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Master's thesis in mathematics

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Quasidiagonality, AF-embeddability and the Blackadar-Kirchberg conjectures

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Abstract

This thesis examines the notions of quasidiagonality and AF-embeddability for C^*-algebras as well as the related Blackadar-Kirchberg conjectures. The C^*-algebraic approximation property of quasidiagonality is examined in detail, including permanence properties, obstructions as well as the representation theoretic formulation. The Tikuisis-White-Winter theorem, which states that, on separable, exact C^*-algebras satisfying the UCT, every faithful, amenable tracial state is quasidiagonal, is proved following the extension theoretic proof of Schafhauser. Some of the consequences of the theorem are studied, and connections to both the Blackadar-Kirchberg conjectures as well as Elliott's classification programme are established. Moreover, in a recent paper, Gabe showed that the Blackadar-Kirchberg conjectures hold true for traceless, exact C^*-algebras. A part of the necessary background information including the concept of primitive ideal spaces as well as Rørdam's ASH-algebra $\mathcal{A}_{[0,1]}$ will be examined, and Gabe's proof will be reproduced.

Contents

Introduction

The purpose of this thesis is, roughly speaking, to study the C^*-algebraic approximation property known as quasidiagonality, some of the important results on the topic as well as related concepts. Historically, quasidiagonality was first studied within single operator theory, where Halmos defined an operator T to be quasidiagonal if there exists a sequence of finite-rank projections strongly converging to 1 and asymptotically commuting with T. We can hence view quasidiagonal operators as those which are almost blockdiagonal, which explains the terminology. Using this definition, it is natural to define that a C^*-algebra A is quasidiagonal if it has a faithful representation π such that $\pi(A)$ is a quasidiagonal set of operators, i.e., if we can faithfully realise A as a quasidiagonal set of operators on some Hilbert space. It is not immediate how this translates into an approximation property for C^*-algebras, but Voiculescu [72] proved that this representation theoretic definition is equivalent to the existence of a net of asymptotically multiplicative and asymptotically isometric c.c.p. maps into finite-dimensional C^*-algebras. In this way, quasidiagonal C^*-algebras can be understood as those which can be asymptotically embedded into finite-dimensional C^*-algebras. This description of quasidiagonality shows the finiteness of the property, and, in fact, it can be shown that quasidiagonality implies stably finiteness. While not all stably finite C^*-algebras are quasidiagonal with $C_r^*(\mathbb{F}_2)$ being the standard counterexample using Rosenberg's theorem [31], no counterexamples has been found in the class of nuclear C^*-algebras. Indeed, it is conjectured by Blackadar-Kirchberg, see [4, Question 7.3.1], that all separable, nuclear, stably finite C^*-algebras are quasidiagonal.

Another property closely related to quasidiagonality is AF-embeddability of C^*-algebras, that is, which C^*-algebras can be realised as a C^*-subalgebra of an AF-algebra. It is easy to see that AF-embeddable C^*-algebras are separable, quasidiagonal and exact, and in [4, Question 7.3.3] Blackadar and Kirchberg conjectured that the converse is true. Both of the mentioned Blackadar-Kirchberg conjectures remain unsolved, but there are on-going progress on resolving them. For example, the Tikuisis-White-Winter theorem [70, Theorem A] provided the machinery to prove that nuclearity, quasidiagonality and AF-embeddability are equivalent properties for the class of reduced group C^*-algebras, and that these properties are satisfied exactly when the underlying group is amenable.

Another consequence of the Tikuisis-White-Winter theorem is that it resolves Rosenberg's conjecture. After proving that quasidiagonality of the reduced group C^*-algebra implies amenability of the underlying group G [31], Rosenberg conjectured that the converse is true, which remained unknown until the Tikuisis-White-Winter theorem provided a machinery with which one can easily prove that it is true. This is not the only consequence of the theorem; it can be used to prove that stably finiteness and quasidiagonality are equivalent properties for separable, nuclear, simple C^*-algebras satisfying the UCT, and it allowed for a quasidiagonality assumption to be rendered superfluous in Elliott's classification program [19], completing the classification.

As alluded to in the previous paragraphs, the significance of the Tikuisis-White-Winter theorem lies in the breadth and importance of its corollaries. For this reason, the theorem shall be a central cornerstone for the entire thesis, and therefore we shall mention more precisely what it entails. The theorem states that if A is a separable, exact C^*-algebra satisfying the UCT and admitting a faithful, amenable tracial state, then this tracial state is quasidiagonal. As the existence of a faithful, quasidiagonal tracial state implies quasidiagonality of the C^*-algebra, this provides a tool for proving quasidiagonality of certain C^*-algebras. This is not the original result proved by Tikuisis-White-Winter in [70], but instead a refined result due to Gabe [24]. We shall follow an extension theoretic proof due to Schafhauser [66], and since the underlying idea of his proof is quite simple to state, let us mention the proof structure.

Denote by $\mathcal{Q}$ the universal UHF-algebra, which is the unique UHF-algebra with $K_0(\mathcal{Q}) = \mathbb{Q}$, and let $\mathcal{R}$ be the unique hyperfinite II_1-factor. Quasidiagonality, respectively amenability, of tracial states can be characterised by the existence of a trace-preserving *-homomorphism into $\mathcal{Q}_\omega$ with a c.c.p. lift $A \to \ell^\infty(\mathcal{Q})$, respectively a trace-preserving *-homomorphism $\mathcal{R}^\omega$ with a c.c.p. lift $A \to \ell^\infty(\mathcal{R})$. Let A be a separable, exact C^*-algebra in the UCT-class with an amenable tracial state τ, and let $\varphi\colon A \to \mathcal{R}^\omega$ be a trace-preserving *-homomorphism with a c.c.p. lift $A \to \ell^\infty(\mathcal{R})$. One can show that there exists a short exact sequence $0 \to J \to \mathcal{Q}_\omega \to \mathcal{R}^\omega \to 0$, where J is the so-called trace-kernel ideal. By considering the pullback, we can obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \varphi & & \\ 0 & \longrightarrow & J & \longrightarrow & \mathcal{Q}_\omega & \longrightarrow & \mathcal{R}^\omega & \longrightarrow & 0 \end{array}$$

where E is the pullback. If we manage to prove the existence of a *-homomorphism $A \to E$ splitting the upper row, then the map φ can be lifted to a map $\psi\colon A \to \mathcal{Q}_\omega$, and proving quasidiagonality of τ then boils down to showing ψ can be lifted to a c.c.p. map $A \to \ell^\infty(\mathcal{Q})$. In other words, answering the following question in the affirmative gives a proof of the Tikuisis-White-Winter theorem: Does the short exact sequence split in such a way that the induced *-homomorphism $A \to \mathcal{Q}_\omega$ can be lifted to a c.c.p. map $A \to \ell^\infty(\mathcal{Q})$? We shall see that, up to some separability issues that need to be resolved, faithfulness of the tracial state along with an assumption that A satisfies the UCT imply the existence of such a map $\psi\colon A \to \mathcal{Q}_\omega$, and exactness will provide the fact that ψ has a c.c.p. lift $A \to \ell^\infty(\mathcal{Q})$.

On the topic of the Blackadar-Kirchberg conjectures, Gabe recently published a preprint of a paper [26] resolving the conjectures for the class of separable, exact and traceless C^*-algebras. In fact, he characterised all C^*-subalgebras of a specific C^*-algebra $\mathcal{A}_{[0,1]}$, which can be realised as an inductive limit of cones over increasingly larger matrix algebras. This characterisation is based upon the primitive ideal space, which is a certain non-commutative generalisation of the spectrum for Abelian C^*-algebras.

The above paragraphs should give the reader a taste of the contents of this thesis, and of some of the ongoing C^*-algebraic research in this area. To better introduce the reader to the thesis, we provide a chapter-by-chapter overview.

- In *Chapter 1*, we establish some of the elementary facts on C^*-algebras and von Neumann-algebras, including K-theory and group C^*-algebras, that are needed for understanding the thesis. Moreover, we include a section on the classification theory of certain classes of C^*-algebras to form a historical perspective on Elliott's classification program, which motivates the idea of classifying C^*-algebras by their K-theoretical data. The chapter is only meant as a reference point, and most of the results are assumed to be well-known.
- In *Chapter 2*, we begin actually analysing quasidiagonality in depth. This chapter has two purposes: Understand quasidiagonality as the interesting approximation property it is, but also establish the background theory needed to understand and follow Schafhauser's proof of the Tikuisis-White-Winter theorem. Of course, these are not disjoint purposes by any means, but the reader should be aware that not everything studied in this chapter is inherently related to our interest in the Tikuisis-White-Winter theorem. Beyond quasidiagonality, we also briefly discuss AF-embeddability with a focus on the associated Blackadar-Kirchberg conjecture.
- Then, in *Chapter 3*, we are setting the stage for Schafhauser's proof of the Tikuisis-White-Winter theorem. As the introduction has shown, the proof is extension theoretic at its core, and a good starting point would therefore be to understand this subject. We shall develop the theory of extensions of C^*-algebras, both in general by developing the Ext-semigroup structure, and the specifics needed for the aforementioned proof.
- In *Chapter 4*, we finally turn our attention to the Tikuisis-White-Winter theorem and its proof, and the chapter can thus be seen as the culmination of the previous chapters. While we have laid the foundational work in the previous chapters, and the proof has been sketched in the introduction, there are several results needed, some more subtle than others, which we shall resolve. After proving the Tikuisis-White-Winter theorem, we then study a few of its corollaries. More precisely, we show that the Rosenberg's conjecture is true, and we look at connections to both the Blackadar-Kirchberg conjectures on quasidiagonality and AF-embeddability as well as Elliott's classification program.
- Lastly, in *Chapter 5*, we study a recent paper of Gabe [26] in which he proves the Blackadar-Kirchberg conjectures to be true for traceless C^*-algebras. We also study some of the background material, most notably the primitive ideal space of C^*-algebras and the topological connection to the ideal lattice as well as Rørdam's ASH-algebra $\mathcal{A}_{[0,1]}$.

Notation and terminology

Since virtually every single mathematical paper differs in notation in some way, we shall establish some of the notation used in this thesis. This is, obviously, not a complete list of the notation in the thesis, as it only concerns the foundational notation that may be different from other papers — some of the notation will be introduced in the thesis at due time and, consequently, we shall not mention this here.

We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the natural numbers, i.e., we do not consider 0 a natural number. If we wish to adjoin 0, we shall denote the set by $\mathbb{N}_0$.

Given any set X with a subset S, we denote by $\chi_S \colon X \to \{0, 1\}$ the characteristic function on S. Moreover, we denote the identity function on X by id_X.

C^*-algebras are usually denoted as A or B. We do not assume that C^*-algebras are unital unless explicitly mentioned. If H denotes a Hilbert space, then we denote by $\mathbb{B}(H)$ and $\mathbb{K}(H)$ the bounded linear operators and the compact operators, respectively, on H, and we denote by $\mathscr{C}(H)$ the Calkin algebra $\mathscr{C}(H) = \mathbb{B}(H)/\mathbb{K}(H)$. Whenever we refer to matrix algebras, we mean C^*-algebras of the form $M_n(\mathbb{C})$ for some integer $n \geq 1$, unless we explicitly mention otherwise. We denote by Tr_n the unique *normalised* tracial state on $M_n(\mathbb{C})$.

If A, B are C^*-algebras, we denote by $A \odot B$ the algebraic tensor product of A and B. The minimal, or spatial, tensor product is denoted $A \otimes B$, and the maximal tensor product is denoted $A \otimes_{\max} B$. Moreover, all ideals of C^*-algebras are two-sided, and if not explicitly mentioned otherwise, e.g., by writing *algebraic ideal*, we assume that they are closed.

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1 Preliminaries

It is impossible to start a thesis at square one; we must assume some things to be well-known. For this thesis in particular, we assume elementary knowledge about C^*-algebras, including group C^*-algebras, and knowledge about K-theory, and for motivational purposes it might be of interest to know of Elliott's classification program and its history. However, to ensure that everybody starts at roughly the same page, and in order for the thesis to be somewhat self-contained, we shall briefly discuss these subjects in this chapter. Consequently, this chapter is almost entirely void of proofs, and we refer to elementary textbooks for them; the specific references will be mentioned at the start of each section.

1.1 Elementary results on C^*-algebras and von Neumann-algebras

The theory of C^*-algebra is, as anyone who has ever studied it knows, very rich indeed, and it is thus virtually impossible to include all the relevant background knowledge in one section. Consequently, the topics discussed below are hand-picked by their relevance to the rest of the thesis. Most of the proofs for the statements herein can be found in elementary textbooks on the subject, e.g., [47, 74], and for statements about c.c.p. and u.c.p. maps and finite-dimensional approximation properties such as nuclearity we refer to [11]. Another good reference containing a lot of the theory, although without many proofs, is [3].

Let A be an arbitrary C^*-algebra. An element $x \in A$ is called *positive* if $x = y^*y$ for some $y \in A$, and the collection of positive elements of A is denoted by A_+. We can define a partial order on A by $x \leq y$ if and only if $y - x \in A_+$. An element $p \in A$ is called a *projection* if $p^2 = p = p^*$. Denote the set of projections in A by $\mathcal{P}(A)$. We equip $\mathcal{P}(A)$ with the equivalence relation that $p \sim q$ if and only if there exists $v \in A$ with $v^*v = p$ and $vv^* = q$; this is called the *Murray-von Neumann equivalence relation*. If $v \in A$ is an element such that both v^*v and vv^* are projections, then we say that v is a *partial isometry*, and we call v^*v the *support projection* and vv^* the *range projection* of v. Assume now that A is unital with unit 1. An *isometry* is an element $v \in A$ with $1 = v^*v$. We further say that v is *unitary* if v is invertible with $v^{-1} = v^*$. For any C^*-algebra A, unital or not, there exists a unital C^*-algebra $A^\dagger$, which contains A as an ideal, and with the quotient $A^\dagger/A \cong \mathbb{C}$, i.e., it fits into a short exact sequence $0 \to A \to A^\dagger \to \mathbb{C} \to 0$. It is often the case that properties of C^*-algebras, which are inherently unital in nature, are extended to non-unital C^*-algebras by considering their unitisations.

No matter if A is unital or not, we can still approximate a unit in the sense that there exists an *approximate unit*, that is, an increasing net $(e_\alpha)_{\alpha\in\Lambda}$ of positive contractions such that $\|xe_\alpha - x\| \to 0$ for each $x \in A$. In fact, if I is an ideal in A, then there exists an approximate unit $(e_\alpha)_{\alpha\in\Lambda}$ in I which is quasicentral for A, meaning that $\|e_\alpha a - ae_\alpha\| \to 0$ for all $a \in A$; see [53, Theorem 3.12.14]. If the C^*-algebras are separable, these approximate units can be assumed to be sequences.

Suppose that A is a unital C^*-algebra with unit 1. A *state* on A is a linear functional $\rho\colon A \to \mathbb{C}$ satisfying $\rho(1) = 1$. A *tracial state* is a state τ satisfying the tracial property $\tau(ab) = \tau(ba)$ for all $a, b \in A$. Note that there are many unital C^*-algebras, which do not admit tracial states, but which still admit trace-like maps. Consider for example a separable, infinite-dimensional Hilbert space H with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, then $\mathbb{B}(H)$ admits no tracial state as it is a properly infinite C^*-algebra; the proof may also be found in [47, Remark 6.2.2]. Nonetheless, the usual trace by $\mathrm{Tr}(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle$ is still resembling of a trace, even though it is not defined everywhere. For the next definition, we shall use the terminology that a subset S of a C^*-algebra A is called *symmetric* if $x^*x \in S$ implies $xx^* \in S$.

Definition 1.1. A *trace* on a C^*-algebra A is a positive linear map $\tau\colon I \to \mathbb{C}$, where I is a symmetric, self-adjoint algebraic ideal in A, and where $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$ with $x^*x \in I$.

We call I the *domain* of τ, but occasionally we shall just say that τ is a trace on I; this should not be any cause of confusion. Alternatively, traces on a C^*-algebra A can be defined as additive, homogeneous maps $\tau\colon A_+ \to [0, \infty]$ such that $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. The reader is warned that the two definitions of traces are not equivalent, although we shall always make another assumption on both types of traces, with which the two definitions agree. Define for each positive $a \in A$ and $\varepsilon > 0$ the ε-cutoff element $(a - \varepsilon)_+$ in A by continuous functional calculus using the continuous function $f_\varepsilon\colon [0,\infty) \to [0,\infty)$ by $f_\varepsilon(t) = \max\{t-\varepsilon, 0\}$ for $t \geq 0$. Note that if I is hereditary in A and $a \in I$ is positive, then $(a-\varepsilon)_+ \in I$ for all $\varepsilon > 0$. We say that a trace τ, in either picture, on A is *lower semi-continuous* if $\tau(a) = \sup_{\varepsilon>0} \tau((a-\varepsilon)_+)$ for all positive a in the suitable domain of τ. Under the assumption that the traces are assumed to be lower semi-continuous, the two definitions agree, see [63, Proposition 2.10] as well as the following discussion in the same paper.

One important class of C^*-algebras, which shows up quite often throughout the thesis as well as in the study of C^*-algebra in general, is the class of Abelian C^*-algebras. They can be easily classified by the following proposition.

Proposition 1.2 (Gelfand)**.** *If A is an Abelian C^*-algebra, then A is is *-isomorphic $C_0(X)$ for some locally compact Hausdorff space X. An Abelian C^*-algebra is unital if and only if it can be realised as $C(X)$ for some compact Hausdorff space X. Two Abelian C^*-algebras $C_0(X)$ and $C_0(Y)$ are *-isomorphic if and only if X is homeomorphic to Y.*

There is an explicit description of the underlying space for Abelian C^*-algebras, which we shall briefly sketch — the details may be found in [47, Chapter 1.3]. Let A be an Abelian C^*-algebra and consider the set $\hat{A}$ of *-homomorphisms $A \to \mathbb{C}$, then $\hat{A}$ is a locally compact Hausdorff space in the weak*-topology and, if A is unital, $\hat{A}$ is compact. The map $\Gamma\colon A \to C_0(\hat{A})$ by $\Gamma(a)(\varphi) = \varphi(a)$ for $a \in A$ and $\varphi \in \hat{A}$ is a *-isomorphism; we call $\hat{A}$ the *spectrum* of A, and Γ the Gelfand transform.

Another important class is the class of finite-dimensional C^*-algebras, which also admits a very nice classification.

Proposition 1.3. *If A is a finite-dimensional C^*-algebra, then $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ for some suitable integers $r, n_i > 0$.*

The class of finite-dimensional C^*-algebras is the smallest class containing $\mathbb{C}$ and which is closed under finite direct sums and tensor products. However, it is not closed under inductive limits. We define the class of AF-algebras to be the smallest class of C^*-algebras containing $\mathbb{C}$ and which is closed under direct sums, tensor products and sequential inductive limits, and in this way AF-algebras is a natural and small generalisation of finite-dimensional C^*-algebras. A perhaps more tangible definition of AF-algebras is the following, which is closer to the original definition by Bratteli [5].

Definition 1.4. A C^*-algebra is said to be *approximately finite-dimensional*, usually shortened AF-algebra, if it is the norm-closure of an increasing sequence of finite-dimensional C^*-algebras.

Note that with our definition, all AF-algebras are assumed to be separable — one can define a notion of non-separable AF-algebras by considering arbitrary increasing unions of finite-dimensional C^*-algebras, but we shall not do this here. Another characterisation of AF-algebras, with which it is easy to verify that the class of AF-algebras is closed under inductive limits, is that AF-algebras are precisely those arising as inductive limits of sequences of finite-dimensional C^*-algebras. Yet another characterization is the following local one, see [64, Proposition 7.2.2]

Proposition 1.5 (Bratteli)**.** *A separable C^*-algebra A is an AF-algebra if and only if for each finite number of elements $a_1, \ldots, a_n \in A$ and tolerance $\varepsilon > 0$, there exist a finite-dimensional C^*-subalgebra B of A and elements $b_1, \ldots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$ for all $i = 1, \ldots, n$.*

All of the three characterizations are useful to have in mind when studying AF-algebras.

An important AF-algebra is the C^*-algebra $\mathbb{K}(H)$ of compact operators on some separable, infinite-dimensional Hilbert space H. One can realise this as the inductive limit of the sequence

$$\mathbb{C} \xrightarrow{\varphi_1} M_2(\mathbb{C}) \xrightarrow{\varphi_2} M_3(\mathbb{C}) \xrightarrow{\varphi_3} \cdots$$

where $\varphi_n\colon M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ is given by $\varphi(a) = \operatorname{diag}(a, 0)$ for $a \in M_n(\mathbb{C})$. If A is a C^*-algebra such that $A \otimes \mathbb{K}(H) \cong A$, we say that A is *stable*, and it is easily verified that $\mathbb{K}(H)$ itself is stable.

If A is an arbitrary C^*-algebra, then, for each $n \in \mathbb{N}$, we can construct the matrix algebra $M_n(A)$ over A, which turns out to be a C^*-algebra. If $\varphi\colon A \to B$ is a *-homomorphism, the n-amplification $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ by $\varphi^{(n)}([a_{ij}]) = [\varphi(a_{ij})]$ is again a *-homomorphism. In fact, the n-amplification of any linear map is again linear. Note that *-homomorphisms are always contractive, that is, $\|\varphi\| \leq 1$ for all *-homomorphisms $\varphi\colon A \to B$, and they take positive elements to positive elements.

Definition 1.6. Let A and B be C^*-algebras. A linear map $\varphi\colon A \to B$ is called:

- *contractive* if $\|\varphi\| \leq 1$;
- *completely contractive* if $\left\|\varphi^{(n)}\right\| \leq 1$ for all $n \in \mathbb{N}$;
- *positive* if $\varphi(A_+) \subseteq B_+$;
- *completely positive* if $\varphi^{(n)}(M_n(A)_+) \subseteq M_n(B)_+$ for all $n \in \mathbb{N}$.
- *unital* if, under the additional assumptions that A and B are unital, $\varphi(1_A) = 1_B$.

We shall abbreviate "completely positive" by c.p., "complete positive and contractive" by c.c.p., and "unital and completely positive" by u.c.p..

Note that while one might be cautious of the above notation, since c.c.p. could mean either completely contractive, positive or contractive, completely positive, we only adopt the latter meaning and, hence, there is no ambiguity. Note moreover that we do not assume any contractive property on u.c.p. maps, since this is automatic. It is also the case that *-homomorphisms are c.c.p. and, in the unital case, u.c.p..

Proposition 1.7. *Let A, B be C^*-algebras, and let $\varphi\colon A \to B$ be a c.c.p. map. Then the set*

$$A_\varphi = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$

is a C^*-subalgebra of A. For any $a, b \in A$ and $x \in A_\varphi$, we have

$$\varphi(axb) = \varphi(a)\varphi(x)\varphi(b)$$

We call the C^*-subalgebra A_φ defined above the *multiplicative domain* of A, and it is the largest C^*-subalgebra C of A on which $\varphi|_C$ is a *-homomorphism. The latter property in Proposition 1.7 will be referred to as the *bimodule property* of c.c.p. maps. There is a generalisation, see [11, Proposition 1.5.7], but the above shall suffice for our purposes.

The following proposition gives a description of c.p. maps with codomain in $\mathbb{B}(H)$ for some Hilbert space H.

Proposition 1.8 (Stinespring)**.** *Let A be a unital C^*-algebra, let H be a Hilbert space, and suppose $\varphi\colon A \to \mathbb{B}(H)$ is a c.p. map. Then there exists a Hilbert space H' , an operator $V\colon H \to H'$ and a *-homomorphism $\pi\colon A \to \mathbb{B}(H')$ such that*

$$\varphi(a) = V^*\pi(a)V, \qquad a \in A.$$

*The norm of φ is given by $\|\varphi\| = \|\varphi(1)\| = \|V^*V\|$. Conversely, if $\varphi\colon A \to \mathbb{B}(H)$ is of the form $\varphi(a) = V^*\pi(a)V$ for some operator $V\colon H \to H'$ and a *-homomorphism $\pi\colon A \to \mathbb{B}(H')$, then φ is c.p..*

A proof may be found in [11, Theorem 1.5.3]. We shall actually need a more general result later by weakening the conditions on the codomain of φ; this is of importance in extension theory.

One important fact about c.c.p. and u.c.p. maps is that we can often extend them in certain ways. In order to make this statement precise, we need a definition.

Definition 1.9. An *operator system* is a closed subspace of a unital C^*-algebra A, which is self-adjoint and contains the unit of A.

The following proposition shows that we can extend c.c.p. and u.c.p. maps from an operator system E to the C^*-algebra A, which E is an operator subsystem of, if we map into bounded operators on a Hilbert space. A proof can be found in [11, Theorem 1.6.1].

Proposition 1.10 (Arveson's extension theorem)**.** *Let A be a unital C^*-algebra A, and suppose $E \subseteq A$ is an operator subsystem. Then every c.c.p. (or u.c.p.) map $\varphi\colon E \to \mathbb{B}(H)$ extends to a c.c.p. (or u.c.p.) map $\overline{\varphi}\colon A \to \mathbb{B}(H)$.*

Some of the most important approximation properties of C^*-algebra are the notions of nuclearity and exactness, which studies whether certain maps — the identity map or some faithful *-representation, respectively — can be factored through finite-dimensional C^*-algebras.

Definition 1.11. Let A and B be C^*-algebras. A map $\theta\colon A \to B$ is *nuclear* if there exist nets of c.c.p. maps $\varphi_\alpha\colon A \to M_{k_\alpha}(\mathbb{C})$ and $\psi_\alpha\colon M_{k_\alpha}(\mathbb{C}) \to A$ for some integers $k_\alpha \in \mathbb{N}$ such that

$$\lim_\alpha \|\theta(a) - \psi_\alpha \circ \varphi_\alpha(a)\| = 0$$

for all $a \in A$.

In other words, nuclear maps are those factorised by c.c.p. maps through finite-dimensional C^*-algebras.

Definition 1.12. Let A be a C^*-algebra. We say that

(i) A is *nuclear* if the identity map $\mathrm{id}_A\colon A \to A$ is nuclear.

(ii) A is *exact* if there exists a faithful *-representation $\pi\colon A \to \mathbb{B}(H)$, which is nuclear. Equivalently, A is exact if and only if there exists a C^*-algebra B and an injective *-homomorphism $\varphi\colon A \to B$, which is nuclear.

Note that this is not the original algebraic formulation of exactness, which we shall encounter soon, and note also that the equivalence of being nuclearly embeddable and being exact is a deep result due to Kirchberg, see [3, IV.3.4.18].

There is another description of nuclearity and exactness, which is based on tensor products of C^*-algebras. We shall not define neither the minimal tensor product $\otimes$ nor the maximal tensor product $\otimes_{\max}$ in this thesis; the details can be found in [11, Chapter 3]. One thing we point out is that given a short exact sequence $0 \to I \to A \to A/I \to 0$ of C^*-algebras and a C^*-algebra B, the sequence

$$0 \to I \otimes_{\max} B \to A \otimes_{\max} B \to A/I \otimes_{\max} B \to 0$$

is always exact. However, this is not the case for the minimal tensor product; generally, exactness of the sequence may fail in the middle in the sense that the kernel of the map $A \otimes B \to A/I \otimes B$ may be strictly larger than the image of the map $I \otimes B \to A \otimes B$.

Theorem 1.13. *A C^*-algebra A is*

(i) *exact if and only if the functor $A \otimes -$ preserves short exact sequences;*

(ii) *nuclear if and only if, for any C^*-algebra B, the algebraic tensor product $A \odot B$ has a unique C^*-norm.*

This characterisation of exactness is the original one, and the terminology coincides with exactness as a term in homological algebra. Nuclearity and exactness are both pleasant properties of C^*-algebras, as they limit the exoticness that a general C^*-algebra may have, and this is also the reason why, for classification purposes, one often only considers nuclear C^*-algebras. While exactness passes to C^*-subalgebras, which is easily seen using the notion of nuclear embeddability, the same does not hold for nuclear C^*-algebras. Hence one has to be careful when passing to C^*-subalgebras; passing to hereditary C^*-subalgebras, in particular ideals, does, however, preserve nuclearity.

Given a C^*-algebra A, it is desirable to understand its *-representations, that is, the *-homomorphisms $A \to \mathbb{B}(H)$ for Hilbert spaces H. An important fact is that any C^*-algebra admits a faithful, non-degenerate representation on $\mathbb{B}(H)$ for some Hilbert space H, which entails that we can characterise C^*-algebras as norm-closed *-subalgebras of $\mathbb{B}(H)$. The canonical way of showing this is through the GNS-construction.

Theorem 1.14 (Gelfand-Neimark-Segal)**.** *If φ is a state on a C^*-algebra A, then there exists a Hilbert space H_φ, a non-degenerate *-homomorphism $\pi_\varphi\colon A \to H_\varphi$ and a cyclic unit vector $\xi_\varphi \in H_\varphi$ such that $\varphi(x) = \langle \pi_\varphi(x)\xi_\varphi, \xi_\varphi\rangle$ for all $x \in A$. We refer to the tripe $(H_\varphi, \pi_\varphi, \xi_\varphi)$ as the* GNS-representation *associated to φ.*

By cyclicity of ξ_φ, we mean that $\pi_\varphi(A)\xi_\varphi$ is norm-dense in H_φ. Observe that the state φ is faithful if and only if the induced representation π_φ is faithful. Taking the direct sum of all GNS-representations associated to states on A, one achieves a faithful *-representation of A on some Hilbert space as desired.

To end the discussion on C^*-algebras, we define a quite useful tool, whose construction is due to Pedersen, see [53, Theorem 5.6.1].

Proposition 1.15. *For each C^*-algebra A, there exists a unique minimal norm-dense algebraic ideal in A.*

The ideal is called the *Pedersen ideal* and is denoted $\mathrm{Ped}(A)$. If $A = C_0(X)$ is an Abelian C^*-algebra, then $\mathrm{Ped}(A) = C_c(A)$ is the collection of continuous, compactly supported functions $X \to \mathbb{C}$, see [3, II.5.2.5]. One fact that we need to know is the following, the proof of which follows from the construction in [53, Theorem 5.6.1].

Proposition 1.16. *Let A be a C^*-algebra. Then for each positive element $x \in A$ and $\varepsilon > 0$, the element $(x - \varepsilon)_+$ belongs to $\mathrm{Ped}(A)$.*

Now we turn our attention to the theory of von Neumann-algebras. Recall that a von Neumann-algebra M is a non-degenerate *-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H, which is closed in the strong operator topology. There are many equivalent characterisations of von Neumann-algebras, the most important being the following, which is part of the von Neumann bicommutant theorem. Recall that we for any set $M \subseteq \mathbb{B}(H)$ denote by $M' = \{T \in \mathbb{B}(H) \,|\, TS = ST \text{ for all } S \in M\}$ the *commutant* of M.

Theorem 1.17. *Let M be a non-degenerate *-subalgebra of $\mathbb{B}(H)$. Then the following are equivalent:*

(i) *$M = M''$;*

(ii) *M is closed in the strong operator topology;*

(iii) *M is closed in the weak operator topology.*

We refrain from defining the topologies in detail; the definitions may be found in [3, Chapter I.3]. A linear map $\varphi\colon M \to N$ between von Neumann-algebras is called *normal* if it is ultrastrong-ultrastrong continuous, see [3, Proposition III.2.2.2].

Definition 1.18. A von Neumann-algebra M is called a *factor* if the center $\mathcal{Z}(M) = M \cap M'$ is trivial, that is, if $\mathcal{Z}(M) = \mathbb{C}$.

A von Neumann-algebra being a factor is similar to a C^*-algebra being simple. Most of the von Neumann-algebras appearing in this thesis will be factors, as they admit a lot of pleasant structures. For example, the possible properties of the projections in a factor give rise to a type decomposition.

Definition 1.19. Let M be a von Neumann-algebra, and let $p \in M$ be a projection. We say that:

(i) p is *finite* if $p \sim q \leq p$ implies that $p = q$;

(ii) p is *infinite* if p is not finite;

(iii) p is *Abelian* if pMp is Abelian;

(iv) p is *minimal* if $pMp = \mathbb{C}$.

We call M *finite* if its unit 1 is finite, and we call M *infinite* if 1 is infinite.

Note that minimal projections are always Abelian. A projection $p \in M$ is *central* if $p \in M \cap M'$. We shall only explain the type decomposition for factors and disregard the general definitions, which may be found in [74, Definition 26.3].

Definition 1.20. Assume that M is a factor. We say that:

(i) M is of type I if M admits a non-zero minimal projection. Equivalently, M is isomorphic to $\mathbb{B}(H)$ for some, possibly finite-dimensional, Hilbert space H. If M is isomorphic to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, we say that M is of type I_n.

(ii) M is of type II if there are no minimal projections, but there are non-zero finite projections. Moreover, if M is finite, we say that M is of type II_1, and if M is infinite, it is of type II_∞.

(iii) M is of type III, if M is not of type I or type II.

By construction, this is a type decomposition in the sense that any factor is of either type I, type II_1, type II_∞ or type III. Any von Neumann-algebra factor of type III is necessarily infinite, and the only finite factors are those of type I_n for some $n \in \mathbb{N}$ or of type II_1. Another important fact regarding factors is that we can always compare projections. If M is a von Neumann-algebra, then we write $p \precsim q$ for projection $p, q \in M$ if p is Murray-von Neumann equivalent to a subprojection of q. If $p \precsim q$, but $p \not\sim q$, we write $p \prec q$.

Proposition 1.21. *Let M be a von Neumann-algebra factor, and let $p, q \in M$ be projections. Then either $p \prec q$, $p \sim q$ or $p \succ q$.*

Tracial states preserve this ordering in the sense that if $p \precsim q$ and τ is a tracial state, then $\tau(p) \leq \tau(q)$.

As stated before, the von Neumann-algebras of interest for the purposes in this thesis are mostly factors, and specifically we are interested in II_1-factors. It is hence advantageous to have some more tangible characterisations of II_1-factors.

Proposition 1.22. *If M is an infinite-dimensional factor with a faithful tracial state, then M is of type II_1.*

The condition that M admits a faithful tracial state implies that M is finite, such that M is either of type I_n for some $n \in \mathbb{N}$, or of type II_1, and the infinite-dimensionality assumption provides that the former cannot hold. In fact, the converse also holds; a II_1-factor is always infinite-dimensional with a faithful tracial state, and the tracial state is actually unique.

The next proposition shows how projections in a II_1-factor may effectively be halved, which is an important property of such von Neumann-algebras.

Proposition 1.23. *If M is a II_1-factor and $p \in M$ is a projection, then there exists n mutually orthogonal and Murray-von Neumann equivalent projections $q_1, \dots, q_n \in M$ such that $q_1 + \cdots + q_n = p$. Moreover, if τ denotes the unique tracial state on M, then $\tau(\mathcal{P}(M)) = [0, 1]$.*

The following definition is closely related to that of AF-algebras, cf. Definition 1.4.

Definition 1.24. A von Neumann-algebra M is called *hyperfinite* if there exists an increasing sequence of finite-dimensional von Neumann-algebras $M_1 \subseteq M_2 \subseteq \cdots$ such that the union $\bigcup_{n\in\mathbb{N}} M_n$ is strongly dense in M.

For this thesis, the single-most important von Neumann algebra is the unique hyperfinite II_1-factor $\mathcal{R}$, see [3, Theorem III.3.4.3].

1.2 Brief introduction to K-theory

Briefly stated, the idea behind K-theory is to introduce, for each C^*-algebra, a pair of Abelian groups, which carry some of the information of the original C^*-algebra. They do not carry all the information — K-theory is not a complete classification tool by itself — but being an isomorphism invariant, if two C^*-algebras have non-isomorphic K-theory, they are non-isomorphic C^*-algebras. This is true for all isomorphism invariants, but the success of K-theory lies in the breadth of topics, which can advantageously be studied using K-theory. For instance, the AF-algebras are classified completely by their ordered K_0-groups, and K-theory played an important role in the classification of essentially normal operators within single operator theory. We shall discuss the former later on in the thesis, and the latter can be studied in [33]. Since we shall extensively use K-theory in this thesis, we present below an overview of the most important aspects. The proofs can, along with further details, be found in [64].

Let A be an arbitrary C^*-algebra, and denote for each $n \in \mathbb{N}$ the collection of projections on $M_n(A)$ by $\mathcal{P}_n(A)$. Let $\mathcal{P}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n(A)$ and define an equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ by the following: Let $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, then we say that $p \sim_0 q$ if and only if $p = v^*v$ and $q = vv^*$ for some $v \in M_{m,n}(A)$. Denote the equivalence class of $p \in \mathcal{P}_\infty(A)$ by $[p]_0$, and let $\mathcal{D}(A)$ be the collection of equivalence classes. This is easily verified to be an Abelian semigroup with addition defined by $[p]_0 + [q]_0 = [p \oplus q]_0$, and with the equivalence class of the zero projection being the identity. By the Grothendieck construction, which is a generalisation of the construction of $\mathbb{Z}$ from $\mathbb{N}_0$, we get an Abelian group $\mathcal{K}_0(A)$. If $0 \to A \to A^\dagger \to \mathbb{C} \to 0$ is the short exact sequence associated to the unitisation of A, one can verify that the corresponding sequence $\mathcal{K}_0(A) \to \mathcal{K}_0(A^\dagger) \to \mathcal{K}_0(\mathbb{C})$ is an exact sequence of Abelian groups. Now we define the Abelian group $K_0(A) = \ker(\mathcal{K}_0(A^\dagger) \to \mathcal{K}_0(\mathbb{C}))$. If A is already unital, one can check that $K_0(A) = \mathcal{K}_0(A)$, but in the non-unital case, the two Abelian groups may be widely different.

In order to describe the elements of $K_0(A)$ for a general C^*-algebra, we define the map $s\colon A^\dagger \to A^\dagger$ by $s(a + \lambda 1) = \lambda 1$ for $a \in A$ and $\lambda \in \mathbb{C}$; this map is known as the *scalar map*.

Proposition 1.25 (The standard picture of K_0)**.** *If A is a C^*-algebra, unital or not, then*

$$K_0(A) = \{[p]_0 - [s(p)]_0 \,|\, p \in \mathcal{P}_\infty(A^\dagger)\}.$$

*Also, if $\varphi\colon A \to B$ is a *-homomorphism, then it induces a group homomorphism $K_0(\varphi)\colon K_0(A) \to K_0(B)$ by*

$$K_0(\varphi)([p]_0 - [s(p)]_0) = [\varphi^\dagger(p)]_0 - [s(\varphi^\dagger(p)]_0,$$

*for all $p \in \mathcal{P}_\infty(A^\dagger)$, where $\varphi^\dagger\colon A^\dagger \to B^\dagger$ is the *-homomorphism given by $\varphi^\dagger(a+\lambda 1) = \varphi(a) + \lambda 1$ for $a \in A$ and $\lambda \in \mathbb{C}$.*

In the case where the C^*-algebra in question is unital, we do not have to go through the unitisation to make a sensible functor and as such we have the following description of K_0:

Proposition 1.26 (The standard picture of K_0, unital case)**.** *If A is a unital C^*-algebra, then*

$$K_0(A) = \{[p]_0 - [q]_0 \,|\, p, q \in \mathcal{P}_\infty(A)\},$$

*and if $\varphi\colon A \to B$ is a *-homomorphism between unital C^*-algebras, then it induces a group homomorphism $K_0(\varphi)\colon K_0(A) \to K_0(B)$ given by*

$$K_0(\varphi)([p]_0 - [q]_0) = [\varphi(p)]_0 - [\varphi(q)]_0$$

for all $p, q \in \mathcal{P}_\infty(A)$.

In both Proposition 1.25 and Proposition 1.26, we have abused notation and denoted by φ the induced map on matrix algebras.

Let us discuss some properties of the functor K_0.

Proposition 1.27 (Properties of K_0)**.** *The functor K_0 from the category of C^*-algebras to the category of Abelian groups satisfy the following:*

(i) *K_0 is half-exact;*

(ii) *K_0 is split-exact;*

(iii) *K_0 is a homotopy invariant, i.e., if A and B are homotopically equivalent C^*-algebras, then $K_0(A) = K_0(B)$;*

(iv) *K_0 preserves direct sums, i.e., $K_0(A \oplus B) = K_0(A) \oplus K_0(B)$ for all C^*-algebras A and B;*

(v) *K_0 is stable in the sense that for each $n \in \mathbb{N}$ and C^*-algebra A, the inclusion $A \hookrightarrow M_n(A)$ induces a group isomorphism $K_0(M_n(A)) \cong K_0(A)$. In fact, for any Hilbert space H, the inclusion $A \hookrightarrow A \otimes \mathbb{K}(H)$ induces a group isomorphism $K_0(A) \cong K_0(A \otimes \mathbb{K}(H))$;*

(vi) *K_0 is continuous in the sense that it commutes with inductive limits.*

Note that (iii) and (v) explicitly implies that K_0 cannot be a classification invariant, since there exists non-isomorphic C^*-algebras which are homotopically equivalent. For instance, if A is any C^*-algebra, then the cone $CA = C_0((0,1]) \otimes A$ is homotopically equivalent to the zero C^*-algebra, see [64, Example 4.1.5]. Nevertheless, the K_0-functor plays a role in classification theory, but we need to use the fact that the K_0-group naturally can admit more structure than what we have established so far.

Definition 1.28. A pair (G, G^+), where G is an Abelian group, and G^+ is a subset of G, is called an *ordered Abelian group* if

(i) $G^+ + G^+ \subseteq G^+$;

(ii) $G^+ \cap (-G^+) = \{0\}$;

(iii) $G = G^+ - G^+$.

If (G, G^+) only satisfies (i) and (iii), we call it a *preordered Abelian group.*

The terminology should not be the cause of any confusion, since if (G, G^+) is a (pre)ordered group, it induces a (pre)order on G by $x \leq y$ if $y - x \in G^+$.

Definition 1.29. If (G, G^+) is an ordered Abelian group and $u \in G^+$ satisfies that for each $h \in G$, there exists $n \in \mathbb{N}$ with $-nu \leq h \leq nu$, then u is called an *order unit.* The triple (G, G^+, u) is then called an *ordered Abelian group with a distinguished order unit.*

It turns out that $K_0(A)$ is often an ordered Abelian group. By considering that constructing the group $K_0(A)$ mimics the construction of $\mathbb{Z}$ from $\mathbb{N}_0$, and as the canonical positive cone of $\mathbb{Z}$ is exactly $\mathbb{N}_0$, it is natural to define the *positive cone* of $K_0(A)$ to be

$$K_0(A)^+ = \{[p]_0 \mid p \in \mathcal{P}_\infty(A)\}.$$

Note that $K_0(A)^+$ coincides with $\mathcal{D}(A)$.

Proposition 1.30. *If A is a unital C^*-algebra, then $(K_0(A), K_0(A)^+, [1]_0)$ is a preordered Abelian group with a distinguished order unit. If, in addition, A is stably finite, the triple is an ordered Abelian group.*

The above triple is precisely a classification invariant for unital AF-algebras, see Theorem 1.39. However, note that non-unital C^*-algebra may have an ordered K_0-group, as any AF-algebra A, unital or not, satisfy that the pair $(K_0(A), K_0(A)^+)$ is an ordered Abelian group. We shall discuss the classification and the structure of the ordered K_0-groups of AF-algebras in a later section.

We now turn our attention to the K_1-functor. Let A be a unital C^*-algebra for now, and denote for each $n \in \mathbb{N}$ the collection of unitary elements on $M_n(A)$ by $\mathcal{U}_n(A)$. Define moreover $\mathcal{U}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n(A)$. Define on $\mathcal{U}_\infty(A)$ the equivalence relation $\sim_1$ by the following: If $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, then $u \sim_1 v$ if and only if there exists an integer k such that $u \oplus 1_{k-n}$ is homotopically equivalent to $v \oplus 1_{k-m}$ inside $\mathcal{U}_k(A)$. Denote the equivalence classes of $u \in \mathcal{U}_\infty(A)$ by $[u]_1$. For any C^*-algebra A, we define $K_1(A) = \{[u]_1 \mid u \in \mathcal{U}_\infty(A^\dagger)\}$, which is an Abelian group when equipped with the addition $[u]_1 + [v]_1 = [u \oplus v]_1$. One finds that K_1 is a functor from the category of C^*-algebras to the category of Abelian groups. It turns out that K_1 satisfies all the same properties of Proposition 1.27 as K_0, and hence we shall not repeat them here.

One viewpoint, which is sometimes advantageous to consider, is that $K_1(A)$ can be seen as the K_0-group of the suspension of A. If A is any C^*-algebra, define the suspension $SA = C_0(\mathbb{R}, A)$.

Proposition 1.31. *For any C^*-algebra A, we have $K_1(A) \cong K_0(SA)$.*

The reason this viewpoint is helpful, is that it phrases K_1 in terms of K_0 which, in some circumstances, can be useful. Generalising this idea, we can define $K_n(A) = K(S^n A)$ for all C^*-algebras A, which on the surface introduces a countably infinite number of different K-groups. But we need not worry; there are only two K-groups for C^*-algebras as the following proposition shows.

Proposition 1.32 (Bott periodicity)**.** *For every C^*-algebra A, we have an isomorphism $K_1(SA) = K_0(A)$.*

Bott periodicity implies that, for any C^*-algebra A, we have

$$K_n(A) = \begin{cases} K_0(A) & \text{if } n \text{ is even} \\ K_1(A) & \text{if } n \text{ is odd} \end{cases}.$$

Consequently, there are only two K-groups. Recall that neither of the functors K_i for $i = 0, 1$ preserve short exact sequences in general, but that they are half-exact. The following proposition generalises this result by showing that any short exact sequence of C^*-algebras induces a six-term exact sequence in K-theory, which is highly useful in calculating the K-theory of C^*-algebras fitting in a short exact sequence.

Proposition 1.33 (Six-term exact sequence in K-theory)**.** *Let $0 \to I \to E \to A \to 0$ be a short exact sequence of C^*-algebras. Then there exists a six-term exact sequence in K-theory:*

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ {\scriptstyle \partial_1}\uparrow & & & & \downarrow{\scriptstyle \partial_0} \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(I) \end{array}$$

The exponential map ∂_0 and the index map ∂_1 in the above diagram are called the *boundary maps*; the precise definitions of both maps can be found in [64].

1.3 Groups and C^*-algebras

One way of constructing C^*-algebras with specific properties is by considering C^*-algebras constructed from groups in natural ways. The question of how the structure of a group C^*-algebra is influenced by the structure of the underlying group is an important question. We emphasise that this section by no means is a thorough study of the subject, as we shall only briefly discuss the construction of the reduced and the full group C^*-algebras and study the notion of amenability of groups. The go-to textbook for the subject of discrete groups within C^*-algebraic theory is [11].

Let G be a discrete group with unit e, and denote by $\lambda\colon G \to \mathbb{B}(\ell^2(G))$ the left regular representation $\lambda_g(\delta_h) = \delta_{gh}$, where $\{\delta_g \mid g \in G\}$ is the canonical orthonormal basis for $\mathbb{B}(\ell^2(G))$. Define the group ring $\mathbb{C}[G]$ as the collection of formal sums $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{C}$ non-zero only for finitely many $g \in G$. Equip the group ring with multiplication and involution given by

$$\Big(\sum_{g \in G} a_g g\Big)\Big(\sum_{h \in G} b_h h\Big) = \sum_{g,h \in G} a_g b_h gh \qquad \text{and} \qquad \Big(\sum_{g \in G} a_g g\Big)^* = \sum_{g \in G} \overline{a_g} g^{-1}.$$

Equipped with these two algebraic operations and the obvious addition, the group ring $\mathbb{C}[G]$ becomes a *-algebra, and we can extend the left regular representation to an injective *-homomorphism $\lambda\colon \mathbb{C}[G] \to \mathbb{B}(\ell^2(G))$ in an obvious way. There are two canonical ways of completing the group ring. The *full group C^*-algebra* of G is denoted by $C^*(G)$ and is the completion of the group ring $\mathbb{C}[G]$ with respect to the norm $\|x\| = \sup_\pi \|\pi(x)\|$ with the supremum taken over all cyclic *-representations of the group ring. For the purposes of this thesis, we shall mostly be interested in the reduced group C^*-algebra.

