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AF-algebras and their invariants

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Abstract

This project is a study of AF-algebras, which are defined as inductive limits of finite-dimensional C^*-algebras. First, the unital AF-algebras are classified using their K-theoretic invariants following Elliott's proof, using results such as the classification of finite-dimensional C^*-algebras and the fact that the functor K_i preserves inductive limits for $i = 0, 1$. The classification invariants turn out to have an intrinsic characterization as countable, unperforated ordered groups with Riesz' interpolation property, and this characterization is proved following the original papers of Shen and Effros-Handelman-Shen. Lastly, the tracial state spaces of AF-algebras are examined, and it is proved using a result of Lazar and Lindenstrauss that any metrizable Choquet simplex can be realized as the tracial simplex of a simple and unital AF-algebra.

Contents

Introduction

The study of AF-algebras started when Bratteli introduced them in his 1972 paper [2]. In this paper, he defined AF-algebras as unital C^*-algebra which could be realized as the closure of the union of an increasing sequence of finite-dimensional C^*-algebras. This generalized the UHF-algebras, which were classified by Glimm in 1960 using supernatural numbers [8], by expanding the dense subalgebra from a union of matrix algebras under certain conditions to a union of arbitrary finite-dimensional C^*-algebras. This generalization allowed for more exotic behaviour; for instance, AF-algebras need not be simple, which UHF-algebras necessarily are. One key point of Bratteli's paper was the introduction of Bratteli diagrams, which he used both as a way to graphically represent AF-algebras in an easy way, as well as a classification tool. Later on the definition of AF-algebras was expanded to be the class of inductive limits of finite-dimensional C^*-algebras, allowing for non-unital cases and sometimes — depending on one's definition of inductive limits — non-separability.

Throughout the 1970's, AF-algebras and their K-theoretical invariants were studied closely [4, 16], including Elliott's 1976 classification of unital AF-algebras via their ordered K_0-groups [5]. This was among the first classes of C^*-algebras to be classified via K-theory and marked the beginning of Elliott's classification program. He conjectured that unital, separable, nuclear, simple C^*-algebras could be classified by their K-theory, tracial simplex structure and a natural pairing of the two; this invariant is often called the *Elliott invariant*. In the early 2000's, however, examples of non-isomorphic nuclear and separable C^*-algebras with the same Elliott invariant were constructed, disproving Elliott's conjecture. Even though the conjecture was false in general, it was proven for a wide class of C^*-algebras that the Elliott invariant could be used in classification. In fact, it was proven recently by Elliott, Gong, Lin and Niu [7] that if one restricts these C^*-algebras to the ones satisfying a certain regularity property and the so-called UCT, this class of C^*-algebras is classifiable by the Elliott invariant, and with these further assumptions, the Elliott conjecture is hence a theorem.

The idea of AF-algebras is thus not new whatsoever, as they have been studied and completely classified for more than 40 years. Still, there are several reasons to study AF-algebras. For one the AF-algebras actually satisfy Elliott's conjecture with a very simple invariant, since portion of Elliott's invariant concerning the K_0-group is used, giving a very hands-on demonstration of the usefulness of the sometimes very abstract notion of K-theory. Moreover, AF-algebras are in many ways uncomplicated C^*-algebras to examine; they do not behave too exotically, such that they are difficult to study, nor do they behave too uninterestingly. For instance, AF-algebras are all nuclear and stably finite, but they still exhibit almost arbitrary tracial simplices. All of these results will be shown in this project. For a more current reason to be interested in AF-algebras, one can look at the question of AF-embeddability, that is, the question: When can a C^*-algebra be identified as a subalgebra of an AF-algebra? This is still unresolved, and a more intrinsic characterization would lead to a better understanding of AF-algebras. More details regarding AF-embeddability can be found in [3, Chapter 8].

The structure of this project is as follows:

- In *Chapter 1*, we establish some facts regarding C^*-algebras, elementary K-theory, and convex analysis, which are assumed well-known and mostly included in the project for the purpose of it being self-contained. Moreover, the chapter contains a classification of finite-dimensional C^*-algebras, as well as a study of ordered Abelian groups, including under which conditions K_0-groups are ordered. Both of these results are vital in the classification of AF-algebras.
- In *Chapter 2*, the notions of inductive and inverse limits in general categories are established. The most important parts are the proofs of the categories of C^*-algebras and (ordered) Abelian groups admitting inductive limits as well as the continuity of the K_0-functor, meaning that it preserves inductive limits. Throughout the chapter we encounter many small propositions and lemmas about the structure of inductive limits in C^*-algebras, some of which are used in proving continuity of K_0, and others that are used in the study and classification of AF-algebras. In some sense, this chapter lacks many of the interesting properties of inverse limits, as they are not directly necessary for the rest of the project. The chapter ends with a proof of continuity of the suspension of C^*-algebras as well as K_1, neither of which are particularly surprising, nor particularly useful in this project and are mostly added for completeness in showing continuity of the K_i-functors for $i = 0, 1$, as well as to show that AF-algebras have trivial K_1-groups.

- In *Chapter 3*, we finally define and study AF-algebras. Our first goal is a justification of the name approximately finite-dimensional C^*-algebras using several notions of finiteness, for example having finite approximation properties and being stably finite in particular. The main goal of the section is the classification of unital AF-algebras using ordered K_0-groups following the proof in [10]. We end the chapter with a quick perspective on similar classes of C^*-algebras, and how the classification of AF-algebras can be used to achieve Glimm's classification of UHF-algebras by supernatural numbers.

- Having classified the unital AF-algebras, it is natural to ask what structure these K-theoretical invariants may have. This question shall be answered fully in *Chapter 4*, where we define *dimension groups* as inductive limits of simplical groups, which turn out to be precisely the ordered K_0-groups of AF-algebras. In fact, these dimension groups have an intrinsic characterization being countable unperforated groups with Riesz' interpolation. This connection is established by following the proofs of Effros, Handelman and Shen in [4] as well as Shen in [16].

- Following the classification and study of the invariants in the previous two chapters, *Chapter 5* serves as a study of the tracial simplices on AF-algebras. More precisely we shall, using a theorem of Lazar and Lindenstrauss in [11], prove that any infinite-dimensional metrizable Choquet simplex can be realized as the tracial simplex of a simple AF-algebra. We end the section with a few specific examples of interest.

Notation

Since virtually every single mathematical paper differs notationally in some way, we shall establish some of the notation used in this project.

We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the natural numbers, and $\mathbb{Z}^+ = \mathbb{N}_0 = \{0, 1, \ldots\}$ are the natural numbers with 0 adjoined; we shall use both $\mathbb{Z}^+$ and $\mathbb{N}_0$ depending on the scenario. The letters A, B are C^*-algebras and the letters G, H denote groups unless otherwise stated. The letter H will sometimes refer to a Hilbert space, in which case $\mathbb{B}(H)$ and $\mathbb{K}(H)$ will denote the bounded linear operators and the compact operators respectively on H. A C^*-algebra is not assumed to be unital unless explicitly stated as such. Whenever we refer to matrix algebras in the project, we mean C^*-algebras of the form $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, unless otherwise stated.

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I would like to thank my advisor, Mikael Rørdam, for help and guidance throughout the last months, both on this project and in general by introducing me to further subjects within the field of C^*-algebras.

1 Preliminaries

This chapter is, admittedly, rather peculiar. It is a recap of the theory of C^*-algebras and K-theory as well as some basis results on convex analysis, most of which are assumed well-known in the project but included in order to have a reference for whenever we need the results. Proofs for many of the statements may be found in any introductory textbook on C^*-algebras, e.g., [12, 17], and K-theory, e.g., [10, 12]. The chapter also includes a study of a somewhat basic classification result, namely the classification of finite-dimensional C^*-algebras, which is (obviously) needed later on in the project, but which is often regarded as well-known and consequently stated without proof. As the result, but not the proof, was known to the author, a complete proof has been added to this chapter. Lastly we shall delve into ordered Abelian groups, which is a more sophisticated type of structure a K_0-group may have.

1.1 Elementary results on C^*-algebras

Throughout this section we let A denote an arbitrary C^*-algebra, not necessarily unital unless otherwise stated. For $n \in \mathbb{N}$, define $M_n(A)$ to be the matrix algebra with entries in A in the usual sense, and if $\varphi\colon A \to B$ is a *-homomorphism, then the amplified map $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ is given by $\varphi^{(n)}((a_{ij})) = (\varphi(a_{ij}))$ and is a *-homomorphism. Note that we can consider A as a C^*-subalgebra of $M_2(A)$ etc. via the injection $A \hookrightarrow M_2(A)$ given by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. The unit of a unital C^*-algebra is denoted by 1 or 1_A, whichever is more useful.
For a general C^*-algebra A, unital or not, we can construct its *unitization* $\tilde{A}$ that contains A as an ideal with $\tilde{A}/A \cong \mathbb{C}$, i.e. such that we have the short exact sequence $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$. If $\pi\colon \tilde{A} \to \mathbb{C}$ is the quotient map, then the map $\lambda\colon \mathbb{C} \to \tilde{A}$ by $\lambda(z) = z1_{\tilde{A}}$ satisfies $\pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$, i.e., the sequence is split exact. We get the mapping $s = \lambda \circ \pi\colon \tilde{A} \to \tilde{A}$, which maps $s(a + \alpha 1_{\tilde{A}}) = \alpha 1_{\tilde{A}}$, and is named the *scalar mapping* for this reason. Unitization is functorial in the sense that if $\varphi\colon A \to B$ is a *-homomorphism, then the induced map $\tilde{\varphi}\colon \tilde{A} \to \tilde{B}$ by $\tilde{\varphi}(a + \alpha 1_{\tilde{A}}) = \varphi(a) + \alpha 1_{\tilde{B}}$ is a *-homomorphism.

