm{id}_A} \mathbb{Z} \;\cong\; C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z}.$$

6.3 Extending injective *-homomorphisms to crossed products

In the following it will be shown how, on certain assumptions, a *-homomorphism between C^*-algebras A and B can be extended to a *-homomorphism from $A \rtimes_\alpha \mathbb{Z}$ into B. For our purposes it will also be necessary to extend injective *-homomorphisms without loosing injectivity, and for this we need the notion of faithful expectations.

Definition 6.3.1. Let A be a C^*-algebra and let B be a sub-C^*-algebra of A. An *expectation* of A onto B is a positive linear map $E\colon A \to B$ which is surjective and satisfies $E^2 = E$. An expectation E is said to be *faithful* if

$$E(a) = 0 \quad \Longrightarrow \quad a = 0 \qquad \text{for all positive } a \in A.$$

The following remark is extracted from Theorem VIII.2.1 in [Da] and its proof.

Remark 6.3.2. Let A be a C^*-algebra, let α be a *-automorphism on A and let u be a unitary in $\mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ which implements the action α. In the following we exploit the fact that u can be replaced by tu for any $t \in \mathbb{T}$: For each $t \in \mathbb{T}$ define a map $\varrho_t\colon A \rtimes_\alpha \mathbb{Z} \to A \rtimes_\alpha \mathbb{Z}$ by

$$\varrho_t\Big(\sum_{n\in\mathbb{Z}} a_n u^n\Big) = \sum_{n\in\mathbb{Z}} t^n a_n u^n \qquad \text{for all } \sum_{n\in\mathbb{Z}} a_n u^n \in A\mathbb{Z}.$$

Then each ϱ_t is a *-automorphism on $A \rtimes_\alpha \mathbb{Z}$, and the map

$$\varrho\colon \mathbb{T} \to \mathrm{Aut}(A \rtimes_\alpha \mathbb{Z}),\ t \mapsto \varrho_t$$

is a group homomorphism. Moreover, ϱ is *point-wise continuous*, i.e. the map

$$\mathbb{T} \to A \rtimes_\alpha \mathbb{Z},\ t \mapsto \varrho_t(x)$$

is continuous for each $x \in A \rtimes_\alpha \mathbb{Z}$. Hence we can define a map E on $A \rtimes_\alpha \mathbb{Z}$ by

$$E(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \varrho_t(x)dt \qquad \text{for all } x \in A \rtimes_\alpha \mathbb{Z}. \tag{6.3.1}$$

Notice that E satisfies

$$E\big(\sum_{n\in\mathbb{Z}} a_n u^n\big) = a_0 \qquad \text{for all } \sum_{n\in\mathbb{Z}} a_n u^n \in A\mathbb{Z},$$

because

$$\int_{\mathbb{T}} t^n dt = \begin{cases} 2\pi, \text{ if } n = 0 \\ 0\ , \text{ else} \end{cases}. \tag{6.3.2}$$

It is shown in the reference mentioned above that E defines a faithful expectation of $A \rtimes_\alpha \mathbb{Z}$ onto A.

We can now prove the following extension result:

Lemma 6.3.3. Let A and B be C^*-algebras, let α be a *-automorphism on A, let u be a unitary in $\mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ which implements the action α and let $\varphi\colon A \to B$ be a *-homomorphism. Suppose there is a unitary $v \in \mathcal{M}(B)$ such that

$$v\varphi(a)v^* = \varphi(\alpha(a)) \qquad \text{for all } a \in A. \tag{6.3.3}$$

Then there is a *-homomorphism $\psi\colon A \rtimes_\alpha \mathbb{Z} \to B$ defined by

$$\psi\Big(\sum_{n\in\mathbb{Z}} a_n u^n\Big) = \sum_{n\in\mathbb{Z}} \varphi(a_n) v^n \qquad \text{for all } \sum_{n\in\mathbb{Z}} a_n u^n \in A\mathbb{Z}. \tag{6.3.4}$$

If, additionally, there is a group homomorphism $\gamma\colon \mathbb{T} \to \mathrm{Aut}(B),\ t \mapsto \gamma_t$ such that

(i) the map $\mathbb{T} \to B,\ t \mapsto \gamma_t(b)$ is continuous for each $b \in B$,

(ii) $\gamma_t(\varphi(a)v^n) = t^n \varphi(a) v^n$ for all $t \in \mathbb{T},\ a \in A$ and all $n \in \mathbb{Z}$,

then ψ is injective if φ is.

Proof. Assumption (6.3.3) enables us to define a covariant representation (φ, π) of $(A, \mathbb{Z}, \alpha)$ on $\mathcal{M}(B)$ where π is defined by $\pi(u) = v$. By the universal property of crossed products this yields a *-homomorphism $\psi\colon A \rtimes_\alpha \mathbb{Z} \to \mathcal{M}(B)$ defined by (6.3.4) above. To see that $\mathrm{Im}(\psi) \subseteq B$ note that $\psi(A\mathbb{Z}) \subseteq B$ as B is an ideal in $\mathcal{M}(B)$, and therefore

$$\psi(A \rtimes_\alpha \mathbb{Z}) = \psi\left(\overline{A\mathbb{Z}}\right) \subseteq \overline{\psi(A\mathbb{Z})} \subseteq B.$$

Thus, we have obtained the desired *-homomorphism $\psi\colon A \rtimes_\alpha \mathbb{Z} \to B$.

Assume now that φ is injective and that there is a group homomorphism $\gamma\colon \mathbb{T} \to \mathrm{Aut}(B)$, $t \mapsto \gamma_t$ satisfying the assumptions in (i) and (ii). We use the maps γ_t to define a map on B which corresponds to the faithful expectation E on $A \rtimes_\alpha \mathbb{Z}$ as given in (6.3.1): Define

$$F\colon B \to B,\ b \mapsto \frac{1}{2\pi} \int_{\mathbb{T}} \gamma_t(b) dt.$$

We show that the diagram

$$\begin{tikzcd} A \rtimes_\alpha \mathbb{Z} \arrow[r, "\psi"] \arrow[d, "E"'] & B \arrow[d, "F"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

commutes: Let $\sum_{n\in\mathbb{Z}} a_n u^n \in A\mathbb{Z}$ and use assumption (ii) and $E\left(\sum_{n\in\mathbb{Z}} a_n u^n\right) = a_0$ to compute

$$\begin{aligned}
(F \circ \psi)\big(\sum_{n\in\mathbb{Z}} a_n u^n\big) &= F\big(\sum_{n\in\mathbb{Z}} \varphi(a_n) v^n\big) = \sum_{n\in\mathbb{Z}} F(\varphi(a_n) v^n) \\
&= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}} \gamma_t\left(\varphi(a_n) v^n\right) dt = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}} t^n \varphi(a_n) v^n dt \\
&= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \varphi(a_n) v^n \int_{\mathbb{T}} t^n dt \\
&\overset{(6.3.2)}{=} \varphi(a_0) \\
&= (\varphi \circ E)\big(\sum_{n\in\mathbb{Z}} a_n u^n\big),
\end{aligned}$$

which, as both $F \circ \psi$ and $\varphi \circ E$ are norm-decreasing, implies $F \circ \psi = \varphi \circ E$. Let now $x \in A \rtimes_\alpha \mathbb{Z}$ with $\psi(x) = 0$. Then also $\psi(x^* x) = 0$ and $(F \circ \psi)(x^* x) = 0$, and hence $(\varphi \circ E)(x^* x) = 0$. As φ is injective, this implies $E(x^* x) = 0$, and as E is a faithful expectation this forces x to be zero, which completes the proof. $\square$

Corollary 6.3.4. Let A and B be C^*-algebras such that A embeds into B. Let $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$ be *-automorphisms with $\beta|_A = \alpha$. Then there exists an injective *-homomorphism from $A \rtimes_\alpha \mathbb{Z}$ into $B \rtimes_\beta \mathbb{Z}$.