Definition 1.34. The *reduced group C^*-algebra* of G, denoted $C^*_r(G)$, is the completion of the group ring $\mathbb{C}[G]$ with respect to the norm $\|x\| = \|\lambda(x)\|$, with the right-hand norm being the usual norm on $\mathbb{B}(\ell^2(G))$.

In this way, the reduced group C^*-algebra is the completion of the group ring when viewed as a *-subalgebra of $\mathbb{B}(\ell^2(G))$; this viewpoint is meaningful as the left regular representation is an embedding $\mathbb{C}[G] \hookrightarrow \mathbb{B}(\ell^2(G))$. Both of these group C^*-algebras have interesting structure based upon the structure of the underlying discrete group. We shall study the connection between amenability of the group and quasidiagonality of the corresponding reduced group C^*-algebra, as it turns out that the two are equivalent. One important fact regarding the reduced group C^*-algebra is that it always admits faithful tracial states.

Proposition 1.35. *The function $\tau\colon C^*_r(G) \to \mathbb{C}$ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful tracial state on $C^*_r(G)$.*

Let us now see how the structure of the underlying group can impact the structure of a corresponding group C^*-algebra by way of an example, which is both of great importance in general C^*-algebraic study as well as in this thesis. We define an action of G on $\ell^\infty(G)$ by $s.f(t) = f(s^{-1}t)$ for all $s, t \in G$ and $f \in \ell^\infty(G)$.

Definition 1.36. A group G is *amenable* if there exists a left-invariant mean on G, i.e., if there exists a state $\mu\colon \ell^\infty(G) \to \mathbb{C}$ such that $\mu(s.f) = \mu(f)$ for all $f \in \ell^\infty(G)$ and $s \in G$.

There are several equivalent formulations of amenability. We only mention a few C^*-algebraic characterizations here out of interest.

Proposition 1.37. *If G is a discrete group, then the following are equivalent.*

(i) *G is amenable;*

(ii) *$C_r^*(G)$ is nuclear;*

(iii) *$C_r^*(G)$ admits a one-dimensional representation.*

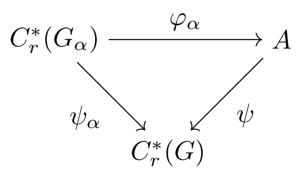
Even more characterisations can be found in [11, Theorem 2.6.8]. After proving the Tikuisis-White-Winter theorem, we shall add another equivalent formulation, namely that $C_r^*(G)$ is quasidiagonal. What this means, and how the notion of amenability and quasidiagonality are related, will be established in the later chapters.

We end this section with a result stating that the functor $C_r^*(\cdot)$ commutes with taking inductive limits.

Proposition 1.38. *If $(G_\alpha)_{\alpha\in\Lambda}$ is an inductive system of groups with the assumption that there exists an embedding $G_\alpha \hookrightarrow G_\beta$ whenever $\alpha \leq \beta$, then we have an isomorphism of C^*-algebras*

$$C_r^*(\varinjlim G_\alpha) \cong \varinjlim C_r^*(G_\alpha).$$

Proof. Define G to be the limit of the inductive system $(G_\alpha)_{\alpha\in\Lambda}$. Note that if H is a subgroup of H', then we have a natural embedding $C_r^*(H) \hookrightarrow C_r^*(H')$, see [11, Proposition 2.5.9]. In particular, this implies that whenever $\alpha \leq \beta$, we have an injective *-homomorphism $\varphi_{\beta,\alpha}\colon C_r^*(G_\alpha) \to C_r^*(G_\beta)$. Hence we have constructed an inductive system $(C_r^*(G_\alpha))_{\alpha\in\Lambda}$ with injective connecting maps in the category of C^*-algebras. Denote by A the inductive limit of this system, and let $\varphi_\alpha\colon C_r^*(G_\alpha) \to A$ be the boundary maps; these are all injective by injectivity of each $\varphi_{\alpha,\beta}$. Since we can also realise each G_α as a subgroup of G, there exist injective *-homomorphisms $\psi_\alpha\colon C_r^*(G_\alpha) \to C_r^*(G)$. Since the embeddings are canonical, it follows that $\psi_\alpha = \psi_\beta \circ \varphi_{\beta,\alpha}$. By the universal property of inductive limits, there exists a *-homomorphism $\psi\colon A \to C_r^*(G)$ such that the following diagram commutes for each $\alpha \in \Lambda$:



We claim that ψ is a *-isomorphism. First we show that ψ is injective by showing that it is isometric. Note that we can realise A as the closure of $\bigcup_{\alpha\in\Lambda} C_r^*(G_\alpha)$ since each φ_α is injective. Let $a \in A$ be arbitrary, then we can realise a as the limit of a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in C_r^*(G_{\alpha_n})$ for each n. Invoking injectivity of ψ_α for each α as well as the above commutative diagram, we get

$$\|\psi(a)\| = \lim_{n\to\infty} \|\psi_{\alpha_n}(a_n)\| = \lim_{n\to\infty} \|a_n\| = \|a\|$$

proving injectivity of ψ.

Lastly, we prove that ψ is surjective. Using that G is the inductive limit of the inductive sequence $(G_\alpha)_{\alpha\in\Lambda}$, we can realise each element in $C_r^*(G)$ as the norm-limit of linear combinations of elements in $\mathbb{C}[G_\alpha]$ for $\alpha \in \Lambda$. Since the latter C^*-algebras naturally embeds into A, as A is the closure of $\bigcup_{\alpha\in\Lambda} C_r^*(G_\alpha)$, one easily verifies, using the above commutative diagram as well as continuity and linearity of the maps involved, that ψ is surjective. □

1.4 Classification of C^*-algebras

A prevalent idea throughout many mathematical disciplines is the idea of classifying things such as algebraic structures, their elements or certain morphisms. For C^*-algebras in particular it is no different, and the classification theory of C^*-algebras have a rich history indeed. Some of the more elementary classification results are those for Abelian C^*-algebras, Proposition 1.2, and finite-dimensional C^*-algebras, Proposition 1.3. In both of these cases, the classification results are not particularly deep and can be proved with relative ease, but this is not always the case. In the last couple of decades, there has been an on-going classification programme due to Elliott, which seeks to classify a large class of C^*-algebras via their K-theoretic data as well as their tracial state simplices. In this section, we shall discuss the historical context of this classification programme and in a later chapter, we shall see how this classification is, in some sense, almost complete.

The first class of C^*-algebras to be classified explicitly by their K-theoretic data was the class of AF-algebras. Historically, this was not the first classification of AF-algebras, since Bratteli in a paper [5] from 1972, in which he defined the AF-algebras, classified them combinatorially by introducing certain graphs now known as Bratteli diagrams. A few years later, Elliott provided a different classification of the AF-algebras, this time precisely by using their K-theoretic data. This was effectively the first time K-theory had been used explicitly as a classification tool, and it marked the beginning of Elliott's classification program. Before we can state the classification, we need to define what is turning out to be a crucial part of the classification invariant. If A is any C^*-algebra, we define the *dimension range* $\mathcal{D}_0(A)$ by $\mathcal{D}_0(A) = \{[p]_0 \mid p \in \mathcal{P}(A)\}$. The following theorem is formulated as in [59, Theorem 1.3.3], and the original proof can be found in [17].

Theorem 1.39 (Elliott, 1976)**.** *Let A and B be AF-algebras.*

(i) *If there exists a group isomorphism $\alpha\colon K_0(A) \to K_0(B)$ such that $\alpha(\mathcal{D}_0(A)) = \mathcal{D}_0(B)$, then there exists a *-isomorphism $\varphi\colon A \to B$ with $K_0(\varphi) = \alpha$.*

(ii) *In the case A and B are both unital, then if there exists a group isomorphism $\alpha\colon K_0(A) \to K_0(B)$ with $\alpha(K_0(A)^+) = K_0(B)^+$ and $\alpha([1_A]_0) = [1_B]_0$, then there exists a *-isomorphism $\varphi\colon A \to B$ with $K_0(\varphi) = \alpha$.*

The classification invariant, at least in the unital case, actually has a very nice and intrinsic structure, as the classification invariant are Riesz groups.

Definition 1.40. Let (G, G^+) be an ordered Abelian group. We say that

(i) G is *unperforated* if $ng \geq 0$ implies $g \geq 0$ for all $n \in \mathbb{N}$ and $g \in G$;

(ii) G has *Riesz interpolation* if for every $g_1, g_2, h_1, h_2 \in G$ with $g_i \leq h_j$ for $i, j = 1, 2$, there exists $k \in G$ with $g_i \leq k \leq h_j$ for $i, j = 1, 2$.

(iii) G is *almost unperforated* if, whenever $(n+1)g \leq nh$ for some $g, h \geq 0$ and $n \in \mathbb{N}_0$, we have $g \leq h$;

(iv) G is *weakly unperforated* if $ng > 0$ implies $g > 0$ for all $n \in \mathbb{N}$ and $g \in G$.

If (G, G^+) satisfy (i) and (ii), we call the pair a *Riesz group.* Note that a weakly unperforated ordered Abelian group is unperforated if and only if it is torsion-free.

Proposition 1.41 (Effros-Handelmann-Shen, 1980)**.** *If A is an AF-algebra, then $(K_0(A), K_0(A)^+)$ is a Riesz group. Conversely, every countable Riesz group can be realised as $(K_0(A), K_0(A)^+)$ for some AF-algebra A. If, additionally, (G, G^+) has an order unit u, then there exists a unital AF-algebra A such that $(K_0(A), K_0(A)^+, [1]_0) \cong (G, G^+, u)$.*

For the proof, we refer to [16, Theorem 2.2] and [64, Proposition 7.2.8]. The case in which the ordered Abelian group admits an order unit follows from the previous results by invoking [29, Corollary 3.18].

When we say that AF-algebras were the first class of C^*-algebras to be classified explicitly by their K-theoretic data, there are valid objections to be made, as there is a subclass of AF-algebras, namely UHF-algebras, which was classified prior to AF-algebras, and where the classification invariant is in fact based on K-theoretic data, although this is somewhat anachronistic.

Definition 1.42. A C^*-algebra A is a *UHF-algebra* if it is isomorphic to a sequential inductive limit of simple finite-dimensional C^*-algebras with unital connecting maps.

An important fact about UHF-algebras is that they always admit a unique tracial state, which is the one induced by the unique tracial state on simple finite-dimensional C^*-algebras. Observe that a trace on a UHF-algebra is always faithful by simplicity.

UHF-algebras were first introduced by Glimm in 1960 [28], where he classified them by finding a way to associate to each UHF-algebra a unique supernatural number, that is, a number of the form $\prod_p p^{n_p}$ with $n_p \in \mathbb{N}_0 \cup \{\infty\}$, and the product to be taken over the prime numbers; note that this generalises the idea behind the uniqueness of prime factorisation. As UHF-algebras are clearly AF-algebras, we can classify them completely by Theorem 1.39, and one can derive Glimm's classification by supernatural numbers via Elliott's classification — for more details on this, we refer to [64, Chapter 7.4]. One UHF-algebra of particular importance in this thesis is the universal UHF-algebra $\mathcal{Q}$, which is the unique UHF-algebra with $K_0(\mathcal{Q}) = \mathbb{Q}$ or, using the notion of supernatural numbers, it is the UHF-algebra associated to the supernatural number $\prod_p p^\infty$. This is the universal UHF-algebra in the sense that it contains an isomorphic copy of all UHF-algebras and hence, in a sense, contains the information of each. Another way of expressing the universal UHF-algebra is as an infinite tensor product $\mathcal{Q} = \bigotimes_{n=1}^\infty M_n(\mathbb{C})$, which is to be understood as the inductive limit of the tensor products $\bigotimes_{n=1}^N M_n(\mathbb{C})$. Since all UHF-algebras admit unique tracial states, there exists a unique tracial state $\tau_\mathcal{Q}$ on $\mathcal{Q}$, induced by the unique tracial states on matrix algebras over $\mathbb{C}$. There exists, for any $n \in \mathbb{N}$, a conditional expectation $E_n \colon \mathcal{Q} \to M_n(\mathbb{C})$, which is trace-preserving in the sense that $\tau_\mathcal{Q} = \mathrm{Tr}_n \circ E_n$.

As stated several times in this section, AF-algebras were classified completely by their K-theoretic data — in fact, only by the data of their ordered K_0-groups. In [18], Elliott classified a larger class of C^*-algebras, and, again, the classification was only by K-theoretic data, this time including the K_1-groups. We follow the definition given in [59, Definition 3.2.1].

Definition 1.43. An *A$\mathbb{T}$-algebra* is a C^*-algebra, which can be realised as an inductive limit of C^*-algebras of the form $C(\mathbb{T}) \otimes F$, where F is a finite-dimensional C^*-algebra.

In order to understand the classification theorem on A$\mathbb{T}$, we need to briefly define what is known as graded K-theory. For any C^*-algebra, we define the *graded K-group* $K_*(A) = K_0(A) \oplus K_1(A)$, as well as the *graded dimension range*

$$\mathcal{D}_*(A) = \{([p]_0, [u]_1) \mid p \in \mathcal{P}(A), u \in \mathcal{U}(pAp)\}.$$

Note that $\mathcal{D}_*(A)$ clearly lies in $K_*(A)$. We say that a map $\alpha\colon K_*(A) \to K_*(B)$ is a *graded group homomorphism*, if it is a group homomorphism such that $\alpha(K_i(A)) \subseteq K_i(B)$ for $i = 0, 1$. With this terminology in mind, we mention the classification result on A$\mathbb{T}$-algebras due to Elliott. We express it as formulated in [59, Theorem 3.2.6], where it is stated without proof; a proof can be found in the original paper [18, Theorem 7.1].

Theorem 1.44 (Elliott, 1989)**.** *Let A and B be A$\mathbb{T}$-algebras of real rank zero. Then A and B are isomorphic as C^*-algebras if and only if there exists a graded group isomorphism $\alpha\colon K_*(A) \to K_*(B)$ satisfying that $\alpha(\mathcal{D}_*(A)) = \mathcal{D}_*(B)$. In the affirmative case, there exists a *-isomorphism $\varphi\colon A \to B$ such that $K_*(\varphi) = \alpha$.*

Note the similarities to the classification of AF-algebras, Theorem 1.39. This led to the conjecture that one can use K-theory as a classification tool, and Elliott's classification program was exactly to classify C^*-algebras by the data from their K-theory and tracial simplex. We shall, in a later chapter, examine the classification program in higher detail and explicitly mention what the classification invariant, known as Elliott's invariant, actually is, and how it is supposed to be understood. Moreover, we shall mention how this all relates to the Tikuisis-White-Winter theorem.

2 Quasidiagonality and related concepts

Having recapped the most central aspects of C^*-algebraic theory for this thesis, we are now ready to study new material. More specifically, and as the title suggests, we shall study quasidiagonality of C^*-algebras. It turns out that, in studying quasidiagonality, it is a fruitful idea to understand ultrafilters and ultrapowers, since quasidiagonality can be understood in terms of the ultrapower $\mathcal{Q}_\omega$ of the universal UHF-algebra $\mathcal{Q}$. For this reason, we shall dedicate a section to study ultrafilters and corresponding ultrapowers. Subsequently, we shall examine quasidiagonality in greater detail, including establishing a link between the approximation property for C^*-algebras and the concept in single operator theory from where the terminology originates. The concept of AF-embeddability is also discussed with the intention of introducing the reader to this area of on-going research, which is examined in more depth in later chapters. Lastly, we study quasidiagonality and amenability of tracial states, which are closely related to the similarly named properties of C^*-algebras, and which are essential for the Tikuisis-White-Winter theorem.

2.1 Ultrafilters and ultrapowers

In both the original as well as in Schafhauser's proof of the Tikuisis-White-Winter theorem, ultrapowers play an important role, partially due to appearances in important characterisations of quasidiagonality, which arises from the fact that approximative properties, informally speaking, become exact when passing to ultrapowers. We shall formalise this idea by giving a few examples in the following, but first we shall define the concept of an ultrafilter and subsequently of ultraproducts of C^*-algebras.

Definition 2.1. A *filter* ω on $\mathbb{N}$ is a non-empty family of subsets with the following properties:

(i) $\emptyset \notin \omega$;

(ii) For each $A, B \in \omega$, there exists $C \in \omega$ with $C \subseteq A \cap B$;

(iii) For each $A \in \omega$ and $A \subseteq B \subseteq \mathbb{N}$, we have $B \in \omega$.

We say that a filter ω is an *ultrafilter* if it is maximal in the sense that if ω' is a filter such that $\omega \subseteq \omega'$, then $\omega = \omega'$. A *free filter* is a filter ω for which $\bigcap_{A\in\omega} A = \emptyset$.

Condition (ii) above is called the *finite intersection property*, and (iii) is called *upwards directedness.* The following proposition gives a different and often useful characterisation of maximality of filters in terms of partitions of $\mathbb{N}$.

Proposition 2.2. *Let ω be a filter on $\mathbb{N}$. The following are equivalent:*

(i) *ω is an ultrafilter;*

(ii) *If $A \subseteq \mathbb{N}$, then either $A \in \omega$ or $A^c \in \omega$;*

(iii) *For all partitions $\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_n$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, there exists a unique $j \in \{1, \ldots, n\}$ with $A_j \in \omega$.*

Proof. (i)⇒(ii): Let $A \subseteq \mathbb{N}$ be arbitrary and suppose that $A^c \notin \omega$. Note that, in particular, we have $A \neq \emptyset$. Consider the set

$$\omega' = \{B \subseteq \mathbb{N} \mid \exists A_0 \in \omega : A_0 \cap A \subseteq B\}.$$

One easily verifies that ω' is a filter and that $\omega \subseteq \omega'$, and hence by assumption $\omega = \omega'$. Since $A \in \omega'$ and $\omega' = \omega$, we find that $A \in \omega$.

(ii)⇒(i): Suppose that for any $A \subseteq \mathbb{N}$, either $A \in \omega$ or $A^c \in \omega$, and let ω' be a filter containing ω. If $\omega \neq \omega'$, then there exists $A \in \omega' \setminus \omega$, and consequently $A^c \in \omega$. However, then $A^c \in \omega'$, which is impossible, as this in conjunction with the finite intersection property would imply that $\emptyset = A \cap A^c \in \omega'$.

(ii)⇒(iii): Let $\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_n$ be a partition and assume that $A_j \notin \omega$ for all $j \in \{1, \ldots, n\}$. Then $A_j^c \in \omega$ for all j, and as

$$\mathbb{N} = A_1 \cup A_2 \cup \cdots \cup A_n = (A_1^c \cap A_2^c \cap \cdots \cap A_n^c)^c$$

we find that $\emptyset \in \omega$, which is impossible. Consequently, there exists at least one j with $A_j \in \omega$, and the finite intersection property along with the fact that the empty set does not belong to any filter proves that this j is unique.

(iii)⇒(ii): This is immediate, just take $n = 2$. □

Let us look at some examples.

Example 2.3. The following families of subsets on $\mathbb{N}$ are all filters on $\mathbb{N}$.

(i) For each fixed $n \in \mathbb{N}$, the *principal filter* defined by $\omega_n = \{A \subseteq \mathbb{N} \mid n \in A\}$ is an ultrafilter. It is not free.

(ii) The *Fréchet filter*, also known as the *cofinite filter*, defined by $\omega_\infty = \{A \subseteq \mathbb{N} \mid A^c \text{ is finite}\}$ is a free filter. In fact, one can easily verify that a filter is free if and only if it contains the cofinite filter.

The following proposition is an easy application of Zorn's lemma.

Proposition 2.4. *All filters on $\mathbb{N}$ are contained in an ultrafilter. In particular, there exists a free ultrafilter on $\mathbb{N}$.*

The reason we are interested in filters is that they give us a way of generalising convergence of sequences or, in general, of nets. We only phrase this in terms of metric spaces.

Definition 2.5. Let (M, d) be a metric space, and let ω be a filter on $\mathbb{N}$. We say that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ *converges along* ω if there exists $x \in M$ such that for any $\varepsilon > 0$, we have $\{n \in \mathbb{N} \mid d(x_n, x) < \varepsilon\} \in \omega$. We denote this by $\lim_{n\to\omega} x_n = x$.

Note that the limit point of a sequence, if it exists, is unique by the finite intersection property. Convergence along a free filter coincide with the usual notion of convergence whenever the latter exists by the remark in Example 2.3(ii). Convergence of sequences along ultrafilters is particularly pleasant by the following proposition.

Proposition 2.6. *If (M, d) is a compact metric space and ω is an ultrafilter on $\mathbb{N}$, then for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$, the limit $\lim_{n\to\omega} x_n$ exists.*

Proof. Assume that the limit along ω does not exist, that is, for all $y \in M$ there exists some $\varepsilon > 0$ such that $\{n \in \mathbb{N} \mid d(x_n, y) < \varepsilon\} \notin \omega$. Using upwards directedness of filters, we can phrase this as saying that, for any $y \in M$, there exists an open neighbourhood U_y of y such that $\{n \in \mathbb{N} \mid x_n \in U_y\} \notin \omega$. Since $(U_y)_{y\in M}$ is an open cover of M, it follows by compactness that there exists a finite subcover, say, $U_{y_1}, \ldots, U_{y_m}$. For each $n \in \mathbb{N}$, we can find some $i = 1, \ldots, m$ such that $x_n \in U_{y_i}$; if $x_n \in U_{y_i} \cap U_{y_j}$, choose the minimum of i and j. Then we have constructed a partition of $\mathbb{N}$ where each part, by construction, cannot lie in ω; but such a construction is impossible by the equivalences of (i) and (iii) in Proposition 2.2. □

Since any free filter contains the cofinite filter ω_∞, and as convergence along ω_∞ is precisely the same as normal sequential convergence, we immediately get the following corollary.

Corollary 2.7. *If $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a bounded sequence, then its limit along any ultrafilter exists. Moreover, if ω is a free ultrafilter and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\omega} x_n = x$.*

We now turn our attention to examining how we can use ultrafilters in the theory of operator algebras. *For the remainder of this thesis, we let ω be a fixed free ultrafilter ω on $\mathbb{N}$.* Let $A_1, A_2, \ldots$ be C^*-algebras and consider the *product algebra*

$$\ell^\infty(A_n, \mathbb{N}) = \left\{ (a_n)_{n\in\mathbb{N}} \mid a_n \in A_n, \sup_{n\in\mathbb{N}} \|a_n\| < \infty \right\}.$$

This is easily verified to be a C^*-algebra with the norm $\|(a_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \|a_n\|$. Let $c_\omega(A_n, \mathbb{N})$ be the set

$$c_\omega(A_n, \mathbb{N}) = \left\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A_n, \mathbb{N}) \mid \lim_{n\to\omega} \|a_n\| = 0 \right\}.$$

Then $c_\omega(A_n, \mathbb{N})$ is a closed two-sided ideal in $\ell^\infty(A_n, \mathbb{N})$; for example, if $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in c_\omega(A_n, \mathbb{N})$, then as

$$0 \leq \|a_n + b_n\| \leq \|a_n\| + \|b_n\|$$

for all $n \in \mathbb{N}$, one easily finds that $\lim_{n\to\omega} \|a_n + b_n\| = 0$ and hence $c_\omega(A_n, \mathbb{N})$ is additively closed, and in a similar fashion one can show that $c_\omega(A_n, \mathbb{N})$ is multiplicatively closed as well as a two-sided ideal in $\ell^\infty(A_n, \mathbb{N})$. In order to show that $c_\omega(A_n, \mathbb{N})$ is norm-closed, assume that $((a_n^{(k)})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ is a sequence in $c_\omega(A_n, \mathbb{N})$ converging in norm to some element $(a_n)_{n\in\mathbb{N}} \in \ell^\infty(A_n, \mathbb{N})$. Fix $\varepsilon > 0$ and find $K \in \mathbb{N}$ such that $\left\| a_n^{(k)} - a_n \right\| < \varepsilon$ for all $k \geq K$ and $n \in \mathbb{N}$. Then,

$$0 \leq \|a_n\| \leq \left\| a_n^{(k)} - a_n \right\| + \left\| a_n^{(k)} \right\| < \left\| a_n^{(k)} \right\| + \varepsilon$$

for $k \geq K$ and $n \in \mathbb{N}$. Consequently, by taking the limit along ω, we see that $0 \leq \lim_{n\to\omega} \|a_n\| < \varepsilon$ for all $\varepsilon > 0$ and hence $(a_n)_{n\in\mathbb{N}} \in c_\omega(A_n, \mathbb{N})$.

All in all, this entails that we can take the quotient

$$\ell_\omega(A_n, \mathbb{N}) = \ell^\infty(A_n, \mathbb{N}) / c_\omega(A_n, \mathbb{N})$$

which we call the *ultraproduct* of the sequence. Note that, in general, limits along ultrafilters depend on the choice of ultrafilter, and hence an ultraproduct depends on the choice of ultrafilter — this is the reason for keeping ω fixed throughout the thesis. The case where the sequence of C^*-algebras is constant, i.e., where $A_n = A$ for all n, is of great interest to us. We introduce the simplified notation

$$\ell^\infty(A) = \ell^\infty(A, \mathbb{N}), \qquad c_\omega(A) = c_\omega(A, \mathbb{N}), \qquad A_\omega = \ell_\omega(A, \mathbb{N})$$

and call A_ω the *ultrapower* of A.

As mentioned before, one of the primary interests in ultrapowers is the fact that approximate properties of C^*-algebras hold exactly in the ultrapower. Let us give an explicit example to show what we mean; later in the thesis we shall use this to see how an ultrafilter formulation encompasses the approximative behaviour of quasidiagonality as an exact property. Suppose that $\varphi_n \colon A \to B$ is a sequence of c.c.p. map between separable C^*-algebras, which is approximately multiplicative, that is, for all $a, b \in A$, we have

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0.$$

We can take the direct products of the c.c.p. maps φ_n, i.e., define a c.c.p. map $\varphi \colon A \to \ell^\infty(B)$ by $\varphi(a) = (\varphi_n(a))_{n\in\mathbb{N}}$ for $a \in A$. This, in turn, defines a map $\varphi_\omega \colon A \to B_\omega$ by $\varphi_\omega = \pi_\omega \circ \varphi$, where $\pi_\omega \colon \ell^\infty(B) \to B_\omega$ is the quotient map. Then it follows directly from the approximate multiplicativity of the sequence $(\varphi_n)_{n\in\mathbb{N}}$ that φ_ω is a multiplicative c.c.p. map, that is, φ_ω is a *-homomorphism. This example is just one of many such approximate properties, which are exact in the ultrapower.

One very nice, and quite abstract, result on ultrapowers is the following proposition called Kirchberg's ε-test, which in some sense is a generalisation of diagonal arguments. The reader should note the high level of generality in the proposition, which is part of its beauty. The proof follows that of [40, Lemma 3.1].

Proposition 2.8 (Kirchberg's ε-test)**.** *Let $X_1, X_2, \ldots$ be a sequence of non-empty sets. Suppose that, for each $k \in \mathbb{N}$, there exists a sequence $(f_n^{(k)})_{n\in\mathbb{N}}$ of functions $f_n^{(k)} \colon X_n \to [0, \infty)$. Define for each $k \in \mathbb{N}$ the function $f_\omega^{(k)} \colon \ell^\infty(X_n, \mathbb{N}) \to [0, \infty]$ by*

$$f_\omega^{(k)}(s_1, s_2, \ldots) = \lim_{n\to\omega} f_n^{(k)}(s_n).$$

Suppose that, for each $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an element $s = (s_1, s_2, \ldots) \in \ell^\infty(X_n, \mathbb{N})$ such that $f_\omega^{(k)}(s) < \varepsilon$ for all $k = 1, \ldots, m$, then there exists an element $t = (t_1, t_2, \ldots) \in \ell^\infty(X_n, \mathbb{N})$ such that $f_\omega^{(k)}(t) = 0$ for all $k \in \mathbb{N}$.

Proof. Observe first of all that the functions $f_\omega^{(k)}$ exist by Proposition 2.6. For each $n \in \mathbb{N}$, we define a sequence of subsets $(X_{n,m})_{m\in\mathbb{N}_0}$ of X_n by the following: Put $X_{n,0} = X_n$ and, for $m \geq 1$, define

$$X_{n,m} = \left\{ s \in X_n \mid \max\{f_n^{(1)}(s), \dots, f_n^{(m)}(s)\} < \frac{1}{m} \right\}.$$

Note that $(X_{n,m})_{m\in\mathbb{N}_0}$ is a decreasing sequence of subsets of X_n. Define the map $m\colon \mathbb{N} \to \mathbb{N}$ by $m(n) = \max\{\ell \leq n \,|\, X_{n,\ell} \neq \emptyset\}$.

Let $k \geq 1$ be arbitrary, then, by assumption, there exists some $s = (s_n)_{n\in\mathbb{N}} \in \ell^\infty(X_n, \mathbb{N})$ satisfying that $f_\omega^{(j)}(s) < \frac{1}{k}$ for $j = 1, \dots, k$. Hence, for each $k \in \mathbb{N}$, we find that the set

$$Z_k = \{n \in \mathbb{N} \,|\, \max\{f_n^{(1)}(s_n), \dots, f_n^{(k)}(s_n)\} < \frac{1}{k}\}.$$

necessarily belongs to ω, and, in particular, is non-empty. Observe that if $n \in Z_k$, then $s_n \in X_{n,k}$ and, hence $X_{n,k} \neq \emptyset$ for all k whenever $n \in Z_k$. This gives the inequalities $\min\{k, n\} \leq m(n) \leq n$ for $n \in Z_k$. Define, for each $k \in \mathbb{N}$, the set $Y_k = \{n \in \mathbb{N} \,|\, k \leq m(n)\}$, then we get the inclusion

$$Z_k \setminus \{1, \dots, k-1\} \subseteq Y_k.$$

Since ω is a free filter, it contains the cofinite filter, which — along with the fact that ω is an ultrafilter — implies that $Y_k \in \omega$ for each $k \geq 1$; in particular, Y_k is never empty. Since $k \leq m(n)$ for each $n \in Y_k$, we calculate explicitly that

$$\lim_{n\to\omega} \frac{1}{m(n)} = \liminf_{n\to\omega} \frac{1}{m(n)} \leq \limsup_{n\to\omega} \frac{1}{m(n)} = \inf_{A\in\omega} \sup_{n\in A} \frac{1}{m(n)} \leq \inf_{k\in\mathbb{N}} \sup_{n\in Y_k} \frac{1}{m(n)} \leq \inf_{k\in\mathbb{N}} \frac{1}{k} = 0.$$

By definition of the function $n \mapsto m(n)$, we find that for each $n \in \mathbb{N}$ there exists some element $t_n \in X_{n,m(n)} \subseteq X_n$. Put $t = (t_n)_{n\in\mathbb{N}} \in \ell^\infty(X_n, \mathbb{N})$, then one finds that

$$0 \leq f_\omega^{(k)}(t) = \lim_{n\to\omega} f_n^{(k)}(t_n) \leq \lim_{n\to\omega} \frac{1}{m(n)} = 0$$

which completes the proof. □

We shall use this result shortly to show that approximately unitarily equivalent maps $\varphi, \psi\colon A \to B_\omega$, where A is a separable C^*-algebra, and B is a unital C^*-algebra, are actually unitarily equivalent; this is yet another approximate property, which is exact by passing to ultrapowers. However, before we are able to prove this result, we need a few lemmas, which are also of independent interest.

Lemma 2.9. *Let A be a C^*-algebra, and let $\pi_\omega : \ell^\infty(A) \to A_\omega$ be the ultrapower quotient map. Then for each $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$, we have $\|\pi_\omega(a)\| = \lim_{n\to\omega} \|a_n\|$.*

Proof. The map $\pi_\omega(a) \mapsto \lim_{n\to\omega} \|a_n\|$ defines a C^*-norm on A_ω, and uniqueness of C^*-norms implies the desired result. □

Recall for the next proposition that a surjection need not lift projections, isometries or unitaries to elements with similar properties.

Lemma 2.10. *Let A be a C^*-algebra, and let $\pi_\omega\colon \ell^\infty(A) \to A_\omega$ be the quotient map onto the ultrapower. Then each projection in A_ω lifts to a projection on $\ell^\infty(A)$. If, moreover, A is unital, then each isometry and each unitary element in A_ω lifts to an isometry, respectively, a unitary element in $\ell^\infty(A)$.*

Proof. We only prove it for the unitaries, as the other statements are proved in similar ways. Let $u \in A_\omega$ be a unitary element, and let $a = (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ be any lift. Then,

$$0 = \|\pi_\omega(a^*a) - 1\| = \lim_{n\to\omega} \|a_n^* a_n - 1\|$$

and, similarly, $\lim_{n\to\omega} \|a_n a_n^* - 1\| = 0$. Let $X \in \omega$ be some element in the free ultrafilter for which $\|a_n^* a_n - 1\| < 1$ and $\|a_n a_n^* - 1\| < 1$ for all $n \in X$. In particular, $a_n \in A$ is invertible for all $n \in X$, and hence, by polar decomposition of invertible elements in a C^*-algebra, we can write $a_n = w_n |a_n|$ for each $n \in X$, where w_n is unitary in A. If $n \notin X$ put $w_n = 1$. Then $w = (w_n)_{n\in\mathbb{N}} \in \ell^\infty(A)$ is unitary, and a standard continuous functional calculus trick proves that $\lim_{n\to\omega} \|w_n - a_n\| = 0$, and hence $\pi_\omega(w) = u$ as desired. □

Definition 2.11. Let A, B be C^*-algebras and assume that B is unital. Let $\varphi, \psi\colon A \to B$ be *-homomorphisms. We say that φ and ψ are *approximately unitarily equivalent* if there is a net of unitaries $(u_\alpha)_{\alpha\in\Lambda}$ in B such that $\mathrm{Ad}(u_\alpha) \circ \varphi \to \psi$ in the point-norm topology.

In the separable case, we can restrict ourselves to considering sequences of unitaries. The following proposition is the promised showcase of Kirchberg's ε-test.

Proposition 2.12. *Let A be a separable C^*-algebra, and let B be a unital C^*-algebra. Suppose that $\varphi, \psi\colon A \to B_\omega$ are approximately unitarily equivalent *-homomorphisms, then they are unitarily equivalent.*

Proof. We shall use Kirchberg's ε-test, see Proposition 2.8. By separability of A, let $\{x^{(1)}, x^{(2)}, \ldots\}$ be a countable, dense subset of A, and find for each $k \in \mathbb{N}$ lifts $(a_n^{(k)})_{n\in\mathbb{N}} \in \ell^\infty(B)$ of $\varphi(x^{(k)})$ and $(b_n^{(k)})_{n\in\mathbb{N}} \in \ell^\infty(B)$ of $\psi(x^{(k)})$. Let $X_n = \mathcal{U}(B)$ for each $n \in \mathbb{N}$, and let $f_n^{(k)}\colon X_n \to [0,\infty)$ be given by

$$f_n^{(k)}(u) = \left\|\mathrm{Ad}(u) \circ a_n^{(k)} - b_n^{(k)}\right\|, \qquad u \in X_n$$

for each $k, n \in \mathbb{N}$. Using Lemma 2.9, we find that

$$f_\omega^{(k)}(u) = \lim_{n\to\omega} \left\|\mathrm{Ad}(u_n) \circ a_n^{(k)} - b_n^{(k)}\right\| = \left\|\mathrm{Ad}(\pi_\omega(u)) \circ \varphi(x^{(k)}) - \psi(x^{(k)})\right\|$$

for all $u = (u_n)_{n\in\mathbb{N}} \in \ell^\infty(X_n, \mathbb{N})$. Since φ and ψ are approximately unitarily equivalent, and since unitaries in B_ω lift to unitaries in $\ell^\infty(B)$ by Lemma 2.10, we have for each $\varepsilon > 0$ and $m \in \mathbb{N}$ that there exists $u = (u_n)_{n\in\mathbb{N}} \in \ell^\infty(X_n, \mathbb{N})$ such that $f_\omega^{(k)}(u) < \varepsilon$ for all $k = 1, \ldots, m$. Then Kirchberg's ε-test, Proposition 2.8, states that there exists some element $v = (v_n)_{n\in\mathbb{N}} \in \ell^\infty(X_n, \mathbb{N})$ such that

$$0 = f_\omega^{(k)}(v) = \left\|\mathrm{Ad}(\pi_\omega(v)) \circ \varphi(x^{(k)}) - \psi(x^{(k)})\right\|$$

for all $k \in \mathbb{N}$. Since the collection $\{x^{(1)}, x^{(2)}, \ldots\}$ is dense in A, we conclude that φ and ψ are unitarily equivalent. $\square$

We end the examination of ultrapowers of C^*-algebras by examining some specific properties of the ultrapower $\mathcal{Q}_\omega$, which we shall use later on in the thesis. If we denote by $\tau_{\mathcal{Q}}$ the unique tracial state on $\mathcal{Q}$, then we can induce a tracial state τ_ω in $\mathcal{Q}_\omega$ by $\tau_\omega(x) = \lim_{n\to\omega} \tau_{\mathcal{Q}}(x_n)$ for $x \in \mathcal{Q}_\omega$ where $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{Q})$ is a lift of x. In fact, this is the only tracial state on $\mathcal{Q}_\omega$ by [50, Theorem 8]. The next result states that any corner of a matrix algebra over $\mathcal{Q}$ (or $\mathcal{Q}_\omega$) is automatically isomorphic to $\mathcal{Q}$ (or $\mathcal{Q}_\omega$).

Proposition 2.13. *Let $n \in \mathbb{N}$ be arbitrary, then the following hold:*

(i) *If $p \in M_n(\mathcal{Q})$ is any non-zero projection, then $pM_n(\mathcal{Q})p \cong \mathcal{Q}$;*

(ii) *If $p \in M_n(\mathcal{Q}_\omega)$ is any non-zero projection, then $pM_n(\mathcal{Q}_\omega)p \cong \mathcal{Q}_\omega$.*

Proof. (i): As UHF-algebras are simple, any non-zero projection in an UHF-algebra, in particular p, is necessarily full. In other words, $pM_n(\mathcal{Q})p$ is a full corner in the separable C^*-algebra $M_n(\mathcal{Q})$, and [6, Corollary 2.6] provides an isomorphism $pM_n(\mathcal{Q})p \otimes \mathbb{K}(H) \cong M_n(\mathcal{Q}) \otimes \mathbb{K}(H)$. Note that, being the universal UHF-algebra, we have the isomorphism $M_n(\mathcal{Q}) \cong \mathcal{Q}$, and hence stability of K-theory, see Proposition 1.27(v), provides the following chain of isomorphisms:

$$K_0(pM_n(\mathcal{Q})p) \cong K_0(pM_n(\mathcal{Q})p \otimes \mathbb{K}(H)) \cong K_0(M_n(\mathcal{Q}) \otimes \mathbb{K}(H)) \cong K_0(\mathcal{Q} \otimes \mathbb{K}(H)) \cong K_0(\mathcal{Q}) = \mathbb{Q}.$$

Identify $[p]_0$ with some non-zero $q \in \mathbb{Q}$ under this identification, then the linear map $\mathbb{Q} \to \mathbb{Q}$ defined by $1 \mapsto q$ provides an isomorphism of the pairs $(K_0(\mathcal{Q}), [1]_0)$ and $(K_0(pM_n(\mathcal{Q})p), [p]_0)$, but such pairs form a classification invariant of UHF-algebras by [64, Theorem 7.4.5].

(ii): Use Lemma 2.10 and the isomorphism $M_n(\mathcal{Q}) \cong \mathcal{Q}$ to find a sequence of projections $(p_n)_{n\in\mathbb{N}}$ in $\ell^\infty(\mathcal{Q})$ lifting the projection p. Since taking ultrapowers commutes with taking matrix algebras, we can use (i) to obtain the following chain of isomorphisms

$$\mathcal{Q}_\omega \cong \ell_\omega(M_n(\mathcal{Q})) \cong \ell_\omega(p_k M_n(\mathcal{Q}) p_k, \mathbb{N}) \cong \ell_\omega(M_n(p_k \mathcal{Q} p_k), \mathbb{N}) \cong M_n(\ell_\omega(p_k \mathcal{Q} p_k, \mathbb{N})) \cong pM_n(\mathcal{Q}_\omega)p$$

as desired. $\square$

While these isomorphisms may seem inconsequential, we shall use them in an important characterisation of quasidiagonal tracial states in a later chapter.

We now turn our attention to ultrapowers of von Neumann-algebras, as the construction changes slightly compared to the C^*-algebraic analogue, and as there are some specific results we need to study. Let M be a von Neumann-algebra with a faithful tracial state τ. Define the tracial norm $\|\cdot\|_2$ on M by $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in M$ and consider the closed two-sided ideal

$$c^\omega(M) = \left\{ (x_n) \in \ell^\infty(M) \mid \lim_{n\to\omega} \|x_n\|_2 = 0 \right\}.$$

We define the *tracial ultrapower* M^ω as the quotient $M^\omega = \ell^\infty(M)/c^\omega(M)$ and the map $\tau^\omega \colon M^\omega \to \mathbb{C}$ by $\tau^\omega(x) = \lim_{n\to\omega} \tau(x_n)$, where $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(M)$ is any lift of x. Then τ^ω defines a faithful tracial state on M^ω.

Proposition 2.14. *If M is a von Neumann-algebra with a faithful tracial state τ, then the tracial ultrapower M^ω is a von Neumann-algebra equipped with the normal faithful tracial state τ^ω.*

Proof. It suffices to prove that M^ω is isometrically isomorphic to the von Neumann-algebra generated by the GNS-representation $(H_\tau, \pi_\tau, \xi_\tau)$ associated to τ^ω; then normality of τ^ω follows automatically. Since $\|x\|_2 = \|\pi_\tau(x)\xi_\tau\|$ for all $x \in M^\omega$, it suffices by [74, Corollary 19.6] to show that the closed unit ball in M^ω is complete in the tracial norm $\|\cdot\|_2$.

Let $(x^{(k)})_{k\in\mathbb{N}}$ be any Cauchy sequence in the closed unit ball of M^ω with respect to the norm $\|\cdot\|_2$. By passing to a subsequence, we may as well assume that $\left\|x^{(k+1)} - x^{(k)}\right\|_2 < 2^{-k}$ for all $k \in \mathbb{N}$. Find, for each $k \in \mathbb{N}$, a lift $(x_n^{(k)})_{n\in\mathbb{N}}$ of $x^{(k)}$ inside the closed unit ball of $\ell^\infty(M)$. It follows that the set

$$F_k = \left\{ n \in \mathbb{N} \,\middle|\, \left\|x_n^{(j+1)} - x_n^{(j)}\right\|_2 < 2^{-j} \text{ for } j = 1, 2, \ldots, k \right\}$$

belongs to ω for each $k \in \mathbb{N}$. Put $G_k = F_k \setminus \{1, 2, \ldots, k\}$ for $k \in \mathbb{N}$, and let $G_0 = \mathbb{N}$. Note that since ω is a free filter, it contains the cofinite filter ω_∞ and, hence, $G_k \in \omega$ for every $k \in \mathbb{N}_0$. Moreover, we clearly have a descending chain $G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ with $\bigcap_{k\in\mathbb{N}_0} G_k = \emptyset$. For each $n \in \mathbb{N}$, we have two options: Either $n \notin G_1$, in which case we put $x_n = 0$, or there exists $k \in \mathbb{N}$ such that $n \in G_k \setminus G_{k-1}$, in which case we put $x_n = x_n^{(k)}$. By construction, the image of the element $x = (x_n)_{n\in\mathbb{N}}$ inside M^ω belongs to the closed unit ball in M^ω. Moreover, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \left\|x - x^{(k)}\right\|_2 &\le \sup_{n\in G_k} \left\|x_n - x_n^{(k)}\right\|_2 = \sup_{m\ge k} \sup_{n\in G_m\setminus G_{m-1}} \left\|x_n - x_n^{(k)}\right\|_2 \\ &= \sup_{m\ge k} \sup_{n\in G_m\setminus G_{m-1}} \left\|x_n^{(m)} - x_n^{(k)}\right\|_2 \le 2^{-k+1}. \end{aligned}$$

This proves that $\lim_{n\to\omega} \left\|x - x^{(k)}\right\|_2$, which completes the proof. □

Having established that the tracial ultrapower of a von Neumann-algebra is, again, a von Neumann-algebra, we now wish to determine how ultrapowers preserve von Neumann-algebraic structures. Our goal is to prove that if M is a II_1-factor, then so is its ultrapower M^ω; in particular, if $\mathcal{R}$ denotes the unique hyperfinite II_1-factor, then the ultrapower $\mathcal{R}^\omega$ is, again, a II_1-factor. First, we need a lemma which characterises factors:

Lemma 2.15. *Let M be a von Neumann-algebra with a faithful tracial state τ. Then M is a factor if and only if for any non-zero projection $p \in \mathcal{P}(M)$ with $\tau(p) \le \frac{1}{2}$ we have $p \preceq 1 - p$.*

Proof. Note that as M admits a faithful tracial state, it is a finite von Neumann-algebra. Suppose that M is a factor and let $p \in \mathcal{P}(M)$ be a projection with $\tau(p) \le \frac{1}{2}$. By comparability of projections in a factor, see Proposition 1.21, at least one of $1 - p \preceq p$ and $p \preceq 1 - p$ must be true. We claim that $1 - p \prec p$ is never true, and hence it necessarily follows that $p \preceq 1 - p$. Assume hence for contradiction that $1 - p \prec p$, then there exists a proper subprojection $q \le p$ such that $1 - p \sim q$. Consider first the case that $\tau(p) = \frac{1}{2}$, then $\tau(p) = \tau(1 - p) = \tau(q)$ and faithfulness implies that $p = q$. However, then p is Murray-von Neumann equivalent to a proper subprojection of itself, which is impossible by finiteness of M. On the other hand, if $\tau(p) < \frac{1}{2}$, then since τ preserves the order induced by $\preceq$, we easily reach a contradiction.

Conversely, suppose that M is not a factor and find a non-trivial central projection $p \in \mathcal{P}(M)$. We can, without loss of generality, assume that $\tau(p) \leq \frac{1}{2}$, because if $\tau(p) > \frac{1}{2}$, then we can just consider the non-trivial central projection $1-p$, whose trace is $\tau(1-p) < \frac{1}{2}$, instead. Suppose that $q \in \mathcal{P}(M)$ is a projection Murray-von Neumann equivalent to p, and let $v \in M$ be a partial isometry such that $v^*v = p$ and $vv^* = q$. Then, using centrality of p, we find

$$q = q^2 = vv^*vv^* = vpv^* = vp^2v^* = pvv^*p \leq p.$$

Since $q \leq p$, it is immediate that q is not a subprojection of $1-p$ which implies that p cannot be Murray-von Neumann equivalent to a subprojection of $1-p$. □

Proposition 2.16. *If M is a factor, then so is the ultrapower M^ω. If M, moreover, is of type* II_1, *then so is M^ω.*

Proof. Assume that M is a factor. Let $p \in \mathcal{P}(M^\omega)$ be an arbitrary non-zero projection with $\tau(p) \leq \frac{1}{2}$. Since $\|a\|_2 \leq \|a\|$ for all $a \in M$ by contractivity of linear functionals, we can invoke Lemma 2.10 to lift p to a sequence $(p_n)_{n\in\mathbb{N}} \in \ell^\infty(M)$ of projections in M. In particular, the set

$$G = \left\{ n \in \mathbb{N} \,|\, \tau(p_n) \leq \frac{1}{2} \right\}$$

belongs to ω. For each $n \in G$, we can use Lemma 2.15 to find a partial isometry $v_n \in M$ such that $p_n = v_n^*v_n$ and $v_nv_n^* \leq 1-p$. We can extend this to a partial isometry $w_n \in M$ for each $n \in \mathbb{N}$ by

$$w_n = \begin{cases} v_n & \text{if } n \in G \\ 1 & \text{else} \end{cases}$$

and the induced partial isometry w inside M^ω implements that $w^*w = p$ and $ww^* \leq 1-p$ as desired. We conclude by another use of Lemma 2.15 that M^ω is a factor.