Given an element $a \in A$, we denote the spectrum by $\sigma(a)$. An element $v \in A$ in a unital C^*-algebra is an *isometry* if $v^*v = 1$, and $u \in A$ is *unitary* if $u^*u = 1_A = uu^*$, and we let $\mathcal{U}(A)$ denote the collections of unitary elements on A. For any $u \in \mathcal{U}(A)$ the map $\operatorname{Ad} u\colon A \to A$ by $\operatorname{Ad} u(a) = uau^*$ for $a \in A$ is a *-homomorphism. An element $p \in A$ is called a *projection* if $p^2 = p = p^*$, and two projections $p, q \in A$ are said to be *Murray-von Neumann equivalent* if there exists a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$. In this case we write $p \sim q$; this is an equivalence relation on $\mathcal{P}(A)$, the collection of projections on A.

We now establish a couple of lemmas, which we shall use in proving continuity of the K_0-functor in Chapter 2.

Lemma 1.1. *If $a \in A$ is self-adjoint and $\delta = \|a^2 - a\| < \frac{1}{4}$, then there exists a projection $p \in A$ with $\|p - a\| \leq 2\delta$.*

Proof. Since a is self-adjoint, we know that $\sigma(a) \subseteq \mathbb{R}$. Note that $\sigma(a^2 - a) \subseteq [-\delta, \delta]$, and that if $t \in \sigma(a)$, then $t^2 - t \in \sigma(a^2 - a)$. Moreover, if $|t^2 - t| \leq \delta$, then we find that $t \in [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$, as $\delta < \frac{1}{4}$. Therefore, if $t \in \sigma(a)$, then $t \in [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$. All in all this implies that

$$\sigma(a) \subseteq [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta].$$

Since $\delta < \frac{1}{4}$, the right-hand side consists of two disjoint intervals. The function $f\colon \sigma(a) \to \mathbb{C}$ given by

$$f(t) = \begin{cases} 0 & \text{if } t \in [-2\delta, 2\delta] \\ 1 & \text{if } t \in [1 - 2\delta, 1 + 2\delta] \end{cases} \tag{1.1}$$

is thus well-defined and continuous. By continuous functional calculus, put $p = f(a)$ and note that p is a projection, since f is idempotent and real-valued. Moreover, as $|t - f(t)| \leq 2\delta$ for all $t \in \sigma(a)$, it follows that $\|a - p\| \leq 2\delta$ as desired. □

The next lemma states that if two projections are approximately Murray-von Neumann equivalent in a specific sense, then they actually are Murray-von Neumann equivalent.

Lemma 1.2. *If $p, q \in A$ are projections such that there exists $x \in A$ satisfying*

$$\|x^*x - p\| < \frac{1}{2} \qquad \text{and} \qquad \|xx^* - q\| < \frac{1}{2},$$

then $p \sim q$.

Proof. Define $\delta = \frac{1}{2}\max\{\|x^*x - p\|, \|xx^* - q\|\} < \frac{1}{4}$ and let $X = \sigma(x^*x) \cup \sigma(xx^*)$. Since x^*x and xx^* are self-adjoint elements of distance at most δ to a projection, we find that $X \subseteq [-2\delta, 2\delta] \cup [1 - 2\delta, 1 + 2\delta]$. Define $f \in C(X)$ as in (1.1) and projections $p_0 = f(x^*x)$ and $q_0 = f(xx^*)$. Note that $\|x^*x - p_0\| \leq 2\delta$ and $\|xx^* - q_0\| \leq 2\delta$, hence the triangle inequality implies that

$$\|p - p_0\|, \|q - q_0\| \leq 4\delta < 1$$

and thus $p \sim p_0$ and $q \sim q_0$. If we show that $p_0 \sim q_0$, then we are done.
If $h \in C(X)$ is a polynomial, then one easily verifies that we have the identity

$$xh(x^*x)x^* = h(xx^*)xx^*, \tag{1.2}$$

again by continuous functional calculus. Hence, by the Stone-Weierstrass theorem, it holds for all $h \in C(X)$. Define the continuous function $g\colon X \to \mathbb{C}$ by

$$g(t) = \begin{cases} \sqrt{\frac{f(t)}{t}} & \text{if } t \geq 1 - 2\delta \\ 0 & \text{if } t \leq 2\delta \end{cases},$$

which is well-defined and continuous as $f(t) \geq 0$ and $1 - 2\delta > 0$. Note that g satisfies the identity $tg(t)^2 = f(t)$ for all $t \in X$. Define $v = xg(x^*x)$, then using this identity along with (1.2), one finds that

$$v^*v = p_0, \qquad \text{and} \qquad vv^* = q_0,$$

completing the proof. □

We end this discussion on elementary C^*-algebraic definitions and results with the notion of finite C^*-algebras. This is based on the idea of an infinite set being one, which contains a proper subset of the same cardinality.

Definition 1.3. Let A be a C^*-algebra. A projection $p \in A$ is said to be *infinite* if there exists a projection $q \in A$ such that $p \sim q$ and $q < p$. If p is not infinite, we say that p is *finite*. A unital C^*-algebra is called *finite* if 1_A is finite. Otherwise A is said to be *infinite*. Moreover, we say A is *stably finite* if $M_n(A)$ is finite for each $n \in \mathbb{N}$.
If A is not unital, then we say that A is finite/infinite/stably finite if the unitization $\tilde{A}$ is finite/infinite/stably finite. Recall that $s \in A$ is an isometry if $s^*s = 1$.

The following proposition gives several characterizations of finite unital C^*-algebras.

Proposition 1.4. *Let A be a unital C^*-algebra. The following are equivalent:*

(i) *A is finite.*

(ii) *All isometries on A are unitary.*

(iii) *All projections in A are finite.*

(iv) *All left-invertible elements in A are invertible.*

(v) *All right-invertible elements in A are invertible.*

Proof. (i)⇒(ii): If $s \in A$ is an isometry, then

$$1 = s^*s \sim ss^* \leq 1$$

implying that $ss^* = 1$ by finiteness of A.
(ii)⇒(i): Suppose $p \sim 1$, and let $1 = s^*s$ and $p = ss^*$. Then s is an isometry and by assumption thus unitary, such that $p = ss^* = 1$ proving that A is finite.

(ii)⇒(iii): Suppose $p \sim q \leq p$ and let $v \in A$ be the partial isometry implementing the equivalence, i.e., such that $p = v^*v$ and $q = vv^*$. Note that $v^*(1-p) \leq vv^*(1-p)v = q(1-p)v$ such that $v^*(1-p) = 0$. Similarly one finds $(1-p)v = 0$, $(1-p)v^* = 0$ and $v(1-p) = 0$. Define $u = v + (1-p)$, then

$$u^*u = v^*v + (1-p) = p + (1-p) = 1$$

such that u is an isometry. By assumption, u is hence a unitary, and thus we find that

$$1 = uu^* = q + (1-p)$$

such that $q = p$.
(iii)⇒(ii): If $u \in A$ is an isometry, i.e. $u^*u = 1$, then $1 \leq uu^* \sim 1$ and finiteness of A imply that u is unitary.
(iv)⇒(v): Suppose a is right-invertible, then a^* is left-invertible, hence invertible, and thus a is invertible.
(v)⇒(iv): Analogous to the previous implication.
(iv)⇒(ii): If $u^*u = 1$, then as left-invertibility implies invertibility by assumption, we find that $u^*u = 1$ immediately, and hence all isometries are unitaries.
(ii)⇒(iv): Suppose $a \in A$ is left-invertible, then a^*a is invertible and positive. Put $u = a(a^*a)^{-1/2}$, then

$$u^*u = (a^*a)^{-1/2}a^*a(a^*a)^{-1/2} = 1$$

such that u is an isometry. By (ii), u is unitary, in particular invertible, and thus $a = u(a^*a)^{1/2}$ is invertible. □

1.2 Brief recap of K-theory

The following is a recap of the assumed background knowledge in K-theory, which is stated for convenience of the reader. We follow the exposition in [10] and refer to this for proofs of the claims in this section.