Proof. We apply Lemma 6.3.3 to the embedding $\iota\colon A \to B \rtimes_\beta \mathbb{Z}$. Let v be a unitary in $\mathcal{M}(B \rtimes_\beta \mathbb{Z})$ which implements the action β, i.e. $vbv^* = \beta(b)$ for all $b \in B$. It follows that

$$v\iota(a)v^* = \beta(\iota(a)) = \alpha(a) = \iota(\alpha(a)),$$

and hence Lemma 6.3.3 yields a *-homomorphism $\varphi\colon A \rtimes_\alpha \mathbb{Z} \to B \rtimes_\beta \mathbb{Z}$ which extends ι. By Remark 6.3.2 there exists a group homomorphism $\gamma\colon \mathbb{T} \to \mathrm{Aut}(B \rtimes_\beta \mathbb{Z}),\ t \mapsto \gamma_t$ such that the map $t \mapsto \gamma_t(x)$ is continuous for each $x \in B \rtimes_\beta \mathbb{Z}$ and such that

$$\gamma_t(bv^n) = t^n bv^n \qquad \text{for all } t \in \mathbb{T},\ b \in B,\ n \in \mathbb{Z}.$$

In particular,

$$\gamma_t\left(\iota(a)v^n\right) = t^n \iota(a)v^n \qquad \text{for all } t \in \mathbb{T},\ a \in A,\ n \in \mathbb{Z},$$

and it follows from Lemma 6.3.3 that φ is injective. □

When we apply Lemma 6.3.3 in the proof of Kirchberg's Embedding Theorem we shall need the following property of $C(\mathbb{T})$:

Lemma 6.3.5. There exists a group homomorphism

$$\varrho\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T})),\ t \mapsto \varrho_t$$

such that the map

$$\mathbb{T} \to C(\mathbb{T}),\ t \mapsto \varrho_t(f)$$

is continuous for each $f \in C(\mathbb{T})$.

Proof. By the Stone-Weierstraß Theorem the C^*-algebra $C(\mathbb{T})$ is generated by the embedding of $\mathbb{T}$ into $\mathbb{C}$, which we shall denote by z, and its adjoint $z^*\colon \mathbb{T} \to \mathbb{C},\ t \mapsto \bar{t}$. We can set $\varrho_t(z) = tz$ and $\varrho_t(z^*) = \bar{t}z^*$ for each $t \in [0,1]$ and extend this to a *-homomorphism $\varrho_t\colon C(\mathbb{T}) \to C(\mathbb{T})$. For all $s, t \in [0,1]$, we have $\varrho_{st} = \varrho_s \circ \varrho_t$, and hence the inverse of ϱ_t is given by $\varrho_{1/t} = \varrho_{\bar{t}}$. This shows that

$$\varrho\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T})),\ t \mapsto \varrho_t$$

is well-defined and a group homomorphism. For pointwise continuity on $C(\mathbb{T})$ it is by Lemma 1.1.27 sufficient that $t \mapsto \varrho_t(z) = tz$ and $t \mapsto \varrho_t(z^*) = \bar{t}z^*$ are continuous, which is clearly the case. □

6.4 Crossed products and tensor products

In this section it will be shown that the crossed product of a minimal tensor product $A \otimes_{\min} B$ by $\mathbb{Z}$ equals the minimal tensor product of $A \rtimes_\alpha \mathbb{Z}$ and B. In the proof we shall need a corollary to the following statement about minimal tensor products of essential ideals:

Lemma 6.4.1. Let A and B be C^*-algebras and let I and J be essential ideals in A and B, respectively. Then $I \otimes_{\min} J$ is an essential ideal in $A \otimes_{\min} B$.

Proof. It is easy to show that $I \otimes_{\min} J$ is an ideal in $A \otimes_{\min} B$. By Lemma 1.1.18 there exist faithful representations (H_A, π_A) of A and (H_B, π_B) of B with

$$[\pi_A(I)H_A] = H_A, \qquad [\pi_B(I)H_B] = H_B. \tag{6.4.1}$$

Lemma 1.5.1 on representations of minimal tensor products yields a faithful *-representation

$$\pi_A \otimes \pi_B \colon A \otimes_{\min} B \to B(H_A \otimes H_B)$$

such that

$$(\pi_A \otimes \pi_B)(a \otimes b) = \pi_A(a) \otimes \pi_B(b) \qquad \text{for all } a \in A,\ b \in B.$$

Again by Lemma 1.1.18, $I \otimes_{\min} J$ is essential in $A \otimes_{\min} B$ if

$$[(\pi_A \otimes \pi_B)(I \otimes_{\min} J)(H_A \otimes H_B)] = H_A \otimes H_B.$$

As the left hand side is a closed linear subspace of $H_A \otimes H_B$ it suffices to show that any elementary tensor $x \otimes y \in H_A \otimes H_B$ is contained in $[(\pi_A \otimes \pi_B)(I \otimes_{\min} J)(H_A \otimes H_B)]$. Let thus $x \otimes y \in H_A \otimes H_B$ and let $\varepsilon > 0$. Choose $\delta > 0$ such that $(\|x\| + \|y\| + \delta)\delta < \varepsilon$. By (6.4.1) there exist $n \in \mathbb{N}$ and elements $a_1, \dots, a_n \in I,\ b_1, \dots b_n \in J$ and $\xi_1, \dots, \xi_n \in H_A,\ \eta_1, \dots \eta_n \in H_B$ such that

$$\|x - \sum_{i=1}^n \pi_A(a_i)\xi_i\| < \delta, \qquad \|y - \sum_{i=1}^n \pi_B(b_i)\eta_i\| < \delta.$$

Then

$$\sum_{i=1}^n \pi_A(a_i)\xi_i \otimes \sum_{j=1}^n \pi_B(b_j)\eta_j = \sum_{i,j=1}^n (\pi_A \otimes \pi_B)(a_i \otimes b_j)(\xi_i \otimes \eta_j)$$

is contained in the linear span of $(\pi_A \otimes \pi_B)(I \otimes_{\min} J)(H_A \otimes H_B)$, and

$$\begin{aligned}
&\|x \otimes y - \sum_{i=1}^n \pi_A(a_i)\xi_i \otimes \sum_{j=1}^n \pi_B(b_j)\eta_j\| \\
&\leq \|x\|\|y - \sum_{j=1}^n \pi_B(b_j)\eta_j\| + \|x - \sum_{i=1}^n \pi_A(a_i)\xi_i\|\|\sum_{j=1}^n \pi_B(b_j)\eta_j\| \\
&< \|x\|\delta + \delta(\|y\| + \delta) < \varepsilon,
\end{aligned}$$

which completes the proof. □

Corollary 6.4.2. Let A and B be C^*-algebras. Then there exists a unital *-homomorphism from $\mathcal{M}(A) \otimes_{\min} \mathcal{M}(B)$ into $\mathcal{M}(A \otimes_{\min} B)$ that extends the inclusion of $A \otimes_{\min} B$ into $\mathcal{M}(A \otimes_{\min} B)$.