Now suppose further that M is a II_1-factor. Since M^ω admits a tracial state τ^ω, it is a finite von Neumann-algebra, and hence it is either of type I_n for some integer $n \in \mathbb{N}$ or of type II_1. We claim that it cannot be the former. Let $n \in \mathbb{N}$ be an arbitrary integer, then we can find $n+1$ non-trivial mutually orthogonal projections $p_1, \ldots, p_{n+1}$ inside M, since M is a II_1-factor. Mapping these to M^ω gives $n+1$ mutually orthogonal projections inside M^ω, which implies that M^ω cannot be of type I_n. Since this holds for any $n \in \mathbb{N}$, we conclude that M^ω is necessarily of type II_1. □

2.2 Quasidiagonal C^*-algebras and properties thereof

We now study quasidiagonality, which is an important C^*-algebraic approximation property and one of the central concepts in this thesis. The property of quasidiagonality was, as we mentioned in the introduction of the thesis, originally not related to C^*-algebras, but was a property in single operator theory, and this concept was later translated to an approximation property of C^*-algebras by Voiculescu. We shall consider the C^*-algebraic approximation property of quasidiagonality and both study this concept and establish the link connecting quasidiagonality of C^*-algebras and quasidiagonality of operators, demonstrating the historical context of both quasidiagonality as a property as well as the terminology. We do this to study quasidiagonality for its own sake, but note that our general study of quasidiagonality has a specific goal in mind, namely the Tikuisis-White-Winter theorem. For the purpose of proving this theorem, we also introduce some ultrapower-theoretic notions of quasidiagonality, which also gives a link between the previous and the present section. Our main reference for quasidiagonality is [11, Chapter 7].

Our starting point for this section is the definition of quasidiagonality as an approximation property for C^*-algebras.

Definition 2.17. A C^*-algebra A is *quasidiagonal* if there exists a net of c.c.p. maps $\varphi_\alpha \colon A \to M_{n_\alpha}(\mathbb{C})$ for $n_\alpha \in \mathbb{N}$ which is asymptotically multiplicative and asymptotically isometric, that is, for all $a, b \in A$, we have

$$\lim_\alpha \|\varphi_\alpha(ab) - \varphi_\alpha(a)\varphi_\alpha(b)\| = 0, \quad \text{and} \quad \lim_\alpha \|\varphi_\alpha(a)\| = \|a\|.$$

One can intuitively think of quasidiagonality of C^*-algebras as being the existence of asymptotic embeddings in finite-dimensional C^*-algebras. Indeed, since the only C^*-algebraic structure not preserved by c.c.p. maps are multiplication and the norm, which are preserved asymptotically, this point of view makes sense in an intuitive and informal fashion. Hence, we can view quasidiagonality as a finite concept for C^*-algebras. In fact, as we shall see later, quasidiagonal C^*-algebras are always stably finite. Before we show this, however, let us look a bit more in depth at quasidiagonality in itself. It should come as no surprise that quasidiagonality can be characterised locally.

Proposition 2.18. *A C^*-algebra A is quasidiagonal if and only if for any finite subset $F \subseteq A$ and $\varepsilon > 0$ there exists a c.c.p. map $\varphi\colon A \to M_n(\mathbb{C})$ such that*

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \varepsilon \quad \text{and} \quad \|\varphi(a)\| > \|a\| - \varepsilon$$

for all $a, b \in F$.

Proof. The proof is similar to other local characterisations of finite-dimensional approximation properties, so we only sketch the proof. Suppose A is quasidiagonal, and let $\varphi_\alpha\colon A \to M_{n_\alpha}(\mathbb{C})$ be a net of c.c.p. maps indexed by Λ witnessing the quasidiagonality. Let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$ be an arbitrary tolerance. As the net $(\varphi_\alpha)_{\alpha\in\Lambda}$ is asymptotically multiplicative and asymptotically isometric, it is easily seen that there exists some $\alpha \in \Lambda$ such that

$$\|\varphi_\alpha(ab) - \varphi_\alpha(a)\varphi_\alpha(b)\| < \varepsilon \quad \text{and} \quad \|\varphi_\alpha(a)\| > \|a\| - \varepsilon$$

for all $a, b \in F$. For the opposite direction, assume that A is locally quasidiagonal in the above sense. Consider the family $\mathcal{F}$ of pairs (F, ε) with finite subsets $F \subseteq A$ and arbitrary tolerances $\varepsilon > 0$ and direct this family with the lexicographical ordering. Find for each pair (F, ε) a c.c.p. map $\varphi_{(F,\varepsilon)}\colon A \to M_{k(F,\varepsilon)}(\mathbb{C})$ which is both multiplicative and isometric on F up to an ε tolerance. Then the net $\varphi_{(F,\varepsilon)}$ is asymptotically multiplicative and asymptotically isometric on A. □

Since quasidiagonality is actually a local property, the following proposition follows immediately.

Proposition 2.19. *If A is a separable C^*-algebra, then A is quasidiagonal if and only if there exists a sequence of asymptotically multiplicative and asymptotically isometric c.c.p. maps $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$.*

Whenever we work with unital C^*-algebras, we can always consider u.c.p. maps instead of general c.c.p. maps. The proof is a quite cumbersome one and uses a lot of continuous functional calculus and spectral theory.

Proposition 2.20. *If A is a unital C^*-algebra, then A is quasidiagonal if and only if there exists a net of asymptotically multiplicative and asymptotically isometric u.c.p. maps $\varphi_\alpha\colon A \to M_{k_\alpha}(\mathbb{C})$.*

Proof. Note that as u.c.p. maps are always contractive, we only need to show one direction. We prove it in the separable case — the non-separable case follows by a similar argument. By quasidiagonality, we can find a sequence of asymptotically multiplicative and asymptotically isometric c.c.p. maps $\psi_n\colon A \to M_{k_n}(\mathbb{C})$. Observe that $\left\|\psi_n(1)^2 - \psi_n(1)\right\| < \varepsilon_n$ for each $n \in \mathbb{N}$, where $\varepsilon_n \to 0$ as $n \to \infty$. Since each ψ_n is c.c.p., we find that the spectrum of each $\psi_n(1)$ is contained in $[0, 1]$. Moreover, we have the following equality for every $n \in \mathbb{N}$ by invoking the spectral mapping theorem with the continuous function $t \mapsto t^2 - t$:

$$\sigma(\psi_n(1)^2 - \psi_n(1)) = \sigma(\psi_n(1))^2 - \sigma(\psi_n(1))$$

This implies that if $\lambda \in \sigma(\psi_n(1))$, then $\left|\lambda^2 - \lambda\right| < \varepsilon_n$. Hence there exists some δ_n such that $\lambda \in [0, \delta_n) \cup (1 - \delta_n, 1]$, and as $\varepsilon_n \to 0$ as $n \to \infty$, it necessarily follows that $\delta_n \to 0$ as $n \to \infty$. By passing to another subsequence, we may as well assume that $\delta_n < \frac{1}{2}$ for all n. In particular, $[0, \delta_n) \subseteq [0, 1/2)$ and $(1 - \delta_n, 1] \subseteq (1/2, 1]$, and hence $\sigma(\psi_n(1)) \subseteq [0, 1/2) \cup (1/2, 1]$ for all $n \in \mathbb{N}$.

Denote by f the restriction of the indicator function $\chi_{[1/2,1]}$ onto $\sigma(\psi_n(1))$. If $\lambda \in \sigma(\psi_n(1))$, then

$$|\lambda - f(\lambda)| = |\lambda - 1|\,\chi_{[1/2,1]}(\lambda) + |\lambda|\,\chi_{[0,1/2)}(\lambda) < \delta_n.$$

Define the projection $P_n = \chi_{[1/2,1]}(\psi_n(1))$ of $\psi_n(1)$ corresponding to the interval $[1/2, 1]$; for this we use that $\chi_{[1/2,1]}$ is a continuous, positive and idempotent function on $\sigma(\psi_n(1))$. Another use of the continuous functional calculus gives us that $\|\psi_n(1) - P_n\| < \delta_n$, which implies that

$$\|\psi_n(1)P_n - P_n\| \le \|P_n\|\,\|\psi_n(1) - P_n\| < \delta_n.$$

Since $\delta_n < 1$, we find that $\psi_n(1)P_n$ is a positive and invertible element in $P_n M_{k_n}(\mathbb{C})P_n$ for each $n \in \mathbb{N}$, meaning that we can consider the element $(\psi_n(1)P_n)^{-1/2}$ in this C^*-algebra. Another use of the continuous functional calculus on the function $\lambda \mapsto \lambda^{-1/2}$ implies the inequality $\|(\psi_n(1)P_n)^{-1/2} - P_n\| < \delta_n$, and, in particular, the norms $\|(\psi_n(1)P_n)^{-1/2}\|$ are uniformly bounded by, say, 2. Let ℓ_n be the rank of the projection P_n, then the maps $\varphi_n : A \to M_{\ell_n}(\mathbb{C})$ given by

$$\varphi_n(a) = (\psi_n(1)P_n)^{-1/2}\psi_n(a)(\psi_n(1)P_n)^{-1/2}$$

can be shown to be a sequence of asymptotically multiplicative and asymptotically isometric u.c.p. maps as desired. □

All the characterisations of quasidiagonality mentioned at this point will be used in different ways throughout the thesis without explicitly mentioning the different propositions, since most of the results are just technicalities. In fact, we shall right away see how these different characterisations can be used in different contexts in order to prove various permanence properties of quasidiagonality.

Proposition 2.21. *The following permanence properties hold:*

(i) *C^*-subalgebras of quasidiagonal C^*-algebras are again quasidiagonal;*

(ii) *If $(A_\alpha)_{\alpha\in\Lambda}$ is an inductive system of quasidiagonal C^*-algebras with the property that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, then $A = \overline{\bigcup_{\alpha\in\Lambda} A_\alpha}$ is quasidiagonal. In particular, quasidiagonality passes to inductive limits under the assumption that the connecting maps are all injective;*

(iii) *If $\{A_\alpha\}_{\alpha\in\Lambda}$ is a collection of quasidiagonal C^*-algebras, then the product $\ell^\infty(A_\alpha, \Lambda)$ is quasidiagonal;*

(iv) *Quasidiagonality passes to unitisations;*

(v) *If A and B are quasidiagonal C^*-algebras, then the minimal tensor product $A \otimes B$ is quasidiagonal.*

Proof. (i): Trivial, just take restrictions of the c.c.p. maps witnessing the quasidiagonality.

(ii): Extend by Arveson's extension theorem, Proposition 1.10, for each $\alpha \in \Lambda$ the c.c.p. map $A_i \to M_{k(i,\alpha)}(\mathbb{C})$ to a c.c.p. map $A \to M_{k(i,\alpha)}(\mathbb{C})$; these are asymptotically multiplicative and asymptotically isometric.

(iii): Follows by the same line of thinking as in (ii).

(iv): If $\varphi_\alpha : A \to M_{n_\alpha}(\mathbb{C})$ are c.c.p. maps witnessing quasidiagonality of A, then the net of u.c.p. maps $\varphi_\alpha^\dagger : A^\dagger \to M_{n_\alpha}(\mathbb{C})$ by $\varphi_\alpha^\dagger(a+\lambda 1) = \varphi(a)+\lambda 1$ for $a \in A$ and $\lambda \in \mathbb{C}$ is easily seen to be asymptotically multiplicative and asymptotically isometric.

(v): We postpone the proof, as the representation theoretic notion of quasidiagonality provides machinery with which the proof becomes almost trivial. □

The reader is asked to note that we, for example, only explicitly mentioned inductive limits with *injective* connecting maps. The reason for this is not just that the proof is easier in this case — it fails in the general sense. In the following remark, we establish a few constructions under which quasidiagonality is not preserved, or where the question remains unanswered. Note that we refer to later results and constructions below.

Remark 2.22.

(i) Quasidiagonality does not pass to quotients. The cone and the suspension of any C^*-algebra can be shown to be quasidiagonal by homotopy invariance of quasidiagonality, Theorem 2.24, and if A is any non-quasidiagonal C^*-algebra, the isomorphism $A \cong CA/SA$ provides a counterexample.

(ii) Quasidiagonality does not pass to extensions. Let H be a separable, infinite-dimensional Hilbert space with orthonormal basis $(e_n)_{n\in\mathbb{N}}$, and define the unilateral shift $u \in \mathbb{B}(H)$ by $ue_n = e_{n+1}$. Define the *Toeplitz algebra* $\mathcal{T} = C^*(u)$, which can be realised as an extension of $C(\mathbb{T})$ by $\mathbb{K}(H)$.

Both $C(\mathbb{T})$ and $\mathbb{K}(H)$ are quasidiagonal — the former as it is Abelian, the latter as it is an AF-algebra, see Proposition 2.23 — but $\mathcal{T}$ is not stably finite, which all quasidiagonal C^*-algebras are by Proposition 2.26.

(iii) If A is an inductive limit of a sequence of quasidiagonal C^*-algebras with non-injective connecting maps, then A need not be quasidiagonal. A counterexample may be found in [11, Section 17.3].

(iv) It is unknown if quasidiagonality is preserved by the maximal tensor product.

Of course, knowing permanence properties of quasidiagonality is of no use if we do not have any examples of quasidiagonal C^*-algebras to build new examples from.

Proposition 2.23. *The following C^*-algebras are all quasidiagonal.*

(i) *Finite-dimensional C^*-algebras;*

(ii) *Abelian C^*-algebras;*

(iii) *AF-algebras.*

Proof. (i): If A is a finite-dimensional C^*-algebra, we can, by the classification of finite-dimensional C^*-algebras, identify $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ for suitable integers $r, n_i > 0$. Let $N = \sum_{i=1}^{r} n_i$, then the canonical embedding $A \hookrightarrow M_N(\mathbb{C})$ is an injective *-homomorphism witnessing the quasidiagonality of A.

(ii): If A is an Abelian C^*-algebra, it follows from Proposition 1.2 that $A = C_0(X)$ for some locally compact Hausdorff space X. Define for each $x \in X$ the *-homomorphism $\mathrm{ev}_x \colon C_0(X) \to \mathbb{C}$ to be the evaluation map at x, i.e., $\mathrm{ev}_x(f) = f(x)$ for all $f \in C_0(X)$. For each finite subset $F \subseteq X$, define the *-homomorphism $\varphi_F \colon C_0(X) \to M_F(\mathbb{C})$ by $\varphi_F = \bigoplus_{x \in F} \mathrm{ev}_x$. Direct the family $\mathscr{F}$ of finite subsets of X by set inclusion, then one readily verifies that the net $(\varphi_F)_{F \in \mathscr{F}}$ is asymptotically multiplicative and asymptotically injective.

(iii): All AF-algebras are of the form $\overline{\bigcup_{n\in\mathbb{N}} A_n}$, where $A_1 \subseteq A_2 \subseteq \cdots$ are nested finite-dimensional C^*-algebras, hence it follows by (i) and Proposition 2.21(ii) that all AF-algebras are quasidiagonal. ☐

These examples and permancence properties are highly elementary. A deep result is homotopy invariance of quasidiagonality. Recall that two *-homomorphisms $\varphi, \psi \colon A \to B$ are said to be *homotopic* if there exist *-homomorphisms $\sigma_t \colon A \to B$ for $t \in [0,1]$ such that $\sigma_0 = \varphi$ and $\sigma_1 = \psi$, and which satisfy that the map $t \mapsto \sigma_t(a)$ is continuous for all $a \in A$. In this case we write $\varphi \sim_h \psi$. We then say that two C^*-algebras A, B are *homotopy equivalent* if there exist *-homomorphism $\varphi \colon A \to B$ and $\psi \colon B \to A$ such that $\varphi \circ \psi \sim_h \mathrm{id}_B$ and $\psi \circ \varphi \sim_h \mathrm{id}_A$.

Theorem 2.24 (Voiculescu, 1991)**.** *If A, B are homotopy equivalent C^*-algebras and A is quasidiagonal, then B is also quasidiagonal. In particular, if A is any C^*-algebra, then the cone CA of A and the suspension SA of A are quasidiagonal.*

The proof of the homotopy invariance is quite technical, and we refer to [11, Theorem 7.3.6] or the original paper by Voiculescu [72, Theorem 5] for the proof. The latter part of the theorem follows by the well-known fact that, for any C^*-algebra A, the cone CA is homotopy equivalent to zero, see [64, Example 4.1.5]. Observe that the theorem implies that any C^*-algebra A, quasidiagonal or not, can be realised as a quotient of quasidiagonal C^*-algebras by the isomorphism $A \cong CA/SA$.

At this point we have no tools to actually verify the existence of non-quasidiagonal C^*-algebras — we have only alluded to their existence in Remark 2.22, where we stated that all quasidiagonal C^*-algebras are stably finite. We shall now develop some intrinsic properties of quasidiagonal C^*-algebras. Our first goal is to show that all quasidiagonal C^*-algebras are stably finite.

Lemma 2.25. *If a net $a_\alpha \in M_{n_\alpha}(\mathbb{C})$ satisfies that $\|a_\alpha^* a_\alpha - 1\| \to 0$, then $\|a_\alpha a_\alpha^* - 1\| \to 0$.*

Proof. By polar decomposition in finite-dimensional C^*-algebras, we can write $a_\alpha = u_\alpha |a_\alpha|$, where each u_α is unitary. Then $\mathrm{Ad}(u_\alpha)(a_\alpha^* a_\alpha) = a_\alpha a_\alpha^*$, and consequently we have

$$\|a_\alpha a_\alpha^* - 1\| = \|\mathrm{Ad}(u_\alpha)(a_\alpha^* a_\alpha - 1)\| = \|a_\alpha^* a_\alpha - 1\| \to 0$$

as desired. ☐

Proposition 2.26. *If a C^*-algebra A is quasidiagonal, then it is stably finite.*

Proof. If A is quasidiagonal, then the unitisation $A^\dagger$ as well as all matrix algebras $M_n(A) = M_n(\mathbb{C}) \otimes A$ over A are all quasidiagonal. It thus suffices to show that all quasidiagonal C^*-algebras are finite. Let A be an arbitrary unital quasidiagonal C^*-algebra, and assume the existence of a proper isometry, that is, an element $s \in A$ with $s^*s = 1$ and $ss^* \neq 1$. Let $(\varphi_\alpha)_{\alpha \in I}$ be a net of u.c.p. maps witnessing the quasidiagonality of A. Then,

$$\|1 - \varphi_\alpha(s^*)\varphi_\alpha(s)\| = \|\varphi_\alpha(s^*s) - \varphi_\alpha(s^*)\varphi_\alpha(s)\| \to 0.$$

However, this implies by Lemma 2.25 that

$$\|1 - \varphi_\alpha(s)\varphi_\alpha(s^*)\| \to 0.$$

which cannot be possible, as $ss^* \neq 1$ and hence $\varphi_\alpha(ss^*) \neq 1$. □

This is one of only two known obstructions of quasidiagonality — the other being that unital, quasidiagonal C^*-algebras admit amenable tracial states, which is a property of tracial states to be defined in Definition 2.47. For the sake of not postponing this obstruction of quasidiagonality, we state and prove the result here.

Proposition 2.27. *Any unital, quasidiagonal C^*-algebra has an amenable tracial state.*

Proof. Let $\varphi_\alpha \colon A \to M_{n_\alpha}(\mathbb{C})$ be a net of u.c.p. maps witnessing quasidiagonality of A. Consider for each α the state $\rho_\alpha = \mathrm{Tr}_{n_\alpha} \circ \varphi_\alpha$ on A. One can then verify that if τ is any accumulation point of the net ρ_α in the weak*-topology — the existence of an accumulation point is guaranteed by the compactness of the state space — then τ is an amenable tracial state. □

Finally, we can provide some examples of non-quasidiagonal C^*-algebras.

Example 2.28 (Examples of non-quasidiagonal C^*-algebras)**.** The following C^*-algebras are not quasidiagonal.

(i) The Toeplitz algebra $\mathcal{T}$ as defined in Remark 2.22 is obviously not finite, as it is generated by a proper isometry, and, consequently, is not quasidiagonal. Since the Toeplitz algebra can be realized as an extension of quasidiagonal algebras

$$0 \to C^*(\mathbb{T}) \to \mathcal{T} \to \mathbb{K}(H) \to 0,$$

this example shows that quasidiagonality is not preserved by taking extensions.

(ii) For each $n \geq 2$ define the *Cuntz algebra* $\mathcal{O}_n$ by the following construction: Let H be a separable infinite-dimensional Hilbert space, find n isometries $s_1, \ldots, s_n \in \mathbb{B}(H)$ with the constraint that $\sum_{i=1}^n s_i s_i^* = 1$, and define $\mathcal{O}_n = C^*(s_1, \ldots, s_n)$. Then $\mathcal{O}_n$ is not finite, and hence it is not quasidiagonal.

Note that the converse of Proposition 2.26 is not true, as there exist stably finite C^*-algebras, which are not quasidiagonal. The canonical counterexample is $C_r^*(\mathbb{F}_2)$, which, as $\mathbb{F}_2$ is not amenable, is not quasidiagonal by Rosenberg's theorem, see Theorem 4.25 later in the thesis. However, $C_r^*(\mathbb{F}_2)$ is stably finite. In order to see this, we prove that the existence of a faithful tracial state implies stably finiteness.

Lemma 2.29. *If A, B are unital C^*-algebras and τ_A, τ_B are tracial states on A and B, respectively, then there exists a tracial state τ on $A \otimes B$ extending the product $\tau_A \otimes \tau_B$ on $A \odot B$. Moreover, if both τ_A and τ_B are faithful, then so is $\tau_A \otimes \tau_B$.*

Proof. One easily verifies that τ is a tracial state on $A \otimes B$ by continuity. Assume that τ_A and τ_B are faithful tracial states, but that τ is not faithful. Consider the non-zero ideal $I = \{a \in A \mid \tau(a^*a) = 0\}$ in $A \otimes B$. Invoking Kirchberg's slicing lemma, see [59, Lemma 4.1.9], we find a non-zero element $z \in A \otimes B$ such that $z^*z = a \otimes b$ for some $a \in A$ and $b \in B$, and such that $zz^* \in I$. However, then faithfulness of τ_A and τ_B implies that

$$0 = \tau(zz^*) = \tau(z^*z) = \tau(a \otimes b) = \tau_A(a)\tau_B(b) > 0,$$

which is a clear contradiction. □

Lemma 2.30. *If A is a unital C^*-algebra admitting a faithful tracial state τ, then A is stably finite.*

Proof. Since $\tau \otimes \mathrm{Tr}_n$ is a faithful tracial state on $M_n(A)$ by Lemma 2.29, it suffices to show that the existence of a faithful tracial state implies finiteness. Suppose $s \in A$ is a proper isometry, i.e., assume that it satisfies that $s^*s = 1$ and $ss^* \neq 1$. Then faithfulness of τ implies that

$$0 < \tau(1 - ss^*) = \tau(1) - \tau(ss^*) = \tau(s^*s) - \tau(ss^*) = 0$$

and as such we reach a contradiction. We conclude that A is stably finite. □

Since $C_r^*(G)$ admits a faithful tracial state for any discrete group G by Proposition 1.35, all reduced group C^*-algebras, in particular $C_r^*(\mathbb{F}_2)$, are stably finite.

The above arguments imply that quasidiagonality is an inherently stronger property than stably finiteness. Note that as $\mathbb{F}_2$ is not amenable, the reduced group C^*-algebra $C_r^*(\mathbb{F}_2)$ is not nuclear by Proposition 1.37. There are no known examples of separable, nuclear, stably finite and non-quasidiagonal C^*-algebras. This led Blackadar-Kirchberg, see [4, Question 7.3.1] to conjecture that there in fact exists no such C^*-algebra.

Conjecture 2.31 (Blackadar-Kirchberg, 1997)**.** In the class of separable, nuclear C^*-algebra, being stably finite is equivalent to being quasidiagonal.

In Chapter 4, we shall provide some partial answers to this conjecture by invoking the Tikuisis-White-Winter theorem.

We now turn our attention to understanding the connection between quasidiagonality, as we have phrased it as an abstract approximation property, to its origins as a property in single operator theory. Let us first properly define the notion of quasidiagonality in operator theory; we shall consider quasidiagonality of *sets* of operators, which generalises the concept in single operator theory, and which is more closely related to the C^*-algebraic approximation property. The following definition could be called a *local* characterisation of quasidiagonality of operators, but we shall take it as our definition of quasidiagonality of operators in general.

Definition 2.32. Let H be a Hilbert space, and let $\Omega \subseteq \mathbb{B}(H)$ be an arbitrary set of bounded linear operators on H. We say that the set Ω is *quasidiagonal* if for each finite set $F \subseteq \Omega$, for each finite set $V \subseteq H$ and for each tolerance $\varepsilon > 0$, there exists a finite-rank projection $P \in \mathbb{B}(H)$ such that

$$\|[P, T]\| < \varepsilon \qquad \text{and} \qquad \|Pv - v\| < \varepsilon$$

for all $T \in F$ and $v \in V$.

We shall refer to the conditions in the definition as P *almost commuting* with T, and as P *almost fixing* v.

In order to alleviate the issues with the terminology clash of quasidiagonality for C^*-algebras and sets of operators, we shall exclusively refer to the latter as quasidiagonality of operators. Unfortunately, the two notions do not agree exactly, but we shall see how they are related. As we have mentioned several times in this chapter, the connection is representation theoretic in nature, and it is related to the following natural definition.

Definition 2.33. A representation $\pi\colon A \to \mathbb{B}(H)$ is said to be *quasidiagonal* if $\pi(A)$ is a quasidiagonal set of operators.

Again, and unfortunately, the terminology is not as nice as one could have hoped, since π being a quasidiagonal representation is not equivalent to $\pi(A)$ being a quasidiagonal C^*-algebra. We shall see that A being quasidiagonal as a C^*-algebra is equivalent to the existence of a *faithful* quasidiagonal representation, which gives the desired link. However, we shall not get ahead of ourselves, as we still have to understand several things including the structure of quasidiagonal sets of operators before we can attempt to prove this connection. The first result is, a priori, not related to quasidiagonality of operators, but is instead a purely C^*-algebraic result on projections.

Lemma 2.34. *Let A be a unital C^*-algebra, and let $p, q \in A$ be projections.*

(i) *If* $\|p-q\| < 1$*, then there exists* $u \in \mathcal{U}(A)$ *such that* $uqu^* = p$ *and* $\|1-u\| \leq 4\|p-q\|$*;*

(ii) *If* $\|q-pq\| < 1/4$*, then there exists* $u \in \mathcal{U}(A)$ *such that* $uqu^* \leq p$ *and* $\|1-u\| \leq 10\|q-pq\|$*.*

Proof. The proof is quite technical with loads of calculations, so we shall skip most of these.

(i): Since $\|p-q\| < 1$, it is a well-known fact, see for instance [64, Proposition 2.2.4 and Proposition 2.2.7], that $p \sim q$. Let $v \in A$ be a partial isometry such that $v^*v = q$ and $vv^* = p$. Likewise, we can find a partial isometry $w \in A$ such that $w^*w = 1-q$ and $ww^* = 1-p$. Put $u = v + w$, then u is a unitary element in A satisfying $uqu^* = p$ and $\|1-u\| \leq 4\|p-q\|$.

(ii): Suppose that $\|q-pq\| < 1/4$ and put $y = pq$. Then $y^*y = qAq$, and one easily verifies that $\|y^*y - q\| < 1$. Consequently, the element $|y| = (y^*y)^{1/2}$ is invertible in qAq, say with inverse $|y|_q^{-1}$. Let $v \in A$ be a partial isometry implementing the polar decomposition of y, i.e., put $v = y\,|y|_q^{-1}$, then $v^*v = q$ and $vv^* \leq p$. Define the projection $p_0 = vv^*$. A brief calculation shows that $\|q-v\| \leq 2\|q-pq\|$, which in turn implies that $\|q-p_0\| \leq 4\varepsilon < 1$. Since the distance between q and p_0 is the same as the distance between $1-q$ and $1-p_0$, we can find a partial isometry $w \in A$ such that $w^*w = 1-q$ and $ww^* = 1-p_0$. Then $u = v + w$ is a unitary element in A satisfying that $uqu^* = p_0 \leq p$ and $\|1-u\| \leq 10\|q-pq\|$ as desired. □

The above technical result is used in the following proposition, which gives a global characterisation of quasidiagonality of operators in terms of a quasicentral-esque approximate unit of projections. The following characterisation is a generalisation of Halmos' definition of quasidiagonality to arbitrary sets of operators, see [32].

Proposition 2.35. *Let* $\Omega \subseteq \mathbb{B}(H)$ *be a separable set of operators on a separable Hilbert space* H*. Then* Ω *is quasidiagonal in the sense of Definition 2.32 if and only if there exists an increasing sequence of finite-rank projections* $P_1 \leq P_2 \leq \cdots$ *in* $\mathbb{B}(H)$ *converging strongly to* 1 *such that* $\|[P_n, T]\| \to 0$ *as* $n \to \infty$ *for all* $T \in \Omega$*.*

Proof. Sufficiency is trivial, so let us focus on necessity. Using the perturbations provided in Lemma 2.34, we shall actually prove that the definition of quasidiagonal sets of operators can be slightly refined without loss of generality. In fact, we shall prove that the condition that P almost fixes the elements of the set V in Definition 2.32 can be altered to ensure that P, in fact, does fix all $v \in V$. In other words, we claim that for any finite sets $F \subseteq \Omega$ and $V \subseteq H$ and tolerance $\varepsilon > 0$, there exists a finite-rank projection $P \in \mathbb{B}(H)$ such that $\|[P,T]\| < \varepsilon$ for all $T \in F$, and such that $Pv = v$ for all $v \in V$. If we were to show this, then one can just use separability of Ω and H to construct the desired sequence without further complications.

Suppose that $F \subseteq \Omega$ and $V \subseteq H$ are finite subsets and let $\varepsilon > 0$. Let K denote the span of V, and let $Q\colon H \to K$ be the orthogonal projection. Since V is assumed to be finite, K is finite-dimensional and, hence, the closed unit ball $(K)_1$ of K is compact. Let $\delta > 0$ be some, as of now unspecified, number, and consider the open cover $(B(v,\delta))_{v\in(K)_1}$ of $(K)_1$. Using compactness, we can pick finitely many vectors $v_1, \ldots, v_n \in (K)_1$ such that $(B(v_i,\delta))_{i=1,\ldots,n}$ covers $(K)_1$. Since Ω is a quasidiagonal set of operators, there exists a projection $P \in \mathbb{B}(H)$ such that $\|Pv_i - v_i\| < \delta$ for $i = 1, \ldots, n$ along with $\|[P,T]\| < \delta$ for all $T \in F$. If $e \in H$ is any unit vector, then, by the finite sub-cover we constructed above, there exists some $i_0 \in \{1, \ldots, n\}$ such that $\|Qe - v_{i_0}\| < \delta$. A simple application of the triangle inequality provides the norm inequality $\|PQ - Q\| < 3\delta$.

Suppose that $\delta < 1/12$, then Lemma 2.34(ii) entails the existence of a unitary $U \in \mathbb{B}(H)$ such that $Q \leq UPU^*$, and which also satisfies the norm inequality $\|1-U\| \leq 10\|PQ-Q\| < 30\delta$. Let $R = UPU^*$, then R dominates Q and, hence, fixes every $v \in V$, since Q is the orthogonal projection of H onto the linear span of V. Moreover, for any $T \in F$, another easy application of the triangle inequality gives us

$$\|[R,T]\| \leq \|R-P\|\,\|T\| + \|[P,T]\| + \|T\|\,\|R-P\| < (120\,\|T\| + 1)\delta.$$

Putting $M = \max\{\|T\| \mid T \in F\}$ and letting $\delta < \min\{\frac{1}{12}, \frac{\varepsilon}{120M+1}\}$, one obtains the desired result. □

We shall need both the local and global characterisations of quasidiagonality of operators in order to prove the representation theoretic connection between quasidiagonality of C^*-algebras and of operators. In order to state and prove the theorem, we need some more notation. Recall that for a Hilbert space H, the associated Calkin algebra $\mathscr{C}(H)$ is defined as the quotient $\mathbb{B}(H)/\mathbb{K}(H)$.

Definition 2.36. A *-representation $\pi\colon A \to \mathbb{B}(H)$ is called *ample* if the induced *-homomorphism $A \to \mathscr{C}(H)$ is injective.

If $\pi\colon A \to \mathbb{B}(H)$ is any faithful *-representation of a C^*-algebra A, then the infinite direct product defines an ample representation, and as such any C^*-algebra admits a faithful, ample *-representation.

The next definition weakens the conditions on π above in a way that it is not necessarily multiplicative, but that it is still an injective *-homomorphism when mapped to the Calkin algebra $\mathscr{C}(H)$.

Definition 2.37. Let $\pi\colon A \to \mathbb{B}(H)$ be a u.c.p. map. We say that π is a *faithful representation modulo the compacts* if the mapping $q \circ \pi\colon A \to \mathscr{C}(H)$ is an injective *-homomorphism.

In other words, π is faithful modulo the compacts if $\pi(ab) - \pi(a)\pi(b) \in \mathbb{K}(H)$ for all $a, b \in A$, and if $\pi(a) - \pi(b) \in \mathbb{K}(H)$ implies $a = b$. In this case, we define for each $a \in A$ the quantity

$$\eta_\pi(a) = 2\max\{\|\pi(a^*a) - \pi(a^*)\pi(a)\|^{1/2}, \|\pi(aa^*) - \pi(a)\pi(a^*)\|^{1/2}\}$$

which, using the notion of multiplicative domains from Proposition 1.7, measures how far π is from being multiplicative in a. We need this number for the next theorem due to Voiculescu, which shows that $\eta_\pi(a)$ provides a pointwise boundary for the distance a faithful representation modulo the compacts can have to a representation with several pleasant properties. We omit the proof and refer the reader to [11, Theorem 1.7.6].

Proposition 2.38 (Voiculescu)**.** *Let A be a unital and separable C^*-algebra, and let $\pi\colon A \to \mathbb{B}(H)$ be a faithful representation modulo the compacts on a separable Hilbert space H. If $\sigma\colon A \to \mathbb{B}(K)$ is a faithful, unital, ample representation on a separable Hilbert space K, then there exists a sequence of unitaries $U_n\colon H \to K$ such that*

$$\limsup_{n\to\infty} \|\sigma(a) - U_n\pi(a)U_n^*\| \leq \eta_\pi(a)$$

for each $a \in A$.

We can finally prove the representation theorem of quasidiagonality. We shall state it for the unital, separable case, but do note that the theorem holds in full generality.

Theorem 2.39. *Let A be a separable and unital C^*-algebra. The following are equivalent.*

(i) *A is quasidiagonal as a C^*-algebra;*

(ii) *A admits a faithful quasidiagonal representation on a separable Hilbert space;*

(iii) *Every faithful, unital, ample representation of A on a separable Hilbert space is quasidiagonal.*

Proof. (i)⇒(iii): Let $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$ be u.c.p. maps witnessing the quasidiagonality of A, and let $\pi\colon A \to \mathbb{B}(H)$ be any faithful, ample, unital representation. By identifying $\ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})$ with $\mathbb{B}(K)$ for $K = \bigoplus_{n\in\mathbb{N}} \mathbb{C}^{k_n}$, we get a u.c.p. map $\varphi\colon A \to \mathbb{B}(K)$ by $\varphi(a) = (\varphi_n(a))_{n\in\mathbb{N}}$ for $a \in A$. Using that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative and asymptotically isometric, one can check that φ is faithful modulo the compacts.

Let $F \subseteq \Omega$ and $V \subseteq H$ be finite subsets, and let $\varepsilon > 0$ be arbitrary. By asymptotic multiplicativity, we can, for any $\delta > 0$, find $N \in \mathbb{N}$ such that

$$\|\varphi_n(aa^*) - \varphi_n(a)\varphi_n(a^*)\|, \|\varphi_n(a^*a) - \varphi_n(a^*)\varphi_n(a)\| < \delta$$

for all $n \geq N$ and $a \in F$. We can hence, without loss of generality, assume that $\eta_\varphi(a) < \frac{\varepsilon}{2}$ for all $a \in F$. Then, by Voiculescu's theorem, we can find a unitary $U\colon H \to K$ providing the inequality

$$\|\pi(a) - U^*\varphi(a)U\| \leq \eta_\varphi(a) < \frac{\varepsilon}{2}$$

for all $a \in F$. Let $P_n \in \mathbb{B}(K)$ denote the projections onto the first n components for each $n \in \mathbb{N}$, and consider the projections $Q_n = U^*P_nU$ on H attained from unitary equivalence. Since $(P_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite-rank projections strongly tending to the identity operator, so is $(Q_n)_{n\in\mathbb{N}}$. A simple calculation shows that $\|[Q_n,\pi(a)]\| < \varepsilon$ for all $a \in F$, and that $\|Q_n v - v\| < \varepsilon$ for all $v \in V$ for sufficiently large $n \in \mathbb{N}$. This proves that π is a quasidiagonal representation.

(iii)⇒(ii): This implication is trivial since all unital C^*-algebras admit faithful, unital and ample *-representations.

(ii)⇒(i): Let $\pi\colon A \to \mathbb{B}(H)$ be a faithful, quasidiagonal representation. Find by Proposition 2.35 an increasing sequence $P_1 \leq P_2 \leq \cdots$ of finite-rank projections in H such that P_n tends strongly to the identity, and such that $\|[P_n,\pi(a)]\| \to 0$ as $n \to \infty$ for all $a \in A$. Note that $P_n\mathbb{B}(H)P_n$ is isomorphic to $M_{k_n}(\mathbb{C})$ for some integer $k_n \in \mathbb{N}$, and we can therefore define the u.c.p. maps $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$ by $\varphi_n(a) = P_n\pi(a)P_n$. One can verify that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is asymptotically multiplicative and asymptotically isometric, which proves quasidiagonality of the C^*-algebra A. □

Having proved the representation theorem for quasidiagonality, let us show how one can use this to prove the as of now unproven statement in Proposition 2.21(v) stating that the minimal tensor product of two quasidiagonal C^*-algebra is again quasidiagonal; we only sketch the proof in the unital, separable case, as we have only shown the representation theorem for this class of C^*-algebras, but the argument holds in general. Let A and B be separable, unital, quasidiagonal C^*-algebras and let $\pi_A\colon A \to \mathbb{B}(H)$ and $\pi_B\colon B \to \mathbb{B}(K)$ be two faithful quasidiagonal representations. Then $\pi_A \otimes \pi_B\colon A \otimes B \to \mathbb{B}(H \otimes K)$ is a faithful representation of the minimal tensor product $A \otimes B$. Since the tensor product of two finite-rank projections is again of finite rank, it is routine to verify that $\pi_A \otimes \pi_B$ is in fact a quasidiagonal representation, which proves quasidiagonality of A by the above representation theorem.

We have at this point seen two different ways of viewing quasidiagonality: Either as the approximation property of C^*-algebras as in Definition 2.17, or as the representation theoretic flavoured characterisation of Theorem 2.39. However, there is yet another characterisation, which in essence is a way of transforming the former into something perhaps more tangible. Since taking ultrafilters, generally speaking, takes approximate properties to exact ones, and quasidiagonality can be phrased in terms of c.c.p. maps into matrix algebras asymptotically tending to an injective *-homomorphism, we can view quasidiagonality in terms of embeddings into the ultrapower $\mathcal{Q}_\omega$.

Proposition 2.40. *A separable, unital and nuclear C^*-algebra A is quasidiagonal if and only if there exists a unital embedding $A \hookrightarrow \mathcal{Q}_\omega$.*

The proof uses the Choi-Effros lifting theorem, which we shall state below.

Proposition 2.41 (Choi-Effros)**.** *Let A be a separable C^*-algebra, and let $J \trianglelefteq B$ be a closed two-sided ideal in a C^*-algebra B. For every nuclear c.c.p. map $\varphi\colon A \to B/J$ there exists a nuclear c.c.p. lift $\overline{\varphi}\colon A \to B$ with $\rho \circ \overline{\varphi} = \varphi$, where $\rho\colon B \to B/J$ is the quotient map. In particular, if A in addition is nuclear, then every c.c.p. map $\varphi\colon A \to B/J$ can be lifted to a nuclear c.c.p. map $\overline{\varphi}\colon A \to B$.*

A proof of the Choi-Effros lifting theorem may be found in [11, Theorem C.3] or in the original paper by Choi and Effros [12, Theorem 3.10].

Proof of Proposition 2.40. Suppose there exists a unital embedding $\iota\colon A \to \mathcal{Q}_\omega$. We can, by Choi-Effros lifting theorem, find a c.c.p. map $\varphi\colon A \to \ell^\infty(\mathcal{Q})$ which lifts ι. This, in turn, gives us a sequence of c.c.p. maps $\varphi_n\colon A \to \mathcal{Q}$, where $\varphi(a) = (\varphi_1(a), \varphi_2(a), \ldots)$ for $a \in A$. This sequence of c.c.p. maps is asymptotically multiplicative and asymptotically isometric, since ι is a unital *-homomorphism. Let $E_n\colon \mathcal{Q} \to M_n(\mathbb{C})$ be conditional expectations onto matrix algebras, then the sequence of c.c.p. maps $E_n \circ \varphi_n\colon A \to M_{k_n}(\mathbb{C})$ is asymptotically multiplicative and asymptotically isometric and, consequently, A is quasidiagonal.

Conversely, suppose that A is quasidiagonal, and let $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$ be the sequence c.c.p. maps witnessing the quasidiagonality. Since $\mathcal{Q}$ is the universal UHF-algebra, we have embeddings $M_{k_n}(\mathbb{C}) \subseteq \mathcal{Q}$ for all $n \in \mathbb{N}$, and hence we can view φ_n as maps from A into $\mathcal{Q}$. Since the sequence φ_n is asymptotically multiplicative and asymptotically isometric, it induces an isometric, hence injective, *-homomorphism $A \to \mathcal{Q}_\omega$, which was what we wanted to construct. □

Note that the nuclearity assumption on A is in order to invoke the Choi-Effros lifting theorem, Proposition 2.41, to construct a sequence of c.c.p. maps witnessing the quasidiagonality. Consequently, we can actually weaken the assumptions, which is going to be used in a proof later.

Corollary 2.42. *Suppose that A is a separable, unital C^*-algebra, and that there exists a unital embedding $\iota\colon A \to \mathcal{Q}_\omega$ which is liftable to a sequence of c.c.p. maps $\varphi_n\colon A \to \ell^\infty(\mathcal{Q})$. Then A is quasidiagonal. In particular, if ι is a unital and nuclear embedding into $\mathcal{Q}_\omega$, then A is quasidiagonal.*

In a later section, we shall examine quasidiagonal tracial states, which are tracial states that can be witnessed by quasidiagonal approximations. However, we can also view quasidiagonal tracial states as being those which can be realised as the tracial state on $\mathcal{Q}_\omega$ in a certain liftable sense, see Proposition 2.49 for the precise meaning. These results entail a great link between quasidiagonality and the ultrapower $\mathcal{Q}_\omega$, and this viewpoint is exactly the starting point for understanding the Tikuisis-White-Winter theorem.

2.3 Introduction to AF-embeddability

While AF-algebras have been studied for almost half a century and have been classified completely both combinatorically by their Bratteli diagrams as well as by their ordered K_0-groups, there are still aspects of AF-algebras which are unknown. Current interest in AF-algebra includes the question of AF-embeddability — when can we identify a C^*-algebra as a C^*-subalgebra of an AF-algebra? Perhaps surprisingly, it does not hold in general that C^*-subalgebras of AF-algebras are again AF-algebras. The typical counterexample is as follows: Consider the Cantor set $\mathcal{C}$, which we can realise as an inverse limit of finite sets, such that $C(\mathcal{C})$ is a sequential inductive limit of finite-dimensional C^*-algebras, i.e., it is an AF-algebra. As all compact metric spaces can be identified as continuous images of the Cantor set, we, in particular, get an injection $C([0,1]) \hookrightarrow C(\mathcal{C})$. However, $C([0,1])$ is not an AF-algebra, as $[0,1]$ is a connected compact Hausdorff space and, consequently, it only has trivial projections. We can clearly generalise the above argument to prove that the C^*-algebra of continuous functions on some locally compact, Hausdorff space is AF-embeddable, but it is an AF-algebra only when the space is of dimension zero.

The point of the above example is that the question of classifying subalgebras of AF-algebras, either in general or for a specific AF-algebra, is not as easy as one might have hoped. The question of AF-embeddability was prominently studied by Pimsner and Voiculescu in [54], in which they proved that the irrational rotation C^*-algebras are AF-embeddable, and it has since then been an active topic of research. There are very few known obstructions to being AF-embeddable; in fact, we only have three such properties, and they are easily seen to be obstructions.

Proposition 2.43. *If A is an AF-embeddable C^*-algebra, then A is separable, quasidiagonal and exact.*

Proof. AF-algebras are separable, quasidiagonal and exact and all three properties pass to C^*-subalgebras. □

In [4, Question 7.3.3], Blackadar and Kirchberg asked whether the converse is true.

Conjecture 2.44 (Blackadar-Kirchberg, 1997)**.** A C^*-algebra A is AF-embeddable if and only if it is separable, exact and quasidiagonal.

As for any interesting open problems, there are some partial results, and some of the classes of C^*-algebras, for which the conjecture has been shown to be true, can be found in the following list. We give references to the original papers, but for the first two examples the reader can also consult [11, Theorem 8.5.3 and Theorem 8.5.4].

(i) $C(X) \rtimes \mathbb{Z}$ for compact metric spaces X, see [55, Theorem 9];

(ii) $A \rtimes \mathbb{Z}$ for AF-algebras A, see [9, Theorem 0.2];

(iii) $C^*_r(G)$ for countable, discrete groups G, see [70, Corollary 6.6];

(iv) Traceless[1] C^*-algebra, see [26, Corollary C].

[1] This concept is a bit more subtle than what one might think; we shall define it properly in Chapter 5, when we reproduce the result.

We shall in this thesis concentrate on (iii) and (iv), which are also the most recent contributions to the subject in the above list. In fact, Schafhauser proves in [67, Theorem D] that if G is a countable, discrete group, then $C_r^*(G)$ embeds into the universal UHF-algebra $\mathcal{Q}$ if and only if G is amenable. Note that this gives an embedding into a specific AF-algebra, and, in fact, $\mathcal{Q}$, in contrast to the similar corollary of the Tikuisis-White-Winter theorem, see (iii) above, where the embedding is into a non-specific AF-algebra.

Intuitively, since AF-algebras are, as the name suggests, approximately finite-dimensional, one should think of AF-embeddability as a property of a finite nature; this is further motivated by the fact that AF-embeddability implies quasidiagonality. However, there are C^*-algebras, which admit some very infinite properties, but still are AF-embeddable. In [49, Theorem 1], Ozawa proved an analogue of Voiculescu's proof of homotopy invariance of quasidiagonality for AF-embeddability, cf. Theorem 2.24.

Theorem 2.45 (Ozawa, 2003)**.** *AF-embeddability is a homotopic invariant of separable, exact C^*-algebras. Moreover, the cone CA and the suspension SA of any separable, exact C^*-algebra A are AF-embeddable.*

We shall later see how the latter half of the theorem can be generalised to a certain property of the so-called primitive ideal space, see Chapter 5.

Lastly, let us mention a recent result due to Schafhauser, see [67, Theorem A], which gives a complete characterisation of C^*-subalgebras of simple, unital AF-algebras under the additional assumption that the C^*-subalgebras satisfies the UCT. Just as in Proposition 2.27, the theorem mentions the notion of an amenable tracial state — we shall define this concept in the next section.

Theorem 2.46. *Let A be a separable C^*-algebra satisfying the UCT. Then A embeds into a simple, unital AF-algebra if and only if A is exact and has a faithful, amenable tracial state.*

2.4 Amenability and quasidiagonality of tracial states

In this section we examine two important properties of tracial states, namely amenability and quasidiagonality. These both appear in the Tikuisis-White-Winter theorem, and for this alone a survey is relevant, but there are also results of independent interest within this topic. We shall hence not only study aspects directly related to the proof of the Tikuisis-White-Winter theorem.

Given a C^*-algebra A with a tracial state τ, we can naturally construct a seminorm $\|\cdot\|_2$ by $\|a\|_2 = \tau(a^*a)^{1/2}$ for $a \in A$. This is a norm if and only if τ is faithful. Note that, given distinct tracial states, one might get completely different (semi)norms, so unless the C^*-algebra is monotracial, the norm in itself need not be natural; in a sense, this norm studies the properties of the tracial state moreso than of the C^*-algebra.