Again, we let A denote an arbitrary C^*-algebra, which is not unital unless otherwise stated. For each $n \in \mathbb{N}$, denote by $\mathcal{P}_n(A)$ the collection of projections on $M_n(A)$. Then define $\mathcal{P}_\infty(A) = \bigcup_{n=1}^\infty \mathcal{P}_n(A)$ to be the collection of projections on the matrix algebras of A. Define the equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ by $p \sim_0 q$ if $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ and there exists $v \in M_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$. The equivalence class of $p \in \mathcal{P}_\infty(A)$ shall be denoted $[p]_0$. Note that if $p, q \in \mathcal{P}_n(A)$, then $p \sim_0 q$ if and only if $p \sim q$ on $M_n(A)$. Denote by $\mathcal{D}(A)$ the equivalence classes of this equivalence relation. The set $\mathcal{D}(A)$ becomes an Abelian monoid when equipped with the binary operation $p \oplus q = \mathrm{diag}(p,q)\ \mathcal{D}(A)$, and using the Grothendieck construction we get an Abelian group $\mathcal{K}_0(A)$. This is a functor from the category of C^*-algebras to the category of Abelian groups. It is not half-exact, though, which is a fixable problem. If $0 \to A \to \tilde{A} \xrightarrow{\pi} \mathbb{C} \to 0$ is the split exact sequence coming from unitization of a C^*-algebra A, then it follows that the sequence $0 \to \mathcal{K}_0(A) \to \mathcal{K}_0(\tilde{A}) \xrightarrow{\mathcal{K}_0(\pi)} \mathcal{K}_0(\mathbb{C}) \to 0$ is also exact. Define $K_0(A) = \ker(\mathcal{K}_0(\pi))$; if A is already unital, then $K_0(A) = \mathcal{K}_0(A)$. The functor K_0 is then a homotopy-invariant and half-exact functor from the category of C^*-algebras to the category of Abelian groups with the following properties, which we call the *standard picture for* K_0:

Proposition 1.5 (Standard picture for K_0)**.** *Let A be a C^*-algebra, unital or not, and let $s\colon \tilde{A} \to \tilde{A}$ be the corresponding scalar mapping. Then*

$$K_0(A) = \{[p]_0 - [s(p)]_0 \mid p \in \mathcal{P}_\infty(\tilde{A})\}$$

and the following holds:

(i) *For $p, q \in \mathcal{P}_\infty(\tilde{A})$ the following are equivalent:*

  (a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$*;*

  (b) *There exist integers $k, l \in \mathbb{N}$ such that $p \oplus 1_k \sim_0 q \oplus 1_l$;*

  (c) *There exist scalar projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$.*

(ii) *If* $[p]_0 - [s(p)]_0 = 0$ *for some* $p \in \mathcal{P}_\infty(A)$*, then* $p \oplus 1_n \sim s(p) \oplus 1_n$ *for some* $n \in \mathbb{N}$*.*

(iii) *If* $\varphi\colon A \to B$ *is a *-homomorphism, then*

$$K_0(\varphi)([p]_0 - [s(p)]_0) = [\tilde{\varphi}(p)]_0 - [\tilde{\varphi}(s(p))]_0, \qquad p \in \mathcal{P}_\infty(\tilde{A}).$$

In the case where A is unital, the standard picture is much simpler as we do not need to consider the unitization at all.

Proposition 1.6 (Standard picture for K_0 in the unital case)**.** *Let* A *be a unital* C^**-algebra. Then*

$$K_0(A) = \{[p]_0 - [q]_0 \,|\, p, q \in \mathcal{P}_\infty(A)\}$$

and the following properties hold:

(i) $[p \oplus q]_0 = [p]_0 + [q]_0$ *for all* $p, q \in \mathcal{P}_\infty(A)$*;*

(ii) *If* $p, q \in \mathcal{P}_n(A)$ *and there exists a homotopy* ψ *on* $\mathcal{P}_n(A)$ *with* $\psi(0) = p$ *and* $\psi(1) = q$*, then* $[p]_0 = [q]_0$*;*

(iii) *If* $pq = qp = 0$ *in* $\mathcal{P}_n(A)$*, then* $[p + q]_0 = [p]_0 + [q]_0$*;*

(iv) *For* $p, q \in \mathcal{P}_\infty(A)$ *we have* $[p]_0 = [q]_0$ *if and only if there exists* $r \in \mathcal{P}_\infty(A)$ *such that* $p \oplus r \sim_0 q \oplus r$*.*

The following proposition is a consequence of the standard picture of K_0, which is used in proving continuity of K_0.

Proposition 1.7. *Suppose* A *and* B *are* C^**-algebras, and that* $\varphi\colon A \to B$ *is a *-homomorphism. Let* $g \in \ker K_0(\varphi)$*, then there exist* $k \in \mathbb{N}$ *and a projection* $p \in M_k(\tilde{A})$ *such that* $g = [p]_0 - [s(p)]_0$ *and* $\tilde{\varphi}^{(k)}(p) \sim \tilde{\varphi}^{(k)}(s(p))$*.*

There are other pleasant properties of K_0; the following shall become very useful when examining AF-algebras.

Proposition 1.8. *If* A *and* B *are* C^**-algebras, then* $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$ *by the following map: If* $\iota_A\colon A \hookrightarrow A \oplus B$ *and* $\iota_B\colon B \to A \oplus B$ *are the canonical inclusion maps, then the map* $K_0(\iota_A) \oplus K_0(\iota_B)\colon K_0(A) \oplus K_0(B) \to K_0(A \oplus B)$ *is an isomorphism of Abelian groups.*

For a C^*-algebra, we define the *suspension algebra* $SA := C_0((0,1), A)$, which is again a C^*-algebra. It is functorial in the sense that if $\varphi\colon A \to B$ is a *-homomorphism, then $S\varphi\colon SA \to SB$ defined by $S\varphi(f)(t) = \varphi(f(t))$ for $f \in C_0((0,1), A)$ and $t \in (0,1)$ is a *-homomorphism. For a C^*-algebra A, we can define $K_1(A) := K_0(SA)$. Another characterization of the K_1-group is via unitaries in a way similar to the construction of K_0 via projections. Denote by $\mathcal{U}_\infty(\tilde{A})$ the collection of unitary elements on $M_n(\tilde{A})$ for all $n \in \mathbb{N}$. Define an equivalence relation $\sim_1$ on $\mathcal{U}_\infty(\tilde{A})$ by $u \sim_1 v$ if and only if there exist k, k' such that $u \oplus 1_m$ and $v \oplus 1_,$ are homotopically equivalent within $\mathcal{U}_k(\tilde{A})$ for some integer k; the equivalence class of $u \in \mathcal{U}_\infty(A)$ will be denoted $[u]_1$. Then one can define $K_1(A) = \mathcal{U}_\infty(\tilde{A})/{\sim_1}$, which is equivalent to the above definition via suspension. In the case where A is unital, one can verify that $K_1(A) = \mathcal{U}_\infty(A)/{\sim_1}$. Note that K_1 is a split exact functor and that it preserves direct sums.

1.3 The basics of convex analysis

In this section, we mention some elementary definitions and results of convex analysis, which can be found in most introductory textbooks on e.g. functional analysis. A subset K of a (complex or real) vector space X is said to be *convex* if for any $x, y \in K$ and $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in K$. We call such a combination a *convex combination*, and a map $f\colon K_1 \to K_2$ of convex sets is said to be *affine* if it preserves convex combinations. If X is a topological vector space, which has a neighbourhood base at 0 consisting of convex sets, we say that X is *locally convex.*

Let $K \subseteq X$ be a non-empty convex set. An element $x \in K$ is called an *extreme point* if x can only trivially be written as a convex combination of elements in K, that is, if $x = \lambda y + (1-\lambda)z$ for some $y, z \in K$ and $\lambda \in (0,1)$, then $x = y = z$. The set of all extreme points of K is denoted $\partial_e K$ and is called the *extremal boundary*. The *convex hull* of a subset $A \subseteq X$, denoted $\text{conv}(A)$, is the smallest convex set in X containing A. One can easily verify that

$$\text{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i \,\middle|\, x_i \in A, \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

A *face* F in K is a convex subset satisfying the following property: If $x, y \in K$ and $\lambda \in (0,1)$ satisfies that $\lambda x + (1-\lambda)y \in F$, then $x, y \in F$.

The following theorem gives a description of compact convex sets in terms of their extreme points.

Theorem 1.9 (Krein-Milman). *Let K be a non-empty convex compact subset of a locally convex Hausdorff topological vector space X. Then $\partial_e K \neq \emptyset$ and $K = \overline{\text{conv}(\partial_e K)}$.*

There is a partial converse to the Krein-Milman theorem.

Theorem 1.10. *If X is a locally convex Hausdorff topological vector space and $K \subseteq X$ is a non-empty convex compact subset, and $F \subseteq K$ is a subset satisfying that $K = \overline{\text{conv}(F)}$, then $\partial_e K \subseteq \overline{F}$.*

Now consider $\mathbb{R}^{n+1}$ for some $n \in \mathbb{N}$ and let $\{e_0, e_1, \dots, e_n\}$ be the canonical basis. We define the n-simplex Δ_n as

$$\Delta_n = \text{conv}\{e_0, e_1, \dots, e_n\} \subseteq \mathbb{R}^{n+1}.$$

The n-simplex Δ_n is easily seen to be a compact, convex subset of $\mathbb{R}^{n+1}$. Properties of simplices such as n-simplices, especially related to tracial simplices of unital C^*-algebras, shall be studied in greater detail in Chapter 5.