Proof. By Remark 1.1.16 we know that $A \overset{\text{ess}}{\lhd} \mathcal{M}(A)$ and $B \overset{\text{ess}}{\lhd} \mathcal{M}(B)$, which by Lemma 6.4.1 implies that

$$A \otimes_{\min} B \overset{\text{ess}}{\lhd} \mathcal{M}(A) \otimes_{\min} \mathcal{M}(B).$$

Lemma 1.1.17 yields a unital embedding of $\mathcal{M}(A) \otimes_{\min} \mathcal{M}(B)$ into $\mathcal{M}(A \otimes_{\min} B)$ as required. □

Lemma 6.4.3. Let A and B be C^*-algebras and let α be a *-automorphism on A. Then

$$(A \otimes_{\min} B) \rtimes_{\alpha\otimes \mathrm{id}_B} \mathbb{Z} \;\cong\; (A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B.$$

Proof. Notice that $\alpha \otimes \mathrm{id}_B$ is a *-automorphism on $A \otimes_{\min} B$ by Lemma 1.5.2. Let ι denote the embedding of A into the crossed product $A \rtimes_\alpha \mathbb{Z}$ and put

$$\varphi = \iota \otimes \mathrm{id}_B \colon A \otimes_{\min} B \to (A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B.$$

Again by Lemma 1.5.2, φ is an injective *-homomorphism. Let u be a unitary in $\mathcal{M}(A \rtimes_\alpha \mathbb{Z})$ which implements the action α on A, and let $v' = u \otimes 1_{\mathcal{M}(B)}$. Then v' is a unitary element in $\mathcal{M}(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} \mathcal{M}(B)$, and, via the unital embedding

$$\kappa \colon \mathcal{M}(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} \mathcal{M}(B) \longrightarrow \mathcal{M}((A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B)$$

from Corollary 6.4.2, the element $v = \kappa(v')$ is a unitary in $\mathcal{M}\left((A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B\right)$. Since κ extends the inclusion of $(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B$ into $\mathcal{M}((A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B)$, we can for each elementary tensor $a \otimes b \in A \otimes_{\min} B$ calculate

$$\begin{aligned} v\varphi(a\otimes b)v^* &= \kappa\left((u \otimes 1_{\mathcal{M}(B)})(a \otimes b)(u^* \otimes 1_{\mathcal{M}(B)})\right) = \kappa\left(uau^* \otimes b\right) = uau^* \otimes b \\ &= \alpha(a) \otimes b = (\iota \otimes \mathrm{id}_B)(\alpha(a) \otimes b) = \varphi(\alpha(a) \otimes b) = \varphi\left((\alpha \otimes \mathrm{id}_B)(a \otimes b)\right), \end{aligned}$$

which extends to

$$v\varphi(x)v^* = \varphi\left((\alpha \otimes \mathrm{id}_B)(x)\right) \qquad \text{for all } x \in A \otimes_{\min} B.$$

Let w be a unitary element in $\mathcal{M}(A \otimes_{\min} B)$ which implements the action $\alpha \otimes \mathrm{id}_B$. By Lemma 6.3.3 there exists a *-homomorphism

$$\psi \colon (A \otimes_{\min} B) \rtimes_{\alpha\otimes\mathrm{id}_B} \mathbb{Z} \to (A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B,$$

given by

$$\psi\left((a \otimes b)w^n\right) = \varphi(a \otimes b)v^n \qquad \text{for all } a \in A,\ b \in B,\ n \in \mathbb{Z}.$$

We show next that ψ is injective. By Remark 6.3.2 there exists a group homomorphism $\varrho \colon \mathbb{T} \to \mathrm{Aut}(A \rtimes_\alpha \mathbb{Z}),\ t \mapsto \varrho_t$ such that the map $\mathbb{T} \to A \rtimes_\alpha \mathbb{Z},\ t \mapsto \varrho_t(x)$ is continuous for each $x \in A \rtimes_\alpha \mathbb{Z}$ and

$$\varrho_t(au^n) = t^n au^n \qquad \text{for all } t \in \mathbb{T},\ a \in A,\ n \in \mathbb{Z}.$$

Define $\gamma_t = \varrho_t \otimes \mathrm{id}_B$ for each $t \in \mathbb{T}$. Then each γ_t is a *-automorphism as ϱ_t and id_B are, and

$$\gamma \colon \mathbb{T} \to \mathrm{Aut}((A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B),\ t \mapsto \gamma_t$$

is a group homomorphism. To check that the map $t \mapsto \gamma_t(z)$ is continuous for each z in $(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B$ it suffices to show this for any elementary tensor $z = a \otimes b$, because the set

$$\left\{z \in (A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B \;\middle|\; t \mapsto \gamma_t(z) \ \text{ is continuous}\right\}$$

is a sub-C^*-algebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B$ by Lemma 1.1.27. Thus, let $a \in A \rtimes_\alpha \mathbb{Z}$ and $b \in B$. Then for all $t, s \in \mathbb{T}$ we have

$$\|\gamma_t(a \otimes b) - \gamma_s(a \otimes b)\| = \|\varrho_t(a) \otimes b - \varrho_s(a) \otimes b\| = \|\varrho_t(a) - \varrho_s(a)\|\|b\|,$$

and hence continuity of $t \mapsto \gamma_t(a \otimes b)$ follows from continuity of $t \mapsto \varrho_t(a)$. Moreover, we have

$$\varphi(a \otimes b)v^n = \kappa\left(\varphi(a \otimes b)(u \otimes 1_{\mathcal{M}(B)})^n\right) = \kappa\left(au^n \otimes b\right) = au^n \otimes b, \tag{6.4.2}$$

and thus

$$\gamma_t\left(\varphi(a \otimes b)v^n\right) = \left(\varrho_t \otimes \mathrm{id}_B\right)\left(au^n \otimes b\right) = t^n au^n \otimes b = t^n(au^n \otimes b) = t^n \varphi(a \otimes b)v^n$$

for all $t \in \mathbb{T}$, for all elementary tensors $a \otimes b \in A \otimes_{\min} B$ and for all $n \in \mathbb{Z}$. As $\gamma_t(\varphi(\cdot)v^n)$ and $t^n\varphi(\cdot)v^n$ are norm-decreasing for all $t \in \mathbb{T}$ and all $n \in \mathbb{Z}$, this extends to

$$\gamma_t\left(\varphi(x)v^n\right) = t^n \varphi(x)v^n \qquad \text{for all } t \in \mathbb{T},\ x \in A \otimes_{\min} B,\ n \in \mathbb{Z}.$$

Now it follows from Lemma 6.3.3 that ψ is injective, and we proceed to show that ψ is surjective. For this it suffices to show that

$$\left\{au^n \otimes b \;\middle|\; a \in A,\ n \in \mathbb{Z},\ b \in B\right\} \subseteq \mathrm{Im}(\psi),$$

because the set on the left hand side generates $(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B$ and $\mathrm{Im}(\psi)$ is a closed sub-C^*-algebra of $(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B$ as ψ is a *-homomorphism. Let now $a \in A,\ n \in \mathbb{Z}$ and $b \in B$ and calculate

$$au^n \otimes b \overset{(6.4.2)}{=} \varphi(a \otimes b)v^n = \psi((a \otimes b)w^n)$$

to see that $au^n \otimes b$ is an element of $\mathrm{Im}(\psi)$. Altogether we have shown that ψ is a *-isomorphism between $(A \otimes_{\min} B) \rtimes_{\alpha \otimes \mathrm{id}_B} \mathbb{Z}$ and $(A \rtimes_\alpha \mathbb{Z}) \otimes_{\min} B$ which completes the proof. □

6.5 A non-zero projection in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$

In this section we perform the first step of the proof of Kirchberg's Embedding Theorem, namely the embedding of a C^*-algebra A into $(C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A$, where τ is the left shift on $C_0(\mathbb{R})$. To obtain this embedding we prove the existence of a non-zero projection in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$.

Lemma 6.5.1. Let A be a C^*-algebra and let $\tau\colon C_0(\mathbb{R}) \to C_0(\mathbb{R}),\ f \mapsto f(\cdot + 1)$ be the left-shift on $C_0(\mathbb{R})$. Then there exists a non-zero projection in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$.