Definition 2.47. Let A be a unital and separable C^*-algebra with a tracial state τ. We say that:

(i) τ is *amenable* if there exists a sequence of u.c.p. maps $\varphi_n \colon A \to M_{k_n}(\mathbb{C})$ such that

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 = 0 \qquad \text{and} \quad \lim_{n\to\infty} \mathrm{Tr}_{k_n}(\varphi_n(a)) = \tau(a)$$

for all $a, b \in A$.

(ii) τ is *quasidiagonal* if there exists a sequence of u.c.p. maps $\varphi_n \colon A_n \to M_{k_n}(\mathbb{C})$ such that

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0 \qquad \text{and} \quad \lim_{n\to\infty} \mathrm{Tr}_{k_n}(\varphi_n(a)) = \tau(a)$$

for all $a, b \in A$.

We shall think of quasidiagonal tracial states as being witnessed by quasidiagonal approximations — the connection between quasidiagonality as a property of C^*-algebras and quasidiagonality as a property of tracial states will be examined throughout this section.

Amenability and quasidiagonality of tracial states are stated in quite similar manners, but it is unknown to what degree they coincide. Nevertheless, it is easily seen that quasidiagonal tracial states are always amenable.

Proposition 2.48. *For each $k \in \mathbb{N}$ and $a \in M_k(\mathbb{C})$, we have the inequalities*

$$\|a\|_2 \leq \|a\| \leq \sqrt{k}\,\|a\|_2$$

where $\|a\|_2 = \mathrm{Tr}_k(a^*a)^{1/2}$. In particular, quasidiagonal tracial states are amenable.

Proof. The first inequality is immediate, as linear functionals are contractive, and the second inequality follows from the fact that $\|a\|_2 = \sqrt{\frac{1}{k}\sum_{i,j=1}^{k} |a_{ij}|^2}$. □

It is still unknown whether or not amenable tracial states are quasidiagonal. There are some specific classes of C^*-algebras where this is resolved — for example, the Tikuisis-White-Winter theorem, to be proved later, proves that amenability and quasidiagonality are equivalent for faithful tracial states on exact, separable C^*-algebras in the UCT-class.

Interestingly, we can express amenability and quasidiagonality of tracial states in terms of certain lifting properties. The reader should note the similarities to Proposition 2.40 both in terms of the statement as well as the proof. For the remainder of the thesis, we shall denote by τ_ω and τ^ω the unique, faithful tracial states on $\mathcal{Q}_\omega$ and $\mathcal{R}^\omega$, respectively, unless otherwise mentioned.

Proposition 2.49. *Let τ be a tracial state on a separable C^*-algebra A.*

(i) *The tracial state τ is amenable if and only if there exists a *-homomorphism $\varphi^\omega\colon A \to \mathcal{R}^\omega$ with a c.c.p. lift $A \to \ell^\infty(\mathcal{R})$ such that $\tau^\omega \circ \varphi^\omega = \tau$.*

(ii) *The tracial state τ is quasidiagonal if and only if there exists a *-homomorphism $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ with a c.c.p. lift $A \to \ell^\infty(\mathcal{Q})$ such that $\tau_\omega \circ \varphi_\omega = \tau$.*

Proof. We only prove (ii); the proof of (i) is analogous. Suppose first that τ is quasidiagonal and let $\varphi_n\colon A \to M_{k_n}(\mathbb{C})$ be a sequence of c.c.p. maps witnessing its quasidiagonality. Since $\mathcal{Q}$ is the universal UHF-algebra, we can embed each $M_{k_n}(\mathbb{C})$ inside $\mathcal{Q}$ and, hence, we can view φ_n as a map $A \to \mathcal{Q}$. Let $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ be the induced c.c.p. map, i.e., $\varphi_\omega(a) = \lim_{n\to\omega} \varphi_n(a)$ for $a \in A$. Since the sequence φ_n is asymptotically multiplicative, it follows that φ_ω is a *-homomorphism. Moreover, as $\tau(a) = \lim_{n\to\infty} \mathrm{Tr}_{k_n}(\varphi_n(a))$, we see that $\tau = \tau_\omega \circ \varphi_\omega$. Denote by $\varphi\colon A \to \ell^\infty(\mathcal{Q})$ the direct product of the maps φ_n, i.e., $\varphi(a) = (\varphi_1(a), \varphi_2(a), \ldots)$ for $a \in A$. Then φ is a c.c.p. lift of φ_ω, which proves one direction.

Now suppose that $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ is a *-homomorphism with a c.c.p. lift $\varphi\colon A \to \ell^\infty(\mathcal{Q})$ such that $\tau_\omega \circ \varphi_\omega = \tau$. Let $\varphi_n\colon A \to \mathcal{Q}$ be the composition of φ with the projection of $\ell^\infty(\mathcal{Q})$ onto the nth copy of $\mathcal{Q}$. Let $E_n\colon \mathcal{Q} \to M_n(\mathbb{C})$ be the canonical trace-preserving conditional expectation for each $n \in \mathbb{N}$ and let $\psi_n = E_n \circ \varphi_n\colon A \to M_n(\mathbb{C})$; note that each ψ_n is a c.c.p. map. Denote by $\iota_n\colon M_n(\mathbb{C}) \to \mathcal{Q}$ the canonical inclusion, and observe that $\iota_n \circ E_n = \rangle\lceil_{\mathcal{Q}}$. Then, for each $a, b \in A$ and $n \in \mathbb{N}$, we have

$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| = \|\iota_n(\psi_n(ab) - \psi_n(a)\psi_n(b))\| = \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|\,,$$

and asymptotic multiplicativity of the sequence $(\varphi_n)_{n\in\mathbb{N}}$ provides asymptotic multiplicativity of the sequence $(\psi_n)_{n\in\mathbb{N}}$. Moreover, for any $a \in A$, we have

$$\mathrm{Tr}_n(\psi_n(a)) = \mathrm{Tr}_n(E_n \circ \varphi_n(a)) = \tau_{\mathcal{Q}}(\varphi_n(a)),$$

and therefore

$$\tau(a) = \tau_\omega(\varphi_\omega(a)) = \lim_{n\to\omega} \tau_{\mathcal{Q}}(\varphi_n(a)) = \lim_{n\to\omega} \mathrm{Tr}_n(\psi_n(a)),$$

where $\pi_\omega\colon \ell^\infty(\mathcal{Q}) \to \mathcal{Q}_\omega$ is the canonical quotient map. This completes the proof. □

Note that this characterises quasidiagonality of tracial states in terms of a lifting problem $A \to \mathcal{Q}_\omega$, which is similar to the descriptions of quasidiagonality of C^*-algebra from the proof of Proposition 2.40 and, more explicitly, the statement of Corollary 2.42. These lifting characterisations shall become a vital starting point for the proof of the Tikuisis-White-Winter theorem, as they allow us to examine these concepts in terms of extension theory, which we shall examine in the next chapter. However, there are other characterisations of quasidiagonality of tracial states, which are important as well. First, we need a lemma regarding nuclearity of maps $A \to \ell^\infty(B_n, \mathbb{N})$; it can also be found in [14, Proposition 3.3].

Lemma 2.50. *Let A be an exact C^*-algebra, and let B_n be a sequence of C^*-algebras. Suppose that $\varphi_n\colon A \to B_n$ are nuclear c.c.p. maps. Then the induced c.c.p. map $\varphi\colon A \to \ell^\infty(B_n, \mathbb{N})$ is nuclear.*

Proof. Since A is exact, we can identify A as a C^*-subalgebra of a nuclear C^*-algebra C. Let $F \subseteq A$ be a finite subset, and let $\varepsilon > 0$ be an arbitrary tolerance. By nuclearity of each φ_n, we can find c.c.p. maps $\theta_n\colon A \to M_{k_n}(\mathbb{C})$ and $\psi_n\colon M_{k_n}(\mathbb{C}) \to B_n$ such that

$$\|\varphi_n(a) - \psi_n \circ \theta_n(a)\| < \frac{\varepsilon}{2}$$

for all $a \in F$. By Arveson's extension theorem, Proposition 1.10, there exist c.c.p. maps $\overline{\theta}_n\colon C \to M_{k_n}(\mathbb{C})$ extending the c.c.p. maps θ_n. Now let $\eta\colon C \to \ell^\infty(B_n, \mathbb{N})$ be the c.c.p. map defined by $\eta(a) = (\psi_n \circ \overline{\theta}_n(a))_{n\in\mathbb{N}}$. Then, for any $a \in A$, we see that

$$\|\varphi(a) - \eta(a)\| = \sup_{n\in\mathbb{N}} \left\|\varphi_n(a) - \psi_n \circ \overline{\theta}_n(a)\right\| = \sup_{n\in\mathbb{N}} \|\varphi_n(a) - \psi_n \circ \theta_n(a)\| \le \frac{\varepsilon}{2}.$$

We claim that this implies that φ is nuclear. Since C is a nuclear C^*-algebra, we can find c.c.p. maps $\alpha_n\colon A \to M_{\ell_n}(\mathbb{C})$ and $\beta_n\colon M_{\ell_n}(\mathbb{C}) \to C$ such that

$$\|a - \beta_n \circ \alpha_n(a)\| < \frac{\varepsilon}{2}$$

for all $a \in F$. In particular, we find for each $a \in F$ that

$$\|\varphi(a) - \eta \circ \beta_n \circ \alpha_n(a)\| \le \|\varphi(a) - \eta(a)\| + \|\eta(a) - \eta \circ \beta_n \circ \alpha_n(a)\| \le \varepsilon$$

proving nuclearity of the map φ. □

Using nuclearity of $\mathcal{Q}$, we get the following corollary.

Corollary 2.51. *If A is an exact C^*-algebra, then any c.c.p. map $A \to \ell^\infty(\mathcal{Q})$ is nuclear.*

With this we can further characterise quasidiagonality for the class of separable, unital and exact C^*-algebras. Note that these are all still quite similar to the characterisations in Proposition 2.49.

Proposition 2.52. *Let A be a separable, unital, exact C^*-algebra. The following are equivalent for any tracial state τ on A:*

(i) *The tracial state τ is quasidiagonal;*

(ii) *There exists a unital, nuclear *-homomorphism $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ such that $\tau_\omega \circ \varphi_\omega = \tau$;*

(iii) *There exists $\gamma \in (0, 1]$ such that, for any finite subset $F \subseteq A$ and tolerance $\varepsilon > 0$, there exists a nuclear map $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ such that*

$$\|\varphi_\omega(ab) - \varphi_\omega(a)\varphi_\omega(b)\| < \varepsilon \qquad \text{and} \qquad \tau_\omega \circ \varphi_\omega(a) = \gamma\tau(a)$$

for all $a, b \in F$.

Proof. (i)⇔(ii): Analogous to the proof of Proposition 2.49; use Corollary 2.51 to ensure nuclearity of the c.c.p. lift $A \to \ell^\infty(\mathcal{Q})$ in the direction (i)⇒(ii).

(ii)⇒(iii): This implication is trivial with $\gamma = 1$.

(iii)⇒(ii): This direction is an application of Kirchberg's ε-test, see Proposition 2.8. By separability of A, we can find a countable dense subset $\{a_1, a_2, \dots\}$ of A. Define for each $k \in \mathbb{N}$ the finite subset $F_k = \{a_1, a_2, \dots, a_k\}$ of A. Find the scalar $\gamma \in (0, 1]$ satisfying the condition (iii). We now proceed with using Kirchberg's ε-test, Proposition 2.8, and we shall use the notation from this proposition.

For each $n \in \mathbb{N}$, let X_n be the set of c.c.p. maps $A \to \mathcal{Q}$. For each fixed $n \in \mathbb{N}$, we define the maps $f_n^{(k)}\colon X_n \to [0, \infty)$ by

$$f_n^{(1)}(\varphi) = \|\tau_{\mathcal{Q}} \circ \varphi - \gamma\tau\| \qquad \text{and,}$$
$$f_n^{(k)}(\varphi) = \max_{i,j\le k} \|\varphi(a_ia_j) - \varphi(a_i)\varphi(a_j)\|, \qquad \text{for } k \ge 2$$

for all $\varphi \in X_n$. Observe that, for any $k \in \mathbb{N}$ and $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \ell^\infty(X_n, \mathbb{N})$,

$$f_\omega^{(1)}(\varphi) = \|\tau_\omega \circ \varphi_\omega - \gamma\tau\| \qquad \text{and,}$$
$$f_\omega^{(k)}(\varphi) = \max_{i,j\le k} \|\varphi_\omega(a_i a_j) - \varphi_\omega(a_i)\varphi_\omega(a_j)\|, \qquad \text{for } k \ge 2,$$

where $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ is the map induced by $\varphi\colon A \to \ell^\infty(\mathcal{Q})$. Consider an arbitrary integer $m \in \mathbb{N}$ and tolerance $\varepsilon > 0$. By assumption, there exists a nuclear map $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ such that, for each $k \le m$, $\|\varphi_\omega(a_i a_j) - \varphi_\omega(a_i)\varphi_\omega(a_j)\| < \varepsilon$ and $\tau_\omega \circ \varphi_\omega(a_i) = \gamma\tau(a_i)$ for all $i, j = 1, \ldots, k$. By nuclearity of φ_ω, we can invoke the Choi-Effros lifting theorem, Proposition 2.41, to find a c.c.p. lift $\varphi\colon A \to \ell^\infty(\mathcal{Q})$. Let φ_n be the nth component of φ. Then, for each $2 \le k \le m$, we have the equalities

$$f_\omega^{(k)}(\varphi) = \lim_{n\to\omega}\left(\max_{i,j\le k} \|\varphi_n(a_i a_j) - \varphi_n(a_i)\varphi_n(a_j)\|\right) = \max_{i,j\le k} \|\varphi_\omega(a_i a_j) - \varphi_\omega(a_i)\varphi_\omega(a_j)\| < \varepsilon,$$

and, moreover,

$$f_\omega^{(1)}(\varphi) = \|\tau_\omega \circ \varphi_\omega - \gamma\tau\| = 0.$$

By Kirchberg's ε-test, Proposition 2.8, there exists a sequence of nuclear maps $\psi_n\colon A \to \mathcal{Q}$ such that $f_\omega^{(k)}(\psi_1, \psi_2, \ldots) = 0$ for all $k \in \mathbb{N}$. The direct product $\psi\colon A \to \ell^\infty(\mathcal{Q})$ is nuclear by Corollary 2.51, and hence so is the induced map $\psi_\omega\colon A \to \mathcal{Q}_\omega$.

For each $k \ge 2$, we see that

$$\max_{i,j\le k} \|\psi_\omega(a_i a_j) - \psi_\omega(a_i)\psi_\omega(a_j)\| = f_\omega^{(k)}(\psi_1, \psi_2, \ldots) = 0$$

and density of $\{a_1, a_2, \ldots\}$ in A hence implies that ψ_ω is multiplicative. We have now shown that $\psi_\omega\colon A \to \mathcal{Q}_\omega$ is a *-homomorphism. We shall effectively make a cut-off of ψ_ω using the isomorphism $p\mathcal{Q}_\omega p \cong \mathcal{Q}_\omega$ from Proposition 2.13 for all non-zero projections $p \in \mathcal{Q}_\omega$ in order to construct a unital *-homomorphism, and it turns out that this construction also entails that we can remove γ.

Consider the projection $p = \psi_\omega(1)$ in $\mathcal{Q}_\omega$; this is a non-zero projection, since

$$\tau_\omega(p) = \tau_\omega \circ \psi_\omega(1) = \gamma\tau(1) = \gamma > 0.$$

Since the corner $p\mathcal{Q}_\omega p$ is *-isomorphic to $\mathcal{Q}_\omega$ by Proposition 2.13(ii), we can define the C^*-algebra $B = p\mathcal{Q}_\omega p$ and regard ψ_ω as a unital *-homomorphism $A \to B$. Using that $\mathcal{Q}_\omega$ is monotracial, the isomorphism $p\mathcal{Q}_\omega p \cong \mathcal{Q}_\omega$ entails that

$$\tau_B \circ \psi_\omega = \frac{1}{\tau_\omega(p)}\tau_\omega \circ \psi_\omega = \frac{1}{\gamma}\tau_\omega \circ \psi_\omega = \tau$$

completing the proof. □

This characterisation allows us to easily verify another link between quasidiagonality of tracial states and C^*-algebras. Do note the assumption in the following corollary that the tracial state is faithful.

Corollary 2.53. *If A is a separable, unital, exact C^*-algebra with a faithful, quasidiagonal tracial state τ, then A is quasidiagonal.*

Proof. Find by the equivalence of (i) and (ii) in Proposition 2.52 a unital, nuclear *-homomorphism $\varphi_\omega\colon A \to \mathcal{Q}_\omega$ with $\tau_\omega \circ \varphi_\omega = \tau$. Faithfulness of τ implies injectivity of φ_ω, and then Corollary 2.42 proves quasidiagonality of A. □

Just as quasidiagonality of tracial states has several different characterisations, the same occurs for amenability. For a large list of equivalences, we refer to [10, Theorem 3.1.6]; the proofs are quite technical, and the results are not of high importance for our purposes, and we shall hence omit mentioning most of them. However, we do need one result proving amenability of tracial states on nuclear C^*-algebras, which shows how the refinement due to Gabe is a generalisation of the original theorem of Tikuisis, White and Winter.

First, we need some notation. If τ is a tracial state, then we can consider the left regular representation $\pi_\tau\colon A \to \mathbb{B}(L^2(A,\tau))$ and the right regular representation $\pi_\tau^{\mathrm{op}}\colon A^{\mathrm{op}} \to \mathbb{B}(L^2(A,\tau))$, see [11, Section 6.1]. A proof of the following theorem can be found in the long list of equivalences in [10, Theorem 3.1.6].

Proposition 2.54. *A tracial state τ on a C^*-algebra A is amenable if and only if the product homomorphism $\pi_\tau \times \pi_\tau^{\mathrm{op}}\colon A \odot A^{\mathrm{op}} \to \mathbb{B}(L^2(A,\tau))$ is continuous with respect to the minimal tensor product.*

Equipped with the above tool, we can easily show that tracial states on nuclear C^*-algebras are amenable.

Corollary 2.55. *Let A be a nuclear C^*-algebra, and let τ be a tracial state on A. Then τ is amenable.*

Proof. Since A is nuclear, there exists a unique C^*-norm on $A\odot A^{\mathrm{op}}$. Since the product homomorphism $\pi_\tau \times \pi_\tau^{\mathrm{op}}\colon A \odot A^{\mathrm{op}} \to \mathbb{B}(L^2(A,\tau))$ is continuous in the maximal tensor norm by definition of the maximal tensor product, we conclude from Proposition 2.54 that τ is amenable. ☐

Since nuclearity is equivalent to amenability for C^*-algebras, see [3, Theorem IV.3.13 and Theorem IV.3.14], and all tracial states on nuclear C^*-algebras are amenable, this can motivative the terminology of amenable tracial states. In fact, we can use Proposition 1.37 to show that amenability of a discrete group G implies that all tracial states on $C_r^*(G)$ are amenable.

Corollary 2.56. *Let G be a discrete group. If G is amenable, then all tracial states on $C_r^*(G)$ are amenable.*

Proof. If G is an amenable group, then $C_r^*(G)$ is nuclear by the equivalence of (i) and (ii) in Proposition 1.37. Hence, by Corollary 2.55, we see that every tracial state on $C_r^*(G)$ is amenable. ☐

In particular, the canonical tracial state $x \mapsto \langle x\delta_e, \delta_e\rangle$ on $C_r^*(G)$ is faithful and amenable by Corollary 2.56 and Proposition 1.35. Actually, every tracial state on $C_r^*(G)$ being amenable is a necessary and sufficient condition for G being an amenable group, and this is equivalent to the existence of just one amenable tracial state on $C_r^*(G)$, see [11, Proposition 6.3.3].

3 Setting the extension theoretic stage

We now start studying a rich algebraic theory of C^*-algebras, namely the theory of extensions, which is a crucial ingredient in Schafhauser's proof of the Tikuisis-White-Winter theorem, as it provides the entire framework of the proof. First, we introduce the multiplier algebra and prove a variety of useful results and subsequently use these in order to build the theory of extensions. Hence this chapter has two goals: Build up the theory for its own right, and discuss how we can and will use it in Schafhauser's proof.

3.1 Preliminary discussion on multiplier algebras

Before we can reasonably study extensions, we first need to discuss what are known as multiplier algebras. It turns out that the information in an extension can be encoded in a single *-homomorphism called the *Busby invariant* whose codomain is a quotient of a multiplier algebra, and in order to define an algebraic structure on the set of extensions, we need to have a decent grasp on the theory of multiplier algebras. The starting point for understanding multiplier algebras is understanding essential ideals. We shall define the multiplier algebra using a universal property following [11, Proposition 8.4.2], but there exists a description in terms of centralisers, which can be studied in [47, 53].

Definition 3.1. An ideal I in a C^*-algebra A is called *essential* if the orthogonal complement

$$I^{\perp} = \{a \in A \,|\, aI = Ia = 0\}$$

is trivial, i.e., if $I^{\perp} = \{0\}$.

There is another equivalent formulation of essential ideals, which occasionally is useful.

Proposition 3.2. *An ideal $I \trianglelefteq A$ is essential if and only if it non-trivially intersects any non-zero closed two-sided ideal $J \trianglelefteq A$.*

Proof. Suppose $I \trianglelefteq A$ is an essential ideal, and let $J \trianglelefteq A$ be a closed two-sided ideal in A satisfying that $I \cap J = 0$. Since $I \cap J = IJ$, we find that $IJ = JI = 0$, and as I is an essential ideal in A, we conclude that $J = 0$. On the other hand, assume that I non-trivially intersects any non-zero closed two-sided ideal in A, then we want to show that I has a trivial orthogonal complement. Let $a \in A$ be arbitrary with $aI = Ia = 0$, and let J be the ideal generated by a in A. Then $J \cap I = JI = 0$, and hence $a = 0$. □

One of the foundational theorems in extension theory is the following, which is the definition of the multiplier algebra. We give both the concrete construction as well as a more abstract description in the form of the universal property of multiplier algebras. Let us also note that there exists another characterisation using centralisers — the reader is asked to consult [53, Chapter 3.12] on this matter. The proof of the following result is a generalisation of [11, Proposition 8.4.2], but follows the same structure.

Theorem 3.3. *Let I be a C^*-algebra. Then there exists a unique unital C^*-algebra $\mathcal{M}(I)$ containing I as an essential ideal with the following universal property: If I sits as an ideal in a C^*-algebra A, then there exists a unique *-homomorphism $A \to \mathcal{M}(I)$ extending the inclusion $I \subseteq \mathcal{M}(I)$. Moreover, this *-homomorphism is injective if and only if I is an essential ideal in A.*

Proof. Uniqueness follows easily by invoking the universal property twice. Let $\pi\colon I \to \mathbb{B}(H)$ be any faithful and non-degenerate representation and define

$$\mathcal{M}(I) = \{T \in \mathbb{B}(H) \,|\, T\pi(x), \pi(x)T \in I, \text{ for all } x \in I\}.$$

It is immediate that I sits as an ideal inside $\mathcal{M}(I)$, and that $\mathcal{M}(I)$ is a unital C^*-algebra. Suppose I is not essential in $\mathcal{M}(I)$, then there exists non-zero $T \in \mathcal{M}(I)$ with $T\pi(x) = \pi(x)T = 0$ for all $x \in I$. In particular, there exists $\xi \in H$ such that $T\xi \neq 0$, and as $\pi(x)T\xi = 0$ for all $x \in I$, this contradicts the fact that the representation π is non-degenerate. Hence $I \trianglelefteq \mathcal{M}(I)$ is essential.

We now prove that $\mathcal{M}(I)$ satisfies the desired universal property. Let $(e_\alpha)_{\alpha\in\Lambda}$ be a quasicentral approximate unit for I with respect to A. Define the map $\pi_0\colon A \to \mathbb{B}(H)$ by

$$\pi_0(a)(\pi(b)\xi) = \pi(ab)\xi, \qquad a \in A$$

for all $b \in I$ and $\xi \in H$; note that π_0 exists by non-degeneracy of π and as I is an ideal in A. Since we for any fixed $a \in A$ and arbitrary $b, b' \in I$ and $\xi \in H$ have that

$$\|\pi_0(a)(\pi(b)\xi) - \pi_0(a)(\pi(b')\xi)\| = \|\pi(ab)\xi - \pi(ab')\xi\| = \lim_\alpha \|\pi(ae_\alpha)\pi(b-b')\xi\| \leq \|a\| \, \|\pi(b-b')\xi\| \, ,$$

the net $\big(\pi_0(a)\pi(e_\alpha)\xi\big)_{\alpha\in\Lambda}$ is Cauchy in $\mathbb{B}(H)$. Completeness guarantees the existence of a limit point, and we can consequently define the map $\rho\colon A \to \mathbb{B}(H)$ by

$$\rho(a)\xi = \lim_\alpha \pi_0(a)\pi(e_\alpha)\xi = \lim_\alpha \pi(ae_\alpha)\xi.$$

It is easily shown that ρ is a *-homomorphism extending π. Moreover, it is unique with this property by the following argument: Suppose $\sigma\colon A \to \mathbb{B}(H)$ is a representation extending π, then, for all $a \in A$, $b \in I$ and $\xi \in H$, we have

$$\sigma(a)(\pi(b)\xi) = \sigma(ab)\xi = \pi(ab)\xi = \rho(ab)\xi = \rho(a)(\pi(b)\xi).$$

Since π is non-degenerate, we conclude that $\sigma(a) = \rho(a)$, and hence ρ is the unique *-homomorphism $A \to \mathbb{B}(H)$ extending π.

We claim that ρ is the desired *-homomorphism in the universal property. In order to show this, we need to establish that ρ extends the inclusion $I \subseteq \mathbb{B}(H)$ and that $\rho(A) \subseteq \mathcal{M}(I)$. Since ρ extends π, we can invoke faithfulness of π to identify $\pi(I)$ with I to see that ρ extends the inclusion $I \subseteq \mathbb{B}(H)$. Hence, we only need show that $\rho(A) \subseteq \mathcal{M}(I)$. Note that for all $a \in A$ and $b \in I$,

$$\rho(a)\pi(b) = \lim_\alpha \pi(ae_\alpha)\pi(b) = \lim_\alpha \pi(a)\pi(e_\alpha b) = \pi(ab)$$

and quasicentrality of the approximate unit $(e_\alpha)_{\alpha\in\Lambda}$ in A implies that

$$\pi(b)\rho(a) = \pi(b)\lim_\alpha \pi(ae_\alpha) = \pi(b)\lim_\alpha \pi(e_\alpha a) = \lim_\alpha \pi(be_\alpha)\pi(a) = \pi(ba).$$

Since $\pi(ab), \pi(ba) \in I$, we have that $\rho(a) \in \mathcal{M}(A)$ by construction such that $\rho(A) \subseteq \mathcal{M}(I)$.

Now assume that I is an essential ideal in A, then we show that ρ is injective. Suppose that $\rho(a) = 0$, then, for any $b \in I$, we have

$$0 = \rho(a)\pi(b) = \pi(ab), \qquad \text{and,} \qquad 0 = \pi(b)\rho(a) = \pi(ba)$$

by the above calculations. Since π is faithful and $b \in I$ was arbitrary, we find that a is orthogonal to I and, as I is an essential ideal in A, we conclude that $a = 0$. On the other hand, suppose that I is not essential in A, then there exists a non-zero element $a \in A$ such that $aI = Ia = 0$. Since the approximate unit $(e_\alpha)_{\alpha\in\Lambda}$ belongs to I, we find for each $\xi \in H$ that

$$\rho(a)\xi = \lim_\alpha \pi(ae_\alpha)\xi = 0$$

and hence $\rho(a) = 0$ proving that ρ is not injective. This proves that I is an essential ideal in A if and only if ρ is injective. □

We call $\mathcal{M}(I)$ the *multiplier algebra of* I, and the quotient $\mathcal{Q}(I) = \mathcal{M}(I)/I$ is called the *corona algebra*. The above construction shows that $\mathcal{M}(I)$ is the maximal C^*-algebra in which I sits as an essential ideal.

There are three common examples to study when discussing multiplier algebras. The first is the fact that the multiplier algebra of a unital C^*-algebra is just the C^*-algebra itself.

Proposition 3.4. *If A is a unital C^*-algebra, then $\mathcal{M}(A) = A$.*

Proof. We already established that $\mathcal{M}(A)$ is a unital C^*-algebra, and that A is an essential ideal in $\mathcal{M}(A)$. Let 1_A denote the unit of A, and let $1_{\mathcal{M}(A)}$ denote the unit of $\mathcal{M}(A)$. Then,

$$(1_A - 1_{\mathcal{M}(A)})A = 0 = A(1_A - 1_{\mathcal{M}(A)}),$$

and as A is an essential ideal in $\mathcal{M}(A)$, we find that $1_A = 1_{\mathcal{M}(A)}$, which implies that $A = \mathcal{M}(A)$. □

While the multiplier algebra of a unital C^*-algebra is not of any interest by the above proposition, there are several non-unital C^*-algebras with interesting multiplier algebras. The following example effectively shows that when we study extensions in the next section, we study a generalisation of extensions by compact operators.

Example 3.5. Let H be an infinite-dimensional Hilbert space, and let $\mathbb{K}(H)$ denote the compact operators on H. We claim that $\mathcal{M}(\mathbb{K}(H)) = \mathbb{B}(H)$, which is easily seen as $\mathbb{K}(H)$ already sits inside $\mathbb{B}(H)$ as an essential ideal. More precisely, for any $T \in \mathbb{B}(H)$ and $S \in \mathbb{K}(H)$, we have $TS, ST \in \mathbb{K}(H)$, which implies that $\mathcal{M}(\mathbb{K}(H)) = \mathbb{B}(H)$ by the explicit construction of multiplier algebras in Theorem 3.3.

The next result shows that the multiplier algebra can be seen as a non-commutative analogue of the Stone-Čech compactification.

Proposition 3.6. *If X is a locally compact Hausdorff space, then $\mathcal{M}(C_0(X))$ is isomorphic to $C(X^\dagger)$, where $X^\dagger$ is the Stone-Čech compactification of X.*

Proof. First, observe that the C^*-algebra of bounded functions on X is isomorphic to $C(X^\dagger)$. Since $C_0(X)$ is an essential ideal in $C(X^\dagger)$, there exists a unique injective *-homomorphism $\varphi\colon C(X^\dagger) \to \mathcal{M}(C_0(X))$ extending the inclusion $C_0(X) \subseteq \mathcal{M}(C_0(X))$. We claim that φ is a *-isomorphism. We only need to show that it is surjective, and for this it suffices to show that the image of φ contains all positive elements in $\mathcal{M}(C_0(X))$. Let $g \in \mathcal{M}(C_0(X))$ be any positive element, and let $(e_\alpha)_{\alpha\in\Lambda}$ be an approximate unit for $C_0(X)$. For each $x \in X$, we find that the net $(ge_\alpha(x))_{\alpha\in\Lambda}$ is an increasing net of real numbers, and it is bounded from above by $\|g\|$, and, thus, it has a limit. Define the function $f\colon X \to \mathbb{C}$ by $f(x) = \lim_\alpha (ge_\alpha)(x)$. We claim that $f \in C(X^\dagger)$ and that $\varphi(f) = g$.

Since $C(X^\dagger)$ is isomorphic to the C^*-algebra of bounded functions $X \to \mathbb{C}$, we shall show that f is exactly such a map. It is immediate that f is bounded by $\|g\|$, so we only need to verify that it is continuous. Since $(e_\alpha)_{\alpha\in\Lambda}$ is an approximate unit for $C_0(X)$, it follows that $fh = gh$, and in particular $fh \in C_0(X)$, for each $h \in C_0(X)$. Let $x \in X$ be arbitrary and suppose that $(x_i)_{i\in I}$ is a net in X converging to x. Find by local compactness of X a compact neighbourhood K of x; we can without loss of generality assume that each x_i belongs to K. Use Urysohn's lemma, [21, Lemma 4.32], to find a function $h \in C_0(X)$ with the property that $h|_K = 1$. Since $fh \in C_0(X)$, we find that

$$f(x) = (fh)(x) = \lim_i (fh)(x_i) = \lim_i f(x_i)$$

proving continuity. We have thus shown that $h \in C(X^\dagger)$.

Finally, for any $h \in C_0(X)$, we can use the fact that φ extends the inclusion $C_0(X) \subseteq \mathcal{M}(C_0(X))$ to see that

$$\varphi(f)h = \varphi(fh) = fh = gh,$$

and this implies $(\varphi(f) - g)C_0(X) = 0$. Since $C_0(X)$ is an essential ideal in $\mathcal{M}(C_0(X))$, we conclude that $\varphi(f) = g$, which completes the proof. □

As we mentioned in the introduction of this section, we need multiplier algebras in order to define an algebraic structure on the set of extensions. It actually turns out that we need an embedding of the Cuntz algebra $\mathcal{O}_2$ inside the multiplier algebra or, rather, inside the corona algebra; this is possible in the stable case, as we show below.

Proposition 3.7. *If A and B are C^*-algebras, then we have the inclusion $\mathcal{M}(A) \otimes \mathcal{M}(B) \subseteq \mathcal{M}(A \otimes B)$.*

Proof. Let $A \subseteq \mathbb{B}(H)$ and $B \subseteq \mathbb{B}(K)$ be non-degenerate faithful representations. Then,

$$\mathcal{M}(A) \otimes \mathcal{M}(B) \subseteq \mathbb{B}(H) \otimes \mathbb{B}(K) \subseteq \mathbb{B}(H \otimes K).$$

Moreover, we can represent $A \otimes B \subseteq \mathbb{B}(H \otimes K)$ faithfully and non-degenerately. Given $\sum_i (T_i \otimes S_i) \in \mathcal{M}(A) \otimes \mathcal{M}(B)$ with $T_i \in \mathcal{M}(A)$ and $S_i \in \mathcal{M}(B)$, we see that for any elementary tensor $a \otimes b \in A \otimes B$,

$$\left(\sum_i (T_i \otimes S_i)\right)(a \otimes b) = \sum_i (T_i a \otimes S_i b) \in A \otimes B$$

and similarly the other way around. This proves that we have the inclusion $\mathcal{M}(A) \otimes \mathcal{M}(B) \subseteq \mathcal{M}(A \otimes B)$ as desired. ☐

Corollary 3.8. *If I is stable and H is a separable infinite-dimensional Hilbert space, then $\mathbb{B}(H) \subseteq \mathcal{M}(I)$.*

Proof. Since I is stable, we have the isomorphism $I \cong I \otimes \mathbb{K}(H)$. Then Proposition 3.7 entails that

$$\mathcal{M}(I) \otimes \mathcal{M}(\mathbb{K}(H)) \subseteq \mathcal{M}(I \otimes \mathbb{K}(H)) \cong \mathcal{M}(I).$$

Note that $\mathcal{M}(\mathbb{K}(H)) = \mathbb{B}(H)$ by Example 3.5, and we hence have the inclusion $\mathbb{B}(H) \subseteq \mathcal{M}(I)$ via the embedding $\mathbb{B}(H) \hookrightarrow \mathcal{M}(I) \otimes \mathbb{B}(H)$ given by $T \mapsto 1 \otimes T$. ☐

Since we have a faithful representation $\mathcal{O}_2 \subseteq \mathbb{B}(H)$ for some Hilbert space H, we see that if I is any stable C^*-algebra, then we have the embedding $\mathcal{O}_2 \subseteq \mathcal{M}(I)$. More generally, we have for any separable infinite-dimensional Hilbert space H an embedding $\mathcal{O}_n \subseteq \mathbb{B}(H)$ for any $n \geq 2$.

Corollary 3.9. *If I is a stable C^*-algebra, then $\mathcal{O}_n \subseteq \mathcal{M}(I)$ for all $n \geq 2$.*

3.2 Extension theory and the group $\mathrm{Ext}^{-1}(A, I)$

In this section, we introduce the notion of extensions and define an algebraic structure on these. While many textbooks on the subject of extension theory examine extensions by the C^*-algebra of compact operators $\mathbb{K}(H)$, in this thesis we are interested in general extensions. There are several reasons to be interested in the special case with extensions by $\mathbb{K}(H)$. Historically, the theory of extensions of C^*-algebras was developed in 1977 by Brown, Douglas and Fillmore [7] to study essentially normal operators. While on the surface one would expect no direct link between extension theory, which is highly operator algebraic, and essentially normal operators, which is a topic within single operator theory, it turns out that there is an overlap — in particular, the question of classifying when an essentially normal operator is a normal operator plus a compact operator can be phrased as a question about extensions by compact operators. We only mention this here to illustrate the breadth of topics that can be advantageously studied using extension theory; for more details on essentially normal operators and extension theory, we refer to [33] as well as the original paper by Brown, Douglas and Fillmore [7].

For the purposes of this thesis, we cannot rely only on extensions by the compact operators, as the general theory of extensions plays an important role in Schafhauser's proof of the theorem of Tikuisis, White and Winter, and as such we need to study general extensions in detail. Our main references are [2, Chapter VII] and [34, Chapter 3], with our exposition mostly following that of the latter. First of all, what do we mean by an extension? For anyone having studied basic homological algebra, the answer is quite obvious:

Definition 3.10. Let I, A be C^*-algebras. We say that a C^*-algebra E is an *extension of A by I* if there exists a short exact sequence $0 \to I \to E \to A \to 0$.

While the definition entails that the extension is the C^*-algebra E, it is often useful to think of extensions as being the short exact sequence in itself. Our first goal is to in some way characterise the extensions, and how we can distinguish between different extensions. Of course, this has no mathematical meaning per se, but the informal statement shall guide our study.

Consider an extension $0 \to I \to E \to A \to 0$ of A by I. Since I sits inside E as an ideal, we can use Theorem 3.3 to prove the existence of a unique *-homomorphism $\sigma\colon E \to \mathcal{M}(I)$, which extends the inclusion $I \subseteq \mathcal{M}(I)$. Let $\rho\colon \mathcal{M}(I) \to \mathcal{Q}(I)$ denote the quotient map onto the corona algebra, and let $\pi\colon E \to A$ denote the quotient map of the extension. Consider the map $\beta\colon A \to \mathcal{Q}(I)$ given by $\beta(a) = \rho(\sigma(e))$ where $\pi(e) = a$. It is easily verified that this map is a well-defined *-homomorphism — we call β the *Busby invariant* of the extension. The Busby invariant of any given extension is clearly unique by the universal property of the multiplier algebra. A more graphical way to express the Busby invariant is as the unique *-homomorphism making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \xrightarrow{\iota} & E & \xrightarrow{\pi} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{\rho} & \mathcal{Q}(I) & \longrightarrow & 0
\end{array}$$

Note that β is injective if and only if σ is injective if and only if I is an essential ideal in E.

The importance of the Busby invariant lies in the fact that we can determine the extension by the Busby invariant in a unique way. But first we need a definition regarding isomorphisms of extensions.

Definition 3.11. Two extensions η_1 and η_2 of A by I are said to be *isomorphic* if there exists a *-homomorphism $\varphi\colon E_1 \to E_2$ making the following diagram commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Note that φ necessarily is a *-isomorphism by an easy diagram chase. This definition of extensions being isomorphic is rather strong, and it turns out to be too restrictive to be useful for later purposes, where we shall relax the conditions to allow for unitary equivalence. However, we need it for the following proposition, which states that the Busby invariant completely characterises extensions up to isomorphisms.

Proposition 3.12. *Let I and A be C^*-algebras. Two extensions η_1 and η_2 of A by I are isomorphic if and only if they admit the same Busby invariant. Moreover, if $\beta\colon A \to \mathcal{Q}(I)$ is a *-homomorphism, then there exists an extension whose Busby invariant is β.*

Proof. Suppose first that the two extensions are isomorphic with isomorphism $\varphi\colon E_1 \to E_2$. Let $\beta_i\colon E_i \to A$ be the Busby invariants of the extensions, i.e., consider the following commutative diagram for $i = 1, 2$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E_i & \xrightarrow{\pi_i} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\sigma_i} & & \downarrow{\scriptstyle\beta_i} & & \\
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{\rho} & \mathcal{Q}(I) & \longrightarrow & 0
\end{array}$$

Using that the two extensions are isomorphic, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E_1 & \xrightarrow{\pi_1} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\
0 & \longrightarrow & I & \longrightarrow & E_2 & \xrightarrow{\pi_2} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\sigma_2} & & \downarrow{\scriptstyle\beta_2} & & \\
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{\rho} & \mathcal{Q}(I) & \longrightarrow & 0
\end{array}$$

Note that $\sigma_2 \circ \varphi = \sigma_1$ by the universal property of the multiplier algebra. It is then easily verified that $\beta_1 = \beta_2$.

Now suppose that $\beta \colon A \to \mathcal{Q}(I)$ is a *-homomorphism. We wish to construct an extension whose Busby invariant is precisely β. Define the C^*-algebra

$$E = \{(a, b) \in A \oplus \mathcal{M}(I) \mid \beta(a) = \rho(b)\}$$

where, again, $\rho \colon \mathcal{M}(I) \to \mathcal{Q}(I)$ is the quotient map. Note that I is an ideal in E. Then if $\iota \colon I \to E$ is the inclusion in the second coordinate, and if $\pi \colon E \to A$ is the projection onto the first coordinate, we get the following short exact sequence.

$$0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0.$$

Note that this extension is constructed precisely such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\iota} & E & \xrightarrow{\pi} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \xrightarrow{\rho} & \mathcal{Q}(I) & \longrightarrow & 0 \end{array}$$

Here we have denoted by $\sigma \colon E \to \mathcal{M}(I)$ the projection onto the second coordinate. It follows easily that the extension $0 \to I \to E \to A \to 0$ has β as its Busby invariant.

Lastly, suppose that E_1 and E_2 are two extensions with the same Busby invariant $\beta \colon A \to \mathcal{Q}(I)$. Denote by $\pi_i \colon E_i \to A$ the quotient maps of the extensions. Consider again the pullback

$$E = \{(a, b) \in A \oplus \mathcal{M}(I) \mid \beta(a) = \rho(b)\}$$

whose corresponding extension has β as its Busby invariant. Define the *-homomorphism $\varphi \colon E_1 \to E$ given by $\varphi(e) = (\pi_1(e), \sigma_1(e))$, which is well-defined, as

$$\beta(\pi_1(e)) = \beta_1(\pi_1(e)) = \rho(\sigma_1(e)).$$

One easily verifies that the extensions E_1 and E are isomorphic with $\varphi \colon E_1 \to E$ witnessing the isomorphism. By considering E_2 instead of E_1, we find that the three extensions considered are all isomorphic, and this completes the proof. □

This classification result states that studying extensions of A by I is equivalent to studying *-homomorphisms $A \to \mathcal{Q}(I)$. For example, we can use this point of view to completely characterise the extensions of C^*-algebras by unital C^*-algebras.

Corollary 3.13. *If A is any C^*-algebra and I is unital, then the only extension up to isomorphism is $0 \to I \to I \oplus A \to A \to 0$.*

Proof. Since I is unital, the corona algebra $\mathcal{Q}(I)$ is trivial by Proposition 3.4, hence the only *-homomorphism $A \to \mathcal{Q}(I)$ is the zero homomorphism. By Proposition 3.12, we see that this implies that there is only one extension up to isomorphism, and as $0 \to I \to I \oplus A \to A \to 0$ is obviously an extension, we are done. □

Hence the only interesting extensions are those by non-unital C^*-algebras. However, it turns out that the notion of isomorphic extensions is too restrictive for many purposes, and as such we need to weaken it slightly.

Definition 3.14. Two extensions η_1 and η_2 of A by I are said to be *unitarily equivalent* if there exists a *-homomorphism $\varphi \colon E_1 \to E_2$ and a unitary $u \in \mathcal{M}(I)$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle\mathrm{Ad}(u)}\downarrow & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

In the same way as isomorphic extensions have the same Busby invariants, unitarily equivalent extensions have unitarily equivalent Busby invariants. We only state the result, as the proof is quite similar to that of Proposition 3.12.

Proposition 3.15. *Two extensions of A and I with Busby invariants $\beta_i\colon A \to \mathcal{Q}(I)$ for $i = 1, 2$ are unitarily equivalent if and only if there exists a unitary $u \in \mathcal{M}(I)$ such that $\beta_1 = \operatorname{Ad}(\rho(u)) \circ \beta_2$.*

An important class of extensions is the class of split extensions, which we shall call trivial, since they turn out to exactly correspond to the neutral element in the extension semigroup, as we shall see later.

Definition 3.16. An extension $0 \to I \to E \xrightarrow{\pi} A \to 0$ of A by I is called *trivial* if it splits, i.e., if there exists a *-homomorphism $\varphi\colon A \to E$ with $\pi \circ \varphi = \mathrm{id}_A$.

Since we have already established that studying extensions and studying Busby invariants are equivalent notions, it should come as no surprise that we can characterise the trivial extensions by certain properties of their Busby invariants.

Proposition 3.17. *Let $0 \to I \to E \xrightarrow{\pi} A \to 0$ be an extension with corresponding Busby invariant $\beta\colon A \to \mathcal{Q}(I)$. Then the extension is trivial if and only if β lifts to a *-homomorphism $A \to \mathcal{M}(I)$, i.e., if and only if there exists a *-homomorphism $\overline{\beta}\colon A \to \mathcal{M}(I)$ with $\rho \circ \overline{\beta} = \beta$, where $\rho\colon \mathcal{M}(I) \to \mathcal{Q}(I)$ is the quotient map.*

Proof. Assume the extension splits and that $\varphi\colon A \to E$ is a *-homomorphism such that $\pi \circ \varphi = \mathrm{id}_A$. Denote by $\sigma\colon E \to \mathcal{M}(I)$ the unique *-homomorphism extending the inclusion $I \subseteq \mathcal{M}(I)$, and define $\overline{\beta} = \sigma \circ \varphi$. Then,

$$\rho \circ \overline{\beta} = \rho \circ \sigma \circ \varphi = \beta \circ \pi \circ \varphi = \beta$$

as desired.

For the other implication, assume that β lifts to a *-homomorphism $\overline{\beta}\colon A \to \mathcal{M}(I)$. Note that we, by Proposition 3.12, can realise E as the C^*-algebra

$$E = \{(a, b) \in A \oplus \mathcal{M}(I) \mid \beta(a) = \rho(b)\},$$

and then $\sigma\colon E \to \mathcal{M}(I)$ by $\sigma(a, b) = b$ is the unique *-homomorphism extending the inclusion $I \subseteq \mathcal{M}(I)$, and $\pi\colon E \to A$ is given by $\pi(a, b) = a$. Now define $\varphi\colon A \to E$ by $\varphi(a) = (a, \overline{\beta}(a))$. This is a well-defined *-homomorphism since $\rho \circ \overline{\beta}(a) = \beta(a)$, and it is easily verified that $\pi \circ \varphi = \mathrm{id}_A$. ☐

We now begin describing the algebraic structure on extensions under the additional assumption that I is assumed to be stable. Since we can embed the Cuntz algebra $\mathcal{O}_2$ inside $\mathcal{M}(I)$ by Corollary 3.9, we can construct an explicit isomorphism $\Theta\colon M_2(I) \to I$ by

$$\Theta\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) = \sum_{i,j=1}^{2} s_i b_{i,j} s_j^*, \qquad b_{i,j} \in I,$$

where the isometries s_1, s_2 are those generating the Cuntz algebra $\mathcal{O}_2$. We can extend Θ to an isomorphism $\mathcal{M}(M_2(I)) \cong M_2(\mathcal{M}(I))$ by the same formula, and that if we have two such isomorphisms induced by a pair of isometries s_1, s_2 and t_1, t_2 generating $\mathcal{O}_2$, then the unitary element $u = s_1 t_1^* + s_2 t_2^*$ in $\mathcal{M}(I)$ implements a unitary equivalence of the isomorphisms. This proves that these isomorphisms are unique up to unitary equivalence. Moreover, if we have such an isomorphism $\Theta\colon M_2(I) \to I$, we can construct an induced *-isomorphism $\tilde{\Theta}\colon M_2(\mathcal{Q}(I)) \to \mathcal{Q}(I)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2(I) & \longrightarrow & M_2(\mathcal{M}(I)) & \longrightarrow & M_2(\mathcal{Q}(I)) & \longrightarrow & 0 \\ & & \Theta\big\downarrow & & \Theta\big\downarrow & & \tilde{\Theta}\big\downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \longrightarrow & \mathcal{Q}(I) & \longrightarrow & 0 \end{array}$$

Let η_1, η_2 be two extensions of A by I with Busby invariants $\beta_1, \beta_2 \colon A \to \mathcal{Q}(I)$. We can, up to unitary equivalence, define an addition $\eta_1 \oplus \eta_2$, to be the extension with the Busby invariant $\beta \colon A \to \mathcal{Q}(I)$ given by

$$\beta = \tilde{\Theta} \circ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

Explicitly, we can express this sum of Busby invariants as

$$\beta(a) = s_1 \beta_1(a) s_1^* + s_2 \beta_2(a) s_2^*, \qquad a \in A$$

which is precisely the formulation used in [66]. Without too much trouble, one can prove that this addition is associative and commutative, such that the set of unitary equivalence classes of Busby invariants becomes an Abelian semigroup. Observe that any extension unitarily equivalent to a trivial extension is again trivial, and that any addition of trivial extensions is again trivial. In other words, the set of trivial extensions is a subsemigroup of the unitary equivalence classes of extensions.