1.4 Classification of finite-dimensional C^*-algebras

In this section, we aim to prove that any finite-dimensional C^*-algebra can be realized as a direct sum of matrix algebras over $\mathbb{C}$ following the proof in [10]. First we need some notation; consider the C^*-algebra $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ for each $k = 1, \dots, r$ and $i, j = 1, \dots, n_k$ and define, for each i, j, k, $e_{ij}^{(k)} \in A$ to be the element in A which is

$$e_{ij}^{(k)} = (0, \dots, 0, e_{n_k}(i,j), 0, \dots, 0),$$

where $e_{n_k}(i,j)$ are the usual matrix units in $M_{n_k}(\mathbb{C})$. The following lemma is a straight-forward exercise in linear algebra.

Lemma 1.11. *With the notation above, we have the following properties:*

(i) $e_{ij}^{(k)} e_{jl}^{(k)} = e_{il}^{(k)}$;

(ii) $e_{ij}^{(k)} e_{lm}^{(n)} = 0$ *if* $k \neq n$ *or* $j \neq l$;

(iii) $\left(e_{ij}^{(k)}\right)^* = e_{ji}^{(k)}$;

(iv) $A = \text{span}\left\{e_{ij}^{(k)}\right\}$.

We call the elements $\{e_{ij}^{(k)}\}$ the *standard matrix units* for A. As they are all non-zero, it follows from (i)-(ii) that they are linearly independent, and hence by (iv) they form a basis for A.
If B is another C^*-algebra, and $\{f_{ij}^{(k)}\}$ is a set of elements of B, then there exists a unique linear map $\varphi\colon A \to B$ satisfying $\varphi(e_{ij}^{(k)}) = f_{ij}^{(k)}$. If the elements $\{f_{ij}^{(k)}\}$ moreover satisfy the analogous of (i)-(iii) above, in which case we call them *matrix units*, φ is actually a *-homomorphism. It is easily verified that if each $f_{ij}^{(k)}$ is non-zero, then φ is injective, and that if $\{f_{ij}^{(k)}\}$ span B, i.e. the elements satisfy the analogous of (iv) above, then φ is surjective. Hence it follows that if we for a C^*-algebra can construct a spanning set of non-zero matrix units, then it is isomorphic to a C^*-algebra of the form

$$M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

for some suitable integers $r \in \mathbb{N}$ and $n_i \in \mathbb{N}$. The classification of finite-dimensional C^*-algebras is done precisely by constructing such a spanning set of non-zero matrix units as we shall see shortly.

First we prove a way of extending a finite number of mutually orthogonal and Murray-von Neumann equivalent projections to a system of matrix units.

Lemma 1.12. *If $\{e_{ii}^{(k)} \,|\, 1 \leq k \leq r, 1 \leq i \leq n_k\}$ is a system of mutually orthogonal projections in a C^*-algebra such that $e_{ii}^{(k)} \sim e_{jj}^{(k)}$ for all i, j, k, then there exists a system of matrix units $\{e_{ij}^{(k)} \,|\, 1 \leq k \leq r, 1 \leq i, j \leq n_k\}$, which extends the original system.*

Proof. Find by Murray-von Neumann equivalence partial isometries $e_{1i}^{(k)}$ such that

$$e_{1i}^{(k)} \left(e_{1i}^{(k)}\right)^* = e_{11}^{(k)} \qquad \text{and} \qquad \left(e_{1i}^{(k)}\right)^* e_{1i}^{(k)} = e_{ii}^{(k)}$$

and define $e_{ij}^{(k)} = \left(e_{1i}^{(k)}\right)^* e_{1j}^{(k)}$, then one easily verifies that the system $\{e_{ij}^{(k)}\}$ satisfies Lemma 1.11(i)-(iii), i.e., it is a system of matrix units. ☐

Recall that a C^*-subalgebra $B \subseteq A$ is called a *maximal Abelian subalgebra*, or a *masa*, if B is an Abelian C^*-algebra, which is not properly contained in any other Abelian C^*-subalgebra of A; these are all self-adjoint. One can show that B is a masa if and only if $B = B'$, where $B' = \{a \in A \,|\, ab = ba \text{ for all } b \in B\}$ is the commutant. Using Zorn's lemma, one can show that all Abelian subalgebras of a C^*-algebra is contained in a masa, and consequently all C^*-algebras contain a masa.

Lemma 1.13. *Let D be a masa in a C^*-algebra A. If 1 is a unit for D, then it is a unit for A.*

Proof. Let $a \in A$ be arbitrary and put $z = a - a1$. It is clear that $zd = 0$ for all $d \in D$, and hence $dz^* = 0$ for all $d \in D$ by self-adjointness of D. Thus $d(z^*z) = 0 = (z^*z)d$ such that z^*z commutes with D, and thus $z^*z \in D$, since D is a masa. In particular, $(z^*z)^2 = 0$, such that

$$\|z\|^4 = \|z^*z\|^2 = \left\|(z^*z)^2\right\| = 0$$

using that z^*z is self-adjoint. This proves $a1 = a$ for all $a \in A$, i.e., 1 is a unit for A. ☐

Lemma 1.14. *Let D be a masa in a C^*-algebra A. If $p \in \mathcal{P}(D)$ satisfies $pDp = \mathbb{C}p$, then $pAp = \mathbb{C}p$.*

Proof. Let $d \in D$, then $pd = dp = pdp = \lambda p$ for some $\lambda \in \mathbb{C}$. Then for any $a \in A$,

$$papd = pa\lambda p = \lambda pap = dpap$$

since p is a projection commuting with D. Hence $pap \in D$ such that $pap \in pDp$ as p is idempotent. We conclude that $pAp = pDp = \mathbb{C}p$. ☐

We are now at a point where we can prove that finite-dimensional C^*-algebras are precisely direct sums of matrix algebras over $\mathbb{C}$.

Theorem 1.15. *Any finite-dimensional C^*-algebra is of the form*

$$M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$$

for some integers $r \in \mathbb{N}$ and $n_i \in \mathbb{N}$ for $i = 1, \ldots, r$.

Proof. Let A be a finite-dimensional C^*-algebra and let $D \subseteq A$ be a masa. Since D is commutative, $D \cong C_0(X)$ for some locally compact Hausdorff space X. As A is finite-dimensional so is D and consequently X is a finite set; say $X = \{x_1, \ldots, x_n\}$. Note that in particular X is compact, so $D = C(X)$ is unital. In particular, A is unital by Lemma 1.13; call the unit 1.
Define for each $j = 1, \ldots, n$ the continuous functions $q_j(x_i) = \delta_{ij}$ on X and let p_j be the corresponding elements in D. It is clear that p_j are projections with $\sum_{j=1}^n p_j = 1$ and $p_j D p_j = \mathbb{C}p_j$. In particular, we get the identities $p_j A p_j = \mathbb{C}p_j$ for each j by Lemma 1.14.
If $p_j A p_i \neq 0$, then find $v_{i,j} \in p_j A p_i$ with $\|v_{i,j}\| = 1$. Since $\left\|v_{i,j}^* v_{i,j}\right\| = \|v_{i,j}\|^2 = 1$ and $v_{i,j}^* v_{i,j} \in p_i A p_i = \mathbb{C}p_i$ with $v_{i,j}^* v_{i,j} \geq 0$, we see that $v_{i,j}^* v_{i,j} = p_i$, and analogously that $v_{i,j} v_{i,j}^* = p_j$. Therefore $p_i \sim p_j$ are Murray-von Neumann equivalent. Moreover, if $a \in p_j A p_i$, then $a = ap_i = (av_{i,j}^*)v_{i,j}$, and we find that $a \in \mathbb{C}v_{i,j}$, since $av_{i,j}^* \in p_j A p_j = \mathbb{C}p_j$ and $p_j v_{i,j} = v_{i,j}$. This implies that for any

i, j, we either have $p_j A p_i = \{0\}$ or $p_j A p_i = \mathbb{C} v_{i,j}$ and $v_{i,j}$ is the partial isometry implementing the Murray-von Neumann equivalence $p_i \sim p_j$.

We can hence take $\{p_1, \ldots, p_n\}$ and partition it into Murray-von Neumann equivalence classes. Let r be the number of equivalence classes, and let n_k be the number of elements in the kth equivalence class for each $1 \leq k \leq r$. Let $\{e_{11}^{(k)}, \ldots, e_{n_k n_k}^{(k)}\}$ be the elements in the kth equivalence class, and note that the system $\{e_{ii}^{(k)}\}$ satisfies the conditions of Lemma 1.12. We can hence extend it to a system of matrix units $\{e_{ij}^{(k)}\}$ in A, which satisfies (i)-(iii) in Lemma 1.11. These are all non-zero, so all that remains to be seen is that (iv) in the same lemma is satisfied, i.e., the matrix units span A. Note that it follows by construction that $e_{ii}^{(k)} A e_{jj}^{(k)} = \mathbb{C} e_{ij}^{(k)}$, that $e_{ii}^{(k)} A e_{jj}^{(l)} = \{0\}$ if $k \neq l$, and that $e_{ii}^{(k)} \sim e_{jj}^{(k)}$.