Proof. We explicitly construct a projection p in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$ as follows: Define two functions f and g in $C_0(\mathbb{R})$ by

$$f\colon \mathbb{R} \to \mathbb{R},\ t \mapsto \begin{cases} 1+t, & t \in [-1,0] \\ 1-t, & t \in [0,1] \\ \quad 0\ \ , & \text{else} \end{cases}$$

and

$$g\colon \mathbb{R} \to \mathbb{R},\ t \mapsto \begin{cases} \sqrt{f(t) - f(t)^2}, & t \in [-1,0] \\ \qquad 0 \qquad , & \text{else} \end{cases}.$$

Then f and g are non-zero positive elements in $C_0(\mathbb{R})$. Let u be the unitary in $\mathcal{M}(C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z})$ with $uhu^* = \tau(h)$ for all $h \in C_0(\mathbb{R})$ and put $p = gu + f + u^*g$. Let $E\colon C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z} \to C_0(\mathbb{R})$ be a faithful expectation as in Remark 6.3.2. Then $E(p) = f \neq 0$, which implies that p is a non-zero element in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$. Using that f and g are self-adjoint one can easily compute that

$$p^* = u^* g + f^* + g^* u = u^* g + f + gu = p,$$

i.e. p is self-adjoint. Before checking that $p^2 = p$ it is helpful to have a closer look at some expressions which are going to appear in the calculation: We have $\tau(g)|_{[-1,0]} = 0$ and therefore $g\tau(g) = 0$, and we have

$$\tau(f)(t) = \begin{cases} 2+t, & t \in [-2,-1] \\ -t\ , & t \in [-1,0] \\ 0\ , & \text{else} \end{cases}$$

and hence $(\tau(f) + f)|_{[-1,0]} \equiv 1$. Note that τ^{-1} is given by $\tau^{-1}(h)(t) = h(t-1)$ for all $h \in C_0(\mathbb{R})$ and all $t \in \mathbb{R}$. We therefore obtain

$$\begin{aligned} \tau^{-1}(g^2)(t) &= g^2(t-1) \\ &= \begin{cases} f(t-1) - f(t-1)^2, & t \in [0,1] \\ 0 & , \text{ else} \end{cases} = \begin{cases} f(t) - f(t)^2, & t \in [0,1] \\ 0 & , \text{ else} \end{cases} \end{aligned}$$

where it is used that

$$f(t-1) - f(t-1)^2 = t - t^2 = (1-t) - (1-t)^2 = f(t) - f(t)^2 \qquad \text{for all } t \in [0,1].$$

Combined with

$$g^2(t) \begin{cases} f(t) - f(t)^2, & t \in [-1,0] \\ 0 & , \text{ else} \end{cases}$$

this yields that $g^2 + f^2 + \tau^{-1}(g^2) = f$. Now we can calculate

$$\begin{aligned} p^2 &= (gu + f + u^* g)(gu + f + u^* g) \\ &= gugu^* u^2 + gufu^* u + guu^* guu^* + fgu + f^2 + u^* ufu^* g + u^* ggu + u^* gf + (u^*)^2 ugu^* g \\ &= g\tau(g)u^2 + g\tau(f)u + g^2 + fgu + f^2 + u^*\tau(f)g + \tau^{-1}(g^2) + u^* gf + (u^*)^2\tau(g)g \\ &= (\tau(f) + f)\, gu + g^2 + f^2 + \tau^{-1}(g^2) + u^* g\, (\tau(f) + f) \\ &= gu + f + u^* g = p, \end{aligned}$$

which shows that p is a projection in $C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}$. □

Corollary 6.5.2. Let A be a C^*-algebra and let $\tau\colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$, $f \mapsto f(\cdot + 1)$ be the left-shift on $C_0(\mathbb{R})$. Then there exists an injective *-homomorphism $\iota\colon A \to (C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A$.

Proof. This follows immediately from the preceding lemma and Remark 1.5.3. □

Chapter 7

Kirchberg's Embedding Theorem

Kirchberg's Exact Embedding Theorem states that a separable C^*-algebra is exact if and only if it admits an embedding into $\mathcal{O}_2$. We prove here that every separable, nuclear C^*-algebra can be embedded into $\mathcal{O}_2$, and discuss in the end of Section 7.1 how this proof would have to be adapted in the exact case. Section 7.2 deals with an important consequence of Kirchberg's Embedding Theorem, namely, the existence of injective *-homomorphisms between any two Kirchberg algebras.

7.1 Kirchberg's Embedding Theorem

Theorem 7.1.1 (Kirchberg's Embedding Theorem)**.** For every separable, nuclear C^*-algebra A there exists an injective *-homomorphism $\iota\colon A \to \mathcal{O}_2$. If A is unital, the embedding ι can be chosen to be unital.

Proof. Let A be a separable, nuclear C^*-algebra. We do not embed A into $\mathcal{O}_2$ directly, but embed A into another C^*-algebra which can be embedded into $\mathcal{O}_2$ using the tools we have collected so far. Let $\tau\colon C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be the *-automorphism given by $\tau(f) = f(\cdot + 1)$ for all $f \in C_0(\mathbb{R})$, and let τ_A be the corresponding *-automorphism on $C_0(\mathbb{R}, A)$. Combining Corollary 6.5.2, Lemma 6.4.3 and Example 6.2.4, we can conclude that A can be embedded into $C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z}$, because we have

$$A \hookrightarrow (C_0(\mathbb{R}) \rtimes_\tau \mathbb{Z}) \otimes A \cong (C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\mathrm{id}_A} \mathbb{Z} \cong C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z}. \tag{7.1.1}$$

Notice that all involved tensor products are automatically minimal, because A (and also $C_0(\mathbb{R})$) is nuclear. We perform one more embedding by considering the unitization of $C_0(\mathbb{R}, A)$, which we denote by B. Let τ_B denote the unique, unital extension of τ_A to B and use Corollary 6.3.4 to see that

$$C_0(\mathbb{R}, A) \rtimes_{\tau_A} \mathbb{Z} \hookrightarrow B \rtimes_{\tau_B} \mathbb{Z}.$$

Combined with (7.1.1) this shows that it suffices to embed $B \rtimes_{\tau_B} \mathbb{Z}$ into $\mathcal{O}_2$. We first verify the assumptions of Lemma 5.3.7 to show that B admits a unital embedding into $\mathcal{O}_2$. It is clear that B is unital. As A and $C_0(\mathbb{R})$ are separable, this also holds for $C_0(\mathbb{R}) \otimes A$, and therefore B is separable as the unitization of a separable C^*-algebra. To see that B is exact consider the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}, A) \xrightarrow{\lambda} B \xrightarrow{\pi} \mathbb{C} \longrightarrow 0\,,$$

where $C_0(\mathbb{R}, A)$ is considered as an ideal in its unitization B and where λ and π denote the inclusion and the quotient map, respectively. Recall the permanence properties of nuclear C^*-algebras from Proposition 1.5.7 to see the following: Being the minimal tensor product of two nuclear C^*-algebras, the C^*-algebra $C_0(\mathbb{R}) \otimes A \cong C_0(\mathbb{R}, A)$ is nuclear. Hence, B is nuclear as the extension of $C_0(\mathbb{R}, A)$ by the nuclear C^*-algebra $\mathbb{C}$, which implies that $B \rtimes_{\tau_B} \mathbb{Z}$ is nuclear and, by Remark 1.5.17, that B is exact. We show next that B is quasidiagonal. Since

$$C_0(\mathbb{R}, A) \ \cong\ C_0((0,1), A) \ \cong\ SA \ \subseteq\ CA$$

by Remark 1.1.29 and by definition of SA and CA we can treat $C_0(\mathbb{R}, A)$ as a sub-C^*-algebra of CA and, doing unitizations, we can consider B to be a sub-C^*-algebra of $\widetilde{CA}$. From Corollary 5.3.4 we know that $\widetilde{CA}$ is quasidiagonal, and hence so is the sub-C^*-algebra B by Remark 5.3.2(i). Thus, Lemma 5.3.7 yields a unital, injective *-homomorphism $\iota_B\colon B \to \mathcal{O}_2$.