Definition 3.18. Two extensions η_1, η_2 of A by I with Busby invariants $\beta_1, \beta_2 \colon A \to \mathcal{Q}(I)$ are said to be *stably equivalent* if there exist trivial extensions ζ_1 and ζ_2 of A by I with Busby invariants $\tau_1, \tau_2 \colon A \to \mathcal{Q}(I)$ such that

$$\tilde{\Theta} \circ \begin{pmatrix} \beta_1 & 0 \\ 0 & \tau_1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\Theta} \circ \begin{pmatrix} \beta_2 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

are unitarily equivalent.

In other words, two extensions η_1 and η_2 are stably isomorphic if there exist trivial extensions ζ_1 and ζ_2 such that $\eta_1 \oplus \zeta_1 = \eta_2 \oplus \zeta_2$ up to unitary equivalence. It is easy to see that stable equivalence is an equivalence relation, and that unitary equivalence implies stable equivalence. We define $\mathrm{Ext}(A, I)$ to be the Abelian semigroup of equivalence classes under stable equivalence with the neutral element being the equivalence class of any trivial extension. The reader should be aware that we, occasionally, will be notationally sloppy and refer to an element in $\mathrm{Ext}(A, I)$ by an extension instead of its equivalence class.

Generally, $\mathrm{Ext}(A, I)$ does not admit more algebraic structure than stated above, i.e., it is not a group. However, we are often interested in the Abelian group $\mathrm{Ext}^{-1}(A, I)$ consisting of the invertible extensions. In order to fully understand this group, we need to characterise the invertible extensions. For this we first need a definition, which turns out to be critical for this purpose.

Definition 3.19. An extension $0 \to I \to E \xrightarrow{\pi} A \to 0$ is said to be *semisplit* if there exists a c.c.p. map $\varphi \colon A \to E$ such that $\pi \circ \varphi = \mathrm{id}_A$.

Note that an extension is semisplit if and only if the corresponding Busby invariant $\beta \colon A \to \mathcal{Q}(I)$ lifts to a c.c.p. map $A \to \mathcal{M}(I)$ by an argument similar to that of Proposition 3.17.

We also need a generalisation of Stinespring's dilation theorem, cf. Proposition 1.8, a proof of which can be found in [34, Theorem 3.2.7]. Observe that this is, in fact, a generalisation, since $\mathcal{M}(\mathbb{K}(H)) = \mathbb{B}(H)$ by Example 3.5.

Proposition 3.20 (Kasparov's Stinespring Theorem)**.** *Suppose that A is a separable C^*-algebra, and that I is a stable C^*-algebra. Let $\varphi \colon A \to \mathcal{M}(I)$ be a c.c.p. map, then there exists a *-homomorphism $\pi \colon A \to M_2(\mathcal{M}(I))$ such that*

$$\begin{pmatrix} \varphi(a) & 0 \\ 0 & 0 \end{pmatrix} = p\pi(a)p, \qquad a \in A.$$

where p is the projection $p = \mathrm{diag}(1, 0)$ in $M_2(\mathcal{M}(I))$.

For the following proposition, the reader is reminded that if $\varphi \colon A \to B$ is a *-homomorphism, the induced *-homomorphism on matrix algebras is denoted $\varphi^{(n)} \colon M_n(A) \to M_n(B)$.

Proposition 3.21. *Assume that A is separable, and that I is stable. Let $\eta \in \mathrm{Ext}(A, I)$ be represented by a Busby invariant $\beta \colon A \to \mathcal{Q}(I)$. The following conditions are equivalent.*

(i) *The extension* η *is invertible;*

(ii) *The extension* η *is semisplit;*

(iii) *There exists a *-homomorphism* $\pi\colon A \to M_2(\mathcal{M}(I))$ *such that*

$$\begin{pmatrix} \beta(a) & 0 \\ 0 & 0 \end{pmatrix} = \rho^{(2)}(p\pi(a)p),$$

where $\rho\colon \mathcal{M}(I) \to \mathcal{Q}(I)$ *is the quotient map, and* p *is the projection* $p = \operatorname{diag}(1,0)$ *in* $M_2(\mathcal{M}(I))$*.*

Proof. (i)⇒(ii): Assume that η is invertible, then there exists an extension η' such that $\eta \oplus \eta'$ is stably equivalent to a trivial extension. Representing the extension η' by the Busby invariant $\beta'\colon A \to \mathcal{Q}(I)$, we see that there exists a *-homomorphism $\alpha\colon A \to \mathcal{M}(I)$ splitting the extension $\eta \oplus \eta'$, i.e., such that $\rho \circ \alpha = \tilde{\Theta} \circ \begin{pmatrix} \beta & 0 \\ 0 & \beta' \end{pmatrix}$. Define the *-homomorphism $\pi = \Theta^{-1} \circ \alpha\colon A \to M_2(\mathcal{M}(I))$ and the projection $p = \operatorname{diag}(1,0)$ in $M_2(\mathcal{M}(I))$. Then, for any $a \in A$, we find that

$$\rho^{(2)}(p\pi(a)p) = \begin{pmatrix} \beta(a) & 0 \\ 0 & 0 \end{pmatrix}$$

Use the identification $pM_2(\mathcal{M}(I))p \cong \mathcal{M}(I)$ to define the c.c.p. map $\gamma\colon M_2(\mathcal{M}(I)) \to \mathcal{M}(I)$ to be the cutoff $\gamma(a) = pap$ for $a \in M_2(\mathcal{M}(I))$ and put $\psi = \gamma \circ \pi$, then it is immediate that $\rho \circ \psi = \beta$.

(ii)⇒(iii): This follows immediately from Kasparov's Stinespring theorem, Proposition 3.20.

(iii)⇒(i): Let $\pi\colon A \to M_2(\mathcal{M}(I))$ be a *-homomorphism such that

$$\begin{pmatrix} \beta(a) & 0 \\ 0 & 0 \end{pmatrix} = \rho^{(2)}(p\pi(a)p), \qquad a \in A.$$

We claim that π commutes with both p and $(1-p)$ up to $M_2(I)$. Let $a, b \in A$ be arbitrary, then as β is a *-homomorphism, we find that

$$\rho^{(2)}(p\pi(ab)p - p\pi(a)p\pi(b)p) = 0.$$

Putting $b = a^*$, we see that $p\pi(a)(1-p)\pi(a)^*p \in M_2(I)$, which implies that $p\pi(a)(1-p) \in M_2(I)$ for all $a \in A$. Consequently, we find that

$$p\pi(a) - \pi(a)p = p\pi(a)(1-p) - (1-p)\pi(a)p \in M_2(I)$$

as desired. The same argument holds under interchanging $(1-p)$ with p.

Noting that $\beta(a) = \rho^{(2)}(p\pi(a)p)$ for all $a \in A$ using an obvious identification, a natural candidate for an inverse would be $\beta'\colon A \to \mathcal{Q}(I)$ given by $\beta'(a) = \rho^{(2)}((1-p)\pi(a)(1-p))$. A priori, β' is only a c.c.p. map, however, since $(1-p)$ commutes with π, we find that β is, in fact, a *-homomorphism. Let η' be the extension associated to the Busby invariant β'. An easy calculation then shows that

$$\tilde{\Theta} \circ \begin{pmatrix} \beta & 0 \\ 0 & \beta' \end{pmatrix} = \rho \circ (\Theta \circ \pi)$$

which proves that the extension $\eta \oplus \eta'$ is trivial and, hence, η is invertible. □

Consider now the Abelian group $\operatorname{Ext}^{-1}(A, I)$ consisting of equivalence classes of invertible extensions. The above shows that the extension group $\operatorname{Ext}^{-1}(A, I)$ consists of equivalence classes of semisplit extensions. Moreover, a semisplit extension η corresponds to the neutral element in $\operatorname{Ext}^{-1}(A, I)$ if and only if there exists another semisplit extension η' such that $\eta \oplus \eta'$ is trivial. Since the Abelian group structure of $\operatorname{Ext}^{-1}(A, I)$ is highly appealing compared to the, in general, Abelian semigroup structure of $\operatorname{Ext}(A, I)$, we often work with the former instead of the latter. In the case where A is nuclear, the Choi-Effros lifting theorem, Proposition 2.41, in conjunction with Proposition 3.21, proves that all extensions of A by I are semi-split, and thus $\operatorname{Ext}(A, I) = \operatorname{Ext}^{-1}(A, I)$.

One of the ingredients of Schafhauser's proof is the use of the Universal Coefficient Theorem, most commonly referred to as the UCT, from KK-theory. While we shall not discuss the KK-theoretic background, as this would be too comprehensive for this thesis, we shall mention how it comes up within the framework of extensions, as the description is quite intuitive in this formulation. For our purposes, the UCT is a way of describing the extension theory of a pair of C^*-algebras by their K-theory. We refer to [2, Chapter 23] for a more in-depth study of the UCT and its importance in KK-theory.

Suppose we have an extension η of A by I with short exact sequence $0 \to I \to E \to A \to 0$, then we get the following six term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ \partial_1 \big\uparrow & & & & \big\downarrow \partial_0 \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(I) \end{array}$$

This six term exact sequence entails the existence of a group homomorphism $\alpha\colon \mathrm{Ext}^{-1}(A, I) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_{*+1}(I))$ given by $\alpha([\eta]) = \partial_0 \oplus \partial_1$, where $[\eta]$ denotes the unitary equivalence class of η in $\mathrm{Ext}^{-1}(A, I)$. Suppose $\alpha([\eta]) = 0$, then we get for $i = 0, 1$ extensions of Abelian groups

$$0 \to K_i(I) \to K_i(E) \to K_i(A) \to 0,$$

which we can denote by $K_i(\eta)$. This canonically defines a map $\gamma\colon \ker \alpha \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(I))$ given by $\gamma([\eta]) = K_0(\eta) \oplus K_1(\eta)$, and it is easily seen that γ is a group homomorphism.

Definition 3.22. A separable C^*-algebra A is said to *satisfy the UCT* if for all separable, stable C^*-algebras I, the map α is surjective, and the map γ is an isomorphism.

That is, if A satisfies the UCT, we get a short exact sequence of Abelian groups

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(I)) \to \mathrm{Ext}^{-1}(A, I) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_{*+1}(I)) \to 0$$

for all separable, stable C^*-algebras I. Note that this allows us to describe extension theory of a pair of C^*-algebras using only the data from the K-theory of that pair. The above formulation is also how, precisely, the UCT will show up in the proof of the Tikuisis-White-Winter theorem.

Unfortunately, not all C^*-algebras satisfy the UCT — Skandalis provided in [68] an example of a separable, exact C^*-algebra, for which the UCT does not hold. Nevertheless, it is an open problem whether all nuclear C^*-algebras satisfy the UCT. We can phrase this in a different way: Let $\mathcal{N}$ denote the smallest class of nuclear C^*-algebras containing the Abelian C^*-algebras, and which is closed under so-called KK-equivalence[2]. One can show that a C^*-algebra A satisfies the UCT if and only if A is KK-equivalent to an Abelian C^*-algebra, see [2, Theorem 23.10.5]; we often refer to $\mathcal{N}$ as the *UCT-class* for this reason. The aforementioned open question about nuclearity and the UCT can now be phrased as follows: Are all nuclear C^*-algebras in $\mathcal{N}$?

Let us give an explicit way that K-theory and extension theory are linked through the UCT. Let A be a separable C^*-algebra and consider some extension of $\mathbb{C}$ by A, i.e., a short exact sequence $0 \to A \to E \to \mathbb{C} \to 0$. Noting that $\mathbb{C}$ is a nuclear C^*-algebra satisfying the UCT, since it is an Abelian C^*-algebra, we get the short exact sequence of Abelian groups

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathbb{C}), K_*(A)) \to \mathrm{Ext}(\mathbb{C}, A) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(\mathbb{C}), K_{*+1}(A)) \to 0.$$

However, since $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$, we find that $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(\mathbb{C}), K_*(A)) = 0$, and hence

$$\mathrm{Ext}(\mathbb{C}, A) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, K_1(A)) \cong K_1(A),$$

[2] KK-theory is, roughly stated, a generalisation of K-theory and K-homology, and takes a pair of C^*-algebras A, B and returns an Abelian group $KK(A, B)$. The bifunctor $KK(\cdot, \cdot)$ is contravariant in the first argument and covariant in the second argument. For two C^*-algebras A and B, being KK-equivalent implies the existence of a certain element in $KK(A, B)$, see [2, Definition 19.1.1]

where the latter isomorphism is given by evaluation at 1. In particular, we can determine the K_1-group of a C^*-algebra by understanding the extensions of $\mathbb{C}$ by A. In fact, one can show, see [2, Chapter 15.14], that

$$K_0(A) \cong K_1(C_0(\mathbb{R}, A)) \cong \mathrm{Ext}(\mathbb{C}, C_0(\mathbb{R}, A)) \cong \mathrm{Ext}(C_0(\mathbb{R}), A).$$

We end this chapter on extension theory with a quite technical splitting result, which is precisely how the UCT appears in Schafhauser's proof of the Tikuisis-White-Winter theorem. In order to state and prove the theorem in any meaningful way, we need to establish a few different extension theoretic concepts. The first concerns semisplit extensions, where the splittings are assumed to have a certain approximation property.

Definition 3.23. Let $0 \to I \to E \xrightarrow{\pi} A \to 0$ be an extension with a c.c.p. splitting $\varphi\colon A \to E$. We say that φ is *weakly nuclear* if, for all $x \in I$, the map $\sigma_x\colon A \to I$ given by $\sigma_x(a) = x\varphi(a)x^*$ is nuclear. An extension is *weakly nuclear* if it admits a weakly nuclear splitting.

Consider for separable C^*-algebra A and I, where I is again assumed to be stable, the subset $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$ consisting of equivalence classes of the weakly nuclear extensions. One can verify that this, in fact, is an Abelian subgroup of $\mathrm{Ext}^{-1}(A, I)$ with the trivial extensions being those admitting a weakly nuclear *-homomorphic split, see [20]. Moreover, if A (or I) is KK-equivalent to a nuclear C^*-algebra, then there exists an isomorphism $\mathrm{Ext}^{-1}(A, I) \cong \mathrm{Ext}_{\mathrm{nuc}}(A, I)$ for all separable C^*-algebras I (or A), see [68, Proposition 3.2]. In particular, this isomorphism exists if A is KK-equivalent to an Abelian C^*-algebra, i.e., if A satisfies the UCT.

Definition 3.24. Let η be an extension of A by a stable C^*-algebra I, then we say that η is *nuclearly absorbing* if for any weakly nuclear extension η' of A by I with a weakly nuclear *-homomorphic splitting, the extensions η and $\eta \oplus \eta'$ are unitarily equivalent.

Since extensions with weakly nuclear *-homomorphic splittings are precisely the neutral elements in the Abelian group $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$, one might think that this definition is nonsensical. However, note that stable equivalence entails that two weakly nuclear extensions η and η' are representative of the same element in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$ if there exists extensions ζ and ζ' with weakly nuclear *-homomorphic splitting such that $\eta \oplus \zeta$ and $\eta' \oplus \zeta'$ are unitarily equivalent. If η is, moreover, nuclearly absorbing such that η and $\eta \oplus \zeta$ are unitarily equivalent, then η and $\eta' \oplus \zeta'$ are unitarily equivalent; it does not necessarily hold that this is true if η is not assumed to be nuclearly absorbing. Specifically, if η is a nuclear absorbing extension, which is a representative of the zero element in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$, then η admits a weakly nuclear *-homomorphic split. We shall see this idea used in proving Proposition 3.27 below.

Definition 3.25. An extension $0 \to I \to E \to A \to 0$ is *full* if the corresponding Busby invariant $\beta\colon A \to \mathcal{M}(I)$ is full, that is, if for any non-zero $a \in A$, the ideal generated by $\beta(a)$ is $\mathcal{M}(I)$. If the unitised extension $0 \to I \to E^\dagger \to A^\dagger \to 0$ is full, we say that the extension is *unitisably full*.

Recall that an Abelian group G is said to be *divisible* if for each $g \in G$ and every $n \in \mathbb{Z}\setminus\{0\}$, there exists $h \in G$ such that $nh = g$, and that an Abelian group is divisible if and only if it is injective in the homological sense by [58, Corollary 3.35]. In particular, if G is an Abelian divisible group, then $\mathrm{Ext}^1_{\mathbb{Z}}(H, G) = 0$ for any Abelian group H.

Definition 3.26. A C^*-algebra I is said to be an *admissible kernel* if I has real rank zero, stable rank one, the Abelian semigroup $\mathcal{D}(I)$ is almost unperforated, $K_0(I)$ is divisible, $K_1(I) = 0$, and every projection in $I \otimes \mathbb{K}(H)$ is Murray-von Neumann equivalent to a projection in I.

Many of these conditions may seem quite arbitrary, but they turn out to be of use in the following proposition, which provides a sufficient condition for an extension to have a weakly nuclear *-homomorphic splitting, see [66, Theorem 2.2].

Proposition 3.27. *Suppose that A and I are separable C^*-algebras and consider the extension η given by $0 \to I \to E \to A \to 0$ and denote the quotient map by $\pi\colon E \to A$. Assume that η is a weakly nuclear, unitisably full extension such that I is an admissible kernel and the index map $K_1(A) \to K_0(I)$ is trivial. Suppose further that A satisfies the UCT. Then there exists a weakly nuclear *-homomorphism $\varphi\colon A \to E$ such that $\pi \circ \varphi = \mathrm{id}_A$, that is, the extension η has a weakly nuclear *-homomorphic splitting.*

Proof. The main structure of the proof is as follows: We first prove that η is a representative of the trivial extension in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$, and then we show that η is nuclearly absorbing. As the equivalence class of η is the zero element in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$, there exists a weakly nuclear extension η' of A by I such that $\eta \oplus \eta'$ has a weakly nuclear *-homomorphic splitting. However, as η is nuclearly absorbing, we find that η and $\eta \oplus \eta'$ are unitarily equivalent and, consequently, η has a weakly nuclear *-homomorphic splitting.

Before we are able to invoke any of the algebraic structure on extensions, we need to ensure that I is stable. Since I has real rank zero and stable rank one, we can use Proposition A.3 and Proposition A.5 in the appendix to see that I has an approximate unit consisting of projections and has cancellation of projections. Moreover, all projections in $I \otimes \mathbb{K}(H)$ are Murray-von Neumann equivalent to a projection in I, implying that the positive cone $K_0(A)^+$ and the dimension range $\mathcal{D}_0(A)$ coincide. Using [62, Proposition 3.4], we find that I is stable.

Since I has been shown to be a stable C^*-algebra, and as η is a weakly nuclear extension, we are able to formally state our first goal, which is to prove that η is a representative of the zero element in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$. As $K_0(I)$ is divisible, and as $K_1(I) = 0$, we see that $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(I)) = 0$. Since A satisfies the UCT and $K_1(I) = 0$, this provides the isomorphism $\mathrm{Ext}^{-1}(A, I) \cong \mathrm{Hom}_{\mathbb{Z}}(K_1(A), K_0(I))$ of Abelian groups. Explicitly, the isomorphism maps the extension η to the corresponding index map $K_1(A) \to K_0(I)$, which is assumed to be trivial, and hence η corresponds to the neutral element in $\mathrm{Ext}^{-1}(A, I)$. Since all Abelian C^*-algebras are nuclear, and as A satisfies the UCT, A is KK-equivalent to a nuclear C^*-algebra. We thus have the isomorphism $\mathrm{Ext}^{-1}_{\mathrm{nuc}}(A, I) \cong \mathrm{Ext}^{-1}(A, I)$ of Abelian groups, and we conclude that η is a representative of the zero element in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$ as desired.

Now we show that η is a nuclear absorbing extension. Observe that I is a separable, stable C^*-algebra of real rank zero, that $\mathcal{D}(I)$ is almost unperforated, and that the extension η is assumed to be unitisably full. We can thus combine [48, Corollary 5.1] and [23, Theorem 2.6] to find that η is nuclearly absorbing.

At this point we know that η is a nuclearly absorbing extension which is a representative of the zero element in $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$. We claim that η has a weakly nuclear *-homomorphic splitting. By construction of $\mathrm{Ext}_{\mathrm{nuc}}(A, I)$, there exists an extension η' with a weakly nuclear *-homomorphic splitting such that the extension $\eta \oplus \eta'$ again admits a weakly nuclear *-homomorphic splitting. However, as η is nuclearly absorbing, we see that η and $\eta \oplus \eta'$ are unitarily equivalent, and therefore η admits a weakly nuclear *-homomorphic splitting. ☐

The intuition behind the plethora of conditions in the proposition is to ensure that the extension group $\mathrm{Ext}^{-1}(A, I)$ exists and is isomorphic to $\mathrm{Hom}(K_1(A), K_0(I))$, and then one proves that the specific extension η is trivial. The splitting result is a vital ingredient in Schafhauser's proof, as it is used in constructing a *-homomorphism $\psi\colon A \to \mathcal{Q}_\omega$ given a *-homomorphism $\varphi\colon A \to \mathcal{R}^\omega$ with a c.c.p. lift $A \to \ell^\infty(\mathcal{R})$, since φ can be shown to be nuclear by exactness of A. As a matter of fact, Proposition 3.27 ensures that the ψ is nuclear, and hence it will be liftable by the Choi-Effros lifting theorem.

4 The Tikuisis-White-Winter theorem and its consequences

At this point in the thesis, we are well-versed in quite a few different areas of C^*-algebraic study, including the finite approximation property that is quasidiagonality and the algebraic notion of extension theory. While they are all topics worth studying for their own sake, and even though they have many usages besides those discussed in depth previously in this thesis, we have had a specific goal in mind throughout the entire thesis, which is to prove the Tikuisis-White-Winter theorem and analyse its consequences. In this chapter, we shall do exactly that; obviously, the theorem does not follow directly from what we have already established, and as such there are still several results to prove. After proving the theorem, we proceed by analysing a few corollaries, including some partial resolutions to the Blackadar-Kirchberg conjectures. The main references for this chapter are Schafhauser's paper [66] and the original paper by Tikuisis, White and Winter [70].

4.1 Schafhauser's proof

Recall that $\mathcal{Q}$ denotes the universal UHF-algebra, and that $\mathcal{R}$ denotes the unique hyperfinite II_1-factor. We wish to show that we can explicitly realise $\mathcal{R}$ as the weak closure of the GNS-representation of $\mathcal{Q}$ with respect to the unique tracial state $\tau_{\mathcal{Q}}$ on $\mathcal{Q}$.

Proposition 4.1. *Let A be an infinite-dimensional UHF-algebra with a faithful tracial state τ, and let $(H_\tau, \xi_\tau, \pi_\tau)$ be the GNS-representation of A corresponding to τ. Then $\pi_\tau(A)''$ is a hyperfinite II_1-factor and is, hence, isomorphic to $\mathcal{R}$.*

Proof. Since A is a UHF-algebra, we can realise A as the norm-closure of an increasing sequence of matrix algebras all containing the unit of A. Consider the C^*-algebra $B = \pi_\tau(A)$, then B is isomorphic to the UHF-algebra A by faithfulness of τ, and hence B'' is hyperfinite. Note that since $\tau(xy) = \tau(yx)$ for all $x, y \in A$, we find that $\langle x'y'\xi_\tau, \xi_\tau\rangle = \langle y'x'\xi_\tau, \xi_\tau\rangle$ for each $x', y' \in B$. Using that B is weak-dense in the von Neumann-algebra B'', we find that this extends to all $x, y \in B''$, and hence the function $\tau' \colon B'' \to \mathbb{C}$ given by $\tau'(x) = \langle x\xi_\tau, \xi_\tau\rangle$ for $x \in B''$ is a normal tracial state. We claim that τ' is faithful, that it is the unique normal tracial state on B'', and that B'' is a II_1-factor.

First we show that τ' is faithful. Suppose that $x \in B''$ satisfies that $\tau'(x^*x) = 0$, then, for every $y \in B''$,

$$\|xy\xi_\tau\|^2 = \tau'(y^*x^*xy) = \tau'(xyy^*x^*) \leq \|y\|^2\,\tau'(x^*x) = 0$$

by a Cauchy-Schwarz-esque result, and cyclicity of the vector ξ_τ implies that $x = 0$. Therefore τ' is faithful. For uniqueness of τ', assume that $\tilde{\tau}$ is another normal tracial state on B''. If $\bigcup_{n\in\mathbb{N}} M_{k_n}(\mathbb{C})$ is strongly dense in B'', then $\tau'(a) = \tilde{\tau}(a)$ for all $a \in M_{k_n}(\mathbb{C})$ and $n \in \mathbb{N}$ by uniqueness of tracial states on matrix algebras over $\mathbb{C}$. Normality of τ' and $\tilde{\tau}$ then implies that $\tau' = \tilde{\tau}$ as desired.

Lastly, we show that B'' is a II_1-factor. If we establish that B'' is a factor, then it is necessarily of type II_1 by Proposition 1.22. Hence we only need to prove that the center $\mathcal{Z}(B'')$ is trivial. Observe that $B''' = B'$. Let $p \in B' \cap B''$ be a projection and define the linear functional τ_p on B'' by $\tau_p(x) = \tau'(px)$. It is easily verified that this is a positive, normal linear functional on B'' satisfying the tracial property, and hence $\tau_p = \tau'(p)\tau'$ by uniqueness of τ'. When restricting to the UHF-algebra B, we find by monotraciality of UHF-algebras that $\tau_p(x) = \tau'(p)\tau'(x)$ for all $x \in B$, which by normality is extended to all $x \in B''$. In particular,

$$\tau'(p)\tau'(1-p) = \tau_p(1-p) = \tau'(p(1-p)) = 0$$

which implies either $\tau'(p) = 0$ or $\tau'(1-p) = 0$. By faithfulness, either $p = 0$ or $p = 1$. Hence, any projection in the center $\mathcal{Z}(B'')$ is trivial, and since a von Neumann-algebra can be realised as the closed span of its projections, see [47, Corollary 4.1.14], we conclude that $\mathcal{Z}(B'')$ is trivial such that B'' is a factor.

Having shown that B'' is a hyperfinite II_1-factor, it follows from uniqueness of such von Neumann-algebras, see [3, III.3.4.3], that B'' is isomorphic to $\mathcal{R}$. ☐

The above proposition implies that $\mathcal{Q}$ can be realised as a C^*-subalgebra of $\mathcal{R}$, and more specifically that $\mathcal{R}$ is the strong closure of the GNS-representation of $\mathcal{Q}$ with respect to the unique, faithful tracial state $\tau_{\mathcal{Q}}$ on $\mathcal{Q}$. The following proposition shows that this inclusion induces a surjective *-homomorphism on the level of ultrapowers.

Proposition 4.2. *Let A be a separable, unital C^*-algebra with a faithful tracial state τ, and let M be the strong closure of the GNS-representation of A with respect to τ. Then the inclusion $A \subseteq M$ induces a surjective *-homomorphism $A_\omega \to M^\omega$.*

Proof. It is obvious that the induced map $\pi\colon A_\omega \to M^\omega$ is well-defined and a *-homomorphism, so we shall only prove that it is surjective. Let $x \in M^\omega$ be arbitrary and lift it to an element $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(M)$. Since the GNS-representation of A is strongly dense in M, it follows from Kaplansky's density theorem that, for each $n \in \mathbb{N}$, there exists $a_n \in A$ with $\|a_n\| \leq \|x_n\|$ and $\|a_n - x_n\|_2 \leq \frac{1}{n}$. The first fact proves that $(a_n)_{n\in\mathbb{N}}$ belongs to $\ell^\infty(A)$, such that we can map it to an element $a \in A_\omega$, and the latter fact proves that $\pi(a) = x$. □

Let $\pi_\omega\colon \mathcal{Q}_\omega \to \mathcal{R}^\omega$ be the canonical surjection as described above and define the *trace-kernel ideal* $J = \ker \pi_\omega$. Then we have the short exact sequence $0 \to J \to \mathcal{Q}_\omega \to \mathcal{R}^\omega \to 0$. Our first objective is to show that J is an admissible kernel. This requires several different proof techniques and constructions, and hence we shall split up the proof in a few separable steps.

Lemma 4.3. *The C^*-algebras J and $\mathcal{Q}_\omega$ have stable rank one and real rank zero.*

Proof. Since having real rank zero and stable rank one are properties that pass to ideals, it suffices to show that $\mathcal{Q}_\omega$ has these properties, which is presented in the appendix. □

Our next step is to calculate the K-theory of J, which involves first understanding the K-theory of $\mathcal{Q}_\omega$. Recall that we have the triple of ordered Abelian groups with distinguished order units

$$(K_0(\mathcal{Q}), K_0(\mathcal{Q})^+, [1]_0) \cong (\mathbb{Q}, \mathbb{Q}^+, 1), \tag{4.1}$$

and that the isomorphism is induced by the unique tracial state $\tau_{\mathcal{Q}}$ on $\mathcal{Q}$. In order to understand the next lemma, we need to briefly discuss ultrapowers of ordered Abelian groups. The concept differs slightly, albeit not terribly, from that of C^*-algebras, and we shall therefore discuss the construction.

Let (G, G^+, u) be an ordered Abelian group with a distinguished order unit u, and denote by $\ell^\infty(G)$ the collection of sequences $(x_n)_{n\in\mathbb{N}}$ in G such that there exists $d \in \mathbb{N}$ with $-du \leq x_n \leq du$ for all $n \in \mathbb{N}$; this is a way of saying that the sequence is uniformly bounded. Then $\ell^\infty(G)$ is an ordered Abelian group with a partial ordering $x \leq y$ if $x_n \leq y_n$ for each $n \in \mathbb{N}$, and the constant sequence u is an order unit. We can define the subgroup $c_\omega(G) \subseteq \ell^\infty(G)$ by

$$c_\omega(G) = \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(G) \mid \{n \in \mathbb{N} \mid x_n = 0\} \in \omega\}.$$

The group $c_\omega(G)$ should be understood as the sequences, which are eventually zero with respect to ω. The *ultrapower* G_ω of G is then the quotient $\ell^\infty(G)/c_\omega(G)$, which is clearly an ordered Abelian group with a distinguished order unit; both the positive cone and the order unit are images of the corresponding elements under the quotient map. Note that we then have the short exact sequence

$$0 \to c_\omega(G) \to \ell^\infty(G) \to G_\omega \to 0.$$

The following lemma comes from [56, Proposition 2.6].

Lemma 4.4. *There exists an isomorphism $K_0(\mathcal{Q}_\omega) \cong K_0(\mathcal{Q})_\omega$ of ordered Abelian groups.*

Proof. For each $n \in \mathbb{N}$, we can identify $M_n(\ell^\infty(\mathcal{Q}))$ and $\ell^\infty(M_n(\mathcal{Q}))$ with one another, which allows us to understand the K_0-group of $\ell^\infty(\mathcal{Q}_\omega)$ by analysing projections on $M_n(\ell^\infty(\mathcal{Q}))$ in a suitable fashion. Construct a group homomorphism $\tilde{\theta}\colon K_0(\ell^\infty(\mathcal{Q})) \to \ell^\infty(K_0(\mathcal{Q}))$ by

$$\tilde{\theta}([(p_n)_{n\in\mathbb{N}}]_0 - [(q_n)_{n\in\mathbb{N}}]_0) = ([p_n]_0 - [q_n]_0)_{n\in\mathbb{N}}$$

where $p_n, q_n \in \mathcal{P}_k(\mathcal{Q})$ for some sufficiently large integer $k \in \mathbb{N}$; note that this k is independent of n. We claim that this is an isomorphism of ordered Abelian groups. It is easily seen that $\tilde{\theta}$ is positive and preserves the order unit. To show that it is injective, suppose that $p = (p_n)_{n\in\mathbb{N}}$ and $q = (q_n)_{n\in\mathbb{N}}$ are projections in $M_k(\ell^\infty(\mathcal{Q}))$ with $\tilde{\theta}([p]_0 - [q]_0) = 0$. This implies that $[p_n]_0 = [q_n]_0$ for all $n \in \mathbb{N}$, and since $\mathcal{Q}$ has the cancellation property, as it has stable rank one, we find that p_n and q_n are

Murray-von Neumann equivalent. Let for each $n \in \mathbb{N}$ the element $v_n \in M_k(\mathcal{Q})$ be the partial isometry implementing this equivalence, such that $v_n^* v_n = p$ and $v_n v_n^* = q$. Then $v = (v_n)_{n\in\mathbb{N}}$ is a partial isometry in $\ell^\infty(M_k(\mathcal{Q})) \cong M_k(\ell^\infty(\mathcal{Q}))$ with $v^*v = p$ and $vv^* = q$, which implies that $[p]_0 = [q]_0$.

Now we show that $\tilde{\theta}$ is surjective. It suffices to show that $\tilde{\theta}$ maps onto all positive elements on $\ell^\infty(K_0(\mathcal{Q}))$. Let $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty(K_0(\mathcal{Q}))^+$ be arbitrary, then there exists by definition a natural number $d \in \mathbb{N}$ with $0 \le x_n \le d[1]_0$ for all $n \in \mathbb{N}$, where 1 here denotes the unit of $\mathcal{Q}$. Since $d[1]_0 = [1_d]_0$, where 1_d denotes the unit of $M_d(\mathcal{Q})$, we can use the fact that UHF-algebras are unital, stably finite with the cancellation property and follow the proof of [45, Lemma 3.15] to ensure the existence of projections $p_n \in M_d(\mathcal{Q})$ with $x_n = [p_n]_0$ for all $n \in \mathbb{N}$. Then $p = (p_n)_{n\in\mathbb{N}} \in \ell^\infty(M_d(\mathcal{Q})) \cong M_d(\ell^\infty(\mathcal{Q}))$ is a projection with $\tilde{\theta}([p]_0) = x$, and hence $\tilde{\theta}$ is surjective. Note that we have also proved that the inverse of $\tilde{\theta}$ is positive, which shows that $\tilde{\theta}$ is an isomorphism of ordered Abelian groups. Using half-exactness of the K_0-functor along with Lemma 2.10 stating that projections on $M_d(\mathcal{Q}_\omega)$ lift to projections on $M_d(\ell^\infty(\mathcal{Q}))$, we get the exact sequence

$$K_0(c_\omega(\mathcal{Q})) \to K_0(\ell^\infty(\mathcal{Q})) \to K_0(\mathcal{Q}_\omega) \to 0$$

of Abelian groups. Moreover, one can show that the image of the restriction $\tilde{\theta}_0$ of $\tilde{\theta}$ onto $K_0(c_\omega(\mathcal{Q}))$ is $c_\omega(K_0(\mathcal{Q}))$. We hence see that there exists an isomorphism $\tilde{\theta}_0\colon K_0(c_\omega(\mathcal{Q})) \to c_\omega(K_0(\mathcal{Q}))$ such that the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc}
 & & K_0(c_\omega(\mathcal{Q})) & \longrightarrow & K_0(\ell^\infty(\mathcal{Q})) & \longrightarrow & K_0(\mathcal{Q}_\omega) & \longrightarrow & 0 \\
 & & \big\downarrow \tilde{\theta}_0 & & \big\downarrow \tilde{\theta} & & \downarrow \theta & & \\
0 & \longrightarrow & c_\omega(K_0(\mathcal{Q})) & \longrightarrow & \ell^\infty(K_0(\mathcal{Q})) & \longrightarrow & K_0(\mathcal{Q})_\omega & \longrightarrow & 0
\end{array}$$

where the existence of the dashed isomorphism $\theta\colon K_0(\mathcal{Q}_\omega) \to K_0(\mathcal{Q})_\omega$ can be proved via a simple diagram chase. This completes the proof. □

Note that the above theorem actually gives an explicit description of the isomorphism θ, but for our purposes the mere existence of θ is sufficient.

Consider now the ultrapower $\mathbb{Q}_\omega = \ell^\infty(\mathbb{Q})/c_\omega(\mathbb{Q})$. Lemma 4.4 along with equation (4.1) shows that we have an isomorphism $K_0(\mathcal{Q}_\omega) \cong \mathbb{Q}_\omega$ as ordered Abelian groups. Let $\lim_\omega$ be the map $\mathbb{Q}_\omega \to \mathbb{R}$ given by the following procedure: Given an element $q \in \mathbb{Q}_\omega$, find a lift $(q_n)_{n\in\mathbb{N}}$ in $\ell^\infty(\mathbb{Q})$ and let $\lim_\omega q$ be the limit of the sequence $(q_n)_{n\in\mathbb{N}}$ along ω in $\mathbb{R}$. Since any real number can be realised as a limit of a sequence of rational numbers, and as convergence along free ultrafilters generalises sequential convergence, we see that $\lim_\omega$ is surjective. Let G_0 be the kernel of this map, which can be defined explicitly by

$$G_0 = \{q \in \ell^\infty(\mathbb{Q}) \mid \lim_{n\to\omega} q_n = 0\}/\{q \in \ell^\infty(\mathbb{Q}) \mid \{n \in \mathbb{N} \mid q_n = 0\} \in \omega\}.$$

Let $\mathbb{Q}_\omega^+$ be the subsemigroup of $\mathbb{Q}_\omega$ consisting of elements representable by sequences in $\mathbb{Q}^+$, and let $G_0^+ = \mathbb{Q}_\omega^+ \cap G_0$.

Lemma 4.5. *The C^*-algebras J, $\mathcal{Q}_\omega$ and $\mathcal{R}_\omega$ all have trivial K_1-groups, and we have the following commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(J) & \longrightarrow & K_0(\mathcal{Q}_\omega) & \xrightarrow{K_0(\pi_\omega)} & K_0(\mathcal{R}^\omega) & \longrightarrow & 0 \\
 & & \big\downarrow K_0(\tau_{\mathcal{Q}})_\omega & & \big\downarrow K_0(\tau_{\mathcal{Q}})_\omega & & \big\downarrow K_0(\tau^\omega) & & \\
0 & \longrightarrow & G_0 & \longrightarrow & \mathbb{Q}_\omega & \xrightarrow[\lim_\omega]{} & \mathbb{R} & \longrightarrow & 0
\end{array}$$

where the vertical maps are isomorphisms of Abelian groups. Moreover, the vertical group isomorphisms restrict to semigroup isomorphisms $\theta_J\colon \mathcal{D}(J) \to G_0^+$, $\theta_{\mathcal{Q}_\omega} : \mathcal{D}(\mathcal{Q}_\omega) \to \mathbb{Q}_\omega^+$ and $\theta_{\mathcal{R}^\omega} : \mathcal{D}(\mathcal{R}^\omega) \to \mathbb{R}^+$.

Proof. Note first of all that $\lim_\omega \circ K_0(\tau_{\mathcal{Q}})_\omega = K_0(\tau^\omega) \circ K_0(\pi_\omega)$ by construction, which proves that the diagram — assuming it exists — commutes. We have already established that $K_0(\tau_{\mathcal{Q}})_\omega$ induces the isomorphism $K_0(\mathcal{Q}_\omega) \to \mathbb{Q}_\omega$. Since $\mathcal{R}^\omega$ is a II_1-factor by Proposition 2.16, one finds that the state $K_0(\tau^\omega)\colon K_0(\mathcal{R}^\omega) \to \mathbb{R}$ induced by the tracial state τ^ω is, in fact, an isomorphism of Abelian groups. Moreover, using the Borel functional calculus for von Neumann-algebras, one can show that the unitary group of a von Neumann-algebra is always connected and, consequently, that the K_1-group is trivial. We sketch the idea: If M is a von Neumann-algebra and $u \in \mathcal{U}(M)$ is unitary, then we can use the Borel functional calculus on a branch of the logarithm to find that $u = \mathrm{e}^{ix}$, where $x = -i \log u$ is self-adjoint. Then $u_t = \mathrm{e}^{itx}$ is a path of unitaries connecting 1 and u, which proves that $\mathcal{U}(M)$ is connected. We hence find that $K_0(\mathcal{R}^\omega) = \mathbb{R}$ and $K_1(\mathcal{R}^\omega) = 0$.

Furthermore, $\mathcal{Q}_\omega$ has a trivial K_1-group. Again, we sketch the idea: Let $u \in M_k(\mathcal{Q}_\omega)$ be an arbitrary unitary element, then we can construct a lift $(x_n)_{n\in\mathbb{N}} \in M_k(\ell^\infty(\mathcal{Q}))$. Since u is unitary, we have $\|x_n^* x_n - 1\| \to 0$ and $\|x_n x_n^* - 1\| \to 0$ along the ultrafilter ω. Using polar decompositions, one can find unitaries $u_n \in M_k(\mathcal{Q})$ with $\|x_n - u_n\| \to 0$ along ω. By finding unitaries sufficiently close in finite-dimensional C^*-subalgebras of $M_k(\mathcal{Q})$, we can assume that each u_n belongs to a finite-dimensional C^*-subalgebra of $M_k(\mathcal{Q})$, and this allows us to realise each u_n as $u_n = \mathrm{e}^{ih_n}$ for $0 \le h_n \le 2\pi$. Then the image of the sequence $(h_n)_{n\in\mathbb{N}}$ in $M_k(\mathcal{Q})$ provides a selfadjoint element h in $\mathcal{Q}_\omega$ satisfying $u = \mathrm{e}^{ih}$, which shows that u is path connected to 1 via unitaries.

Note that since $\lim_\omega$ is a surjective map, so is $K_0(\pi_\omega)$. Looking at the six term exact sequence associated to the short exact sequence $0 \to J \to \mathcal{Q}_\omega \to \mathcal{R}^\omega \to 0$ and using the facts that $K_0(\pi_\omega)$ is surjective along with $\mathcal{Q}_\omega$ and $\mathcal{R}^\omega$ having trivial K_1-groups provides the upper row in the desired commutative diagram and that $K_1(J) = 0$. This in conjunction with the five lemma implies that the restriction $K_0(\tau_{\mathcal{Q}})_\omega\colon K_0(J) \to G_0$ is an isomorphism.

We now only need to prove that the maps $\theta_{\mathcal{R}^\omega}$, $\theta_{\mathcal{Q}_\omega}$ and θ_J are isomorphisms of semigroups. Note that as they are each restrictions of isomorphisms, it suffices to show that they are surjective onto their codomains. Since $\mathcal{R}^\omega$ is a II_1-factor, it follows that for any $t \ge 0$ there exists a projection $p \in \mathcal{P}_\infty(\mathcal{R}^\omega)$ with $K_0(\tau^\omega)([p]_0) = t$, which proves that $\theta_{\mathcal{R}^\omega}$ is surjective.

Let $t \in \mathbb{Q}_\omega^+$ be arbitrary and find a representative $(t_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{Q}^+)$. Find by boundedness of the sequence some integer $d \in \mathbb{N}$ such that $0 \le t_n \le d$ for all $n \in \mathbb{N}$. Since $\mathcal{Q}$ is the universal UHF-algebra, there exists for each $n \in \mathbb{N}$ a projection $p_n \in \mathcal{P}_d(\mathcal{Q})$ such that $\tau_{\mathcal{Q}}(p_n) = t_n$; here we abuse the notation and denote by $\tau_{\mathcal{Q}}$ the trace on both $M_d(\mathcal{Q})$ and $\mathcal{Q}$. Then the image p in $\mathcal{P}_d(\mathcal{Q}_\omega)$ of the sequence $(p_n)_{n\in\mathbb{N}}$ defines a projection with $\theta_{\mathcal{Q}_\omega}([p]_0) = t$ proving surjectivity.

Finally, suppose $q \in G_0^+$ is represented by a sequence $(q_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{Q}^+)$, then the proof of surjectivity of $\theta_{\mathcal{Q}_\omega}$ implies the existence of a sequence of projections $(p_n)_{n\in\mathbb{N}}$ in $M_d(\mathcal{Q})$ for which $\tau_{\mathcal{Q}}(p_n) = q_n$, and such that $\theta_{\mathcal{Q}_\omega}([p]_0) = q$, where p denotes the image in $\mathcal{P}_d(\mathcal{Q}_\omega)$. We hence find that

$$\lim_{n\to\omega} \tau_{\mathcal{Q}}(p_n) = \lim_{n\to\omega} q_n = 0$$

using the fact that $q \in G_0$. This implies that $p \in \mathcal{P}_d(J)$, and as $\theta_{\mathcal{Q}_\omega}([p]_0) = q$, we conclude that $\theta_J([p]_0) = q$ proving surjectivity. This finalises the proof. ☐

Lemma 4.6. *The Abelian group $K_0(J)$ is divisible, and the semigroup $\mathcal{D}(J)$ is almost unperforated.*

Proof. We established in Lemma 4.5 the isomorphisms $K_0(J) \cong G_0$ and $\mathcal{D}(J) \cong G_0^+$. It therefore suffices to show that G_0 is divisible, and that G_0^+ is almost unperforated, but these properties are easily verified. ☐

Lemma 4.7. *If $p \in J \otimes \mathbb{K}(H)$ is a projection, then there exists a projection $q \in J$, which is Murray-von Neumann equivalent to p.*

Proof. Suppose that $p \in J \otimes \mathbb{K}(H)$ is a projection and observe that we can identify p with a projection in $M_d(J)$ for some sufficiently large integer $d \in \mathbb{N}$. Let $(p_n)_{n\in\mathbb{N}} \in \ell^\infty(M_d(\mathcal{Q}))$ be a sequence of projections representing p. Since $p \in M_d(J)$, it follows that $\lim_{n\to\omega} \tau_{\mathcal{Q}}(p_n) = 0$, and, consequently, the set

$$S = \{n \in \mathbb{N} \,|\, 0 \le \tau_{\mathcal{Q}}(p_n) < 1\}$$

belongs to ω. If $n \in S$, find a projection $q_n \in \mathcal{Q}$ with $\tau_{\mathcal{Q}}(q_n) = \tau_{\mathcal{Q}}(p_n)$, and if $n \notin S$, let $q_n = 0$. Consider the sequence $(q_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{Q})$, then its image q in $\mathcal{Q}_\omega$ is a projection satisfying

$$\tau_\omega(q) = \lim_{n\to\omega} \tau_{\mathcal{Q}}(q_n) = \lim_{n\to\omega} \tau_{\mathcal{Q}}(p_n) = 0$$

and, hence, q is a projection in J. This implies that $\theta_J([p]_0) = \theta_J([q]_0)$, and as θ_J is an isomorphism of semigroups, we find that $[p]_0 = [q]_0$. Using the fact that J has the cancellation property of projections, as it has stable rank one, we conclude that p and q are Murray-von Neumann equivalent. □

Combining the past few pages gives us the following result.

Theorem 4.8. *The C^*-algebra J is an admissible kernel*

However, it turns out J is too large to invoke Proposition 3.27 when analysing extensions of C^*-algebras by J — namely, J is not separable. Hence we wish to reduce our problem to consider extensions by some suitable separable C^*-subalgebra J_0 of J in such a way that where we can invoke Proposition 3.27. For this we refer to a principle of C^*-algebraic properties called separable inheritability, which is due to Blackadar.

Definition 4.9 (Blackadar)**.** A property $\mathcal{P}$ of C^*-algebras is said to be *separably inheritable* if the following two properties are satisfied:

(i) If A is a C^*-algebra with property $\mathcal{P}$ and $A_0 \subseteq A$ is a separable C^*-subalgebra, then there exists a separable C^*-subalgebra A_1 of A containing A_0 such that A_1 satisifies $\mathcal{P}$.

(ii) Property $\mathcal{P}$ is preserved by sequential inductive limits with injective connecting maps.