Since we have the identity $1 = \sum_{k,i} e_{ii}^{(k)}$, we find that for any $a \in A$,

$$a = \left(\sum_{k=1}^{r} \sum_{i=1}^{n_k} e_{ii}^{(k)} \right) a \left(\sum_{k=1}^{r} \sum_{i=1}^{n_k} e_{ii}^{(k)} \right) = \sum_{k=1}^{r} \sum_{i,j=1}^{n_k} e_{ii}^{(k)} a e_{jj}^{(k)} = \sum_{k=1}^{r} \sum_{i,j=1}^{n_k} \lambda_{ij}^{(k)} e_{ij}^{(k)}$$

for some scalars $\lambda_{ij}^{(k)} \in \mathbb{C}$. This proves that $A = \operatorname{span}\{e_{ij}^{(k)}\}$ such that $\{e_{ij}^{(k)}\}$ satisfies Lemma 1.11(i)-(iv). By the discussion following this lemma, we see that A is isomorphic to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ completing the proof. □

This is in fact a complete classification of finite-dimensional C^*-algebras, since direct sums of matrix algebras are uniquely determined up to isomorphisms by their standard matrix units.

Corollary 1.16. *Let $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ and $B = M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_s}(\mathbb{C})$ be finite-dimensional C^*-algebras. Then A and B are isomorphic as C^*-algebras if and only if $r = s$ and, up to interchanging the order of direct sums, $n_i = m_i$ for each i.*

Having classified the finite-dimensional C^*-algebras, we are able to deduce some properties of finite-dimensional C^*-algebras. First we calculate their K-theory.

Proposition 1.17. *If A is a finite-dimensional C^*-algebra, then $K_0(A) = \mathbb{Z}^r$ for some suitable $r \in \mathbb{N}$, and $K_1(A) = 0$.*

Proof. Since A is finite-dimensional, we can identify A with $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ for some $r \in \mathbb{N}$ and $n_i \in \mathbb{N}$. As the functor K_i preserves direct sums, we see that

$$K_i(A) = \bigoplus_{j=1}^{r} K_i(M_{n_j}(\mathbb{C})),$$

for $i = 0, 1$. Moreover, by Morita equivalence, we have that $K_i(M_{n_j}(\mathbb{C})) = K_i(\mathbb{C})$ for all $j \in \mathbb{N}$ and $i = 0, 1$, so it suffices to calculate the K-theory of the complex numbers. If we show that $K_0(\mathbb{C}) = \mathbb{Z}$ and $K_1(\mathbb{C}) = 0$, then we are done. This is well-known, but we sketch the proofs below.

In order to calculate $K_0(\mathbb{C})$, the easiest way is to first note that if $p, q \in \mathcal{P}_n(\mathbb{C})$ are projections, then $p \sim q$ if and only if $\operatorname{Tr}(p) = \operatorname{Tr}(q)$, where Tr denotes the usual trace. One can use this fact to prove that the map $K_0(\operatorname{Tr})\colon K_0(\mathbb{C}) \to \mathbb{Z}$ by $K_0([p]_0 - [q]_0) = \operatorname{Tr}(p) - Tr(q)$ for $p, q \in \mathcal{P}_\infty(\mathbb{C})$ is a group isomorphism; in particular, $P_\infty(M_n(\mathbb{C}))_{\sim_0}$ is isomorphic to $\mathbb{Z}^+$ as Abelian monoids. To show that $K_1(\mathbb{C}) = 0$, one can verify that the group of unitary elements $\mathcal{U}(M_n(\mathbb{C}))$ on $M_n(\mathbb{C})$ is path-connected by showing that all unitary elements are homotopic to the identity within $\mathcal{U}(M_n(\mathbb{C}))$, which then implies that $K_1(\mathbb{C}) = 0$. The details for both proofs may be found in Example 3.3.2 and Example 8.1.8 in [10]. □

The following corollary is also easily proven using the classification of Theorem 1.15.

Corollary 1.18. *Let A be a finite-dimensional C^*-algebra. Then A is unital, stably finite and has the cancellation property.*

Proof. Unitality of A is clear, but can also be seen a consequence of the proof of Theorem 1.15. For stably finiteness, note that $M_n(A)$ is finite-dimensional for each $n \in \mathbb{N}$, so it suffices to prove that the direct sum of two unital finite C^*-algebras is again finite, which is immediate by Proposition

1.4(ii). For the last part, note that $M_n(\mathbb{C})$ has the cancellation property for all $n \in \mathbb{N}$. Recall that two projections $p, q \in P(M_n(\mathbb{C}))$ are Murray-von Neumann equivalent if and only if they have the same trace. If $p + r \sim q + r$ for some projections $p, q, r \in P(M_n(\mathbb{C}))$ with $p \perp r$ and $q \perp r$, then

$$\mathrm{Tr}(p) + \mathrm{Tr}(r) = \mathrm{Tr}(p + r) = \mathrm{Tr}(q + r) = \mathrm{Tr}(q) + \mathrm{Tr}(r)$$

implying that $\mathrm{Tr}(p) = \mathrm{Tr}(q)$ and, consequently, that $p \sim q$. Moreover, the direct sum of two C^*-algebras with the cancellation property is easily shown to have the cancellation property, and consequently A has the cancellation property. □

1.5 Ordered Abelian groups

For some Abelian groups there can be additional structure hiding. For example, $\mathbb{Z}$ has both the Abelian group structure as well as an order structure, since we have a partial ordering $\leq$ on $\mathbb{Z}$ by $x \geq y$ if and only if $x - y \geq 0$ or, equivalently, if and only if $x - y \in \mathbb{Z}^+$. This partial ordering hence arises from the subset $\mathbb{Z}^+$ of $\mathbb{Z}$ and gives rise to the following definition.

Definition 1.19. An *ordered Abelian group* is a pair (G, G^+) such that G is an Abelian group and $G^+ \subseteq G$ is a subset satisfying the following three conditions:

(i) $G^+ + G^+ \subseteq G^+$;

(ii) $G^+ \cap (-G^+) = \{0\}$;

(iii) $G = G^+ - G^+$.

If (G, G^+) is an ordered Abelian group, then we write $x \leq y$ if $y - x \in G^+$. We call such a subset G^+ the *positive cone* of G.

Note that a given group G may have several different choices for positive cones.

Definition 1.20. Let (G, G^+) and (H, H^+) be ordered Abelian groups. A map $\varphi\colon G \to H$ is called a *positive group homomorphism* or an *order group homomorphism* if it is a group homomorphism and $\varphi(G^+) \subseteq H^+$. If φ is a group isomorphism satisfying $\varphi(G^+) = H^+$, we say that φ is a *positive group isomorphism* or an *order group isomorphism*.

We shall use the above terms interchangeably. The above discussion shows that $(\mathbb{Z}, \mathbb{Z}^+)$ is an ordered Abelian group. It is easily verified that an ordered Abelian group (G, G^+) induces a partially ordered group; in fact, one only uses (i) and (ii) of Definition 1.19 to prove this. Conversely, if $(G, \leq)$ is an Abelian group with a partial ordering, and we define $G^+ = \{g \in G \mid g \geq 0\}$, then (i) and (ii) above are satisfied. The third condition is to ensure that the group is directed in a specific sense, we shall define and show below.

Definition 1.21. Let $(X, \leq)$ be a partially ordered set. We say that X is *upwards (downwards) directed* if any finite subset of X has an upper (lower) bound in X. We say that X is a *lattice* if every finite subset of X has a supremum and an infimum.

Clearly $\mathbb{Z}^n$ is a lattice for all $n \in \mathbb{N}$ with the positive cone $(\mathbb{Z}^+)^n$, but ordered groups need not be lattices. They are, however, upwards and downwards directed by the following proposition.

Proposition 1.22. *Let $(G, \leq)$ be an Abelian group with a partial ordering. Set $G^+ = \{x \in G \mid x \geq 0\}$. Then the following are equivalent.*

(i) *G is upwards directed;*

(ii) *G is downwards directed;*

(iii) *$G = G^+ - G^+$.*

Proof. (i)⇔(ii): This is immediate as $x \mapsto -x$ is an order-reversing bijection on G.
(i)⇒(iii): Let $a \in G$ be arbitrary and find $x \in G^+$ such that $a \leq x$ and $0 \leq x$. Then $x - a, x \in G^+$ such that $a = x - (x - a)$, proving that $G = G^+ - G^+$.
(iii)⇒(i): Let $a_1, \ldots, a_n \in G$ be a finite number of elements in G and find for each i elements $x_i, y_i \in G^+$ such that $a_i = x_i - y_i$. Then $a_i \leq x_i \leq \sum_{j=1}^n x_j$ for all $i = 1, \ldots, n$ proving that G is upwards directed. □

We need one more definition for ordered Abelian groups, as well as a natural definition for group homomorphisms, before we can establish the connection between these group theoretic notions and the classification programme.

Definition 1.23. Let (G, G^+) be an ordered Abelian group. If $u \in G$ satisfies that for any $g \in G$ there exists $n \in \mathbb{N}$ such that $-nu \leq g \leq nu$, then u is called an *order unit*. The triple (G, G^+, u) is then called an *ordered Abelian group with a distinguished order unit*. If every non-zero element of (G, G^+) is an order unit, then we call G *simple*.
A positive group homomorphism $\varphi\colon (G, G^+, u) \to (H, H^+, v)$ is said to be *unit preserving* if $\varphi(u) = v$. The triples (G, G^+, u) and (H, H^+, v) are said to be *isomorphic* if there exists a unit preserving positive group isomorphism from G to H.