Let ω be a free ultrafilter and let $\iota_{\mathcal{O}_2}\colon \mathcal{O}_2 \to (\mathcal{O}_2)_\omega$ be the canonical embedding of $\mathcal{O}_2$ in its ultrapower. Both ι_B and $\iota_B \circ \tau_B$ are unital, injective *-homomorphisms from B into $\mathcal{O}_2$, and therefore by Theorem 3.4.2(ii) approximately unitarily equivalent. By Corollary 4.5.3 this implies that the maps $\iota_{\mathcal{O}_2} \circ \iota_B$ and $\iota_{\mathcal{O}_2} \circ (\iota_B \circ \tau_B)$ are exactly unitarily equivalent in $(\mathcal{O}_2)_\omega$. Write $\iota = \iota_{\mathcal{O}_2} \circ \iota_B\colon B \to (\mathcal{O}_2)_\omega$ and choose a unitary $v \in (\mathcal{O}_2)_\omega$ with

$$v\iota(b)v^* = \iota\left(\tau_B(b)\right) \qquad \text{for all } b \in B. \tag{7.1.2}$$

Let $1_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{C},\ t \mapsto 1$ and $z\colon \mathbb{T} \to \mathbb{C},\ t \mapsto t$ and define

$$\bar{\iota}\colon B \to (\mathcal{O}_2)_\omega \otimes C(\mathbb{T}),\ b \mapsto \iota(b) \otimes 1_{\mathbb{T}}$$

and $\bar{v} = v \otimes z$. As $1_{\mathbb{T}}$ is the unit in $C(\mathbb{T})$ and z is a unitary, the map $\bar{\iota}$ is a unital, injective *-homomorphism and $\bar{v}$ is a unitary in $(\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$. Moreover, $\bar{\iota}$ and $\bar{v}$ satisfy

$$\bar{v}\bar{\iota}(b)\bar{v}^* = (v \otimes z)(\iota(b) \otimes 1_{\mathbb{T}})(v^* \otimes z^*) = v\iota(b)v^* \otimes 1_{\mathbb{T}} \overset{(7.1.2)}{=} \iota\left(\tau_B(b)\right) \otimes 1_{\mathbb{T}} = \bar{\iota}\left(\tau_B(b)\right)$$

for all $b \in B$, and therefore Lemma 6.3.3 yields a unital *-homomorphism

$$\psi\colon B \rtimes_{\tau_B} \mathbb{Z} \to (\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$$

defined by

$$\psi\Big(\sum_{n\in\mathbb{Z}} b_n u^n\Big) = \sum_{n\in\mathbb{Z}} \bar{\iota}(b_n)\bar{v}^n \qquad \text{for all } \sum_{n\in\mathbb{Z}} b_n u^n \in B\mathbb{Z},$$

where u is the unitary which implements the action τ_B. To show that ψ is injective we now construct a group homomorphism $\gamma\colon \mathbb{T} \to \mathrm{Aut}((\mathcal{O}_2)_\omega \otimes C(\mathbb{T}))$ satisfying the assumptions in Lemma 6.3.3. Take $\varrho\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T}))$ as in Lemma 6.3.5 and define

$$\gamma\colon \mathbb{T} \to \mathrm{Aut}\left((\mathcal{O}_2)_\omega \otimes C(\mathbb{T})\right),\ t \mapsto \gamma_t = \mathrm{id}_{(\mathcal{O}_2)_\omega} \otimes \varrho_t.$$

Each γ_t is a *-automorphism as $\mathrm{id}_{(\mathcal{O}_2)_\omega}$ and ϱ_t are and as the tensor product $(\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$ is minimal, see Lemma 1.5.2, and therefore γ is well-defined. Moreover, γ is a group homomorphism as ϱ is, and as in the proof of Lemma 6.4.3 it follows from point-wise continuity of ϱ that $t \mapsto \gamma_t(x)$ is continuous for any $x \in (\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$. Finally, we can check that

$$\begin{aligned}\gamma_t\left(\bar{\iota}(b)\bar{v}^n\right) &= \gamma_t\big((\iota(b) \otimes 1_{\mathbb{T}})(v \otimes z)^n\big) = \gamma_t\big(\iota(b)v^n \otimes z^n\big)\\ &= \iota(b)v^n \otimes t^n z^n = t^n\big((\iota(b) \otimes 1_{\mathbb{T}})(v \otimes z)^n\big) = t^n\bar{\iota}(b)\bar{v}^n\end{aligned}$$

for all $t \in \mathbb{T},\ b \in B$ and for all $n \in \mathbb{Z}$, and hence ψ is injective by Lemma 6.3.3. Thus, we have obtained a unital embedding of $B \rtimes_{\tau_B} \mathbb{Z}$ into $(\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$. By Lemma 4.6.1, $(\mathcal{O}_2)_\omega \otimes C(\mathbb{T})$ embeds unitally into $(\mathcal{O}_2 \otimes C(\mathbb{T}))_\omega$ which, as $C(\mathbb{T})$ embeds unitally into $\mathcal{O}_2$, embeds unitally into $(\mathcal{O}_2 \otimes \mathcal{O}_2)_\omega$ (by Lemmas 5.1.1 and 4.4.5). As $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ it follows altogether that $B \rtimes_{\tau_B} \mathbb{Z}$ embeds unitally into $(\mathcal{O}_2)_\omega$:

$$B \rtimes_{\tau_B} \mathbb{Z} \hookrightarrow (\mathcal{O}_2)_\omega \otimes C(\mathbb{T}) \hookrightarrow (\mathcal{O}_2 \otimes C(\mathbb{T}))_\omega \hookrightarrow (\mathcal{O}_2 \otimes \mathcal{O}_2)_\omega \cong (\mathcal{O}_2)_\omega.$$

Let φ denote the unital, injective *-homomorphism from $B \rtimes_{\tau_B} \mathbb{Z}$ into $(\mathcal{O}_2)_\omega$ we have just constructed. It was shown above that $B \rtimes_{\tau_B} \mathbb{Z}$ is nuclear, moreover, $B \rtimes_{\tau_B} \mathbb{Z}$ is separable as B is, and hence Corollary 1.5.13 implies that φ is a nuclear map. Therefore, we can apply Choi-Effros' Lifting Theorem (Theorem 1.5.14) to obtain a unital, completely positive map

$$\varrho \colon B \rtimes_{\tau_B} \mathbb{Z} \to \ell^\infty(\mathcal{O}_2)$$

which lifts φ, i.e. $\pi_\omega \circ \varrho = \varphi$. Now, Lemma 5.2.1 yields the existence of a unital, injective *-homomorphism from $B \rtimes_{\tau_B} \mathbb{Z}$ into $\mathcal{O}_2$ which, as A embeds into $B \rtimes_{\tau_B} \mathbb{Z}$, proves that A embeds into $\mathcal{O}_2$.