For an easy example of a separably inheritable property one can consider quasidiagonality, see Proposition 2.21(i,ii). There are many properties of C^*-algebras which are separably inheritable, and the proofs often use the same strategy. It is by far easier to explain the technique by an example than it is to explain it abstractly, and as such we prove separable inheritability of a specific property of C^*-algebras, namely having stable rank one.

Proposition 4.10. *The property of having stable rank one is a separably inheritable property of C^*-algebras.*

Proof. We only prove the unital case. Let A be a C^*-algebra with stable rank one, let B be a separable C^*-subalgebra of A, and let $\{x_1, x_2, \ldots\} \subseteq B$ be a countable dense set. Since A has stable rank one, we can, for each $n \in \mathbb{N}$, find a sequence $(x_n^{(k)})_{k\in\mathbb{N}}$ of invertible elements in A such that $x_n^{(k)} \to x_n$ as $k \to \infty$. Let B_1 denote the C^*-subalgebra of A generated by the set $\{x_n^{(k)}\}_{k,n\in\mathbb{N}}$. Then B_1 is separable, as it is generated by a countable set of elements, and every element in B can be expressed as the limit of invertible elements in B_1. In particular, B is a C^*-subalgebra of B_1. Continuing in this manner, we can inductively construct a sequence of C^*-subalgebras B_n of A with the properties that each B_n is separable, and that every element in B_n can be expressed as the limit of invertible elements in B_{n+1}. Let $B_0 = \overline{\bigcup_{n\in\mathbb{N}} B_n}$, then we claim that B_0 is a separable C^*-subalgebra of A containing B, which has stable rank one. Separability is obvious, and it is also easily seen that B_0 contains B. To prove that B_0 has stable rank one, let $x \in B_0$ be arbitrary and let $\varepsilon > 0$ be arbitrary. Then there exists a sufficiently large natural number $n \in \mathbb{N}$ and an element $y \in B_n$ such that $\|x - y\| < \frac{\varepsilon}{2}$. Since $y \in B_n$, we can find an invertible element $z \in B_{n+1}$, which in particular is invertible in B_0, such that $\|y - z\| < \frac{\varepsilon}{2}$. The triangle inequality then implies that $\|x - z\| < \varepsilon$, and hence every element in B_0 can be approximated by invertible elements. By following the same line of reasoning as in the proof of B_0 having stable rank one, it is easily seen that having stable rank one passes to inductive limits with injective connecting maps. □

Note how the proof structure in the below proposition is quite similar to the one above — given a separable C^*-subalgebra B of a C^*-algebra A with certain properties, construct inductively nested sequences of C^*-subalgebras of A containing B and take the closed union.

Proposition 4.11. *If $\mathcal{P}_1, \mathcal{P}_2, \ldots$ is a list of countably many separably inheritable properties of C^*-algebras, then the meet $\mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \cdots$ is separably inheritable.*

Proof. Let A be a C^*-algebra with property $\mathcal{P}_1, \mathcal{P}_2, \ldots$, and let B be a separable C^*-subalgebra of A. Define inductively a sequence $B_{n,k}$ with $n \geq k$ of separable C^*-subalgebras of A such that

$$B \subseteq B_{1,1} \subseteq B_{2,1} \subseteq B_{2,2} \subseteq \cdots \subseteq B_{n,1} \subseteq B_{n,2} \subseteq \cdots B_{n,n} \subseteq \cdots \subseteq A$$

and such that $B_{n,k}$ has property $\mathcal{P}_k$ for all $k \in \mathbb{N}$ and $n \geq k$. Let $B_0 = \overline{\bigcup_{n,k} B_{n,k}}$, then B_0 is a separable C^*-subalgebra of A, which satisfies property $\mathcal{P}_1, \mathcal{P}_2, \ldots$, as separably inheritable properties are preserved by sequential inductive limits with injective connecting maps. □

Using similar techniques, it can be shown that all the conditions in the definition of an admissible kernel are separably inheritable, see [66, Proposition 4.1], and hence being an admissible kernel is a separably inheritable property by Proposition 4.11. In the next proposition, we mention a few specific separably inheritable properties, which we shall explicitly use to pass to certain separable C^*-subalgebras, and as such it is by no means exhaustive.

Proposition 4.12. *The following properties of C^*-algebras are separably inheritable.*

(i) *Having $K_1(\cdot) = 0$;*

(ii) *Being an admissible kernel;*

(iii) *Being simple.*

Some of the details, along with more examples of separably inheritable properties, may be found in [3, Section II.8.5]. In the next proposition, we shall use the examples of separably inheritable properties of Proposition 4.12; note how the proof structure, again, is quite similar to that of Proposition 4.10.

Proposition 4.13. *Let $0 \to I \to E \xrightarrow{\pi} A \to 0$ be a short exact sequence of C^*-algebras such that I is an admissible kernel and such that E and A are unital. Given a separable, unital C^*-subalgebra A_0 of A, there exists a separable, unital C^*-subalgebra E_0 of E such that $\pi(E_0) = A_0$ and $E_0 \cap I$ is an admissible kernel. In particular, we get the short exact sequence*

$$0 \to E_0 \cap I \to E_0 \xrightarrow{\pi} A_0 \to 0.$$

Proof. By separability of A_0, we can find a countable, dense set S in A_0, and, using surjectivity of π, we can find a countable set $T \subseteq E$ such that $\pi(T) = S$. Let E_1 be the unital C^*-subalgebra of E generated by T. Since *-homomorphisms have closed images, we find that $\pi(E_1) = A_0$. The C^*-algebra $E_1 \cap I$ is separable, since E_1 is separable, and hence by Proposition 4.12 there exists a separable admissible kernel I_1 with $E_1 \cap I \subseteq I_1 \subseteq I$. Let E_2 be the unital C^*-algebra generated by E_1 and I_1, which is again separable, and construct a C^*-subalgebra I_2 of I containing $E_2 \cap I$ such that I_2 is a separable admissible kernel. Note also that $\pi(E_2) = A_0$, since E_2 is generated by E_1 and I_1. Continue this procedure inductively to construct an increasing sequence I_n of separable admissible kernels and an increasing sequence of separable, unital C^*-algebras E_n such that $\pi(E_n) = A_0$ and $E_n \cap I \subseteq I_n \subseteq E_{n+1} \cap I$ for all $n \in \mathbb{N}$. Let $E_0 = \overline{\bigcup_{n \in \mathbb{N}} E_n}$, then $\pi(E_0) = A_0$, and since

$$E_0 \cap I = \overline{\bigcup_{n \in \mathbb{N}} E_n \cap I} = \overline{\bigcup_{n \in \mathbb{N}} I_n},$$

it follows from separable inheritability of being an admissible kernel, Proposition 4.12, that $E_0 \cap I$ is an admissible kernel. □

We want to use this proposition on the extension $0 \to J \to \mathcal{Q}_\omega \to \mathcal{R}^\omega \to 0$, as the trace-kernel ideal J is an admissible kernel by Theorem 4.8. However, in order to do this, we need to find a suitable candidate for a separable, unital C^*-subalgebra of $\mathcal{R}^\omega$, whenever we are given some C^*-algebra A as in the Tikuisis-White-Winter theorem. The below proposition shows how we are able to do precisely that — again due to separable inheritability. Note that we need to assume nuclearity of the map $\varphi\colon A \to \mathcal{R}^\omega$, which turns out to be a consequence of A being exact and φ having a c.c.p. lift.

Proposition 4.14. *Let A be a separable C^*-algebra, and suppose that $\varphi\colon A \to \mathcal{R}^\omega$ is a nuclear *-homomorphism. Then there exists a separable, unital C^*-subalgebra R_0 of $\mathcal{R}^\omega$ such that R_0 is simple, $K_1(R_0) = 0$, $\varphi(A) \subseteq R_0$ and φ is nuclear as a map $A \to R_0$.*

Proof. Find by nuclearity of φ sequences of c.c.p. maps $\theta_n \colon A \to M_{k_n}(\mathbb{C})$ and $\psi_n \colon M_{k_n}(\mathbb{C}) \to \mathcal{R}^\omega$ such that $\|\varphi(a) - \psi_n \circ \theta_n(a)\| \to 0$ as $n \to \infty$ for all $a \in A$. Let R_1 be the unital C^*-subalgebra of $\mathcal{R}^\omega$ generated by the images $\psi_n(M_{k_n}(\mathbb{C}))$, then R_1 is separable. Note that $\varphi(A) \subseteq R_1$ and that φ is nuclear as a map $A \to R_1$ by construction of R_1. We want to use the notion of separable inheritability to construct a separable, unital C^*-subalgebra R_0 of $\mathcal{R}^\omega$, which contains R_1 and satisfies all the properties stated in the proposition. Since simplicity and $K_1(\cdot) = 0$ are separably inheritable properties of C^*-algebras by Proposition 4.12, we find that there exists a separable, unital C^*-subalgebra R_0 of $\mathcal{R}^\omega$ containing R_1 with the properties that R_0 is simple and that $K_1(R_0) = 0$. Moreover, it is obvious that $\varphi(A) \subseteq R_0$, since the image is a C^*-subalgebra of R_1 and, hence, of R_0, and φ is a nuclear map $A \to R_0$. □

Having found a way to construct a certain separable, unital C^*-subalgebra of $\mathcal{R}^\omega$ as we need for Proposition 4.13, let us see how we can take the short exact sequence $0 \to J \to \mathcal{Q}_\omega \to \mathcal{R}^\omega \to 0$ and pass to a short exact sequence consisting of sufficiently large separable C^*-subalgebras. Suppose that A is a separable C^*-algebra and that $\varphi \colon A \to \mathcal{R}^\omega$ is an injective and nuclear *-homomorphism. Find a separable, unital C^*-subalgebra R_0 of $\mathcal{R}^\omega$ as in Proposition 4.14, and find by Proposition 4.13 a separable, unital C^*-subalgebra Q_0 of $\mathcal{Q}_\omega$ with the properties that $\pi(Q_0) = R_0$ and that $Q_0 \cap J$ is an admissible kernel. We now have a short exact sequence

$$0 \to J_0 \to Q_0 \xrightarrow{\pi_0} R_0 \to 0$$

where $J_0 = Q_0 \cap J$ and $\pi_0 \colon Q_0 \to R_0$ is the restriction of $\pi_\omega \colon \mathcal{Q}_\omega \to \mathcal{R}^\omega$. Since $\varphi(A) \subseteq R_0$ by construction of R_0, we can define $\varphi_0 \colon A \to R_0$ by restricting the codomain of φ. Proposition 4.14 implies that φ_0 is nuclear. Consider the pullback diagram

$$\begin{array}{ccc} E_0 & \xrightarrow{\tilde{\pi}_0} & A \\ {\scriptstyle \tilde{\varphi}_0}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_0} \\ Q_0 & \xrightarrow{\pi_0} & R_0 \end{array}$$

where, explicitly,

$$E_0 = \{(a, q) \in A \oplus Q_0 \mid \varphi_0(a) = \pi_0(q)\}$$

and $\tilde{\pi}_0 \colon E_0 \to A$ and $\tilde{\varphi}_0 \colon E_0 \to Q_0$ are the projection maps onto the first coordinate, respectively the second coordinate. Note that if $q \in J_0$, then $\pi_0(q) = 0$ and, hence, we get an inclusion map $\tilde{\iota}_0 \colon J_0 \to E_0$, which is the map $\tilde{\iota}_0(q) = (0, q)$. This gives us the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_0 & \xrightarrow{\tilde{\iota}_0} & E_0 & \xrightarrow{\tilde{\pi}_0} & A & \longrightarrow & 0 \\ & & \big\| & & \big\downarrow{\scriptstyle \tilde{\varphi}_0} & & \big\downarrow{\scriptstyle \varphi_0} & & \\ 0 & \longrightarrow & J_0 & \xrightarrow[\iota_0]{} & Q_0 & \xrightarrow[\pi_0]{} & R_0 & \longrightarrow & 0 \end{array}$$

Denote by η_0 the extension of the lower row, and denote by $\varphi_0^* \eta_0$ the extension of the upper row; we can consider $\varphi_0^* \eta_0$ as a *pullback extension* of η_0. The following proposition shows that the pullback extension $\varphi_0^* \eta_0$ in the nonunital case admits a weakly nuclear *-homomorphic split.

Proposition 4.15. *Let A be a separable C^*-algebra, and let $\varphi \colon A \to \mathcal{R}^\omega$ be an injective and nuclear *-homomorphism. Suppose that A satisfies the UCT, and that A is either non-unital, or that A is unital, but that φ is not unital. Then the extension $\varphi_0^* \eta_0$ satisfies the conditions of Proposition 3.27 and, hence, admits a weakly nuclear *-homomorphic splitting.*

Proof. By naturality of the index map, see [64, Proposition 9.1.5], we find that the index map $K_1(A) \to K_0(J_0)$ factors through the Abelian group $K_1(R_0)$, which is trivial by Proposition 4.14, and hence the index map is trivial. We know, by Theorem 4.8 and Proposition 4.13, that J_0 is an admissible kernel, so we only need to verify that the extension $\varphi_0^* \eta_0$ is weakly nuclear and unitisably full in order to invoke the splitting result of Proposition 3.27.

We first show that $\varphi_0^*\eta_0$ is a weakly nuclear extension. The Choi-Effros lifting theorem, see Proposition 2.41, implies the existence of a nuclear map $\psi_0\colon A \to Q_0$ such that $\pi_0 \circ \psi_0 = \varphi_0$. In particular, the map $\sigma\colon A \to E_0$ defined by $\sigma(a) = (a, \psi_0(a))$ is a well-defined c.c.p. map satisfying $\tilde{\pi}_0 \circ \sigma = \mathrm{id}_A$; we show that it is weakly nuclear. Given any $q \in J_0$, we find that $q\sigma(a)q^* = (0, q\psi_0(a)q^*)$ for all $a \in A$, and as ψ_0 is nuclear, so is the map $a \mapsto q\sigma(a)q^*$. Hence σ is weakly nuclear.

Now we show that $\varphi_0^*\eta_0$ is unitisably full. Consider its unitised extension

$$0 \longrightarrow J_0 \xrightarrow{\tilde{\iota}_0} E_0^\dagger \xrightarrow{\tilde{\pi}_0^\dagger} A^\dagger \longrightarrow 0$$

where $\tilde{\pi}_0^\dagger(e + \lambda 1) = \tilde{\pi}_0(e) + \lambda 1$. Consider also the unitised *-homomorphism $\varphi_0^\dagger\colon A^\dagger \to R_0$ by $\varphi_0^\dagger(a+\lambda 1) = \varphi_0(a) + \lambda 1$. Let $\beta\colon R_0 \to \mathcal{Q}(J_0)$ be the Busby invariant of the extension η_0 and consider the following commutative diagram with exact rows:

$$\begin{CD}
0 @>>> J_0 @>\tilde{\iota}_0>> E_0^\dagger @>\tilde{\pi}_0^\dagger>> A^\dagger @>>> 0\\
@. @| @VV\tilde{\varphi}_0^\dagger V @VV\varphi_0^\dagger V @.\\
0 @>>> J_0 @>>\iota_0> Q_0 @>>\pi_0> R_0 @>>> 0\\
@. @| @VV\sigma_0 V @VV\beta V @.\\
0 @>>> J_0 @>>i_0> \mathcal{M}(J_0) @>>\rho_0> \mathcal{Q}(J_0) @>>> 0
\end{CD}$$

Here $i_0\colon J_0 \to \mathcal{M}(J_0)$ and $\rho_0\colon \mathcal{M}(J_0) \to \mathcal{Q}(J_0)$ are the inclusion and quotient maps, and $\sigma_0\colon Q_0 \to \mathcal{M}(J_0)$ is the unique *-homomorphism extending the inclusion $J_0 \subseteq \mathcal{M}(J_0)$. Commutativity of the diagram ensures that the *-homomorphism $\sigma_0 \circ \tilde{\varphi}_0^\dagger\colon E_0^\dagger \to \mathcal{M}(J_0)$ extends the inclusion $J_0 \subseteq \mathcal{M}(J_0)$ and, consequently, we find that the Busby invariant $\beta^\dagger$ of the unitised extension $\eta^\dagger$ is given by $\beta^\dagger = \beta \circ \varphi_0^\dagger$. We claim that $\beta^\dagger$ is full, i.e., that for all non-zero $a \in A^\dagger$, $\beta^\dagger(a)$ generates $\mathcal{Q}(J_0)$ as a closed two-sided ideal. As φ is injective and non-unital, so is φ_0, and consequently we see that $\varphi_0^\dagger$ is injective. In order to show that $\beta^\dagger$ is full, it suffices by injectivity of $\varphi_0^\dagger$ to show that β is full. Since R_0 is simple, any non-zero element in R_0 necessarily generates R_0 as a closed two-sided ideal, and, consequently, β is full. We conclude that the pullback extension $\varphi_0^*\eta_0$ satisfies the conditions of Proposition 3.27, and hence it admits a weakly nuclear *-homomorphic splitting. ☐

Having proved the existence of a weakly nuclear *-homomorphic split in the separable pullback extension $\varphi_0^*\eta_0$, assuming non-unitality, we shall now use this to induce a nuclear *-homomorphism $A \to \mathcal{Q}_\omega$. In order to do this, we shall need a classification result about trace-agreeing normal *-homomorphisms from a hyperfinite von Neumann-algebra to $\mathcal{R}^\omega$.

Lemma 4.16. *Let M be a hyperfinite von Neumann-algebra, and let N be a II_1-factor with a faithful tracial state τ. Assume that $\varphi, \psi\colon M \to N$ are normal *-homomorphisms with $\tau \circ \varphi = \tau \circ \psi$. Then φ and ψ are approximately unitarily equivalent in the tracial norm $\|\cdot\|_2$ on N.*

Proof. Observe that faithfulness of τ along with the condition $\tau \circ \varphi = \tau \circ \psi$ implies that φ and ψ have the same kernel. Moreover, we can, without loss of generality, assume that φ and ψ are injective, since we can just pass to the induced maps $M/\ker\varphi \to N$, using that $M/\ker\varphi$ is a separable, hyperfinite von Neumann-algebra by normality of φ. Hence we can assume that M is a von Neumann-subalgebra of N, and that $\varphi\colon M \to N$ is the inclusion map.

As M is a hyperfinite von Neumann-algebra, we can realise M as the strong closure of $\bigcup_{n\in\mathbb{N}} A_n$, where A_n is a nested sequence of finite-dimensional von Neumann-algebras. Then $\tau \circ \varphi|_{A_n} = \tau \circ \psi|_{A_n}$ for all $n \in \mathbb{N}$. Since two projections in a II_1-factor are Murray-von Neumann equivalent if and only if they have the same trace, we see that $K_0(\varphi|_{A_n}) = K_0(\psi|_{A_n})$ for each $n \in \mathbb{N}$. We claim that $\varphi|_{A_n}$ and $\psi|_{A_n}$ are unitarily equivalent. Following the notation of [64, Section 7.1], we let $\{e_{ij}^{(k)}\}$ denote the canonical matrix units for A_n. Then, as N is a II_1-factor, we find that $\varphi(e_{11}^{(k)}) \sim \psi(e_{11}^{(k)})$ and $(1-\varphi(1_n)) \sim (1-\psi(1_n))$, where 1_n denotes the unit of A_n. Find partial isometries $v_1, \dots, v_r, w \in N$ implementing these Murray-von Neumann equivalences, i.e.,

$$\begin{aligned} v_k^* v_k &= \varphi(e_{11}^{(k)}) & \qquad v_k v_k^* &= \psi(e_{11}^{(k)}) \\ w^* w &= 1 - \varphi(1_n) & \qquad w w^* &= 1 - \psi(1_n) \end{aligned}$$

Let $s_{i,k} = \psi(e_{i1}^{(k)}) v_k \varphi(e_{1i}^{(k)})$, then one readily verifies that $u = w + \sum_{i,k} s_{i,k}$ is a unitary element in N such that $\psi|_{A_n} = \mathrm{Ad}(u) \circ \varphi|_{A_n}$. We now show that this implies that ψ and φ are approximately unitarily equivalent with respect to the tracial norm. For any finite subset $F = \{x_1, \dots, x_n\}$ of M and for any tolerance $\varepsilon > 0$, there exist some $n \in \mathbb{N}$ and elements $y_1, \dots, y_n \in A_n$ such that $\|x_i - y_i\|_2 < \frac{\varepsilon}{2}$. By an easy application of the triangle inequality, and since ψ preserves the trace, we find that

$$\|\psi(x_i) - \mathrm{Ad}(u_n) \circ \varphi(x_i)\|_2 < \varepsilon$$

for all $i = 1, \dots, n$. One easily checks that this proves that ψ and φ are approximately unitarily equivalent in the tracial norm. □

It is clear that if φ and ψ are approximately unitarily equivalent with respect to the tracial norm, then φ and ψ have the same trace, and this hence completely characterise such normal *-homomorphisms. By passing to the tracial ultrapower N^ω of N, we can invoke Proposition 2.12 to get the following result.

Proposition 4.17. *Let M be a separable hyperfinite von Neumann-algebra, and let N be a II_1-factor with a faithful tracial state τ. Denote by τ^ω the induced tracial state on N^ω. If $\varphi, \psi\colon M \to N^\omega$ are normal *-homomorphisms such that $\tau^\omega \circ \varphi = \tau^\omega \circ \psi$, then φ and ψ are unitarily equivalent.*

Once again, it is immediate that this completely characterises unitarily equivalent normal *-homomorphisms $M \to N^\omega$, which is highly useful in the following theorem proving the desired existence of a nuclear *-homomorphism $A \to \mathcal{Q}_\omega$ lifting $\varphi\colon A \to \mathcal{R}^\omega$.

Theorem 4.18. *Let A be a separable, exact C^*-algebra satisfying the UCT, and suppose that $\varphi\colon A \to \mathcal{R}^\omega$ is an injective and nuclear *-homomorphism. Then there exists a nuclear *-homomorphism $\psi\colon A \to \mathcal{Q}_\omega$ such that $\pi_\omega \circ \psi = \varphi$, where $\pi_\omega\colon \mathcal{Q}_\omega \to \mathcal{R}^\omega$ is the natural quotient map. If both A and φ are unital, then we can choose ψ to be unital.*

Proof. For this proof, we follow the notation of the discussion prior to Proposition 4.15. We first consider the case that either A is non-unital, or that A is unital and $\varphi(1) \neq 1$. By Proposition 4.15, we see that the separable pullback extension $\varphi_0^* \eta_0$ admits a weakly nuclear *-homomorphic splitting $\lambda_0\colon A \to E_0$. Let $\psi\colon A \to \mathcal{Q}_\omega$ be given by $\psi = \tilde{\varphi}_0 \circ \lambda_0$, where we naturally identify Q_0 as a C^*-subalgebra of $\mathcal{Q}_\omega$. We shall prove that ψ is nuclear by showing that the map $\psi_0\colon A \to Q_0$, which is just ψ with a restricted codomain, is nuclear. By [11, Corollary 3.8.8], it suffices to prove that, for any C^*-algebra B, the map $\psi_0 \otimes \mathrm{id}_B\colon A \otimes_{\max} B \to Q_0 \otimes_{\max} B$ factors through $A \otimes B$, i.e., that we have the commutative diagram

$$\begin{array}{ccc} A \otimes_{\max} B & \xrightarrow{\ \psi_0 \otimes \mathrm{id}_B\ } & Q_0 \otimes_{\max} B \\ & {\scriptstyle\rho} \searrow \quad \nearrow {\scriptstyle\Psi} & \\ & A \otimes B & \end{array}$$

where $\rho\colon A \otimes_{\max} B \to A \otimes B$ is the canonical quotient map, and $\Psi\colon A \otimes B \to Q_0 \otimes_{\max} B$ is some c.c.p. map. By surjectivity of ρ, it suffices to show that the kernel of ρ lies in the kernel of $\psi_0 \otimes \mathrm{id}_B$. Suppose that $x \in \ker \rho$ and let $y = (\psi_0 \otimes \mathrm{id}_B)(x)$; we claim that $y = 0$. As J_0 is a closed two-sided ideal in Q_0, it follows from Theorem 3.3 that there exists a unique *-homomorphism $\sigma_0\colon Q_0 \to \mathcal{M}(J_0)$ extending the inclusion $J_0 \subseteq \mathcal{M}(J_0)$. We claim that $\sigma_0 \circ \psi_0\colon A \to \mathcal{M}(J_0)$ is weakly nuclear. Let $b \in J_0$ be arbitrary and consider the c.c.p. map $\alpha_b\colon A \to J_0$ by $\alpha_b(a) = b\sigma_0(\psi_0(a))b^*$. Using that $\psi_0 = \tilde{\varphi}_0 \circ \lambda_0$, we see that $\alpha_b(a) = \tilde{\varphi}_0(b\lambda_0(a)b^*)$, and as $\tilde{\varphi}_0$ is a *-homomorphism and λ_0 is weakly nuclear, we conclude that α_b is nuclear. Hence $\sigma_0 \circ \psi_0$ is weakly nuclear. It now follows from [22, Proposition 3.2] that $\sigma_0 \circ \psi_0$ is, in fact, nuclear, since A is assumed to be exact, and since its codomain is the multiplier algebra of a separable C^*-algebra. It thus follows from [11, Corollary 3.8.8] that $(\sigma_0 \circ \psi_0) \otimes \mathrm{id}_B \colon A \otimes_{\max} B \to \mathcal{M}(J_0) \otimes_{\max} B$ factors through $A \otimes B$ and, consequently, we have

$$(\sigma_0 \otimes \mathrm{id}_B)(y) = ((\sigma_0 \circ \psi_0) \otimes \mathrm{id}_B)(x) = 0.$$

Since φ is nuclear, so is φ_0, and hence the same line of reasoning as above shows that

$$(\pi_0 \otimes \mathrm{id}_B)(y) = ((\pi_0 \circ \psi_0) \otimes \mathrm{id}_B)(x) = (\varphi_0 \otimes \mathrm{id}_B)(x) = 0$$

using that $\varphi_0 = \pi_0 \circ \psi_0$. From the commutative diagram arising from the multiplier algebra and an extension, and as maximal tensor products preserve short exact sequences, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_0 \otimes_{\max} B & \xrightarrow{\iota_0 \otimes \mathrm{id}_B} & Q_0 \otimes_{\max} B & \xrightarrow{\pi_0 \otimes \mathrm{id}_B} & R_0 \otimes_{\max} B & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow \sigma_0 \otimes \mathrm{id}_B & & \Big\downarrow \beta \otimes \mathrm{id}_B & & \\
0 & \longrightarrow & J_0 \otimes_{\max} B & \xrightarrow[i_0 \otimes \mathrm{id}_B]{} & \mathcal{M}(J_0) \otimes_{\max} B & \xrightarrow[\rho_0 \otimes \mathrm{id}_B]{} & \mathcal{Q}(J_0) \otimes_{\max} B & \longrightarrow & 0
\end{array}$$

A diagram chase now shows that $y = 0$ as claimed. We have hence shown that ψ_0 is nuclear using [11, Corollary 3.8.8], and it follows that ψ is also nuclear.

Now assume that both A and φ are unital. Our strategy is to construct a map $\varphi_1 \colon A \to \mathcal{R}^\omega$, which is faithful, nuclear and non-unital by embedding φ into a matrix algebra over $\mathcal{R}^\omega$, which is isomorphic to $\mathcal{R}^\omega$. Then we shall invoke the non-unital case, which we have just proved above, to construct a unital, nuclear *-homomorphism $\psi' \colon A \to \mathcal{Q}_\omega$ with $\tau^\omega \circ \varphi = \tau_\omega \circ \psi'$. Using various results we shall construct a *-homomorphism $\psi \colon A \to \mathcal{Q}_\omega$, which is unitarily equivalent to ψ', and consequently nuclear, and which satisfies $\pi_\omega \circ \psi = \varphi$.

It is easily verified that $M_2(\mathcal{R}^\omega) \cong (M_2(\mathcal{R}))^\omega$, where the map is given by taking the limit along ω in each entry of the matrix algebra. Moreover, since matrix algebras of II_1-factors are, again, II_1-factors, and matrix algebras of hyperfinite von Neumann-algebras are again hyperfinite, we have by uniqueness of the hyperfinite II_1-factor that $(M_2(\mathcal{R}))^\omega \cong \mathcal{R}^\omega$. Denote by $\varphi_1 \colon A \to \mathcal{R}^\omega$ the faithful, non-unital and nuclear *-homomorphism given by the composition

$$A \xrightarrow{\varphi} \mathcal{R}^\omega \hookrightarrow M_2(\mathcal{R}^\omega) \xrightarrow{\cong} (M_2(\mathcal{R}))^\omega \xrightarrow{\cong} \mathcal{R}^\omega,$$

where the embedding $\mathcal{R}^\omega \hookrightarrow M_2(\mathcal{R}^\omega)$ is the non-unital embedding in the upper left entry. In particular, $\tau^\omega \circ \varphi_1 = \frac{1}{2}\tau^\omega \circ \varphi$. It follows from the previous non-unital case that there exists a nuclear *-homomorphism $\psi_1' \colon A \to \mathcal{Q}_\omega$ such that $\pi_\omega \circ \psi_1' = \varphi_1$. We then see that

$$\tau_\omega \circ \psi_1' = \tau^\omega \circ \pi_\omega \circ \psi_1' = \tau^\omega \circ \varphi_1 = \frac{1}{2}\tau^\omega \circ \varphi.$$

Put $\tau = \tau^\omega \circ \varphi$ and observe that it is amenable since φ is liftable to a c.c.p. map $A \to \ell^\infty(\mathcal{R}^\omega)$ by nuclearity and the Choi-Effros lifting theorem, Proposition 2.41, and that it is faithful by faithfulness of τ^ω and injectivity of φ. The equivalence of (ii) and (iii) in Proposition 2.52 implies the existence of a unital, nuclear *-homomorphism $\psi' \colon A \to \mathcal{Q}_\omega$ such that $\tau_\omega \circ \psi' = \tau$. Since A is exact and τ is amenable, we can use [10, Corollary 4.3.6] to prove that $\pi_\tau(A)''$ is hyperfinite, where π_τ refers to the GNS-representation of A with respect to the tracial state τ. Since τ is faithful, π_τ is a faithful representation of A, and hence we can identify A by the image $\pi_\tau(A)$, such that A is strongly dense in $\pi_\tau(A)''$. Since $\mathcal{R}^\omega$ is a von Neumann-algebra, its unit ball $(\mathcal{R}^\omega)_1$ is closed in the tracial norm, and we can use the fact that φ and $\pi \circ \psi'$ are isometric with respect to the tracial norm to prove the existence of maps $(\pi_\tau(A)'')_1 \to (\mathcal{R}^\omega)_1$ extending φ and $\pi \circ \psi'$ on $(A)_1$. By scaling, we can hence define maps $\overline{\varphi}, \overline{\psi'} \colon \pi_\tau(A)'' \to \mathcal{R}^\omega$ extending φ and $\pi \circ \psi'$, respectively, and one can verify that they are normal *-homomorphisms satisfying $\tau^\omega \circ \overline{\varphi} = \tau^\omega \circ \overline{\psi'}$. Using Proposition 4.17, we find that there exists $u \in \mathcal{U}(\mathcal{R}^\omega)$ such that $\overline{\varphi} = \mathrm{Ad}(u) \circ \overline{\psi'}$, and hence we have the identity $\varphi = \mathrm{Ad}(u) \circ \pi_\omega \circ \psi'$. As the unitary group of a von Neumann-algebra is always path connected, the proof of which is sketched in Lemma 4.5, [33, Proposition 4.3.14(a)] ensures the existence of a unitary $v \in \mathcal{U}(\mathcal{Q}_\omega)$ such that $\pi_\omega(v) = u$. Define the *-homomorphism $\psi \colon A \to \mathcal{Q}_\omega$ by $\psi = \mathrm{Ad}(v) \circ \psi'$, then it is immediate that $\pi_\omega \circ \psi = \varphi$. Moreover, nuclearity of ψ' implies that ψ is also nuclear, and this completes the proof. □

At this point, the only issue in guaranteeing the existence of the lift $\psi\colon A \to \mathcal{Q}_\omega$ is the nuclearity assumption on φ imposed in Theorem 4.18. Luckily, this condition is ensured by exactness of A:

Lemma 4.19. *Let A be an exact C^*-algebra, and let $\varphi^\omega\colon A \to \mathcal{R}^\omega$ be a c.c.p. map. If φ^ω has a c.c.p. lift $\varphi\colon A \to \ell^\infty(\mathcal{R})$, then φ^ω is nuclear.*

Proof. Let $\varphi_n\colon A \to \mathcal{R}$ be the nth component of the map $\varphi\colon A \to \ell^\infty(\mathcal{R})$. As $\mathcal{R}$ is a hyperfinite von Neumann-algebra, we can find an increasing sequence of finite-dimensional von Neumann-algebras $B_1 \subseteq B_2 \subseteq \cdots$ such that $\bigcup_{n\in\mathbb{N}} B_n$ is strongly dense in $\mathcal{R}$. Let $E_n\colon \mathcal{R} \to B_n$ be trace-preserving expectations and let $\psi_n\colon A \to B_n$ be the composition $\psi_n = E_n \circ \varphi_n$. As the codomains B_n are finite-dimensional, each ψ_n is nuclear, and hence the induced map $\psi\colon A \to \ell^\infty(B_n, \mathbb{N})$ is nuclear by Lemma 2.50. Let $\pi^\omega\colon \ell^\infty(\mathcal{R}) \to \mathcal{R}^\omega$ be the quotient map, then as $\|\psi_n(a) - \varphi_n(a)\|_2 \to 0$ as $n \to \infty$ for each $a \in A$, we find that $\varphi^\omega = \pi^\omega \circ \varphi = \pi^\omega \circ \psi$. Since ψ is nuclear and π^ω is a *-homomorphism, this implies that φ^ω is nuclear as desired. □

With the machinery of Theorem 4.18 and Lemma 4.19, the Tikuisis-White-Winter theorem follows immediately.

Theorem 4.20 (Tikuisis-White-Winter, 2015)**.** *If A is a separable, exact C^*-algebra satisfying the UCT, then every faithful, amenable tracial state on A is quasidiagonal.*

Proof. Let τ be a faithful, amenable tracial state on A, and let $\varphi^\omega\colon A \to \mathcal{R}^\omega$ be a *-homomorphism with $\tau^\omega \circ \varphi^\omega = \tau$, and which admits a c.c.p. lift $\varphi\colon A \to \ell^\infty(\mathcal{R})$. Then Lemma 4.19 implies that φ^ω is nuclear. Moreover, as τ is faithful and φ^ω is trace-preserving, we find that φ^ω is injective. Consequently, by Theorem 4.18, there exists a nuclear *-homomorphism $\psi_\omega\colon A \to \mathcal{Q}_\omega$ with $\pi_\omega \circ \psi_\omega = \varphi^\omega$, where $\pi_\omega\colon \mathcal{Q}_\omega \to \mathcal{R}^\omega$ is the quotient map. Since $\tau_\omega = \tau^\omega \circ \pi_\omega$, we find that ψ_ω is trace-preserving. Moreover, using the Choi-Effros lifting theorem, Proposition 2.41, we find that ψ_ω has a c.c.p. lift $A \to \ell^\infty(\mathcal{Q})$, which proves that τ is quasidiagonal by Proposition 2.49. □

Using the fact that the existence of a faithful, quasidiagonal tracial state on an exact C^*-algebra A implies quasidiagonality of A, see Corollary 2.53, we see that if A is a separable, exact C^*-algebra satisfying the UCT and admitting a faithful, amenable tracial state τ, then A is quasidiagonal.

As stated in the introduction to this thesis, Theorem 4.20 is a refinement due to Gabe of the original result due to Tikuisis-White-Winter. For the sake of completeness, we include the original result as it is proved in [70, Theorem A].

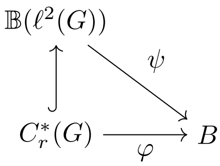
Corollary 4.21 (Tikuisis-White-Winter, 2015, original formulation)**.** *Let A be a separable, nuclear C^*-algebra which satisfies the UCT. Then every faithful tracial state on A is quasidiagonal.*

It is easily seen that this corollary follows from Theorem 4.20, since all tracial states on nuclear C^*-algebras are amenable by Corollary 2.55.

4.2 The Rosenberg theorem and conjecture

We now proceed by analysing some of the corollaries to the Tikuisis-White-Winter theorem. Our first result is Rosenberg's theorem which proves that amenability of the underlying discrete group G is a necessary condition for $C_r^*(G)$ being quasidiagonal. Admittedly, this result is not related to the Tikuisis-White-Winter theorem per se, but the question of whether the converse of Rosenberg's theorem is true remained open until the Tikuisis-White-Winter theorem provided the needed machinery. We shall not follow the original proof of Rosenberg [31], which uses Hilbert-Schmidt operator theory; instead, we shall invoke the more modern notion of ultrapowers as well as the following simple, yet powerful lemma.

Lemma 4.22. *Let G be a discrete group, and let B be a unital C^*-algebra with a tracial state τ. Suppose there exist a *-homomorphism $\varphi\colon C_r^*(G) \to B$ and a u.c.p. map $\psi\colon \mathbb{B}(\ell^2(G)) \to B$ such that the following diagram is commutative:*



Then G is amenable.

Proof. Since $\psi|_{C_r^*(G)} = \varphi$ is a *-homomorphism, we see that $C_r^*(G) \subseteq \mathbb{B}(\ell^2(G))_\psi$, where we use the notation of multiplicative domains from Proposition 1.7. In particular, it follows from the bimodule property of the same proposition that $\psi(axb) = \psi(a)\psi(x)\psi(b)$ for all $a, b \in C_r^*(G)$ and $x \in \mathbb{B}(\ell^2(G))$.

Embed $\ell^\infty(G)$ inside $\mathbb{B}(\ell^2(G))$ diagonally, then a straightforward calculation shows that $g.\alpha = \lambda_g \alpha \lambda_g^*$ for all $g \in G$ and $\alpha \in \ell^\infty(G)$. Consider the map $\rho\colon \ell^\infty(G) \to \mathbb{C}$ defined by $\rho = \tau \circ \psi$; we claim that ρ is a left-invariant mean. It is clear that $\rho(1) = 1$, and that ρ is positive. Moreover, if $g \in G$ and $\alpha \in \ell^\infty(G)$, then using the bimodularity as well as the tracial property of τ, one easily realises that $\rho(g.\alpha) = \rho(\alpha)$. □

This short and sweet lemma is at the heart of our proof of Rosenberg's theorem. It also has a few interesting corollaries in other areas of operator theory, which we shall briefly touch on. These are only of independent interest and are not related to the Tikuisis-White-Winter theorem nor Rosenberg's theorem.

Corollary 4.23. *A discrete group G is amenable if and only if $C_r^*(G)$ has a finite-dimensional representation.*

Proof. One direction follows from Proposition 1.37. For the other direction assume that $C_r^*(G)$ has a finite-dimensional representation $\varphi\colon C_r^*(G) \to M_n(\mathbb{C})$, and note that $M_n(\mathbb{C})$ is a unital C^*-algebra with a tracial state Tr_n. Put $B = M_n(\mathbb{C})$ and find by Arveson's extension theorem, Proposition 1.10, a u.c.p lift $\psi\colon \mathbb{B}(\ell^2(G)) \to M_n(\mathbb{C})$ of φ, then a use of Lemma 4.22 entails that G is amenable. □

For the next corollary, we remind the reader that a von Neumann-algebra $M \subseteq \mathbb{B}(H)$ is injective if and only if there exists a conditional expectation $\mathbb{B}(H) \to M$.

Corollary 4.24. *If the group von Neumann-algebra $L(G)$ is injective, then G is amenable.*

Proof. Recall that $L(G)$ is a unital C^*-algebra admitting a faithful tracial state $x \mapsto \langle x\delta_e, \delta_e\rangle$. Let $E\colon \mathbb{B}(H) \to L(G)$ be a conditional expectation, then Lemma 4.22 provides amenability of G. □

Obviously, these results can be proved using other methods, but the reader should note that they follow almost immediately with the machinery of Lemma 4.22. Nevertheless, if the above does not convince the reader of the strength of the lemma, the following proof of Rosenberg's theorem should.

Theorem 4.25 (Rosenberg, 1987)**.** *Let G be a discrete group. If $C_r^*(G)$ is quasidiagonal, then G is amenable.*

Proof. We first prove it assuming countability of G, and then we extend the result to uncountable groups G by passing to an inductive system of the countable subgroups of G.

So assume that G is countable such that $C_r^*(G)$ is separable. Let $\varphi_n\colon C_r^*(G) \to M_{k_n}(\mathbb{C})$ be a sequence of c.c.p. maps witnessing the quasidiagonality of $C_r^*(G)$. Find by Arveson's extension theorem, Proposition 1.10, for each $n \in \mathbb{N}$ a u.c.p. lift $\psi_n\colon \mathbb{B}(\ell^2(G)) \to M_{k_n}(\mathbb{C})$. Let $\varphi\colon C_r^*(G) \to \ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})$ and $\psi\colon \mathbb{B}(\ell^2(G)) \to \ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})$ be the direct products of the sequences $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$, respectively. Denote by $\rho\colon \ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N}) \to \ell_\omega(M_{k_n}(\mathbb{C}), \mathbb{N})$ the quotient map onto the ultraproduct, then $\rho \circ \varphi$ is a *-homomorphism by asymptotic multiplicity of the sequence $(\varphi_n)_{n\in\mathbb{N}}$. We now only need to verify that the ultraproduct $\ell_\omega(M_{k_n}(\mathbb{C}), \mathbb{N})$ admits a tracial state, but it is easily verified that the map $\tau\colon \ell_\omega(M_{k_n}(\mathbb{C}), \mathbb{N}) \to \mathbb{C}$ by $\tau(x) = \lim_{n\to\omega} \mathrm{Tr}_n(x_n)$, where $(x_n)_{n\in\mathbb{N}} \in \ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})$ lifts x, is a tracial state. It now follows from Lemma 4.22 that G is amenable.

Now we assume that G is uncountable. We can then realise G as the inductive limit of its countable subgroups $(G_\alpha)_{\alpha\in\Lambda}$, and hence Proposition 1.38 states that

$$C_r^*(G) = C_r^*(\varinjlim G_\alpha) \cong \varinjlim C_r^*(G_\alpha).$$

Since each G_α is a subgroup of G, we have natural embeddings $C_r^*(G_\alpha) \subseteq C_r^*(G)$, and as $C_r^*(G)$ is assumed to be quasidiagonal, we find that each of the C^*-subalgebras is quasidiagonal. By the countable case, all the groups G_α are amenable, and as amenability is preserved by inductive limits, we conclude that G is amenable. □

Alternatively, one can prove Rosenberg's theorem by using Proposition 2.27 along with a result saying that $C_r^*(G)$ has an amenable tracial state if and only if G is amenable, see [11, Proposition 6.3.2].

The theorem was first published by Rosenberg in 1987 as an appendix to a paper by Hadwin [31]. In the same paper, he conjectured that the converse is true; this conjecture came to be known as *Rosenberg's conjecture.* As is often the case, there were partial results prior to the complete proof. For instance, Ozawa, Rørdam and Sato proved in 2015 [51] that the conjecture holds for the class of elementary amenable groups[3]. The complete resolution was only achieved after the proof of the Tikuisis-White-Winter theorem, since a confirmation of Rosenberg's conjecture follows almost immediately from Theorem 4.20 and Corollary 2.53

Corollary 4.26 (Rosenberg's conjecture)**.** *If G is an amenable, discrete group, then $C_r^*(G)$ is a quasidiagonal C^*-algebra.*

Proof. Similar to the proof of Rosenberg's theorem, we first prove the countable case and then show how one can extend this to include uncountable groups. So assume that G is a countable, discrete, amenable group. Since G is amenable, we know that $C_r^*(G)$ is nuclear. Moreover, combining the results of [71, Lemma 3.5 and Proposition 10.7], we find that the group C^*-algebras of countable, discrete, amenable groups satisfy the UCT. Finally, note that $C_r^*(G)$ always admits a faithful tracial state by Proposition 1.35. By the Tikuisis-White-Winter theorem, $C_r^*(G)$ admits a faithful and quasidiagonal tracial state and, by Corollary 2.53, we conclude that $C_r^*(G)$ is quasidiagonal.

For the uncountable case, we can realise G as the inductive limit of its countable subgroups $(G_\alpha)_{\alpha\in\Lambda}$. Note that each G_α is amenable, since amenability passes to subgroups, and hence each $C_r^*(G_\alpha)$ is quasidiagonal by the countable case. Using the fact that taking reduced group C^*-algebras commutes with inductive limits, we see that

$$C_r^*(G) = C_r^*(\lim_{\rightarrow} G_\alpha) \cong \lim_{\rightarrow} C_r^*(G_\alpha).$$

As the connecting maps in the inductive system are all injective, and quasidiagonality is preserved by inductive limits of such systems, we conclude that $C_r^*(G)$ is quasidiagonal. □

4.3 The Blackadar-Kirchberg conjectures

We have already mentioned the following two conjectures due to Blackadar-Kirchberg, which were posed in the same paper [4]:

(i) Every separable, nuclear, stably finite C^*-algebra is quasidiagonal;

(ii) Every separable, exact, quasidiagonal C^*-algebra is AF-embeddable.

At this point, we have mentioned the existence of partial results, in particular a few specific classes of C^*-algebras satisfying the latter of the conjectures. We can obviously combine the two Blackadar-Kirchberg conjectures and ask whether, in the class of separable, nuclear C^*-algebras, the notions of stably finiteness, quasidiagonality and AF-embeddability are all equivalent properties. In this section, we shall see how the Tikuisis-White-Winter theorem provides a tool for partially answering these conjectures. More specifically, we shall prove that both conjectures are true for the class of reduced group C^*-algebras, and that every separable, nuclear, stably finite and simple C^*-algebra satisfying the UCT is quasidiagonal. All the corollaries presented in this section can be found in [70].

We commence proving the latter result. An important result is the following deep theorem due to Blackadar-Handelman and Haagerup, a proof of which can be found in [30].

Theorem 4.27 (Blackadar-Handelman, Haagerup)**.** *Every exact, stably finite, unital C^*-algebra admits a tracial state.*

[3]The class of elementary amenable groups is defined as the class of groups which can be built from finite groups and Abelian groups by taking subgroups, extensions, quotients and inductive limits. Since finite groups and Abelian groups are always amenable, and as the mentioned operations all preserve amenability, all elementary amenable groups are amenable, but the converse is not true, see the introduction to [51].

With this, the following corollary is almost trivial when equipped with the tools provided by the Tikuisis-White-Winter theorem.

Corollary 4.28. *Every unital, simple, stably finite and nuclear C^*-algebra in the UCT-class is quasidiagonal.*

Proof. Let A be a unital, simple, stably finite and nuclear C^*-algebra in the UCT-class. By Theorem 4.27 there exists a tracial state τ on A. Since $I = \{a \in A \mid \tau(a^*a) = 0\}$ is a closed two-sided ideal in A, we find by simplicity of A that τ must be faithful. Hence τ is a faithful tracial state on a separable nuclear C^*-algebra in the UCT-class and, consequently, τ is quasidiagonal by Theorem 4.20. As A admits a faithful quasidiagonal tracial state, A is quasidiagonal by Corollary 2.53. □

This resolves the Blackadar-Kirchberg conjecture for the class of *unital*, simple C^*-algebras in the UCT-class. The unitality assumption can, as we mentioned before, be removed, although the proof becomes more involved.

Corollary 4.29. *Every separable, simple, stably finite and nuclear C^*-algebra satisfying the UCT is quasidiagonal. In particular, Conjecture 2.31 is true for the class of simple C^*-algebras satisfying the UCT.*

Proof. Let A be a simple, stably finite and nuclear C^*-algebra satisfying the UCT. There are two possibilities: Either the stabilisation $A \otimes \mathbb{K}(H)$ contains a non-zero projection, or it does not.