As an example, $\mathbb{Z}$ is easily seen to be a simple ordered Abelian group. Not all ordered Abelian groups admit order units, however; one example is the group $c_c(\mathbb{N}, \mathbb{Z})$ of finitely supported sequences $(x_n)_{n\in\mathbb{N}}$ of integers with

$$c_c(\mathbb{N}, \mathbb{Z})^+ = \{(x_n)_{n\in\mathbb{N}} \in c_c(\mathbb{N}, \mathbb{Z}) \,|\, x_n \geq 0 \text{ for all } n \in \mathbb{N}\}\,.$$

This is an ordered Abelian group, but it clearly does not have an order unit. Since K-theory is a way of describing a group to each C^*-algebra, one might ask when the K-theory satisfies any of the previous conditions, e.g. when is it ordered with order units? It turns out that $K_0(A)$ is always an ordered Abelian group for unital and stably finite C^*-algebras A, and even with an order unit $[1_A]_0$. However, before we get ahead of ourselves, we need to define the positive cone of $K_0(A)$.

Definition 1.24. If A is a C^*-algebra, then we define the *positive cone* of $K_0(A)$ to be the subset $K_0(A)^+ = \{[p]_0 \,|\, p \in \mathcal{P}_\infty(A)\}$ of $K_0(A)$.

The definition of the positive cone is a rather natural one. If A is a C^*-algebra, unital or not, we can express $g \in K_0(A)$ as $g = [p]_0 - [s(p)]_0$ for some $p \in \mathcal{P}_\infty(A)$, and consequently the positive cone $K_0(A)^+$ can be viewed as the "positive" part of this difference. In the unital case the positive cone $K_0(A)^+$ can be viewed as the Abelian monoid from which the group $K_0(A)$ is constructed, and since the Grothendieck group generalizes the construction of $\mathbb{Z}$ from $\mathbb{Z}^+$, the definition makes sense. However, it is not always the case that $(K_0(A), K_0(A)^+)$ is an ordered Abelian group; for example, it does not hold for properly infinite, unital C^*-algebras with non-trivial K_0-groups, see [10]. The following proposition gives a sufficient condition for $K_0(A)$ equipped with the positive cone $K_0(A)^+$ to be an ordered Abelian group.

Proposition 1.25. *Let A be a C^*-algebra. Then,*

(i) $K_0(A)^+ + K_0(A)^+ \subseteq K_0(A)^+$*;*

(ii) *If A is unital, then $K_0(A) = K_0(A)^+ - K_0(A)^+$;*

(iii) *If A is stably finite, then $K_0(A)^+ \cap (-K_0(A)^+) = \{0\}$;*

(iv) *If A is unital and stably finite, then $(K_0(A), K_0(A)^+, [1_A]_0)$ is an ordered Abelian group with a distinguished order unit.*

Proof. (i): If $p, q \in \mathcal{P}_\infty(A)$ then $[p]_0 + [q]_0 = [p \oplus q]_0$ proving the desired inclusion.
(ii): Suppose A is unital. Inclusion from the right is trivial, so we prove inclusion from the left. For any $g \in K_0(A)$ there exist projections $p, q \in P_\infty(A)$ such that $g = [p]_0 - [q]_0$ by Proposition 1.6.
(iii): Suppose $g \in K_0(A)^+ \cap (-K_0(A)^+)$ and write $g = [p]_0 = -[q]_0$ for some $p, q \in \mathcal{P}_\infty(A)$. Then $[p \oplus q]_0 = 0$ in $K_0(A)$ and consequently in $K_0(\tilde{A})$, and hence there exists a projection $r \in \mathcal{P}_\infty(\tilde{A})$ with $p \oplus q \oplus r \sim_0 r$. Find a sufficiently large $n \in \mathbb{N}$ such that $p, q, r \in \mathcal{P}_n(\tilde{A})$ and find mutually orthogonal projections $p', q', r' \in \mathcal{P}_n(\tilde{A})$ with $p \sim p'$, $q \sim q'$, $r \sim r'$. Then

$$r' \sim r \sim p \oplus q \oplus r \sim p' + q' + r'.$$

Since A is stably finite, $M_n(\tilde{A})$ is finite and hence all projections on $M_n(\tilde{A})$ are finite by Proposition 1.4. We clearly have that $p' + q' + r' \geq r'$, and this implies that $p' + q' + r' = r'$. Hence $p' = q' = 0$ implying that $p = q = 0$ such that $g = 0$.
(iv) Suppose A is a stably finite C^*-algebra with unit 1, then (i)-(iii) shows that $(K_0(A), K_0(A)^+)$ is an ordered Abelian group. It remains only to be shown that $[1]_0$ is an order unit for $K_0(A)$. Suppose

$g \in K_0(A)$ and find projections $p, q \in \mathcal{P}_n(A)$ such that $g = [p]_0 - [q]_0$. If 1_n denotes the unit on $M_n(A)$, then $[1_n]_0 = n[1]_0$. Moreover, $(1_n - p), (1_n - q) \in \mathcal{P}_n(A)$ and

$$\begin{aligned} -n[1]_0 = -[1_n]_0 = -[q]_0 - [1_n - q]_0 \le -[q]_0 \le [p]_0 - [q]_0 \\ = g \le [p]_0 \le [p]_0 + [1_n - p]_0 = [1_n]_0 = n[1]_0 \end{aligned}$$

proving that $[1]_0$ is an order unit for $(K_0(A), K_0(A)^+)$. □

In the case where $(K_0(A), K_0(A)^+)$ is an ordered Abelian group, we call it the *ordered K_0-group of A*. The conditions above are not necessary in the sense that there exist non-unital C^*-algebras with ordered K_0-groups, for instance non-unital AF-algebras such as the compact operators $\mathbb{K}(H)$ on a separable infinite-dimensional Hilbert space, Example 3.2, as we shall encounter later on in the project.

Proposition 1.26. *If $\varphi\colon A \to B$ is a *-homomorphism, then $K_0(\varphi)\colon K_0(A) \to K_0(B)$ is a positive group homomorphism. Moreover, if φ is a *-isomorphism, then $K_0(\varphi)$ is a positive group isomorphism. If further A and B are unital and φ is unit-preserving, then $K_0(\varphi)$ is unit preserving.*

Proof. The first part is immediate, since $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ for all $p \in \mathcal{P}_\infty(A)$. The second part is also immediate, since if φ is an isomorphism, then $K_0(\varphi)$ is a group isomorphism with $K_0(\varphi)(K_0(A)^+) = K_0(B)^+$. Lastly, in the unital case, we have $K_0(\varphi)([1_A]_0) = [\varphi(1_A)]_0 = [1_B]_0$. □

We can summarize this in the following theorem, which is a theorem due to its importance rather than its difficulty in proving.

Theorem 1.27. *If A is a unital, stably finite C^*-algebra, then $(K_0(A), K_0(A)^+, [1_A]_0)$ is an isomorphism invariant of A.*

One quick use of this isomorphism invariant is to distinguish between different matrix algebras, which is not possible for the usual K_0-group by Morita equivalence. However:

Proposition 1.28. *For any $n \in \mathbb{N}$, we have $(K_0(M_n(\mathbb{C})), K_0(M_n(\mathbb{C}))^+, [1_n]_0) \cong (\mathbb{Z}, \mathbb{Z}^+, n)$.*

Proof. The trace is an order isomorphism $(K_0(M_n(\mathbb{C})), K_0(M_n(\mathbb{C}))^+) \cong (\mathbb{Z}, \mathbb{Z}^+)$, see Proposition 1.17, and $K_0(\mathrm{Tr})([1_n]_0) = n$. □

Consequently, we can distinguish matrix algebras from one another, since they have different ordered K_0-groups with distinguished units. This is, obviously, only a minor result and shall be extended to AF-algebras in Chapter 3.

The following result is an immediate consequence of Proposition 1.8 and the fact that $K_0(\varphi)$ is a positive group homomorphism, if φ is a *-homomorphism.

Proposition 1.29. *For any C^*-algebras A, B, we have $K_0(A \oplus B)^+ = K_0(A)^+ \oplus K_0(B)^+$.*

Note that by Corollary 1.18 and Proposition 1.25, all finite-dimensional C^*-algebras have ordered K_0-groups.