Assume now that A is unital, and let $\iota \colon A \to \mathcal{O}_2$ be an injective *-homomorphism, which exists by the first part of the theorem. Then $p = \iota(1_A)$ is a non-zero projection in $\mathcal{O}_2$, and as $\iota(a) = p\iota(a)p$ for each $a \in A$, we can consider ι to be a unital embedding of A into $p\mathcal{O}_2p$. By Corollary 2.2.7 there exists a *-isomorphism $\eta \colon p\mathcal{O}_2p \to \mathcal{O}_2$ (automatically unital), and hence $\eta \circ \iota$ is a unital embedding of A into $\mathcal{O}_2$, which completes the proof. $\square$

Remark 7.1.2. Theorem 7.1.1 is a modification of Kirchberg's Exact Embedding Theorem which reads as follows: A separable C^*-algebra A is exact if and only if it admits an embedding into $\mathcal{O}_2$. The "if"-part is clear, because if $\iota \colon A \to \mathcal{O}_2$ is an injective *-homomorphism, then A is isomorphic to $\iota(A)$ which is exact as a sub-C^*-algebra of $\mathcal{O}_2$ by Proposition 1.5.18. Assume now that A is exact. Let B be defined as in the proof of Theorem 7.1.1 and note that up to the point where we applied Choi-Effros' Lifting Theorem, nuclearity was not needed. We only used that B was exact which can also be shown in the following way. Notice that the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}, A) \xrightarrow{\lambda} B \underset{\eta}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0$$

is split exact, where $\eta \colon \mathbb{C} \to B,\ \alpha \mapsto \alpha 1_B$. As $C_0(\mathbb{R}, A)$ is exact, being isomorphic to the minimal tensor product of the exact C^*-algebras A and $C_0(\mathbb{R})$, and as $\mathbb{C}$ is exact, Proposition 1.5.18(v) yields that B is exact. The difference in the exact case is that the lifting of the embedding of $B \rtimes_{\tau_B} \mathbb{Z}$ into $(\mathcal{O}_2)_\omega$ gets more involved, see [KP] for a proof in this case.

7.2 Existence of injective *-homomorphisms between Kirchberg algebras

An important consequence of Kirchberg's Embedding Theorem is that, combined with the following lemma, it enables us to prove the existence of non-zero *-homomorphisms from a C^*-algebra A into a C^*-algebra B if A is separable and exact and if B contains a properly infinite, full projection. The following statement is taken from Lemma 4.2.3 (ii) in [R2].

Lemma 7.2.1. Let A be a C^*-algebra and let p be a properly infinite, full projection in A with $[p]_0 = 0$ in $K_0(A)$. Then there exists a unital embedding of $\mathcal{O}_2$ into pAp.

Proof. As p is properly infinite there exist mutually orthogonal projections $p_0, p_1 \in A$ with $p_0 \leq p$, $p_1 \leq p$ and $p_0 \sim p \sim p_1$. We use these to construct isometries in pAp which satisfy the $\mathcal{O}_2$-relation. Let $p_2 = p - p_1$. Using

$$p_2^2 = p - pp_1 - p_1p + p_1 = p_2, \qquad p_2p_0 = (p - p_1)p_0 = p_0, \qquad p - p_2 = p_1 \geq 0$$

it follows that p_2 is a projection and that $p \sim p_0 \leq p_2 \leq p$. Hence,

$$p_2 \oplus p_2 \leq p \oplus p \precsim p \precsim p_2,$$

i.e. p_2 is properly infinite. We now show that p_0 is full and then conclude that p_2 is full as well. Assume that p_0 is not full and choose a proper, closed, two-sided ideal I in A which contains p_0. As $p \sim p_0$ there exists a partial isometry $s_0 \in A$ with $s_0^*s_0 = p$ and $s_0s_0^* = p_0$. Then $p = s_0^*p_0s_0 \in I$ which contradicts the assumption that p is full. Hence, p_0 is full. If p_2 were not full, we could find a proper, closed, two-sided ideal J in A with $p_2 \in J$. This would imply that $p_0 = p_0p_2 \in J$, which is a contradiction as p_0 is full. Thus, p_2 is a properly infinite, full projection. Notice that $p_1p_2 = p_1(p - p_1) = 0$ and therefore

$$[p]_0 = [p_1 + p_2]_0 = [p_1]_0 + [p_2]_0$$

by Proposition 3.1.7(iv) in [R1]. Moreover, $[p_1]_0 = [p]_0 = 0$ as $p_1 \sim p$, and thus

$$[p_2] = [p]_0 - [p_1] = 0 = [p]_0.$$

As both p and p_2 are properly infinite and full, this implies that $p \sim p_2$ by Proposition 1.2.19(ii). We can now choose partial isometries $s_1, s_2 \in A$ such that $s_1^*s_1 = s_2^*s_2 = p$, and $s_1s_1^* = p_1$ and $s_2s_2^* = p_2$. Then $s_1 = pp_1s_1p \in pAp$ and $s_2 = pp_2s_2p \in pAp$, i.e. s_1 and s_2 are isometries in pAp, and as $s_1s_1^* + s_2s_2^* = p_1 + p_2 = p$, these isometries satisfy the $\mathcal{O}_2$-relation in pAp. The universal property of $\mathcal{O}_2$ now yields a unital embedding of $\mathcal{O}_2$ into pAp. □

Corollary 7.2.2. Let A be a separable, exact C^*-algebra and let B be a C^*-algebra which contains a properly infinite, full projection. Then there exists an injective *-homomorphism from A into B.

Proof. By Remark 7.1.2 there exists an embedding $\iota\colon A \to \mathcal{O}_2$. It follows from Proposition 1.2.19(i) that B contains a properly infinite, full projection p with $[p]_0 = 0$ in $K_0(B)$, and thus Lemma 7.2.1 yields a unital embedding $\kappa\colon \mathcal{O}_2 \to pBp$. As pBp is a sub-C^*-algebra of B we can consider κ as an injective *-homomorphism from $\mathcal{O}_2$ into B, and thus $\kappa \circ \iota$ is an injective *-homomorphism from A into B. □

Remark 7.2.3. Recall that a Kirchberg algebra is a simple, separable, nuclear and purely infinite C^*-algebra. As each purely infinite C^*-algebra contains a properly infinite projection, and as each element in a simple C^*-algebra is automatically full, Corollary 7.2.2 implies that there exist injective *-homomorphisms between any two Kirchberg algebras. This was not known before there was Kirchberg's Embedding Theorem.

Appendix A

More on $\mathcal{O}_2$

In this appendix we combine Theorem 3.4.2 on unitary equivalence of *-homomorphisms into $\mathcal{O}_2$ and Kirchberg's Exact Embedding Theorem to present two important properties of the Cuntz algebra $\mathcal{O}_2$. As announced after Theorem 3.4.2 we will show that $\mathcal{O}_2$ is, up to isomorphism, the only unital, separable, exact C^*-algebra which satisfies these two theorems. In Section A.2 we then show that $A \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$ for sufficiently nice C^*-algebras A.

A.1 A uniqueness result for $\mathcal{O}_2$

The following lemma, which is often referred to as "approximate intertwining", shows how approximate unitary equivalence can be used to prove that two given C^*-algebras are isomorphic. Its proof is given in [R2], where it is part of Corollary 2.3.4.

Lemma A.1.1. Let A and B be unital, separable C^*-algebras, and suppose that there are *-homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $\psi \circ \varphi \approx_u \mathrm{id}_A$ and $\varphi \circ \psi \approx_u \mathrm{id}_B$. Then A is isomorphic to B.

Using this lemma we can now prove the following uniqueness result:

Theorem A.1.2. Let A be a unital, separable C^*-algebra. Then A is isomorphic to $\mathcal{O}_2$ if and only if the following hold:

(i) A is exact;

(ii) Each unital *-endomorphism on A is approximately unitarily equivalent to id_A;

(iii) For every unital, separable, exact C^*-algebra B there exists a unital *-homomorphism from B into A.