Suppose there exists a non-zero projection $p \in \mathcal{P}(A \otimes \mathbb{K}(H))$. Since minimal tensor products preserve simplicity, the tensor product $A \otimes \mathbb{K}(H)$ is simple, and hence the projection p is necessarily full. Consider the unital C^*-algebra $B = p(A \otimes \mathbb{K}(H))p$. In the same manner as in the proof of Proposition 2.13, [6, Corollary 2.6] provides the isomorphism $A \otimes \mathbb{K}(H) \cong B \otimes \mathbb{K}(H)$, using that $\mathbb{K}(H)$ is stable. Note that since B is a hereditary C^*-subalgebra of the simple, stably finite and nuclear C^*-algebra $A \otimes \mathbb{K}(H)$, which satisfies the UCT, B inherits all these properties. Hence, Theorem 4.27 implies the existence of a tracial state τ on B. Simplicity of B implies faithfulness of τ, and then the Tikuisis-White-Winter theorem in conjunction with Corollary 2.53 ensures quasidiagonality of B. In particular, since $\mathbb{K}(H)$ is an AF-algebra and, hence, quasidiagonal, the minimal tensor product $B \otimes \mathbb{K}(H)$ is again quasidiagonal by Proposition 2.21(v). We can regard A as a C^*-subalgebra of $A \otimes \mathbb{K}(H)$ via the injection $a \mapsto a \otimes e$, where e is any rank one projection on H, and hence

$$A \subseteq A \otimes \mathbb{K}(H) \cong B \otimes \mathbb{K}(H)$$

implying that A is quasidiagonal, being a C^*-subalgebra of a quasidiagonal C^*-algebra. This proves the result for the case where $A \otimes \mathbb{K}(H)$ admits a non-zero projection.

On the other hand, suppose $A \otimes \mathbb{K}(H)$ does not admit a non-zero projection. By [69, Corollary 2.2], there exists a hereditary C^*-subalgebra B of A, which is algebraically simple and provides the isomorphism $A \otimes \mathbb{K}(H) \cong B \otimes \mathbb{K}(H)$. Observe that since UCT is preserved by stable isomorphisms, see [3, V.1.5.4]. Our goal is to ensure that B is quasidiagonal, then a similar argument as in the projection case proves quasidiagonality of A. Note that $\mathrm{Ped}(B) \subseteq \mathrm{Ped}(B \otimes \mathbb{K}(H)) = \mathrm{Ped}(A \otimes \mathbb{K}(H))$. Since $A \otimes \mathbb{K}(H)$ is stable, projectionless and simple, we can use a result in [36] to ensure the existence of a non-zero lower semi-continuous trace τ on $\mathrm{Ped}(A \otimes \mathbb{K}(H))$; see Definition 1.1. Observe that τ is inherited by $\mathrm{Ped}(B)$. Since B is algebraically simple, $\mathrm{Ped}(B) = B$, and this fact along with separability of B implies by [69, Proposition 2.5] that τ is bounded on all of B_+, hence bounded on all of B. It then follows from [53, Proposition 5.2.2] that τ is, in fact, a tracial state on B, and simplicity of B implies that τ is faithful. Since B is a hereditary C^*-subalgebra of A, it inherits nuclearity. Hence B is a separable, nuclear C^*-algebra admitting a faithful tracial state, so B is quasidiagonal by the Tikuisis-White-Winter theorem. The same argument as in the projection case proves quasidiagonality of A. □

We have previously seen that the Tikuisis-White-Winter theorem in conjunction with Rosenberg's theorem proves that a discrete group G is amenable if and only if the reduced group C^*-algebra $C_r^*(G)$ is quasidiagonal. This in turn implies that if G is a countable, discrete group such that $C_r^*(G)$ is AF-embeddable, then G is amenable. Using the Tikuisis-White-Winter theorem, we shall prove the converse result, that is, we prove that if G is a countable, discrete and amenable group, then $C_r^*(G)$ is AF-embeddable, which hence gives an affirmative answer for the Blackadar-Kirchberg conjectures for the class of reduced group C^*-algebras. Noting that $C_r^*(G)$ is always stably finite and that nuclearity is equivalent to quasidiagonality for reduced group C^*-algebras of discrete groups, this would completely resolve both Blackadar-Kirchberg conjectures for this specific class of C^*-algebras. Ultimately, we aim to show that the following four properties are equivalent for any countable[4], discrete group G.

(i) G is amenable;

(ii) $C_r^*(G)$ is nuclear;

(iii) $C_r^*(G)$ is quasidiagonal;

(iv) $C_r^*(G)$ is AF-embeddable.

The only missing implication is (i)⇒(iv). In order to prove this implication, we shall use tracially AF-algebras, which were first introduced by Lin [42]. We follow the definition given by Rørdam in [59, Definition 3.3.4].

Definition 4.30. A simple, separable, unital C^*-algebra A is a *tracially AF-algebra*, usually denoted TAF-algebra, if it has the following property: For all finite subsets $\{a_1, \dots, a_n\}$ of A, for all tolerances $\varepsilon > 0$ and any non-zero positive element $a \in A$, there exists a finite-dimensional C^*-subalgebra B of A with $1_B = p$ and a subset $\{b_1, \dots, b_n\}$ of B such that the following hold:

(i) $\|pa_j - a_jp\| < \varepsilon$ and $\|pa_jp - b_j\| < \varepsilon$ for all $j = 1, \dots, n$;

(ii) $1 - p$ is Murray-von Neumann equivalent to a projection in the hereditary C^*-subalgebra $\overline{aAa}$ of A.

Let us briefly discuss how to intuitively think of TAF-algebras. If $p \in A$ is a projection and $a \in A$ satisfies that $\|pa - ap\| < \varepsilon$ for some $\varepsilon > 0$, then, when expressed as a matrix indexed by p, a is block-diagonal up an ε-tolerance. We can hence express (i) in Definition 4.30 above as saying that the elements a_j are block-diagonal up to an ε-tolerance, and that its upper-left entry pa_jp can be approximated by elements in finite-dimensional C^*-algebras. Condition (ii) then states that the corner $(1-p)a_j(1-p)$ is small, and we can thus understand TAF-algebras as being C^*-algebras which are locally AF-algebras except in small corners.

One of the important properties of TAF-algebras, which we are going to invoke, is that the K_0-group of a TAF-algebra is weakly unperforated with Riesz interpolation, see Definition 1.40 for the definition and [42, Theorem 6.11] for the result. Moreover, simple, unital AF-algebras are trivially TAF-algebras.

Corollary 4.31. *If A is a separable, simple, unital, monotracial and nuclear C^*-algebra satisfying the UCT, then A embeds unitally into a simple, monotracial AF-algebra in a trace-preserving manner.*

Proof. Since tensoring with the universal UHF-algebra $\mathcal{Q}$ preserves all the aforementioned conditions on A, we can without loss of generality assume that A is $\mathcal{Q}$-stable. Note that since A is simple, its unique tracial state is necessarily faithful. By the Tikuisis-White-Winter theorem, the tracial state is quasidiagonal, and hence A itself is quasidiagonal by Corollary 2.53. Since A is a $\mathcal{Q}$-stable, unital, separable, simple, nuclear and quasidiagonal C^*-algebra, [44, Theorem 6.1] implies that A is actually a TAF-algebra, and hence $K_0(A)$ is weakly unperforated with Riesz interpolation. Since $K_0(\mathcal{Q}) = \mathbb{Q}$ is torsion-free, and moreover $K_1(\mathcal{Q}) = 0$, it follows from the Künneth formula, see [2, Theorem 23.1.3], that

$$K_0(A) \cong K_0(A \otimes \mathcal{Q}) \cong K_0(A) \otimes K_0(\mathcal{Q}) \cong K_0(A) \otimes \mathbb{Q}.$$

In particular, $K_0(A)$ is torsion-free, and $K_0(A)$ is therefore a countable Riesz group. By Proposition 1.41, there exists an AF-algebra B with $K_0(B) = K_0(A)$. In fact, B can be assumed to be unital, since $(K_0(A), K_0(A)^+, [1_A]_0)$ is a Riesz group with an order unit. Moreover, since A is stably finite, as it is quasidiagonal, and simple, we can use [2, Corollary 6.3.6] to see that $K_0(A)$ is a simple ordered Abelian group, and then a use of [59, Corollary 1.5.4] shows that B is actually a simple AF-algebra. Using the Künneth formula, [2, Theorem 23.1.3], twice, we obtain the following sequence of equivalences

$$K_0(B) \cong K_0(A) \cong K_0(A \otimes \mathcal{Q}) \cong K_0(A) \otimes K_0(\mathcal{Q}) \cong K_0(B) \otimes K_0(\mathcal{Q}) \cong K_0(B \otimes \mathcal{Q}).$$

[4]Countability is to ensure separability of $C_r^*(G)$, which is necessary for being AF-embeddable. The equivalences (i)⇔(ii)⇔(iii) hold for any discrete group.

Hence $K_0(B) \cong K_0(B \otimes \mathcal{Q})$ and, in fact, they have the same positive cone by [3, V.2.4.16], and the isomorphism maps $[1_B]_0$ to $[1_{B\otimes\mathcal{Q}}]_0$. Thus Elliott's classification of AF-algebras, Theorem 1.39, entails that B is $\mathcal{Q}$-stable. Since B is an AF-algebra, its K_1-group is trivial, so we consider the trivial group homomorphism $K_1(A) \to K_1(B)$. Let $\alpha\colon K_*(A) \to K_*(B)$ be the group homomorphism as defined above, then a result by Dadarlat, see [15, Theorem 1.1], provides a unital *-homomorphism $\varphi\colon A \to B$ lifting α. Since this *-homomorphism is unital and A is simple, it is necessarily an injection. We can hence realise A as a C^*-subalgebra of the AF-algebra B. The only thing remaining is to show that B has a unique tracial state. Since any state on an ordered K_0-group of a unital, exact C^*-algebra can be realised as $K_0(\tau)$ for a tracial state τ, see [59, Theorem 1.1.11], we find that $K_0(A)$ only has one state by monotraciality of A. In particular, $K_0(B)$ only has one state ρ, which provides monotraciality of B, since the correspondence between states on K-theory and tracial states on C^*-algebras is a homeomorphism for AF-algebras by [59, Proposition 1.5.5]. The tracial state is preserved since the embedding $A \to B$ induces a map between tracial simplices $T(B) \to T(A)$, which are both one-point spaces. □

Let B denote the CAR-algebra, which is the UHF-algebra defined as the inductive limit of the sequence

$$\mathbb{C} \to M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \cdots$$

where the connecting maps are given by $a \mapsto \mathrm{diag}(a,a)$, i.e., B is the UHF-algebra associated to the supernatural number 2^∞. If G is any countable, discrete group, we can define an action of G on the tensor product $\bigotimes_G B$ by left-translation, that is, for any $h \in G$, the action is given by $\bigotimes_{g\in G} a_g \mapsto \bigotimes_{g\in G} a_{h^{-1}g}$. Define for each countable, discrete group G the C^*-algebra $B(G) = (\bigotimes_G B) \rtimes_r G$; see [11, Section 4.1] for the definition of the reduced crossed product. We here give an overview of the necessary properties of $B(G)$, which we are going to use in the following — for the proof, along with more properties, we refer to [51, Proposition 2.1].

Proposition 4.32. *Let G be any countable, discrete, amenable group, then $B(G)$ is a separable, simple, unital, monotracial and nuclear C^*-algebra satisfying the UCT.*

Note that these properties are precisely the conditions on A in Corollary 4.31, and hence $B(G)$ embeds into a simple and monotracial AF-algebra, whenever G is countable and discrete. We can realise $C_r^*(G)$ as a C^*-subalgebra of $B(G)$ to obtain the following corollary.

Corollary 4.33. *If G is any countable, discrete, amenable group, then G embeds into a simple, monotracial AF-algebra. In particular, the Blackadar-Kirchberg conjectures hold for countable, discrete, amenable groups.*

4.4 Elliott's classification program and connections thereto

We end this chapter with a more in-depth study of Elliott's classification program, and how the Tikuisis-White-Winter theorem actually resolves one part of the classification puzzle, in the sense that it shows how it is possible to remove one of the original assumptions. We have already in a previous chapter discussed the history of classifying C^*-algebras by their K-theoretic data, most notably the classification of AF-algebras, and as such we focus on the recent results in Elliott's classification program. We emphasise that this is merely a brief overview of a very rich subject and of work spanning decades.

In Theorem 1.39 and Theorem 1.44, the K-theoretic data was sufficient to classify the examined classes of C^*-algebras. However, only considering K-theory turns out to be too restrictive. Instead, the now named Elliott invariant takes into account both the K-theory and the structure of the tracial state simplex, as well as a map connecting the two. In order to state the Elliott invariant, we need to explain what this map is. Let (G, G^+, u) be a preordered Abelian group with an order unit, and consider the set $S(G)$ of unit-preserving states on G; this is a weak*-compact convex set. For any tracial state τ on a unital C^*-algebra A, we can define a state on $K_0(A)$ denoted $K_0(\tau)\colon K_0(A) \to \mathbb{C}$ by $K_0(\tau)([p]_0 - [q]_0) = \tau(p-q)$ for $p, q \in \mathcal{P}_\infty(A)$. Then $r_A\colon T(A) \to S(K_0(A))$ is the map $\tau \mapsto K_0(\tau)$. Define for a unital and separable C^*-algebra A the *Elliott invariant* $\mathrm{Ell}(A)$ to be

$$\mathrm{Ell}(A) := (K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A).$$

We shall see that for a very large class of C^*-algebras, this is in fact an isomorphism invariant. It is not immediately obvious how we shall consider two C^*-algebras to have isomorphic Elliott invariants, so let us properly define this. If A, B are C^*-algebras, we write that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ if the following properties all hold:

(i) There exists a unit-preserving preorder isomorphism $\varphi\colon K_0(A) \to K_0(B)$;

(ii) There exists a group isomorphism $\psi\colon K_1(A) \to K_1(B)$;

(iii) There exists an affine homeomorphism $\alpha\colon T(B) \to T(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} T(B) & \xrightarrow{\alpha} & T(A) \\ \downarrow{\scriptstyle r_B} & & \downarrow{\scriptstyle r_A} \\ S(K_0(B)) & \xrightarrow{\varphi_*} & S(K_0(A)) \end{array}$$

where the map $\varphi_*\colon S(K_0(B)) \to S(K_0(A))$ is the map $\varphi_*(f) = f \circ \varphi$ induced by φ.

Having established what the Elliott invariant is, and how we are meant to interpret it, we proceed by understanding the classification result with Elliott's invariant as the classification invariant. We still need to introduce some more terminology in order to fully state the classification result.

Definition 4.34. Two elements $a, b \in A$ are called *orthogonal* if $ab = ba = a^*b = ab^* = 0$, in which case we write $a \perp b$.

Definition 4.35. A c.p. map $\varphi\colon A \to B$ between C^*-algebras is said to be of *order zero* if φ preserves orthogonality, that is, if $a \perp b$, then $\varphi(a) \perp \varphi(b)$.

Our interest in order zero maps comes from its usage in the concept of finite nuclear dimension, which is one of the regularity conditions in Elliott's classification. We define nuclear dimension as a local property; this is easily seen to be equivalent to the formulation in [73, Definition 2.1].

Definition 4.36. A C^*-algebra A is said to have *nuclear dimension* n, if n is the smallest positive integer satisfying the following: For any finite subset $F \subseteq A$ and $\varepsilon > 0$ there exists some finite-dimensional C^*-algebra B and some c.p. maps $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that φ is contractive, and such that we have the decompositions

$$B = B_0 \oplus B_1 \oplus \cdots \oplus B_n \qquad \text{and } \psi = \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_n$$

of B and ψ into $n+1$ ideals $B_i \trianglelefteq B$ and order zero maps $\psi_i\colon B_i \to A$ satisfying

$$\|\psi \circ \varphi(a) - a\| < \varepsilon \qquad \text{and} \qquad \|\psi_i\| \leq 1$$

for all $a \in F$ and $i = 0, \ldots, n$. In this case, we write $\dim_{\mathrm{nuc}}(A) = n$.

We say that a C^*-algebra A has *finite nuclear dimension* if $\dim_{\mathrm{nuc}}(A) < \infty$. We can always assume that the composition $\psi \circ \varphi$ is contractive, see [73, Remark 2.2(iv)]. In this way, we see that if $\dim_{\mathrm{nuc}}(A) < \infty$, then, for any finite set $F \subseteq A$ and to any tolerance $\varepsilon > 0$, id_A can be approximated up to an ε tolerance on F by the c.c.p. map $\psi \circ \varphi\colon A \to A$, which has finite rank. Hence, finite nuclear dimension implies nuclearity.

Let us mention one conjecture regarding finite nuclear dimension known as the Toms-Winter conjecture. A construction of Jiang and Su gives an example of a C^*-algebra $\mathcal{Z}$, which has some quite surprising results; we present below a selection of these properties, the proofs of which can be found in [35].

Theorem 4.37 (Jiang-Su, 1999)**.** *The C^*-algebra $\mathcal{Z}$ is a unital, simple, stably finite, separable, nuclear, infinite-dimensional C^*-algebra with a unique tracial state. Moreover, $K_0(\mathcal{Z}) \cong K_0(\mathbb{C})$ as ordered Abelian groups, and $K_1(\mathcal{Z}) = 0$. Further, we have the isomorphisms $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \bigotimes_{\mathbb{N}} \mathcal{Z}$, and $\mathcal{Z}$ satisfies the UCT.*

The Jiang-Su algebra hence has a lot of structure, but the fact that its Elliott invariant is isomorphic to that of $\mathbb{C}$ of course seems like a devastating blow to hopes of classifying C^*-algebras by their Elliott invariants; however, this is rectifiable by restricting the classification to infinite-dimensional C^*-algebras. Having defined the Jiang-Su algebra, we are able to state (part of) the Toms-Winter conjecture.

Conjecture 4.38 (Toms-Winter)**.** A separable, simple, unital, nuclear and infinite-dimensional C^*-algebra has finite nuclear dimension if and only if it is $\mathcal{Z}$-stable.

In the full Toms-Winter conjecture, there is yet another conjectured equivalent property called *strict comparison*; we refrain from discussing this property here. The full Toms-Winter conjecture may be found in [70, Conjecture 6.3]. The Tikuisis-White-Winter actually provides a resolution for C^*-algebras with at most one tracial state, and which satisfy the UCT, see [70, Corollary 6.4].

With the terminology of finite nuclear dimension in place, we are able to state the following classification result. We formulate it as in the original version, and then show how the quasidiagonality assumption is superfluous.

Theorem 4.39 (Elliott-Gong-Lin-Niu, 2015, original formulation)**.** *The class of all unital, separable, simple, infinite-dimensional C^*-algebras satisfying the UCT, with finite nuclear dimension, and with the property that all tracial states are quasidiagonal is classifiable via the Elliott invariant.*

The proof of the classification was finalised by Elliott, Gong, Lin and Niu in [19], and the result can almost be seen as the conclusion to Elliott's classification programme that has spanned several decades.

Let us see how the quasidiagonality assumption is superfluous. Suppose A is a unital, separable, simple C^*-algebra with finite nuclear dimension and let τ be a tracial state on A. Since A has finite nuclear dimension, A is nuclear, and simplicity implies faithfulness of τ in the same manner as in the proof of Corollary 4.28. Hence, the Tikuisis-White-Winter theorem provides that τ is quasidiagonal, and we can phrase Theorem 4.39 as follows:

Theorem 4.40 (Elliott-Gong-Lin-Niu, 2016)**.** *The class of all unital, separable, simple, infinite-dimensional C^*-algebras satisfying the UCT with finite nuclear dimension is classifiable via the Elliott invariant.*

In saying that this almost completes the classification program of Elliott, we mean that it is unknown whether or not the UCT-assumption is superfluous or not. For example, if every separable, nuclear C^*-algebra satisfies the UCT, i.e., there exists an affirmative answer to the UCT-problem, we can remove the UCT-assumption as finite nuclear dimension implies nuclearity.

5 Traceless, exact C^*-algebras and AF-embeddability

We end the thesis with a recent result due to Gabe [26] that the Blackadar-Kirchberg conjectures are true for the class of traceless C^*-algebras. Two important ingredients in his proof are the AF-embeddable C^*-algebra $\mathcal{A}_{[0,1]}$ and the primitive ideal spaces of C^*-algebras, which are certain generalisations of the spectra for Abelian C^*-algebras. In this chapter, we shall reproduce Gabe's result and provide some of the necessary background knowledge.

5.1 The primitive ideal space and the ideal lattice

If A is an Abelian C^*-algebra, it follows from Proposition 1.2 that A is *-isomorphic to $C_0(\hat{A})$ for some locally compact Hausdorff space called the spectrum $\hat{A}$, which can be realised as the set of non-zero *-homomorphisms $A \to \mathbb{C}$. The motivation behind the primitive ideal space is to construct a topological structure generalising the the concept of spectra to non-Abelian C^*-algebras. Recall from representation theory that a *-representation $\pi\colon A \to \mathbb{B}(H)$ on some Hilbert space H is said to be *irreducible* if there exists no non-trivial closed subspace $K \subseteq H$ such that $\pi(A)K \subseteq K$, and that it is said to be *reducible* otherwise.

Definition 5.1. Let A be a C^*-algebra, and let $I \trianglelefteq A$ be a closed two-sided ideal. We say that I is a *primitive ideal* in A if I is the kernel of an irreducible representation of A. We denote the collection of primitive ideals on A by $\mathrm{Prim}(A)$.

If A is a C^*-algebra, then A has an irreducible representation, and hence its primitive ideal space is non-empty. Note that if A is simple, then the only primitive ideal is the zero ideal. We want to topologise the collection of primitive ideals in a way generalising the topology on the spectrum, and for this we shall need a few results on prime ideals and primitive ideals.

Definition 5.2. An ideal I in a C^*-algebra A is called *prime* if for any two ideals I_1, I_2 in A with $I_1 I_2 \subseteq I$, either $I_1 \subseteq I$ or $I_2 \subseteq I$.

We refrain from proving the following two lemmas and refer to [53, Corollary 3.13.8 and Proposition 3.13.10] instead.

Lemma 5.3. *Every closed ideal in a C^*-algebra is the intersection of all the primitive ideals containing it.*

Lemma 5.4. *Every primitive ideal in a C^*-algebra is a prime ideal.*

The converse to Lemma 5.4 holds for separable C^*-algebras, see [53, Proposition 4.3.6].

Let A be a C^*-algebra. For each subset $F \subseteq \mathrm{Prim}(A)$ and $B \subseteq A$, we define

$$\ker F = \bigcap_{I \in F} I \qquad \text{and,}$$
$$\mathrm{hull}\, B = \{I \in \mathrm{Prim}(A) \mid B \subseteq I\}.$$

Consider for each subset $F \subseteq \mathrm{Prim}(A)$ the set $\overline{F} = \mathrm{hull}\ker F$. As the notation suggests, we shall prove that this defines a closure map on the power set of $\mathrm{Prim}(A)$ and, hence, defines a topology. More precisely, we shall prove the following proposition.

Proposition 5.5. *The collection* $\{\mathrm{hull}\, B \mid B \subseteq A\}$ *form the closed sets of a topology on* $\mathrm{Prim}(A)$. *In particular, for any subset $F \subseteq \mathrm{Prim}(A)$, the closure of F is given by $F = \mathrm{hull}\ker F$.*

Proof. We prove that the Kuratowski closure axioms, see [41, p. 38], are satisfied. Let $F, G \subseteq \mathrm{Prim}(A)$ be arbitrary subsets in the following.

(i) We first show that $\overline{F} \cup \overline{G} = \overline{F \cup G}$. If $I \in \overline{F}$, then $\ker(F \cup G) \subseteq \ker F \subseteq I$, proving that $I \in \mathrm{hull}\ker(F \cup G) = \overline{F \cup G}$ and hence $\overline{F} \subseteq \overline{F \cup G}$. By following the same arguments for any $J \in \overline{G}$, one finds that $\overline{F} \cup \overline{G} \subseteq \overline{F \cup G}$. For the other direction, suppose that $I \in \overline{F \cup G}$, then as $\ker(F \cup G) = \ker F \cap \ker G$, we see that $\ker F \cap \ker G \subseteq I$. Since I is a primitive ideal, it is a prime ideal by Lemma 5.4, and hence either $\ker F \subseteq I$ or $\ker G \subseteq I$. In either case, $I \in \overline{F} \cup \overline{G}$. We conclude that $\overline{F \cup G} = \overline{F} \cup \overline{G}$.

(ii) If $I \in F$, then it follows by definition of the kernel that $\ker F \subseteq I$, and hence $I \in \operatorname{hull} \ker F = \overline{F}$, which proves that $F \subseteq \overline{F}$.

(iii) It is immediate that $\overline{\emptyset} = \emptyset$.

(iv) We now aim to prove that $\overline{F} = \overline{\overline{F}}$. In fact, we shall prove that if $F = \operatorname{hull} B$ for some subset $B \subseteq A$, then $F = \overline{F}$, which would imply the desired result. It follows from (ii) that $F \subseteq \overline{F}$, so we prove the other inclusion. Suppose that $I \in \overline{F}$, then $\ker F \subseteq I$, and as

$$\ker F = \ker \operatorname{hull} B = \bigcap_{J \in \operatorname{hull} B} J \supseteq B,$$

we find that $B \subseteq I$. Consequently, $I \in \operatorname{hull} B = F$, proving the desired inclusion.

We conclude by Kuratowski's axioms that the closure map $F \mapsto \overline{F}$ defines a topology on $\operatorname{Prim}(A)$. Note that it follows from (iv) that every closed set in this topology is of the form $\operatorname{hull} B$ for some subset $B \subseteq A$, and that all such subsets of $\operatorname{Prim}(A)$ are closed. This proves the proposition. □

The above topology is often called either the *Jacobson topology* or the *hull-kernel topology.*

Proposition 5.6. *For any C^*-algebra A, the primitive ideal space $\operatorname{Prim}(A)$ is a T_0-space.*

Proof. Suppose that $I_1, I_2 \in \operatorname{Prim}(A)$ are two distinct ideals and assume without loss of generality that $I_1 \not\subseteq I_2$. In particular, $I_2 \notin \operatorname{hull}(I_1)$, and hence the open set $\operatorname{Prim}(A) \setminus \operatorname{hull}(I_1)$ is an open neighbourhood of I_2 not containing I_1. □

However, the space is, in general, not T_1. For a C^*-algebra A, we denote by $\mathcal{I}(A)$ the *ideal lattice* of A, which is the collection of closed two-sided ideals in A, which is a lattice equipped with the usual set inclusions.

Proposition 5.7. *For every C^*-algebra A there exists an order-preserving isomorphism between the open sets in $\operatorname{Prim}(A)$ in the Jacobson topology and the ideal lattice $\mathcal{I}(A)$.*

Proof. Define for each open set $U \subseteq \operatorname{Prim}(A)$ the primitive ideal $I(U) = \ker(\operatorname{Prim}(A) \setminus U)$ and define for each closed two-sided ideal $I \trianglelefteq A$ the open set $U(I) = \operatorname{Prim}(A) \setminus \operatorname{hull} I$ in $\operatorname{Prim}(A)$. We claim that the maps $U \mapsto I(U)$ and $I \mapsto U(I)$ are order-preserving isomorphisms as desired.

It is immediate that they are order-preserving by construction, so let us prove that they are inverses of one another. If $U \subseteq \operatorname{Prim}(A)$ is open, then

$$U(I(U)) = U(\ker(\operatorname{Prim}(A) \setminus U)) = \operatorname{Prim}(A) \setminus (\overline{\operatorname{Prim}(A) \setminus U}) = U,$$

and if $I \trianglelefteq A$ is a closed two-sided ideal, then

$$I(U(I)) = I(\operatorname{Prim}(A) \setminus \operatorname{hull} I) = \ker \operatorname{hull} I = I,$$

where the last equality follows from Lemma 5.3, since $\ker \operatorname{hull} I$ is the intersection of all primitive ideals containing I. This proves that I and U are inverses of one another, which completes the proof. □

As stated before, the primitive ideal space generalises the spectrum for Abelian C^*-algebras — let us establish how this is the case. Suppose that $C_0(X)$ is an Abelian C^*-algebra. By [53, Theorem 3.13.2], any non-zero irreducible representation of $C_0(X)$ is one-dimensional, and hence the spectrum $\hat{A}$ is precisely the set of irreducible representations of $C_0(X)$, and one can show that the canonical map $\hat{A} \to \operatorname{Prim}(A)$ by $\pi \mapsto \ker \pi$ is a homeomorphism. In particular, $\operatorname{Prim}(C_0(X))$ is homeomorphic to X for any locally compact Hausdorff space X.

Using the fact that, in the separable case,

$$\operatorname{Prim}(A \otimes B) \cong \operatorname{Prim}(A) \times \operatorname{Prim}(B)$$

holds whenever either A or B is exact, see [3, Theorem IV.3.4.26], we see that

$$\operatorname{Prim}(C_0(X, A)) \cong \operatorname{Prim}(C_0(X) \otimes A) \cong \operatorname{Prim}(C_0(X)) \times \operatorname{Prim}(A) \cong X \times \operatorname{Prim}(A).$$

Consider $X = (0, 1)$ or $X = (0, 1]$ or, in other words, suppose we want to analyse the primitive ideal space of the suspension or cone of A. Then $\mathrm{Prim}(C_0(X, A))$ admits no non-empty, compact, open subsets, since any such would project onto a subset of X with the same properties, and no such subset of X exists. Gabe's confirmation of the Blackadar-Kirchberg conjectures in the traceless case effectively boils down to showing that a separable exact C^*-algebra A admits an embedding into a specific AF-embeddable C^*-algebra $\mathcal{A}_{[0,1]}$, to be analysed in Section 5.2, if and only if $\mathrm{Prim}(A)$ admits no non-empty, compact, open subsets. Since the topological structure on $\mathrm{Prim}(A)$ by Proposition 5.7 is closely related to the ideal structure of A, it is natural to ask how the topological structure of compactness translate into the ideal lattice in A. For this we shall define a few terms in lattice theory. Recall that a lattice is a partially ordered set for which any *finite* subset has a supremum and an infimum.

Definition 5.8. A *complete* lattice is a lattice for which every subset has a supremum and an infimum.

The ideal lattice $\mathcal{I}(A)$ of any C^*-algebra A is a complete lattice with $\inf_\alpha I_\alpha = \bigcap_\alpha I_\alpha$ and $\sup_\alpha I_\alpha = \overline{\sum_\alpha I_\alpha}$. If the net is increasing, the supremum is given by $\sup_\alpha I_\alpha = \overline{\bigcup_\alpha I_\alpha}$.

Definition 5.9. Let $\mathcal{L}$ be a complete lattice, and let $x, y \in \mathcal{L}$. We say that x is *compactly contained* in y, denoted $x \Subset y$, if for any increasing net $(y_\alpha)_{\alpha\in\Lambda}$ in $\mathcal{L}$ with $y \leq \sup_\alpha y_\alpha$, there exists some α such that $x \leq y_\alpha$. We say that x is *compact* if $x \Subset x$.

Notice that the ideal lattice $\mathcal{I}(A)$ is a complete lattice, and one can show that an ideal I in a C^*-algebra A is compact in the above sense if and only if it corresponds to a compact open subset of $\mathrm{Prim}(A)$.

Definition 5.10. Let $\mathcal{L}$ and $\mathcal{L}'$ be complete lattices, and let $\Phi\colon \mathcal{L} \to \mathcal{L}'$ be an order-preserving map. We say that Φ is a *Cu-morphism* if it preserves arbitrary suprema and compact containment.

The reader may find the terminology confusing and rightfully so, as we have not justified naming such maps *Cu*-morphisms. We shall hence try to develop the necessary theory to understand the terminology, and this will also help in understanding the structure of ideal lattices. First of all, for any C^*-algebra A, we let $M_\infty(A)$ denote the C^*-algebra constructed by the inductive limit

$$A \xrightarrow{\varphi_1} M_2(A) \xrightarrow{\varphi_2} M_3(A) \xrightarrow{\varphi_3} \cdots$$

where the connecting maps are defined by embeddings into the upper-left entries, i.e.,

$$\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \qquad a \in M_n(A),\ n \in \mathbb{N}.$$

Observe that we have an isomorphism $M_\infty(A) \cong A \otimes \mathbb{K}(H)$ for every C^*-algebras A, since $\mathbb{K}(H)$ is nuclear and as inductive limits are preserved by the maximal tensor product by [3, II.9.6.5].

Definition 5.11. Let A be a C^*-algebra, and let $a, b \in A$. We say that a is *Cuntz subequivalent* to b, denoted by $a \precsim b$, if there exists a sequence of elements $(v_n)_{n\in\mathbb{N}}$ in A such that $v_n b v_n^* \to a$ as $n \to \infty$. If $a \precsim b$ and $b \precsim a$, we shall write that $a \sim_c b$.

One can verify that $\sim_c$ defines an equivalence relation on A. Using this fact, we can define the *Cuntz semigroup* $\mathrm{Cu}(A)$ of A to be $\mathrm{Cu}(A) = (A \otimes \mathbb{K}(H))_+ / \sim_c$, see [1, Definition 2.4]. Denote by $[a]_c$ the equivalence class corresponding to an element $a \in (A \otimes \mathbb{K}(H))_+$, and equip $\mathrm{Cu}(A)$ with the addition $[a]_c + [b]_c = [a \oplus b]_c$ as well as the ordering $[a]_c \leq [b]_c$ if and only if $a \precsim b$. Equipped with these operations, $\mathrm{Cu}(A)$ is an ordered Abelian semigroup. Note that since $[a]_c \leq [b]_c$ implies that a belongs to the ideal generated by b, there exists an order-preserving map $\mathrm{Cu}(A) \to \mathcal{I}(A)$ given by $[a]_c \mapsto \overline{AaA}$.

We shall use this in the below definition, which is a categorical theoretical generalisation of some properties that all Cuntz semigroups satisfy.

Definition 5.12. The objects in the category **Cu** are positively[5] ordered Abelian semigroups S with the following properties:

($\mathcal{O}1$) Any increasing sequence $(a_n)_{n\in\mathbb{N}}$ in S admits a supremum $\sup_{n\in\mathbb{N}} a_n$;

[5]An ordered Abelian semigroup S is positively ordered if $x \geq 0$ for all $x \in S$.

($\mathcal{O}2$) For any $a \in A$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in S such that $a_n \Subset a_{n+1}$ for all $n \in \mathbb{N}$, and such that $a = \sup_{n\in\mathbb{N}} a_n$;

($\mathcal{O}3$) If $a' \Subset a$ and $b' \Subset b$ for any $a, a', b, b' \in S$, then $a' + b' \Subset a + b$;

($\mathcal{O}4$) If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are increasing sequences in S, then $\sup_{n\in\mathbb{N}}(a_n + b_n) = \sup_{n\in\mathbb{N}} a_n + \sup_{n\in\mathbb{N}} b_n$.

The morphisms in **Cu** are maps preserving addition, the order, the zero element, compact containment and increasing suprema.

In [13] , Coward, Elliott and Ivanescu showed that, for any C^*-algebra A, the Cuntz semigroup $\mathrm{Cu}(A)$ satisfies the properties ($\mathcal{O}1$)-($\mathcal{O}4$), and this gave rise to the above definition. Observe the similarities between the morphisms in the category **Cu** and the *Cu*-morphisms in Definition 5.10 — both types of morphisms preserve compact containment and increasing suprema. The similarities do not end here, but in order to establish these we need to look a little closer at the ideal lattices of C^*-algebras.

If $\varphi\colon A \to B$ is a *-homomorphism, we claim that the map $\mathcal{I}(\varphi)\colon \mathcal{I}(A) \to \mathcal{I}(B)$ induced on the ideal lattices by $\mathcal{I}(\varphi)(I) = \overline{B\varphi(I)B}$ for $I \in \mathcal{I}(A)$ is a *Cu*-morphism in the sense of Definition 5.10. Further, we claim that if A is separable, then $\mathcal{I}(A)$ is an object in **Cu**, and that if $\varphi\colon A \to B$ is a *-homomorphism between separable C^*-algebras, then $\mathcal{I}(\varphi)$ is a morphism in the category **Cu**.

Lemma 5.13. *Let A be a C^*-algebra, and suppose that I and J are ideals in A. Then $I \Subset J$ if and only if there exists some $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$. In particular, for any $a \in A_+$ and $\varepsilon > 0$, we have $\overline{A(a-\varepsilon)_+A} \Subset \overline{AaA}$.*

Proof. Suppose that there exists $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$. Let $(I_\alpha)_{\alpha\in\Lambda}$ be an increasing net of ideals such that $J \subseteq \overline{\bigcup_{\alpha\in\Lambda} I_\alpha}$. Then there exists a positive element $b \in I_\alpha$ such that $\|a - b\| < \frac{\varepsilon}{2}$ for some α, which implies the inequality $a - \frac{\varepsilon}{2} \leq b$ in the unitisation $A^\dagger$. Let $f\colon [0,\infty) \to [0,\infty)$ be any continuous function satisfying $f(x) = 0$ for $x \in [0, \varepsilon/2)$ and $f(x) = 1$ for $x \in [\varepsilon,\infty)$. Then a use of the continuous functional calculus shows that

$$(a-\varepsilon)_+ \leq f(a)(a - \frac{\varepsilon}{2})f(a) \leq f(a)bf(a),$$

and this gives rise to the inclusion of ideals

$$I \subseteq \overline{A(a-\varepsilon)_+A} \subseteq \overline{AbA} \subseteq I_\alpha.$$

We have hence shown that $I \Subset J$. Note that this also proves the latter half of the lemma, i.e., that $\overline{A(a-\varepsilon)_+A} \Subset \overline{AaA}$ for any $a \in A_+$ and $\varepsilon > 0$.

Conversely, suppose that $I \Subset J$. Let $a_i \in J_+$ and $\varepsilon_i > 0$ for $i = 1, 2$, and let $a \in A_+$ be any positive element. Then the previously shown implication implies that

$$\overline{A(a_i - \varepsilon_i)_+A} \Subset \overline{AaA}.$$

Observing that $\overline{AaA} = \overline{\bigcup_{\varepsilon>0} \overline{A(a-\varepsilon)_+A}}$, we find that there exist $\varepsilon_i' > 0$ for $i = 1, 2$ such that

$$\overline{A(a_i - \varepsilon_i)_+A} \subseteq \overline{A(a - \varepsilon_i')_+A} \subseteq \overline{A(a-\varepsilon)_+A}$$

for $i = 1, 2$, where $\varepsilon = \min\{\varepsilon_1', \varepsilon_2'\}$. We thus see that the family of ideals $(\overline{A(a-\varepsilon)_+A})_{a\in J_+,\varepsilon>0}$ is upwards directed. Since $I \Subset J$, we have for any increasing net $(I_\alpha)_{\alpha\in\Lambda}$ with $J \subseteq \overline{\bigcup_{\alpha\in\Lambda} I_\alpha}$ that there exists some $\alpha_0 \in \Lambda$ with $I \subseteq I_{\alpha_0}$. Observe that we can realise J as

$$J = \overline{\sum_{a\in J_+,\varepsilon>0} \overline{A(a-\varepsilon)_+A}} = \sup_{a\in J_+,\varepsilon>0} \overline{A(a-\varepsilon)_+A}$$

and that if $(a_\alpha)_{\alpha\in\Lambda}$ is a net in a complete lattice with supremum a, then $b_\alpha := \sup_{\beta\leq\alpha} a_\beta$ is an *increasing* net with supremum a. These facts combined with upwards directedness of the family $(\overline{A(a-\varepsilon)_+A})_{a\in J_+,\varepsilon>0}$ as argued before entails the existence of a positive element $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$, which completes the proof. □

This characterisation of compact containment for ideals is a useful tool. We shall use it in the next lemma, which states when the ideal lattice of a C^*-algebra satisfy condition ($\mathcal{O}2$) above.

Lemma 5.14. *Let I be an ideal in a C^*-algebra A. Then I has a full element if and only if there exists a sequence $I_1 \Subset I_2 \Subset \cdots$ of ideals in A with $I = \overline{\bigcup_{n\in\mathbb{N}} I_n}$.*

Proof. If I admits a full element $a \in I$, then the sequence $I_n = \overline{A(a^*a - 1/n)_+A}$ gives a sequence of ideals $I_1 \Subset I_2 \Subset \cdots$ by Lemma 5.13, and moreover $\overline{\bigcup_{n\in\mathbb{N}} I_n} = \overline{AaA} = I$ as desired. Conversely, suppose that $(I_n)_{n\in\mathbb{N}}$ is a sequence of ideals with $I_1 \Subset I_2 \Subset \cdots$ and $I = \overline{\bigcup_{n\in\mathbb{N}} I_n}$. For each $n \in \mathbb{N}$, we can invoke Lemma 5.13 to find a positive element $a_n \in I_n$ with $\|a_n\| \leq 1$ such that $I_n \subseteq \overline{Aa_nA} \subseteq I_{n+1}$. In particular, $I = \overline{\bigcup_{n\in\mathbb{N}} \overline{Aa_nA}}$, and one can verify that the element $h = \sum_{n=1}^{\infty} \frac{1}{2^n} a_n$ is a full element in I. □

The existence of full elements in ideals is important in this discussion due to the following proposition, which gives a complete characterisation of when an ideal lattice is an object in **Cu**.

Proposition 5.15. *The ideal lattice $\mathcal{I}(A)$ of a C^*-algebra A is an object in the category* **Cu** *if and only if all ideals in A admit a full element.*

Proof. Suppose that $\mathcal{I}(A)$ is an object in the category **Cu**, then for any $I \in \mathcal{I}(A)$ there exists a sequence of ideals $I_1 \Subset I_2 \Subset \cdots$ with $I = \overline{\bigcup_{n\in\mathbb{N}} I_n}$, which by Lemma 5.14 implies that I is full. Conversely, if all ideals in $\mathcal{I}(A)$ are full, then it is straightforward to verify that $\mathcal{I}(A)$ is an object in **Cu**; fullness is used to invoke Lemma 5.14, which proves condition ($\mathcal{O}2$). □

Since any ideal of a separable C^*-algebra admits a full element, see [3, II.5.3.10], we get the following result.

Corollary 5.16. *If A is a separable C^*-algebra, the ideal lattice $\mathcal{I}(A)$ is an object in* **Cu**.

We now proceed with proving our second claim that if $\varphi\colon A \to B$ is a *-homomorphism, then the induced map $\mathcal{I}(\varphi)\colon \mathcal{I}(A) \to \mathcal{I}(B)$ is a *Cu*-morphism in the sense of Definition 5.10.

Proposition 5.17. *Suppose that $\varphi\colon A \to B$ is a *-homomorphism. Then $\mathcal{I}(\varphi)\colon \mathcal{I}(A) \to \mathcal{I}(B)$ is an order-preserving morphism which preserves compact containment and suprema. In particular, $\mathcal{I}(\varphi)$ is a Cu-morphism in the sense of Definition 5.10.*

Proof. Clearly, $\mathcal{I}(\varphi)$ preserves the zero element as well as the ordering. Since $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ for any $a \in I$, and the collection of the right-hand side generates $\overline{B\varphi(I_+)B}$, we see that $\varphi(a) \in \overline{B\varphi(I_+)B}$ for all $a \in I$. Consequently, we have the identity

$$\mathcal{I}(\varphi)(I) = \overline{B\varphi(I)B} = \overline{B\varphi(I_+)B}.$$

Moreover, since $(I+J)_+ = I_+ + J_+$ for ideals I, J in A, see [53, Proposition 1.5.9], we have

$$\begin{aligned}\mathcal{I}(\varphi)(I+J) &= \overline{B\varphi((I+J)_+)B} = \overline{B\varphi(I_+ + J_+)B} = \overline{B(\varphi(I_+) \cup \varphi(J_+))B} \\ &= \overline{B\varphi(I_+)B} + \overline{B\varphi(J_+)B} = \mathcal{I}(\varphi)(I) + \mathcal{I}(\varphi)(J)\end{aligned}$$

such that $\mathcal{I}(\varphi)$ is additive.

For compact containment, first note that multiplicativity and continuity of φ implies that

$$\mathcal{I}(\varphi)(\overline{AaA}) = \overline{B\varphi(\overline{AaA})B} = \overline{B\varphi(a)B}.$$

Suppose that I, J are ideals in A with $I \Subset J$. By Lemma 5.13, there exist some positive element $a \in J_+$ and $\varepsilon > 0$ such that $I \subseteq \overline{A(a-\varepsilon)_+A}$, and hence

$$\begin{aligned}\mathcal{I}(\varphi)(I) &\subseteq \mathcal{I}(\varphi)(\overline{A(a-\varepsilon)_+A}) = \overline{B\varphi((a-\varepsilon)_+)B} \\ &= \overline{B(\varphi(a)-\varepsilon)_+B} \Subset \overline{B\varphi(a)B} = \mathcal{I}(\varphi)(\overline{AaA}) \subseteq \mathcal{I}(\varphi)(J).\end{aligned}$$

We hence see that $\mathcal{I}(\varphi)(I) \Subset \mathcal{I}(\varphi)(J)$, and therefore $\mathcal{I}(\varphi)$ preserves compact containment. All that remains to be seen is that $\mathcal{I}(\varphi)$ preserves suprema — first we prove that it preserves increasing suprema. Let I be an ideal in A, and let $(I_\alpha)_{\alpha\in\Lambda}$ be an increasing net of ideals in A with $\sup_\alpha I_\alpha = I$. Note that

$I = \overline{\bigcup_{\alpha\in\Lambda} I_\alpha}$ as the net is increasing. Since $I_\alpha \subseteq I$ for each α, we find that $\mathcal{I}(\varphi)(I_\alpha) \subseteq \mathcal{I}(\varphi)(I)$. This implies in particular that $\sup_{\alpha\in\Lambda} \mathcal{I}(\varphi)(I_\alpha) \subseteq \mathcal{I}(\varphi)(I)$ proving one inclusion. For the other inclusion, let $x \in I$ be arbitrary and find a sequence $(x_n)_{n\in\mathbb{N}}$ in $\bigcup_{\alpha\in\Lambda} I_\alpha$ such that $x_n \to x$. Then $\varphi(x_n) \to \varphi(x)$, and as $(\varphi(x_n))_{n\in\mathbb{N}}$ is a sequence in $\bigcup_{\alpha\in\Lambda} \mathcal{I}(\varphi)(I_\alpha)$, we find that $\varphi(x) \in \overline{\bigcup_{\alpha\in\Lambda} \mathcal{I}(\varphi)(I_\alpha)}$. In particular, $\mathcal{I}(\varphi)(I) \subseteq \overline{\bigcup_{\alpha\in\Lambda} \mathcal{I}(\varphi(I_\alpha))}$ proving the needed inclusion, and we conclude that $\mathcal{I}(\varphi)$ preserves increasing suprema.

We want to generalise this result to *arbitrary* suprema. Let $S \subseteq \mathcal{I}(A)$ be any subset. If $S = \emptyset$, then $\sup S = 0$ and, hence,

$$\mathcal{I}(\varphi)(\sup S) = \mathcal{I}(\varphi)(0) = 0 = \sup \mathcal{I}(\varphi)(S).$$

If S is non-empty, then consider the set

$$T = \left\{ \sum_{I\in S'} I \,|\, S' \subseteq S \text{ is a finite subset} \right\}.$$

It is clear that T is an upwards directed set, and that we can realise it as an increasing net indexed by the finite subsets of S. Moreover, S and T share the same supremum, and therefore

$$\mathcal{I}(\varphi)(\sup S) = \mathcal{I}(\varphi)(\sup T) = \sup \mathcal{I}(\varphi)(T) = \sup \mathcal{I}(\varphi)(S).$$

We conclude that $\mathcal{I}(\varphi)$ is a *Cu*-morphism. □

When we defined the morphisms in **Cu**, the reader may have noticed that they only need preserve suprema of increasing *sequences*, but one can show that a morphism $\varphi\colon S \to T$ between **Cu**-objects S and T necessarily preserves the supremum of any increasing net. Indeed, this is guaranteed by condition ($\mathcal{O}$2), since if $x \in S$ is the supremum of an increasing net $(x_\alpha)_{\alpha\in\Lambda}$, then we know there exists some sequence $y_1 \Subset y_2 \Subset \cdots$ such that $x = \sup_{n\in\mathbb{N}} y_n$, and this implies that

$$\varphi(x) = \varphi(\sup_{\alpha\in\Lambda} x_\alpha) = \varphi(\sup_{n\in\mathbb{N}} y_n) = \sup_{n\in\mathbb{N}} \varphi(y_n) = \sup_{\alpha\in\Lambda} \varphi(x_\alpha).$$

Moreover, if a map $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ between ideal lattices preserves arbitrary suprema, then the zero element is preserved as $\sup \emptyset = 0$, and addition is preserved since the supremum of a finite collection of ideals is the sum. All in all, this implies that, whenever the C^*-algebras in question are separable, a map between ideal lattices is a *Cu*-morphism in the sense of Definition 5.10 precisely when it is one in the sense of the category **Cu**.