Proposition 1.30. *Let $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ be a finite-dimensional C^*-algebra, and let $\{e_{ij}^{(k)}\}$ be the standard matrix units for A. Then we have the isomorphism*

$$(K_0(A), K_0(A)^+, [1_A]_0) \cong (\mathbb{Z}^r, (\mathbb{Z}^+)^r, (n_1, \ldots, n_r))$$

as ordered Abelian groups with distinguished units. More specifically, we have

$$\begin{aligned} K_0(A) &= \mathbb{Z}[e_{11}^{(1)}]_0 + \cdots + \mathbb{Z}[e_{11}^{(r)}]_0 \\ K_0(A)^+ &= \mathbb{Z}^+[e_{11}^{(1)}]_0 + \cdots + \mathbb{Z}^+[e_{11}^{(r)}]_0 \\ [1_A]_0 &= n_1[e_{11}^{(1)}]_0 + \cdots + n_r[e_{11}^{(r)}]_0. \end{aligned}$$

Proof. Immediate using Proposition 1.28 and Proposition 1.29. □

We call the corresponding order isomorphism $\gamma\colon \mathbb{Z}^r \to K_0(A)$ for the *canonical order isomorphism.* Note in particular that the ordered K_0-group of a finite-dimensional C^*-algebra is finitely generated; this is a fact we shall use in preliminary lemmas to the classification of AF-algebras in Chapter 3.

2 Category theoretical constructions

Scattered throughout the entire project we shall encounter several category theoretical constructions, especially concerning inductive and inverse limits. This chapter is dedicated to examining and proving existence of these structures for specific categories, most importantly the categories of C^*-algebras, compact convex spaces and ordered Abelian groups. For the inductive limits of C^*-algebras in particular we shall prove several somewhat technical lemmas, as they will be needed both in proving continuity of K_0 later on in the chapter, as well as in classifying the unital AF-algebras in Chapter 3. Our main reference of this chapter is [10].

2.1 Inductive limits

If we have a sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$ of objects A_n in a category $\mathscr{C}$, wherein $\varphi_n \in \mathrm{Hom}(A_n, A_{n+1})$ are morphisms, in what way can we construct a *limit*, i.e. a way of describing the objects A_n in the limit as n tends to infinity? Intuitively, if we have a sequence of inclusions $A_1 \subseteq A_2 \subseteq \cdots$ of, say, sets, we would expect that such a limit should be the union $A = \bigcup_{n\in\mathbb{N}} A_n$. If we denote by ι_n the inclusion map $A_n \subseteq A_{n+1}$, and by $\iota_{\infty,n}$ the inclusion map $A_n \subseteq A$, it is clear that $\iota_{\infty,n+1} \circ \iota_n = \iota_{\infty,n}$ for each n; this composition along with a universal property turn out to be defining properties of inductive limits in general categories. Note that we are only interested in *sequential* inductive limits in this project; the definition can easily be extended to arbitrary directed sets, see e.g. [13] for a definition.

Definition 2.1. Let $\mathscr{C}$ be a category. An *inductive sequence* is a sequence $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$, where A_n are objects and $\varphi_n \in \mathrm{Hom}(A_n, A_{n+1})$ are morphisms in $\mathscr{C}$. We shall often write $(A_n, \{\varphi_n\})$ for this inductive sequence.

We shall adopt the notation

$$\varphi_{m,n} = \begin{cases} \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_n & \text{if } m > n, \\ \mathrm{id}_{A_n} & \text{if } m = n \\ 0 & \text{if } m < n \end{cases}.$$

Note that we have the easily remembered identity $\varphi_{m,n} \circ \varphi_{n,k} = \varphi_{m,k}$, and that $\varphi_{n+1,n} = \varphi_n$, for all k, m, n.

Definition 2.2. Let $\mathscr{C}$ be a category and let $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$ be an inductive sequence. An *inductive limit* of this sequence is a system $(A, \{\varphi_{\infty,n}\})$, where A is an object in $\mathscr{C}$ and $\varphi_{\infty,n} \in \mathrm{Hom}(A_n, A)$ are morphisms, such that the following properties are satisfied:

(i) For any $n \in \mathbb{N}$, we have $\varphi_{\infty,n+1} \circ \varphi_n = \varphi_{\infty,n}$.

(ii) If $(B, \{\psi_{\infty,n}\})$ is a system, where B is an object in $\mathscr{C}$, and $\psi_{\infty,n} \in \mathrm{Hom}(A_n, B)$ are morphisms satisfying $\psi_{\infty,n+1} \circ \varphi_n = \psi_{\infty,n}$ for each $n \in \mathbb{N}$, then there exists a unique morphism $\lambda \in \mathrm{Hom}(A, B)$ such that $\lambda \circ \varphi_{\infty,n} = \psi_{\infty,n}$ for all $n \in \mathbb{N}$.

We write $(A, \{\varphi_{\infty,n}\}) = \varinjlim (A_n, \{\varphi_n\})$ or, if the maps are understood, just $A = \varinjlim A_n$.

Inductive limits are also known in literature as *direct limits* or — somewhat confusingly — *colimits*, but we shall only use the name inductive limits in this project. The notation $\varphi_{\infty,n} \colon A_n \to A$ is chosen to give the easily remembered identity $\varphi_{\infty,m} \circ \varphi_{m,n} = \varphi_{\infty,n}$ and to suggest the idea of taking the codomain to be the "limit" object. Note that while we may write $\varinjlim A_n$ for an inductive limit, this is somewhat misleading as the inductive limit strongly depends on the connecting morphisms φ_n.

If an inductive sequence admits an inductive limit, then using the universal property in Definition 2.2 twice ensures that it is unique up to isomorphisms, and we can hence talk about *the* inductive limit of a given inductive sequence. Not all categories admit inductive limits; an example of this is the category of finite sets, $\mathbf{Set_F}$. Non-rigorously, the idea behind the counterproof is easy — an infinite union of strictly increasing finite sets is infinite — but to showcase the potential problems with inductive limits, we shall look more closely at the example.

One can easily verify that the category of sets, **Set**, admits inductive limits, and that the inductive limit of the inclusion of sets $A_1 \subseteq A_2 \subseteq \cdots$ is precisely the union $\bigcup_{n\in\mathbb{N}} A_n$ as described in the introduction. Assume A_n has cardinality n for each $n \in \mathbb{N}$ and let $A = \varinjlim A_n$ be the inductive limit. Note that A is infinite, i.e. it is not an object in $\mathbf{Set_F}$. Since $\mathbf{Set_F}$ is a subcategory of **Set**, we see that if $(A_n, \{\iota_n\})$ had an inductive limit B in $\mathbf{Set_F}$, then it would also be an inductive limit in **Set** and consequently A and B would be isomorphic; this is impossible as A is infinite and B finite.

As not all categories admit inductive limits, we must handle this with some care — we cannot just a priori assume that inductive limits exist! Luckily, the categories of C^*-algebras and (ordered) Abelian groups, which we are mostly interested in, are not as strict as $\mathbf{Set_F}$, and they admit inductive limits as we shall see shortly. We first look at the category of C^*-algebras, and we start by looking at products and sums of arbitrary families of C^*-algebras. Let $\{A_i\}_{i\in I}$ be an arbitrary family of C^*-algebras and define the *product* $\prod_{i\in I} A_i$ as the set of all functions $a\colon I \to \bigcup_{i\in I} A_i$ for which $a_i \in A_i$ for each i, and where $\|a\| = \sup_{i\in I} \|a_i\|$ is finite.

Proposition 2.3. *The set $\prod_{i\in I} A_i$ equipped with pointwise addition, multiplication, scalar multiplication and involutions is a C^*-algebra.*

Proof. It is trivial that it is a *-algebra when equipped with these operations. The triangle inequality is also easy to prove, since

$$\|a + b\| = \sup_{i\in I} \|a_i + b_i\| \le \sup_{i\in I} (\|a_i\| + \|b_i\|) \le \|a\| + \|b\|$$

for all $a, b \in \prod_{i\in I} A_i$. Similarly, if $a \in \prod_{i\in I} A_i$, then

$$\|a^* a\| = \sup_{i\in I} \|a_i^* a_i\| = \sup_{i\in I} \|a_i\|^2 = \|a\|^2 .$$

The only difficulty is hence completeness. Let $\{a^{(n)}\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\prod_{i\in I} A_i$. Since $\left\|a^{(n)}\right\| \ge \left\|a_i^{(n)}\right\|$ for each $i \in I$, we have that $\{a_i^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in A_i. Denote for each $i \in I$ the limit in A_i by a_i and consider the element $a = \{a_i\}_{i\in I}$. We claim that $a \in \prod_{i\in I} a_i$ and $a^{(n)} \to a$ in norm. We have that $a \in \prod_{i\in I} A_i$, since

$$\|a_i\| \le \left\|a_i - a_i^{(n)}\right\| + \left\|a_i^{(n)}\right\| \le \left\|a_i - a_i^{(n)}\right\| + \left\|a^{(n)}\right\|$$

and the first term on the right-hand side tends to 0, and the second term is bounded as $\{a^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Again using the fact that $\{a^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we find that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\|a^{(n)} - a^{(m)}\right\| < \varepsilon$$

whenever $n, m \ge N$. For any $n \ge N$ and $i \in I$ we hence find that

$$\left\|a_i^{(n)} - a_i\right\| = \lim_{m\to\infty} \left\|a_i^{(n)} - a_i^{(m)}\right\| \le \lim_{m\to\infty} \left\|a^{(n)} - a^{(m)}\right\| < \varepsilon.$$

This proves completeness of $\prod_{i\in I} A_i$, and consequently that it is a C^*-algebra. □

Denote by $\sum_{i\in I} A_i$ the norm-closure of the set of $a \in \prod_{i\in I} A_i$ for which $a_i = 0$ for all but finitely many $i \in I$. It is easily verified that $\sum_{i\in I} A_i$ is a closed two-sided ideal in $\prod_{i\in I} A_i$, and in particular it is a C^*-algebra. Before we prove the existence of inductive limits of C^*-algebras, we need the following result on how norms in the quotient $\prod_{i\in I} A_i / \sum_{i\in I} A_i$ can be computed in the sequential case.