Proof. Suppose first that $A \cong \mathcal{O}_2$. Then A is exact and simple as $\mathcal{O}_2$ is. Let $\varphi\colon A \to A$ be a unital *-endomorphism, then φ is automatically injective and it follows from $A \cong \mathcal{O}_2$ and Theorem 3.4.2(ii) that $\varphi \approx_u \mathrm{id}_A$. Property (iii) follows from Kirchberg's Exact Embedding Theorem, see Remark 7.1.2.

Conversely, assume that (i),(ii) and (iii) hold. Then A is exact and Kirchberg's Exact Embedding Theorem yields a unital, injective *-homomorphism $\varphi\colon A \to \mathcal{O}_2$. By (iii) there exists a unital *-homomorphism $\psi\colon \mathcal{O}_2 \to A$, which is injective as $\mathcal{O}_2$ is simple. By (ii),

$\psi \circ \varphi \approx_u \mathrm{id}_A$, and by Theorem 3.4.2(ii), $\varphi \circ \psi \approx_u \mathrm{id}_{\mathcal{O}_2}$. By Lemma A.1.1 this implies that $A \cong \mathcal{O}_2$. □

Remark A.1.3. Property (iii) can be replaced by the following requirements: The unit 1_A is a properly infinite, full projection in A and $[1_A]_0 = 0$ in $K_0(A)$. In the "if" part of the proof we can then use Lemma 7.2.1 to obtain a unital embedding of $\mathcal{O}_2$ into A, and the "only if" part follows because $\mathcal{O}_2$ is unital, simple and purely infinite and $K_0(\mathcal{O}_2) = 0$, i.e. $1_{\mathcal{O}_2}$ is a properly infinite, full projection in $\mathcal{O}_2$ with $[1_{\mathcal{O}_2}]_0 = 0$ in $K_0(\mathcal{O}_2)$.

A.2 Kirchberg's $A \otimes \mathcal{O}_2$-Theorem

Theorem A.2.1. Let A be a C^*-algebra. Then $A \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$ if and only if A is unital, separable, simple and nuclear.

Proof. We first give the proof of the "if"-part of the statement. Suppose that A is unital, separable, simple and nuclear. Then $A \otimes \mathcal{O}_2$ also is unital, separable, simple and nuclear, and thus Theorem 7.1.1 yields a unital embedding $\lambda \colon A \otimes \mathcal{O}_2 \to \mathcal{O}_2$. On the other hand we can define a unital embedding $\iota \colon \mathcal{O}_2 \to A \otimes \mathcal{O}_2,\ x \mapsto 1_A \otimes x$. Now $\lambda \circ \iota \colon \mathcal{O}_2 \to \mathcal{O}_2$ is a unital, injective *-homomorphism, and hence $\lambda \circ \iota \approx_u \mathrm{id}_{\mathcal{O}_2}$ by Theorem 3.4.2(ii). Besides, $\iota \circ \lambda \colon A \otimes \mathcal{O}_2 \to A \otimes \mathcal{O}_2$ is a unital, injective *-homomorphism, and Theorem 3.4.2(i) yields that $\iota \circ \lambda \approx_u \mathrm{id}_{A \otimes \mathcal{O}_2}$. By Lemma A.1.1 this implies that $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Assume now that $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. This implies that $A \otimes \mathcal{O}_2$ is unital, separable, simple and nuclear. As A embeds into $A \otimes \mathcal{O}_2$ via $a \mapsto a \otimes 1_{\mathcal{O}_2}$, A is isomorphic to a sub-C^*-algebra of $A \otimes \mathcal{O}_2$ and hence separable. If J were a non-trivial, two-sided, closed ideal in A, then $J \otimes \mathcal{O}_2$ would be a non-trivial, two-sided, closed ideal in $A \otimes \mathcal{O}_2$. Thus, A is simple. To see that A is unital let $(e_n)_{n \in \mathbb{N}}$ be an approximate unit for A. We show that $(e_n)_{n \in \mathbb{N}}$ converges to an element $e \in A$ which then is a unit for A. By Lemma 6.4.1 in [Mu] the sequence $(e_n \otimes 1_{\mathcal{O}_2})_{n \in \mathbb{N}}$ is an approximate unit for $A \otimes \mathcal{O}_2$. Then

$$\|e_n \otimes 1_{\mathcal{O}_2} - 1_{A \otimes \mathcal{O}_2}\| = \|(e_n \otimes 1_{\mathcal{O}_2}) 1_{A \otimes \mathcal{O}_2} - 1_{A \otimes \mathcal{O}_2}\| \to 0$$

as $n \to \infty$, i.e. $(e_n \otimes 1_{\mathcal{O}_2})_{n \in \mathbb{N}}$ is convergent and hence a Cauchy sequence in $A \otimes \mathcal{O}_2$. As the map given by $a \mapsto a \otimes 1_{\mathcal{O}_2}$ is an isometric *-homomorphism, we can conclude that $(e_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in A. Take $e = \lim_{n \to \infty} e_n$ in A and check that

$$\|ae - a\| \leq \|a(e - e_n)\| + \|ae_n - a\| \leq \|a\| \|e - e_n\| + \|ae_n - a\| \to 0$$

as $n \to \infty$ for every $a \in A$, i.e. $ae = a$ for all $a \in A$. Similarly, $ea = a$ for all $a \in A$, and hence e is a unit for A.

To prove that A is nuclear we need a different notion of nuclearity as we've used so far, and thus we only sketch this part of the proof. For every C^*-algebra B there exists a canonical, surjective *-homomorphism from $A \otimes_{\max} B$ onto $A \otimes_{\min} B$, and A is nuclear if and only if this *-homomorphism is injective, and hence a *-isomorphism, for every C^*-algebra B. Let now B be a C^*-algebra and let $\lambda \colon A \otimes_{\max} B \to A \otimes_{\min} B$ be the canonical surjective *-homomorphism. Notice that, because $\mathcal{O}_2$ is nuclear and taking maximal tensor products is associative,

$$\begin{aligned} \mathcal{O}_2 \otimes_{\min} (A \otimes_{\max} B) &\cong \mathcal{O}_2 \otimes_{\max} (A \otimes_{\max} B) \\ &\cong (\mathcal{O}_2 \otimes_{\max} A) \otimes_{\max} B \cong (\mathcal{O}_2 \otimes_{\min} A) \otimes_{\max} B, \end{aligned} \tag{A.2.1}$$

and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_2 \otimes_{\min} (A \otimes_{\max} B) \arrow[r, "\mathrm{id}_{\mathcal{O}_2}\otimes\lambda"] \arrow[d, "\cong"'] & \mathcal{O}_2 \otimes_{\min} (A \otimes_{\min} B) \arrow[d, "\cong"] \\
(\mathcal{O}_2 \otimes_{\min} A) \otimes_{\max} B \arrow[r, "\cong"'] & (\mathcal{O}_2 \otimes_{\min} A) \otimes_{\min} B
\end{tikzcd}$$

We have a *-isomorphism in the second row because $A \otimes_{\min} \mathcal{O}_2$ is nuclear, and there are *-isomorphisms in the vertical direction by (A.2.1) and because taking minimal tensor products is associative. This shows that the map

$$\mathrm{id}_{\mathcal{O}_2} \otimes \lambda \colon \mathcal{O}_2 \otimes_{\min} (A \otimes_{\max} B) \longrightarrow \mathcal{O}_2 \otimes_{\min} (A \otimes_{\min} B)$$

is a *-isomorphism. Let now $x \in A \otimes_{\max} \mathcal{O}_2$ be such that $\lambda(x) = 0$. Then

$$(\mathrm{id}_{\mathcal{O}_2} \otimes \lambda)\,(1_{\mathcal{O}_2} \otimes x) = 1_{\mathcal{O}_2} \otimes \lambda(x) = 0,$$

i.e. $1_{\mathcal{O}_2} \otimes x = 0$ and hence $x = 0$. This shows that λ is a *-isomorphism and that A is nuclear. $\square$

In the context of the $A \otimes \mathcal{O}_2$-Theorem above it also makes sense to consider the following result, which is taken from Theorem 7.2.6 in [R2]:

Theorem A.2.2. Let A be a simple, separable, nuclear C^*-algebra. Then $A \otimes \mathcal{O}_\infty$ is isomorphic to A if and only if A is purely infinite.