Hopefully, the above discussion on *Cu*-morphisms gave the reader some insight into the nature of *Cu*-morphisms and the terminology itself. One important fact we need is the following lemma.

Lemma 5.18. *If A is a separable C^*-algebra with the property that* $\mathrm{Prim}(A)$ *has no non-empty, compact, open subsets, then there exists a Cu-morphism $\Phi\colon \mathcal{I}(A) \to [0,1]$ with $\Phi^{-1}(\{0\}) = \{0\}$.*

The idea behind the proof is simple: Prove the existence of a continuous[6] and order-preserving map $\Psi\colon [0,1] \to \mathcal{I}(A)$ with $\Psi(0) = 0$ and $\Psi(1) = A$ and invoke a classification result, [27, Theorem VI-3.4], to prove the existence of an order-preserving map $\Phi\colon \mathcal{I}(A) \to [0,1]$ satisfying that, for any $(I,t) \in \mathcal{I}(A) \times [0,1]$, $I \subseteq \Psi(t)$ if and only if $\Phi(I) \le t$; such a Φ is called a *lower adjoint* of Ψ, and it can be shown that Φ preserves suprema and compact containment. In particular, Φ is a *Cu*-morphism $\mathcal{I}(A) \to [0,1]$, and since $\Psi(0) = 0$, it is immediate from Φ being a lower adjoint of Ψ that $\Phi^{-1}(\{0\}) = \{0\}$. For the full details, we refer the reader to [26, Lemma 3].

5.2 Constructing a separable, nuclear, $\mathcal{O}_\infty$-stable ASH-algebra $\mathcal{A}_{[0,1]}$

We now construct an AF-embeddable C^*-algebra $\mathcal{A}_{[0,1]}$, whose ideal lattice is isomorphic to the interval $[0,1]$ with the usual ordering. Our exposition follows that of Rørdam [60], but the construction is originally by Mortensen [46]. To be specific, what Mortensen constructed was the C^*-algebra

[6] Of course, continuity only makes sense if the ideal lattice $\mathcal{I}(A)$ is equipped with some topology. There exists such a topology known as the Lawson topology — we shall not study this here, but the details may be found in [27, Section III-1].

$\mathcal{A}_{[0,1]} \otimes \mathcal{O}_2$, but this is isomorphic to $\mathcal{A}_{[0,1]}$ by $\mathcal{O}_2$-stability, which was proved by Kirchberg and Rørdam in [39, Proposition 6.1]. In understanding the C^*-algebra $\mathcal{A}_{[0,1]}$, we generalise the notion of AF-algebras by allowing the C^*-algebras in the inductive limit to be direct sums of C^*-subalgebras of matrix algebras over arbitrary Abelian C^*-algebras. We follow the definition of Gabe and Rørdam, see [26, p. 2] and [60, p. 9].

Definition 5.19. A C^*-algebra A is called an *approximately subhomogeneous algebra*, denoted ASH-algebra, if it is the inductive limit of a sequence of C^*-algebras, which are finite direct sums of C^*-subalgebras of matrix algebras over separable, Abelian C^*-algebras.

In other words, ASH-algebras are inductive limits, where the inductive sequence can be constructed from certain building blocks, namely C^*-subalgebras of matrices over separable, Abelian C^*-algebras. Observe that all AF-algebras are trivially ASH-algebras, since we can just take the underlying Hausdorff space in the building blocks to be one-point spaces. Note that while it is true that $\mathcal{A}_{[0,1]}$ is an ASH-algebra, there is nothing subhomogeneous about the construction below, and Rørdam called it an AH_0-algebra in [60].

While ASH-algebras can have more exotic behaviour than AF-algebras, they are all AF-embeddable.

Proposition 5.20. *Every ASH-algebra is AF-embeddable.*

Proof. One can show, see [60, Proposition 4.1], that if A is a C^*-subalgebra of $M_n(C_0(X))$ for some locally compact Hausdorff space X, then its bidual A^{**} can be realised as $\bigoplus_{k=1}^n M_k(N_k)$ for some Abelian von Neumann-algebras $N_1, \dots, N_n$. We aim to prove that given an Abelian von Neumann-algebra N and any separable C^*-subalgebra A of $M_k(N)$, then there exists an AF-algebra $C \subseteq M_k(N)$ containing A.

In order to verify this claim, let A_0 be the C^*-algebra generated by A and the matrix units of $M_k(\mathbb{C})$, then $A_0 = M_k(B_0)$ for some separable C^*-subalgebra B_0 of N. Note that B_0 is separable. Using the fact that all von Neumann-algebras have real rank zero, see [8, Proposition 1.3], and that the property of having real rank zero is separably inheritable[7], we find that there exists a separable, unital C^*-subalgebra B of N containing B_0 with real rank zero. Since N is Abelian, so is B, and as any Abelian C^*-algebra with real rank zero is an AF-algebra, we find that B, and, hence, $C = M_k(B)$, is an AF-algebra. We conclude that C is an AF-algebra, which contains A and which is a C^*-subalgebra of $M_k(N)$, and this proves the claim.

Let A be any ASH-algebra, and realise A as the inductive limit of a sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \longrightarrow A$$

where each A_n is a finite sum of C^*-subalgebras of matrix algebras of Abelian C^*-algebras. By taking the bidual, we get an inductive sequence of von Neumann-algebras

$$A_1^{**} \xrightarrow{\varphi_1^{**}} A_2^{**} \xrightarrow{\varphi_2^{**}} A_3^{**} \xrightarrow{\varphi_3^{**}} \cdots .$$

By applying the previously described construction on each of the von Neumann-algebras in the inductive sequence, we can find an AF-algebra B_1 with $A_1 \subseteq B_1 \subseteq A_1^{**}$. Using the exact same construction, we can find another AF-algebra B_2 containing the C^*-algebra generated by A_1 and $\varphi_1^{**}(B_1)$, and which is contained in A_2^{**}. Continuing in this fashion, we get an inductive sequence of AF-algebras

$$B_1 \xrightarrow{\varphi_1^{**}} B_2 \xrightarrow{\varphi_2^{**}} B_3 \xrightarrow{\varphi_3^{**}} \cdots \longrightarrow B$$

where the inductive limit B is again an AF-algebra, since the class of AF-algebras is closed under taking sequential inductive limits. Since A_n is a C^*-subalgebra of B_n for each $n \in \mathbb{N}$, we conclude that A can be realised as a C^*-subalgebra of B, and this proves that all ASH-algebras are AF-embeddable. □

[7] This fact is proved similarly to proving that stable rank one is a separably inheritable property of C^*-algebras, cf. Proposition 4.10: Using separability, we pick a countable dense set of the self-adjoints and add the inverses via real rank zero of the larger C^*-algebra. Continuing in this manner provides an inductive sequence, whose limit is a separable C^*-subalgebra of the initial C^*-algebra with real rank zero.

We now follow the exposition in [60, Section 2] and construct a specific family of ASH-algebras. Let $T \subseteq \mathbb{R}$ be any non-empty compact subset and define $t_{\min} = \min T$ and $t_{\max} = \max T$. Put $T_0 = T \setminus \{t_{\max}\}$ and find a sequence $(t_n)_{n\in\mathbb{N}}$ in T_0 such that the tail $(t_n)_{n=k}^{\infty}$ is dense in T_0 for each $k \in \mathbb{N}$. Consider for each $n \in \mathbb{N}$ the C^*-algebra $A_n = C_0(T_0, M_{2^n}(\mathbb{C}))$, and consider also the *-homomorphisms $\varphi_n \colon A_n \to A_{n+1}$ given by

$$\varphi_n(f) = \begin{pmatrix} f & 0 \\ 0 & f \circ \chi_{t_n} \end{pmatrix}, \qquad f \in A_n,$$

where the map $\chi_s \colon T \to T$ is given by $\chi_s(t) = \max\{s,t\}$ for each $s, t \in T$. The C^*-algebra $\mathcal{A}_T$ is then defined to be the inductive limit of the inductive sequence $(A_n, \{\varphi_n\})$. Note that the C^*-algebra $\mathcal{A}_T$ may be dependent on the choice of sequence $(t_n)_{n\in\mathbb{N}}$. The C^*-algebra $\mathcal{A}_{[0,1]}$, which is of greatest interest for our purposes, can be shown to be independent of the sequence $(t_n)_{n\in\mathbb{N}}$, see [39]. However, none of our results depend on the uniqueness of $\mathcal{A}_{[0,1]}$.

It is clear that $\mathcal{A}_T$ is an ASH-algebra for each compact subset $T \subseteq \mathbb{R}$. We shall analyse these in greater detail with a focus on the specific example $\mathcal{A}_{[0,1]}$. We start by analysing the ideal structure of $\mathcal{A}_T$. For each $t \in T$ and $n \in \mathbb{N}$, define the closed two-sided ideal $I_t^{(n)}$ of A_n by

$$I_t^{(n)} = \{f \in A_n \mid f(s) = 0 \text{ for all } s \geq t\} \cong C_0(T_0 \cap [t_{\min}, t), M_{2^n}(\mathbb{C})).$$

It is immediate that $I_{t_{\min}}^{(n)} = 0$ and $I_{t_{\max}}^{(n)} = A_n$ for all $n \in \mathbb{N}$, and if $t \leq s$, then $I_t^{(n)} \subseteq I_s^{(n)}$. Define

$$I_t = \overline{\bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(I_t^{(n)})}$$

for each $t \in T$, where we use the notation that $\varphi_{\infty,n} \colon A_n \to \mathcal{A}_T$ are the boundary maps in the inductive limit. If we denote by $\varphi_{m,n} \colon A_n \to A_m$ the compositions of the connecting maps, then one easily realises that

$$\varphi_{n+k,n}(f) = \begin{pmatrix} f \circ \chi_{\max F_1} & 0 & \cdots & 0 \\ 0 & f \circ \chi_{\max F_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f \circ \chi_{\max F_{2^k}} \end{pmatrix} \tag{5.1}$$

for a certain enumeration $F_1, F_2, \ldots, F_{2^k}$ of the set $\{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$.

Observe that each I_t is a closed two-sided ideal in $\mathcal{A}_T$ satisfying $I_t^{(n)} = \varphi_{\infty,n}^{-1}(I_t)$ for all $n \in \mathbb{N}$, that $I_t \subseteq I_s$ whenever $s \leq t$, and that $I_{t_{\min}} = 0$ and $I_{t_{\max}} = \mathcal{A}_T$.

Proposition 5.21. *Let $T \subseteq \mathbb{R}$ be a non-empty compact set. Each closed two-sided ideal in $\mathcal{A}_T$ is of the form I_t for some $t \in T$. In particular there is an order-preserving isomorphism $\mathcal{I}(\mathcal{A}_T) \cong T$.*

Proof. Suppose that we have shown that each closed two-sided ideal in $\mathcal{A}_T$ is of the form I_t, then we immediately get a bijection from $\mathcal{I}(\mathcal{A}_T)$ to T. Since $I_t \subseteq I_s$ whenever $t \leq s$, we find that this bijection is order-preserving. Hence if we show the first part of the proposition, we are done.

Let us therefore show that the collection $\{I_t\}_{t\in T}$ exhausts the ideal lattice of $\mathcal{A}_T$. Suppose I is a closed two-sided ideal in $\mathcal{A}_T$. For each $n \in \mathbb{N}$, put $I^{(n)} = \varphi_{\infty,n}^{-1}(I)$ and note that $I^{(n)}$ is a closed two-sided ideal in A_n. Define also the set

$$T_n = \bigcap_{f \in I^{(n)}} f^{-1}(\{0\})$$

for all $n \in \mathbb{N}$. Then T_n is a closed subset of T for each $n \in \mathbb{N}$, since each $f \in I^{(n)}$ is continuous and being closed is preserved by arbitrary intersections. One easily verifies that $I^{(n)} = C_0(T \setminus T_n, M_{2^n}(\mathbb{C}))$ for all $n \in \mathbb{N}$. If we were to show that there exists $t \in T$ such that $T_n = T \cap [t, t_{\max}]$ for all $n \in \mathbb{N}$, then $I^{(n)} = C_0(T_0 \cap [t_{\min}, t), M_{2^n}(\mathbb{C})) = I_t^{(n)}$ and, consequently, $I = I_t$.

Fix an arbitrary natural number n. Define for each $k \in \mathbb{N}$ the set

$$X_{n,k} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}.$$

We claim that we have the identity

$$T_n = \bigcup_{F \subseteq X_{n,k}} \chi_{\max F}(T_{n+k}) \tag{5.2}$$

for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be arbitrary and let $T'_{n,k}$ denote the right-hand side of the claimed equation. In order to prove the claim, it suffices to show that a function $f \in C_0(T_0, M_{2^n}(\mathbb{C}))$ vanishes on T_n if and only if f vanishes on $T'_{n,k}$. However, we have the following bi-implications for any $f \in C_0(T_0, M_{2^n}(\mathbb{C}))$:

$$\begin{aligned} f|_{T_n} = 0 \qquad &\iff \qquad f \in I^{(n)} \\ &\iff \qquad \varphi_{n+k,n}(f) \in I^{(n+k)} \\ &\iff \qquad \varphi_{n+k,n}(f)(s) = 0 \text{ for all } s \in T_{n+k} \\ &\iff \qquad f(\chi_{\max F}(s)) = 0 \text{ for all } s \in T_{n+k} \text{ and } F \subseteq X_{n,k} \\ &\iff \qquad f|_{T'_{n,k}} = 0. \end{aligned}$$

In the second-to-last equivalence, we have used (5.1). In particular, we see that $T_n = T_{n+1} \cup \chi_{t_n}(T_{n+1})$, which implies that $\min T_n \leq \min T_{n+1}$. Moreover, we have the inequality

$$\min \chi_{t_n}(T_{n+1}) = \max\{t_n, \min T_{n+1}\} \geq \min T_{n+1}.$$

We have hence proved that $\min T_n = \min T_{n+1}$. Since $n \in \mathbb{N}$ was arbitrary, this proves that the collection $\{T_m\}_{m \in \mathbb{N}}$ has a common minimum; call this element t. We claim that this t is the one we mentioned in the beginning of the proof, i.e., that $T_m = T \cap [t, t_{\max}]$ for all $m \in \mathbb{N}$. Noting that $t \in T_{n+k}$ for all $k \in \mathbb{N}$ and using (5.2), we see that

$$T_n \ni \chi_{t_{n+k}}(t) = \max\{t_{n+k}, t\} = t_{n+k}$$

for each natural number k. This implies that T_n contains the set $\{t_{n+k}\}_{k \in \mathbb{N}} \cap [t, t_{\max}]$. Since the sequence $(t_{n+k})_{k \in \mathbb{N}}$ is dense in T by assumption, and as T_n is closed, we must have the inclusion $T \cap [t, t_{\max}] \subseteq T_n$. However, the other inclusion is trivial, and hence we have the equality of set $T_n = T \cap [t, t_{\max}]$. Since $n \in \mathbb{N}$ was an arbitrary fixed natural number, we are done. □

Since there exists an order-preserving isomorphism from $\mathcal{I}(A)$ to the open subsets of $\mathrm{Prim}(A)$ for each C^*-algebra A, we can use Proposition 5.21 to determine the topological structure of $\mathrm{Prim}(\mathcal{A}_T)$ for all non-empty compact $T \subseteq \mathbb{R}$. In particular, we see that $\mathrm{Prim}(\mathcal{A}_{[0,1]})$ contains no non-empty, compact, open subsets, which plays a crucial role in Gabe's result.

We now show that $\mathcal{A}_{[0,1]}$ is traceless. As pointed out earlier, this notion is a bit more subtle than one might expect. In order to not worry about so-called quasitraces, which are certain generalisations of traces, we only define them for exact C^*-algebras; for a general definition we refer to [38, Definition 4.2].

Definition 5.22. Let A be an exact C^*-algebra. We say that A is *traceless* if A admits no non-zero lower semi-continuous trace.

It is clear that a C^*-algebra cannot have any tracial states if it is traceless. However, do note that being traceless is inherently stronger than just having no tracial states; for example, whenever H is infinite-dimensional, $\mathbb{B}(H)$ has no tracial state, however it is not traceless, since the canonical trace is, as stated before, a trace in the sense of Definition 1.1.

If I is any algebraic ideal of a C^*-algebra B, then I contains $\mathrm{Ped}(\overline{I})$, hence I contains all elements of the form $(x - \varepsilon)_+$ for positive $x \in I$ and $\varepsilon > 0$ by Proposition 1.16. We shall use this in the following proposition, which gives a taste as to the possible exoticness of ASH-algebras; note that it is not used in proving Gabe's resolution of the Blackadar-Kirchberg conjectures. Observe that AF-algebras are never traceless by the following argument: If $p \in \mathcal{P}(A)$ is a non-trivial projection in an AF-algebra, which exists since AF-algebras have an abundance of projections, then the hereditary C^*-subalgebra pAp of A is unital and admits a tracial state — the latter fact can be concluded by, for example, using Theorem 4.27.

Proposition 5.23. *The ASH-algebra $\mathcal{A}_{[0,1]}$ is traceless.*

Proof. Since Abelian and finite-dimensional C^*-algebras are nuclear, and as tensor products and inductive limits preserve nuclearity, we see that $\mathcal{A}_{[0,1]}$ is nuclear and, hence, exact. Suppose that $\mathcal{A}_{[0,1]}$ has an algebraic ideal I admitting a non-zero lower semi-continuous trace τ. We wish to reach a contradiction.

Since I is an algebraic ideal of $\mathcal{A}_{[0,1]}$, its completion is a closed two-sided ideal in $\mathcal{A}_{[0,1]}$, and hence by Proposition 5.21 there exists $t \in [0,1]$ such that $\overline{I} = I_t$. Note that as τ is non-zero, we must have that $I_t \neq \{0\}$ and thus $t > 0$.

For each $n \in \mathbb{N}$, we consider $I_t^{(n)} = \varphi_{\infty,n}^{-1}(I_t)$ and $I^{(n)} = \varphi_{\infty,n}^{-1}(I)$. Whenever $x \in I_t$ is positive and $\varepsilon > 0$, it follows from Proposition 1.16 and the fact that $\operatorname{Ped}(I_t) \subseteq I$ that

$$\varphi_{\infty,n}((x-\varepsilon)_+) = (\varphi_{\infty,n}(x) - \varepsilon)_+ \in I$$

and, hence, $(x-\varepsilon)_+ \in I^{(n)}$ for all $n \in \mathbb{N}$. This proves that, for each $n \in \mathbb{N}$, the positive elements in $I^{(n)}$ are dense in the positive cone of $I_t^{(n)}$ and, consequently, that $\overline{I^{(n)}} = I_t^{(n)}$. In particular, $I^{(n)}$ contains the Pedersen ideal $C_c([0,t), M_{2^n}(\mathbb{C}))$ of $I_t^{(n)}$.

Let $\tau_n = \tau \circ \varphi_{\infty,n}$ be a trace with domain $I^{(n)}$. We claim that we can realise τ_n as

$$\tau_n(f) = 2^n \int_0^t \operatorname{Tr}_{2^n}(f(s))\,\mathrm{d}\mu_n(s) \tag{5.3}$$

for all $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$, where μ_n is some Radon measure on $[0,t)$. By Riesz' representation theorem, we can realise τ_n on the C^*-subalgebra $C_c([0,t),\mathbb{C})$ by

$$\tau_n(f) = 2^n \int_0^t f(s)\,\mathrm{d}\mu_n(s), \qquad f \in C_c([0,t),\mathbb{C})$$

for some Radon measure μ_n on $[0,t)$. Note that this shows that the formulation in (5.3) holds on the C^*-subalgebra $C_c([0,t),\mathbb{C})$. Consider for each $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$ the set

$$U_f = \overline{\operatorname{conv}}\{ufu^* \mid u \in \mathcal{U}(C([0,t], M_{2^n}(\mathbb{C})))\}.$$

Suppose that $f \geq 0$, let $u \in C([0,t], M_{2^n}(\mathbb{C}))$ be unitary, and consider $x = uf^{1/2}$. Then $x \in C_c([0,t), M_{2^n}(\mathbb{C}))$ and $x^*x = f$ and $xx^* = ufu^*$, and hence

$$\tau_n(f) = \tau_n(x^*x) = \tau_n(xx^*) = \tau_n(ufu^*).$$

Thus, by continuity, τ_n is constant on each U_f whenever $f \geq 0$, which can be extended to all $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$. If we show that the intersection $U_f \cap C_c([0,t),\mathbb{C})$ is non-empty for all $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$, we have shown that (5.3) holds. In fact, we shall see that the conditional expectation $E\colon C_c([0,t), M_{2^n}(\mathbb{C})) \to C_c([0,t),\mathbb{C})$ by $E(f)(s) = \operatorname{Tr}_{2^n}(f(s))$ for $s \in [0,t)$ satisfies that $E(f) \in U_f \cap C_c([0,t),\mathbb{C})$ for any $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$.

We shall first consider some matrices known as Voiculescu's matrices. Let $\omega = \mathrm{e}^{2\pi i/2^n}$ be a root of unity of degree 2^n and consider the unitary matrices $u, v \in M_{2^n}(\mathbb{C})$ given by $ue_j = \omega^{j-1}e_j$ and $ve_j = e_{j-1}$ for $j = 1, \ldots, 2^n$, where e_j is the jth standard basis vector. It is easily verified that $vu = \omega uv$. Define the maps $\Phi, \Psi\colon M_{2^n}(\mathbb{C}) \to M_{2^n}(\mathbb{C})$ by

$$\Phi(x) = \frac{1}{n}\sum_{j=1}^{n} u^j x u^{-j}, \qquad \text{and} \qquad \Psi(x) = \frac{1}{n}\sum_{j=1}^{n} v^j x v^{-j}$$

for $x \in M_{2^n}(\mathbb{C})$. A few calculations show that $\Phi(x)$ is the diagonal matrix whose elements are the diagonal elements in x. Moreover, one can verify that $\Psi(\Phi(x)) = \operatorname{Tr}_{2^n}(x)$, and that

$$\Psi(\Phi(x)) = \frac{1}{n^2}\sum_{j,k=1}^{n} v^j u^k x u^{-k} v^{-j}.$$

In particular, if $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$, then we see that

$$E(f)(s) = \mathrm{Tr}_{2^n}(f(s)) = \frac{1}{n^2} \sum_{j,k=1}^{n} v^j u^k f(s) u^{-k} v^{-j}$$

which proves that $E(f) \in U_f \cap C_c([0,t), \mathbb{C})$. This proves that (5.3) holds.

Having established the previous result, we now aim to show that each Radon measure μ_n is the zero measure. In order to do this, we wish to determine the relations between the Radon measures μ_n. Note first of all that $\tau_n = \tau_{n+k} \circ \varphi_{n+k,n}$ for all $n, k \in \mathbb{N}$. Define, as we did in the proof of Proposition 5.21, for each $n, k \in \mathbb{N}$ the set $X_{n,k} = \{t_n, t_{n+1}, \ldots, t_{n+k-1}\}$. Fix $n \in \mathbb{N}$, then, for any $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$, we have

$$\begin{aligned} 2^n \int_0^t \mathrm{Tr}_{2^n}(f(s))\, \mathrm{d}\mu_n(s) &= \tau_n(f) = \tau_{n+k}(\varphi_{n+k,n}(f)) \\ &= 2^n \int_0^t \mathrm{Tr}_{2^n}(\varphi_{n+k,n}(f(s))\, \mathrm{d}\mu_{n+k,n}(s) \\ &= \sum_{F \subseteq X_{n,k}} 2^n \int_0^t \mathrm{Tr}_{2^n}(f \circ \chi_{\max F}(s))\, \mathrm{d}\mu_{n+k,n}(s) \\ &= \sum_{F \subseteq X_{n,k}} 2^n \int_0^t \mathrm{Tr}_{2^n}(f(s))\, \mathrm{d}(\mu_{n+k,n} \circ \chi_{\max F}^{-1})(s). \end{aligned}$$

Since $f \in C_c([0,t), M_{2^n}(\mathbb{C}))$ was arbitrary, we find that

$$\mu_n = \sum_{F \subseteq X_{n,k}} \mu_{n+k,n} \circ \chi_{\max F}^{-1}. \tag{5.4}$$

Having determined the relations between the different Radon measures in question, we now show that $\mu_n = 0$ for all $n \in \mathbb{N}$. Fix some $s \in (0,t)$, and find r such that $0 < s < r < t$. Consider for each $k \in \mathbb{N}$ the two sets

$$Y_{n,k} = X_{n,k} \cap [0,s] \qquad \text{and} \qquad Z_{n,k} = X_{n,k} \cap [0,r].$$

Using (5.4) along with the fact that

$$\chi_u^{-1}([0,v]) = \begin{cases} \emptyset & \text{for } v < u \\ [0,v] & \text{for } v \geq u \end{cases} \tag{5.5}$$

for each $u, v \in [0,1]$, we find that

$$\mu_n([0,r]) = \sum_{F \subseteq X_{n,k}} \mu_{n+k} \circ \chi_{\max F}^{-1}([0,r]) = \sum_{F \subseteq Z_{n,k}} \mu_{n+k}([0,r]) = 2^{|Z_{n,k}|} \mu_{n+k}([0,r])$$

for each $k \in \mathbb{N}$. Similarly, one verifies that $\mu_n([0,s]) = 2^{|Y_{n,k}|}\mu_{n+k}([0,s])$ for all $k \in \mathbb{N}$. Thus, as $s < r$, we see that

$$\mu_n([0,s]) = 2^{|Y_{n,k}|}\mu_{n+k}([0,s]) \leq 2^{|Y_{n,k}|}\mu_{n+k}([0,r]) = 2^{-(|Z_{n,k}|-|Y_{n,k}|)}\mu_n([0,r]).$$

Since $\bigcup_{k\in\mathbb{N}} X_{n,k} = \{t_{n+k}\}_{k\in\mathbb{N}}$ is dense in $[0,1]$ by construction, we have that

$$\lim_{k\to\infty}(|Z_{n,k}| - |Y_{n,k}|) = \lim_{k\to\infty} |X_{n,k} \cap (s,r]| = \infty.$$

As Radon measures have finite measure on compact sets, such that $\mu_n([0,r]) < \infty$, we necessarily conclude that $\mu_n([0,s]) = 0$, which implies that $\mu_n = 0$. In particular, each trace τ_n must be the zero trace by (5.3). We claim that this contradicts the assumption that τ is non-zero.

Assume for contradiction that there exists a positive element $f \in I$ such that $\tau(f) > 0$. Using lower semi-continuity of τ, we find that there exists $\varepsilon > 0$ such that $\tau((f-\varepsilon)_+) > 0$. Since each $I^{(n)}$

is dense in $I_t^{(n)}$, and as I_t is the closure of $\bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(I_t^{(n)})$, we can find a sufficiently large $n \in \mathbb{N}$ and an element $g \in I^{(n)}$ with $\|\varphi_{\infty,n}(g) - f\| < \varepsilon$. Using [38, Lemma 2.2], we can find an element $d \in \mathcal{A}_{[0,1]}$ with $\|d\| \leq 1$ satisfying that $d^*\varphi_{\infty,n}(g)d = (f-\varepsilon)_+$. Putting $x = \varphi_{\infty,n}(g)^{1/2}d$, the tracial property of τ shows that

$$\begin{aligned}\tau_n(g) &= \tau(\varphi_{\infty,n}(g)) \geq \tau(\varphi_{\infty,n}(g)^{1/2}dd^*\varphi_{\infty,n}(g)^{1/2}) = \tau(xx^*)\\ &= \tau(x^*x) = \tau(d^*\varphi_{\infty,n}(g)d) = \tau((f-\varepsilon)_+) > 0.\end{aligned}$$

This contradicts that τ_n was found to be the zero map. We conclude that $\mathcal{A}_{[0,1]}$ is traceless. □

Another interesting and important property of $\mathcal{A}_{[0,1]}$ is the fact that it is $\mathcal{O}_\infty$-stable, meaning that $\mathcal{A}_{[0,1]} \otimes \mathcal{O}_\infty \cong \mathcal{A}_{[0,1]}$; we shall not prove this here, instead we refer the reader to [60, Corollary 5.3]. We conclude our findings in the following theorem.

Theorem 5.24 (Rørdam, 2004)**.** *The C^*-algebra $\mathcal{A}_{[0,1]}$ is a separable, traceless, nuclear, $\mathcal{O}_\infty$-stable ASH-algebra with $\mathcal{I}(A) \cong [0,1]$.*

In fact, $\mathcal{A}_{[0,1]}$ is $\mathcal{O}_2$-stable by [39, Proposition 6.1], which generalises $\mathcal{O}_\infty$-stability. For our purposes, however, the latter will suffice.

5.3 Resolving the Blackadar-Kirchberg conjectures for traceless, exact C^*-algebras

Having constructed Rørdam's ASH-algebra $\mathcal{A}_{[0,1]}$ and examined some of its properties, we now turn our attention to actually proving that the Blackadar-Kirchberg conjectures hold true for traceless C^*-algebras. More specifically, we shall prove that for a separable, exact C^*-algebra A there exists an embedding into the AF-embeddable $\mathcal{A}_{[0,1]}$ if and only if $\mathrm{Prim}(A)$ has no non-empty, compact, open subsets. In fact, we shall prove that this property of $\mathrm{Prim}(A)$ completely characterises both AF-embeddability, quasidiagonality and stably finiteness for exact, traceless C^*-algebras.

Recall from Proposition 5.17 that if $\varphi\colon A \to B$ is a *-homomorphism, then the induced map $\mathcal{I}(\varphi)\colon \mathcal{I}(A) \to \mathcal{I}(B)$ given by

$$\mathcal{I}(\varphi)(I) = \overline{B\varphi(I)B}, \qquad I \in \mathcal{I}(A)$$

is a *Cu*-morphism. One can ask when a *Cu*-morphism between ideal lattices can be lifted to a *-homomorphism between the corresponding C^*-algebras, and the following proposition, the proof of which can be found in [25, Theorem 6.1], gives sufficient conditions for this to be the case.

Theorem 5.25 (Gabe, 2018)**.** *Suppose that A is a separable and exact C^*-algebra, and that B is a separable, nuclear and $\mathcal{O}_\infty$-stable C^*-algebra. Let $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ be an arbitrary Cu-morphism. Then there exists a *-homomorphism $\varphi\colon A \to B$ such that $\Phi = \mathcal{I}(\varphi)$.*

The theorem originates from Gabe's proof of Kirchberg's classification result that if A, B are separable, nuclear, $\mathcal{O}_2$-stable C^*-algebras, which are either both stable or both unital, then A and B are isomorphic if and only if their primitive ideal spaces are homeomorphic, which holds if and only if their ideal lattices are order isomorphic. We shall use Theorem 5.25 in conjunction with Lemma 5.18 in order to prove Gabe's characterisation of embeddings into $\mathcal{A}_{[0,1]}$.

Theorem 5.26 (Gabe, 2018)**.** *Suppose A is a separable, exact C^*-algebra. Then there exists an embedding of A into $\mathcal{A}_{[0,1]}$ if and only if* $\mathrm{Prim}(A)$ *has no non-empty, compact, open subsets.*

Proof. Assume that $\varphi\colon A \to \mathcal{A}_{[0,1]}$ is an injective *-homomorphism, and suppose for contradiction that $\mathrm{Prim}(A)$ contains a non-empty, compact, open subset or, equivalently, that there exists a non-zero compact ideal I in A. Since $\mathcal{I}(\varphi)\colon \mathcal{I}(A) \to \mathcal{I}(\mathcal{A}_{[0,1]})$ is a *Cu*-morphism by Proposition 5.17, $\mathcal{I}(\varphi)(I)$ is a non-zero compact ideal in $\mathcal{A}_{[0,1]}$. Since $\mathcal{I}(\mathcal{A}_{[0,1]}) \cong [0,1]$ by Proposition 5.21, this would imply that there exists a non-zero compact element in the complete lattice $[0,1]$, which is clearly impossible. We hence conclude that $\mathrm{Prim}(A)$ has no non-empty, compact, open subsets.

Conversely, suppose that $\mathrm{Prim}(A)$ has no non-empty, compact, open subsets, then we claim there exists an embedding of A into $\mathcal{A}_{[0,1]}$. By Lemma 5.18 and as $\mathcal{I}(\mathcal{A}_{[0,1]}) \cong [0,1]$ by Proposition

5.21, there exists a *Cu*-morphism $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(\mathcal{A}_{[0,1]})$ such that $\Phi^{-1}(\{0\}) = \{0\}$. Since $\mathcal{A}_{[0,1]}$ is a separable, nuclear and $\mathcal{O}_\infty$-stable C^*-algebra, it follows from Theorem 5.25 that there exists a *-homomorphism $\varphi\colon A \to \mathcal{A}_{[0,1]}$ lifting Φ. We claim that φ is injective. Suppose that there exists a non-zero element $a \in A$ such that $\varphi(a) = 0$. Let I be the ideal generated by a, then $I \neq 0$ and

$$\mathcal{I}(\varphi)(I) = \overline{\mathcal{A}_{[0,1]}\varphi(I)\mathcal{A}_{[0,1]}} = 0,$$

which contradicts the fact that $\mathcal{I}(\varphi)^{-1}(\{0\}) = \{0\}$. We hence find that $\varphi\colon A \to \mathcal{A}_{[0,1]}$ is an injective *-homomorphism, which completes the proof. ☐

We can combine this theorem with Proposition 5.20 to get the following result, which is a generalisation of Theorem 2.45.

Corollary 5.27. *Let A be a separable, exact C^*-algebra such that* Prim(A) *has no non-empty, compact, open subsets. Then A is AF-embeddable. In particular, the cone and the suspension of any separable, exact C^*-algebra is AF-embeddable.*

Proof. Since A is a separable, exact C^*-algebra with the property that Prim(A) has no non-empty, compact, open subsets, we find that A is isomorphic to a C^*-subalgebra of $\mathcal{A}_{[0,1]}$ by Theorem 5.26. Since $\mathcal{A}_{[0,1]}$ is AF-embeddable by Proposition 5.20, we find that A is AF-embeddable.

Now suppose that A is any separable, exact C^*-algebra. Previously in this chapter, we have seen that for any locally compact Hausdorff space X, we have the isomorphism

$$\mathrm{Prim}(C_0(X, A)) \cong X \times \mathrm{Prim}(A).$$

In particular this holds for $X = (0, 1]$, i.e., we have the isomorphism $\mathrm{Prim}(CA) \cong (0, 1] \times \mathrm{Prim}(A)$, where CA denotes the cone over A. If Prim(CA) contained any non-empty, compact, open subset, then $(0, 1]$ would contain a non-zero compact element, which is impossible. Hence the cone CA over A is AF-embeddable by the above, and so is the suspension SA, since SA is a C^*-subalgebra of CA. ☐

Note that Corollary 5.27 generalises the result due to Ozawa that the cone and suspension over any separable, exact C^*-algebra are exact, cf. Theorem 2.45. We wish to use Corollary 5.27 to resolve the Blackadar-Kirchberg conjecture for the class of separable, exact, traceless C^*-algebras. More precisely, we desire to prove the following theorem.

Theorem 5.28 (Gabe, 2018)**.** *Let A be a separable, exact, traceless C^*-algebra. The following are equivalent:*

(i) *A is AF-embeddable;*

(ii) *A is quasidiagonal;*

(iii) *A is stably finite;*

(iv) Prim(A) *has no non-empty, compact, open subsets.*

The only direction we need to show is (iii)⇒(iv), as the implications (i)⇒(ii)⇒(iii) are immediate from Proposition 2.43 and Proposition 2.26, and the implication (iv)⇒(i) follows from Corollary 5.27. We shall end this section with a proof of this implication. Our first result is a lemma, which gives a sufficient condition for the minimal tensor product of two C^*-algebras to be stably finite; we shall use this to ensure that whenever A is stably finite and exact, then $A \otimes \mathcal{Z}$ is stably finite. First a definition, cf. [11, Definition 11.1.6].

Definition 5.29. A separable C^*-algebra is called MF, short for *matricial field*, if there exists a sequence of integers $(k_n)_{n\in\mathbb{N}}$ such that A is a C^*-subalgebra of $\ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})/c_0(M_{k_n}(\mathbb{C}), \mathbb{N})$.

Here we use the notation that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of C^*-algebras, then the C^*-algebra $c_0(A_n, \mathbb{N})$ is given by

$$c_0(A_n, \mathbb{N}) = \left\{ (a_n)_{n\in\mathbb{N}} \in \ell^\infty(A_n, \mathbb{N}) \mid \lim_{n\to\infty} \|a_n\| = 0 \right\}.$$

It is immediate that if A is MF, then so is its unitisation as well as any matrix algebra over A. Moreover, if A is separable and quasidiagonal, then A is MF: Quasidiagonality of A gives rise to a c.c.p. map $A \to \ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})$, namely the amplification of the c.c.p. maps witnessing quasidiagonality, and this map is easily seen to be an injective *-homomorphism when mapped to the quotient $\ell^\infty(M_{k_n}(\mathbb{C}), \mathbb{N})/c_0(M_{k_n}(\mathbb{C}), \mathbb{N})$.

Lemma 5.30. *If A is stably finite and exact, and B is MF, then the minimal tensor product $A \otimes B$ is stably finite.*

Proof. Suppose for contradiction that $A \otimes B$ is not stably finite, then there exist $n \in \mathbb{N}$ and a proper isometry $v \in M_n((A \otimes B)^\dagger)$. Observe that

$$M_n((A \otimes B)^\dagger) \subseteq M_n(A^\dagger \otimes B^\dagger) \cong A^\dagger \otimes M_n(B^\dagger).$$

Since B is MF, so is $M_n(B^\dagger)$, and hence there exists a sequence of integers $(k_m)_{m\in\mathbb{N}}$ such that $M_n(B^\dagger)$ is a C^*-subalgebra of $\ell^\infty(M_{k_m}(\mathbb{C}), \mathbb{N})/c_0(M_{k_m}(\mathbb{C}), \mathbb{N})$. Hence we get an embedding $A^\dagger \otimes M_n(B^\dagger) \subseteq A^\dagger \otimes (\ell^\infty(M_{k_m}(\mathbb{C}), \mathbb{N})/c_0(M_{k_m}(\mathbb{C}), \mathbb{N}))$. Observe that exactness of A implies exactness of $A^\dagger$ and grants the isomorphism

$$A^\dagger \otimes \frac{\ell^\infty(M_{k_m}(\mathbb{C}), \mathbb{N})}{c_0(M_{k_m}(\mathbb{C}), \mathbb{N})} \cong \frac{A^\dagger \otimes \ell^\infty(M_{k_m}(\mathbb{C}), \mathbb{N})}{A^\dagger \otimes c_0(M_{k_m}(\mathbb{C}), \mathbb{N})}.$$

Identifying this C^*-algebra as a C^*-subalgebra of $\ell^\infty(A^\dagger \otimes M_{k_m}(\mathbb{C}), \mathbb{N}))/c_0(A^\dagger \otimes M_{k_m}(\mathbb{C}), \mathbb{N})$, we can realise v as a proper isometry in this C^*-algebra. We claim that we can lift v to a sequence $(v_m)_{m\in\mathbb{N}}$ in $\ell^\infty(A^\dagger \otimes M_{k_m}(\mathbb{C}), \mathbb{N})$, where v_m is a proper isometry for sufficiently large m.

Find a sequence $(w_m)_{m\in\mathbb{N}} \in \ell^\infty(A^\dagger \otimes M_{k_m}(\mathbb{C}), \mathbb{N})$ lifting v. Since v is an isometry, it holds for all sufficiently large m that $\|w_m^* w_m - 1\| < 1$, and these elements are, hence, invertible. Define $v_m = w_m |w_m|^{-1}$ for these m and $v_m = 1$ elsewhere. Then each v_m is an isometry, and a standard continuous functional calculus trick shows that the sequence $(v_m)_{m\in\mathbb{N}}$ lifts v. Moreover, since

$$\lim_{m\to\infty} \|v_m v_m^* - 1\| = \|vv^* - 1\| > 0$$

there exists some $m \in \mathbb{N}$, where v_m is a proper isometry. However, since $v_m \in A^\dagger \otimes M_{k_n}(\mathbb{C}) \cong M_{k_n}(A^\dagger)$, this contradicts that A is stably finite. We conclude that $A \otimes B$ is a stably finite C^*-algebra. □

The proof of the implication (iii)⇒(iv) in Theorem 5.28 boils down to showing that if A is traceless, stably finite and exact, then $A \otimes \mathcal{Z}$ contains an infinite projection, which is not possible by Lemma 5.30. However, in order to understand the proof better, and how we are able to prove the existence of an infinite projection, we need to define properly infinite projections as well as purely infinite C^*-algebras.

Definition 5.31. A projection $p \in A$ is called *properly infinite* if there exists mutually orthogonal subprojections $p_1, p_2 \leq p$ such that $p \sim p_1 \sim p_2$.

Observe that finite C^*-algebras cannot have properly infinite elements. There is an equivalent characterisation of properly infinite projections, see [59, Proposition 1.1.2], which uses the notion of Cuntz subequivalence from Definition 5.11; a projection $p \in A$ is purely infinite if and only if $p \oplus p \precsim p \oplus 0$ holds in $M_2(A)$. This definition easily extends to arbitrary non-zero positive elements, see [37, Definition 3.2]. The following definition also uses the notion of Cuntz subequivalence:

Definition 5.32. A C^*-algebra A is said to be *purely infinite* if there exists no *-homomorphism $A \to \mathbb{C}$, and if $a \precsim b$ holds for $a, b \in A$ if and only if a belongs to the ideal generated by b.

There is a link between purely infinite C^*-algebras and properly infinite elements, which is the content of the next theorem, the proof of which can be found in [37, Theorem 4.16].

Theorem 5.33. *A C^*-algebra A is purely infinite if and only if all non-zero positive elements in A are properly infinite.*

Having the terminology sorted out, we are able to prove Theorem 5.28.

Proof of Theorem 5.28. We have already established that we only need to prove the implication (iii)⇒(iv). Assume that A is stably finite, then we wish to show that $\mathrm{Prim}(A)$ has no non-empty, compact, open subsets. Consider the C^*-algebra $A \otimes \mathcal{Z}$, which is stably finite by Lemma 5.30, since A is exact and stably finite, and $\mathcal{Z}$ is MF[8]. As $\mathcal{Z}$ is nuclear and simple, we get the isomorphisms

$$\mathrm{Prim}(A \otimes \mathcal{Z}) \cong \mathrm{Prim}(A) \times \mathrm{Prim}(\mathcal{Z}) \cong \mathrm{Prim}(A).$$

[8]One way of realising this is to observe that $\mathcal{Z}$ is a separable, nuclear C^*-algebra satisfying the UCT and admitting a faithful tracial state by Theorem 4.37, and hence the Tikuisis-White-Winter theorem in conjunction with Corollary 2.53 provides quasidiagonality of $\mathcal{Z}$.

Assume for the sake of reaching a contradiction that $\mathrm{Prim}(A)$ has a non-empty, compact, open subset, and let I denote the corresponding non-zero, compact ideal in $A \otimes \mathcal{Z}$. Since $\mathcal{Z}$ is, in itself, $\mathcal{Z}$-stable, see Theorem 4.37, we find that $A \otimes \mathcal{Z}$ is $\mathcal{Z}$-stable. Since exactness is preserved by minimal tensor products, $A \otimes \mathcal{Z}$ is exact. Moreover, it is a traceless C^*-algebra, since A is traceless. Using [61, Corollary 5.1], this implies that $A \otimes \mathcal{Z}$ is purely infinite in the sense of Definition 5.32, which by [52, Proposition 2.7] provides the existence of a projection $p \in A \otimes \mathcal{Z}$ generating I. Observe that p is a non-zero projection since I is non-zero, and hence Theorem 5.33 implies that p is properly infinite. However, we know that $A \otimes \mathcal{Z}$ is stably finite, so in particular it cannot have infinite projections, and we thus reach a contradiction. This proves that $\mathrm{Prim}(A)$ does not have any non-empty, compact, open subsets, which completes the proof. □

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Appendices

A Real rank zero and stable rank one

For any classification purposes, one needs some regularity conditions on the C^*-algebras in question in order to ensure that the C^*-algebras do not behave too exotically. This appendix is meant as a quick introduction and overview of two important regularity properties of C^*-algebras, namely real rank zero and stable rank one. We shall not bother with any proofs and only present the most important results. The main references for this appendix are [8, 57]. Denote by A_{sa} the self-adjoint elements on A.

Definition A.1. Let A be a unital C^*-algebra. We say that

(i) A has *real rank zero* if the set of self-adjoint invertible elements in A is dense in A_{sa};

(ii) A has *stable rank one* if the collection of invertible elements in A is dense in A.

If A is a non-unital C^*-algebra, we say that A has the above properties if the unitisation $A^\dagger$ has the respective property.

While the two definitions look very similar, neither implies the other. Let us also mention that there is a more general notion of both the real rank and the stable rank; we shall not bother studying these and only consider our special cases. In order to gain some intuition, however, on these notions, we mention the following proposition due to Brown and Pedersen, see [8, Proposition 1.1].

Proposition A.2 (Brown-Pedersen, 1991)**.** *If X is a compact Hausdorff space, then the real rank of $C(X)$ is equal to the covering dimension of X.*

The concept of real rank for general C^*-algebras is hence a non-commutative generalisation of the dimension of compact Hausdorff spaces. Since a compact Hausdorff space has dimension zero if and only if it is totally disconnected, in which case it has a plethora of projections, it is not unreasonable to guess that real rank zero implies that the C^*-algebra has several projections. The following result, also due to Brown and Pedersen, see [8, Theorem 2.6], states this intuition rigorously.

Proposition A.3 (Brown-Pedersen, 1991)**.** *Let A be a C^*-algebra. The following three conditions are equivalent.*

(i) *A has real rank zero;*

(ii) *The elements in A_{sa} with finite spectrum are dense in A_{sa};*

(iii) *Every hereditary C^*-subalgebra of A has an approximate unit (not necessarily increasing) consisting of projections.*

The proofs of the following facts are scattered throughout [8].

Proposition A.4. *The following hold:*

(i) *If A has real rank zero, then so does $M_n(A)$ for all $n \in \mathbb{N}$;*

(ii) *Real rank zero passes to hereditary C^*-subalgebras, in particular to ideals;*

(iii) *Real rank zero is preserved by inductive limits;*

(iv) *Real rank zero passes to quotients;*

The notion of stable rank can also be seen as a non-commutative generalisation of the covering dimension of compact Hausdorff spaces; details may be found in [57]. We shall need the following facts regarding C^*-algebras with stable rank one, which are presented in [2, Section 6.5].

Proposition A.5. *Let A be a unital C^*-algebra with stable rank one. Then:*

(i) *For some, hence for all, $n \in \mathbb{N}$, the C^*-algebra $M_n(A)$ has stable rank one;*

(ii) *A is stably finite;*

(iii) *A has cancellation of projections.*

We also have the following permanence properties.

Proposition A.6. *The following hold:*

(i) *All finite-dimensional C^*-algebras have stable rank one;*

(ii) *Limits of inductive sequences with unital connecting maps of C^*-algebras with stable rank one have stable rank one;*

(iii) *In the unital, separable case, stable rank one passes to ultrapowers.*

The proofs of (i) and (ii) may be found in [57], while the proof of (iii) can be found in [65, Lemma 2.4]. The idea behind the proof of (iii) is that if B is a separable, unital C^*-algebra of stable rank one, then, using [43, Lemma 19.2.2(i)], any invertible $x \in B_\omega$ admits a polar decomposition $x = u\,|x|$, where $u \in B_\omega$ is unitary. Hence, we can approximate x by the invertible elements $u(|x|+\varepsilon)$ for $\varepsilon > 0$. If, moreover, B has real rank zero and $x \in B_\omega$ is self-adjoint, we can choose u to be self-adjoint and then B_ω has real rank zero. Since the universal UHF-algebra $\mathcal{Q}$ has real rank zero by Proposition A.4 and stable rank one by Proposition A.6, we find that the ultrapower $\mathcal{Q}_\omega$ has these properties.