Remark A.2.3. Theorems A.2.1 and A.2.2 show that, for every unital Kirchberg algebra A, $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and $A \otimes \mathcal{O}_\infty \cong A$, i.e. $\mathcal{O}_2$ acts as a tensorial zero, and $\mathcal{O}_\infty$ acts as a tensorial unit in the class of unital Kirchberg algebras.

Appendix B

Classification of Kirchberg algebras

To demonstrate the significance of Kirchberg's Exact Embedding Theorem and the $A\otimes\mathcal{O}_2$ and $A\otimes\mathcal{O}_\infty$-Theorems for the classification of Kirchberg algebras we state Kirchberg and Phillips' Classification Theorem. As the formulation of this theorem requires some notions from KK-theory it is far beyond the scope of this thesis to give all definitions which are necessary for understanding this theorem in detail, but I think the result is interesting nonetheless. For an introduction to KK-theory the reader is referred to Section 2.4 in [R2], where the following notions are explained in more detail:

To every pair of C^*-algebras A and B one can assign an abelian group $KK(A,B)$ in such a way that every *-homomorphism $\varphi\colon A\to B$ represents an element $KK(\varphi)$ in $KK(A,B)$. For every triple of C^*-algebras A, B, C there exists a bi-additive map

$$KK(A,B)\times KK(B,C)\to KK(A,C),\ (x,y)\mapsto x\cdot y,$$

the so-called *Kasparov product*, with the following properties:

(i) The product is associative, i.e. if A, B, C, D are C^*-algebras, then for all $x\in KK(A,B)$, $y\in KK(B,C)$ and $z\in KK(C,D)$ we have $(x\cdot y)\cdot z=x\cdot(y\cdot z)$;

(ii) For all *-homomorphisms $\varphi\colon A\to B$ and $\psi\colon B\to C$ we have

$$KK(\varphi)\cdot KK(\psi)=KK(\psi\circ\varphi);$$

(iii) $KK(A,A)$ is a ring with unit $KK(\mathrm{id}_A)$.

An element $x\in KK(A,B)$ is said to be *invertible* if there exists $y\in KK(B,A)$ such that $x\cdot y=KK(\mathrm{id}_A)$ and $y\cdot x=KK(\mathrm{id}_B)$. Two C^*-algebras A and B are said to be *KK-equivalent* if $KK(A,B)$ contains an invertible element. It follows from property (ii) of the Kasparov product that any *-isomorphism $\varphi\colon A\to B$ induces an invertible element $KK(\varphi)$ in $KK(A,B)$, i.e. isomorphic C^*-algebras are KK-equivalent.

For every C^*-algebra A the following isomorphisms hold:

$$K_0(A)\cong KK(\mathbb{C},A),\qquad\qquad K_1(A)\cong KK(C_0(\mathbb{R}),A).$$

Using these identifications and the Kasparov product one can for all C^*-algebras A and B define two group homomorphisms

$$\gamma_0\colon KK(A,B)\to\mathrm{Hom}(K_0(A),K_0(B)),$$
$$\gamma_1\colon KK(A,B)\to\mathrm{Hom}(K_1(A),K_1(B))$$

by setting $\gamma_0(x)(z_0) = z_0 \cdot x$ and $\gamma_1(x)(z_1) = z_1 \cdot x$ for all $x \in KK(A, B)$, $z_0 \in KK(\mathbb{C}, A)$ and $z_1 \in KK(C_0(\mathbb{R}), A)$. The Universal Coefficient Theorem (UCT) below provides more information about these maps.

A C^*-algebra A is called *K-abelian* if it is KK-equivalent to an abelian C^*-algebra, and the *UCT class* $\mathcal{N}$ is defined to be the family of all separable, K- abelian C^*-algebras.

Theorem B.0.4 (Universal Coefficient Theorem)**.** Let A and B be separable C^*-algebras.

(i) If A belongs to $\mathcal{N}$, then the group homomorphism

$$\gamma = \gamma_0 \oplus \gamma_1 \colon KK(A, B) \to \mathrm{Hom}\big(K_0(A), K_0(B)\big) \oplus \mathrm{Hom}\big(K_1(A), K_1(B)\big)$$

is surjective.

(ii) If both A and B belong to $\mathcal{N}$, then an element $x \in KK(A, B)$ is invertible if and only if $\gamma_0(x)\colon K_0(A) \to K_0(B)$ and $\gamma_1(x)\colon K_1(A) \to K_1(B)$ are group isomorphisms.

(iii) If both A and B belong to $\mathcal{N}$, then they are KK-equivalent if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$.

Another term which appears in the classification theorem is the following:

Definition B.0.5. A C^*-algebra A is said to be *stable* if it is isomorphic to its *stabilization* $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable, infinite dimensional Hilbert space.

It might be useful to know the following statement, usually referred to as "Zhang's Dichotomy", which was proved in [Zh]:

Proposition B.0.6. Every separable, simple, purely infinite C^*-algebra is either unital or stable.

Theorem B.0.7 (Kirchberg and Phillips)**.** Let A and B be Kirchberg algebras.

(i) If A and B are stable, then they are isomorphic if and only if they are KK-equivalent. Moreover, for every invertible element x in $KK(A, B)$ there exists a *-isomorphism $\varphi\colon A \to B$ with $KK(\varphi) = x$.

(ii) If A and B are stable and belong to the UCT class $\mathcal{N}$, then A is isomorphic to B if and only if $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. Moreover, for each pair of group isomorphisms $\alpha_0\colon K_0(A) \to K_0(B)$ and $\alpha_1\colon K_1(A) \to K_1(B)$ there is a *-isomorphism $\varphi\colon A \to B$ with $K_0(\varphi) = \alpha_0$ and $K_1(\varphi) = \alpha_1$.

(iii) If A and B are unital, then they are isomorphic if and only if there is an invertible element x in $KK(A, B)$ with $\gamma_0(x)([1_A]_0) = [1_B]_0$. For each such element x there is a *-isomorphism $\varphi\colon A \to B$ with $KK(\varphi) = x$.

(iv) If A and B are unital and belong to the UCT class $\mathcal{N}$, then they are isomorphic if and only if there are group isomorphisms $\alpha_0\colon K_0(A) \to K_0(B)$ and $\alpha_1\colon K_1(A) \to K_1(B)$ such that $\alpha_0([1_A]_0) = [1_B]_0$. For each such pair of group isomorphisms there is a *-isomorphism $\varphi\colon A \to B$ with $K_0(\varphi) = \alpha_0$ and $K_1(\varphi) = \alpha_1$.

The proof of this classification theorem was independently obtained by Kirchberg and Phillips. Kirchberg's proof (see [Ki2]) is based on the existence of non-zero *-homomorphisms from unital, separable, exact C^*-algebras to unital, properly infinite C^*-algebras, which we dealt with in Corollary 7.2.2. Phillips' approach (see [Ph]) uses the $A \otimes \mathcal{O}_2$ and $A \otimes \mathcal{O}_\infty$-Theorems from Section A.2 above.

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