Lemma 2.4. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of C^*-algebras and let*

$$\pi\colon \prod_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} A_n / \sum_{n\in\mathbb{N}} A_n$$

be the quotient mapping. For any $a = \{a_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$ we have $\|\pi(a)\| = \limsup_{n\to\infty} \|a_n\|$ and in particular $a \in \sum_{n\in\mathbb{N}} A_n$ if and only if $\lim_{n\to\infty} \|a_n\| = 0$.

Proof. The last part of the lemma is a trivial consequence of the first part. By density of the set

$$\mathcal{I} = \big\{ b \in \prod_{n\in\mathbb{N}} A_n \,|\, b_n = 0 \text{ except for finitely many } n \big\}$$

in $\sum_{n\in\mathbb{N}} A_n$, and continuity of $b \mapsto \|a - b\|$, we have that $\|\pi(a)\| = \inf\{\|a - b\| \mid b \in \mathcal{I}\}$. Then as $b_n = 0$ except for finitely many $n \in \mathbb{N}$, we have

$$\|a - b\| \geq \limsup_{n\to\infty} \|a_n - b_n\| = \limsup_{n\to\infty} \|a_n\|$$

proving one inequality. For the other inequality, define for each $k \in \mathbb{N}$ the element $b^{(k)} \in \mathcal{I}$ by

$$b_n^{(k)} = \begin{cases} a_n & \text{if } n \leq k, \\ 0 & \text{if } n > k \end{cases}.$$

Then

$$\|\pi(a)\| \leq \inf_{k\in\mathbb{N}} \left\| a - b^{(k)} \right\| = \inf_{k\in\mathbb{N}} \sup_{n\geq k} \left\| a_n - b_n^{(k)} \right\| = \limsup_{n\to\infty} \|a_n\|$$

proving the other inequality. □

We are now able to prove existence of inductive limits of C^*-algebras.

Theorem 2.5. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence in the category of C^*-algebras. Then there exists an inductive limit $(A, \{\varphi_{\infty,n}\})$, which satisfies the following:*

(i) $A = \overline{\bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(A_n)}$;

(ii) *For any $n \in \mathbb{N}$ and $a \in A_n$, we have $\|\varphi_{\infty,n}(a_n)\| = \lim_{m\to\infty} \|\varphi_{m,n}(a)\|$.*

(iii) *For any $n \in \mathbb{N}$, we have $a \in \ker \varphi_{\infty,n}$ if and only if $\lim_{m\to\infty} \|\varphi_{m,n}(a)\| = 0$.*

(iv) *Let $(B, \{\psi_{\infty,n}\})$ be a system, where B is a C^*-algebra and $\psi_{\infty,n} : A_n \to B$ are *-homomorphisms satisfying $\psi_{\infty,n+1} \circ \varphi_n = \psi_{\infty,n}$ for each $n \in \mathbb{N}$, and let $\lambda\colon A \to B$ be the unique morphism from Definition 2.2. Then,*

   (a) $\ker \varphi_{\infty,n} \subseteq \ker \psi_{\infty,n}$ *for all $n \in \mathbb{N}$ with equality if and only if λ is injective,*

   (b) *λ is surjective if and only if $B = \overline{\bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(A_n)}$.*

Proof. Let $\pi\colon \prod_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} A_n / \sum_{n\in\mathbb{N}} A_n$ be the quotient mapping and define for each $n \in \mathbb{N}$ the *-homomorphism $\nu_n\colon A_n \to \prod_{m\in\mathbb{N}} A_m$ by $\nu_n(a) = \{\varphi_{m,n}(a)\}_{m\in\mathbb{N}}$ for $a \in A_n$. Now define $\varphi_{\infty,n} = \pi \circ \nu_n$ for each $n \in \mathbb{N}$. We shall use these maps to construct the inductive limit.

For $n \in \mathbb{N}$ and $a \in A_n$, we have

$$\nu_n(a) - \nu_{n+1} \circ \varphi_n(a) = \{a\delta_{nm}\}_{m\in\mathbb{N}} \in \sum_{m\in\mathbb{N}} A_m.$$

where δ_{nm} denotes the Kronecker delta. Hence $\varphi_{\infty,n+1} \circ \varphi_n = \varphi_{\infty,n}$ for all $n \in \mathbb{N}$, and therefore $\{\varphi_{\infty,n}(A_n)\}_{n\in\mathbb{N}}$ is an increasing sequence of C^*-algebras. It is then well-known that

$$A = \overline{\bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(A_n)}$$

is a C^*-algebra. Co-restrict each $\varphi_{\infty,n}$ to be a *-homomorphism $A_n \to A$. We claim that $(A, \{\varphi_{\infty,n}\})$ is the desired inductive limit.

First of all, Definition 2.2(i) is trivially true by construction, so we shall focus on (ii) in the definition. Let $(B, \{\psi_{\infty,n}\})$ be a system such that B is a C^*-algebra and $\psi_{\infty,n}\colon A_n \to B$ are *-homomorphisms satisfying $\psi_{\infty,n+1} \circ \varphi_n = \psi_{\infty,n}$ for all $n \in \mathbb{N}$. Fix a natural number $n \in \mathbb{N}$, then $\psi_{\infty,n} = \psi_{\infty,m} \circ \varphi_{m,n}$

and hence $\|\psi_{\infty,n}\| \leq \|\varphi_{m,n}\|$ for all $m > n$ as *-homomorphisms are contractions. Lemma 2.4 implies that

$$\|\psi_{\infty,n}(a)\| \leq \limsup_{m\to\infty} \|\varphi_{m,n}(a)\| = \|\pi \circ \nu_n(a)\| = \|\varphi_{\infty,n}(a)\|, \qquad a \in A_n.$$

In particular, $\ker \varphi_{\infty,n} \subseteq \ker \psi_{\infty,n}$, whence the map $\psi'_{\infty,n} \colon \varphi_{\infty,n}(A_n) \to B$ given by

$$\psi'_{\infty,n}(\varphi_{\infty,n}(a)) = \psi_{\infty,n}(a), \quad a \in A_n$$

is well-defined and uniquely defined by the relation $\psi'_{\infty,n} \circ \varphi_{\infty,n} = \psi_{\infty,n}$. Since $\{\varphi_{\infty,n}(A_n)\}_{n\in\mathbb{N}}$ is an increasing sequence of C^*-algebras, we find for any $a \in A_n$ that

$$\psi'_{\infty,n+1}(\varphi_{\infty,n}(a)) = \psi'_{\infty,n+1}(\varphi_{\infty,n+1} \circ \varphi_n(a)) = \psi_{\infty,n+1} \circ \varphi_n(a) = \psi_{\infty,n}(a) = \psi'_{\infty,n}(\varphi_{\infty,n}(a))$$

implying that $\psi'_{\infty,n+1}$ extends $\psi'_{\infty,n}$, thus we can define a *-homomorphism $\lambda' \colon \bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(A_n) \to B$ by $\lambda'(a) = \psi'_{\infty,n}(a)$ if $a \in \varphi_{\infty,n}(A_n)$; note that the domain of λ' is not necessarily a C^*-algebra as it generally is not closed. Since each $\psi_{\infty,n}$ is a *-homomorphism between C^*-algebras, they are contractions, and consequently λ' is a contraction. By uniform continuity, we can hence extend λ' to a *-homomorphism $\lambda \colon A \to B$. It follows easily that $\lambda \circ \varphi_{\infty,n} = \psi_{\infty,n}$ Moreover, λ is unique with this property: Suppose $\mu \circ \varphi_{\infty,n} = \psi_{\infty,n}$, then $\mu(a) = \lambda'(a)$ for all $a \in \bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(A_n)$, and as this set is dense in A, we get $\mu = \lambda$. This proves that $(A, \{\varphi_{\infty,n}\})$ is an inductive limit of the inductive sequence $(A_n, \{\varphi_n\})$.

We now prove that the inductive limit $(A, \{\varphi_{\infty,n}\})$ satisfies (i)-(iv) of the theorem.

(i) This is true by construction.

(ii) This follows from Lemma 2.4 using that $\{\|\varphi_{m,n}(a)\|\}_{m\in\mathbb{N}}$ is a decreasing sequence in $[0, \infty)$, since each φ_n is a *-homomorphism between C^*-algebras and hence a contraction.

(iii) This is a direct consequence of (ii).

(iv)(a) The first part has already been proven, so we only need to prove that λ is injective if and only if $\ker \varphi_{\infty,n} = \ker \psi_{\infty,n}$. Note that λ is injective if and only if it is an isometry, and this holds if and only if λ' is an isometry as extensions of isometries are isometries. But λ' is an isometry if and only if each $\psi'_{\infty,n}$ is an isometry or, equivalently, if and only if each $\psi'_{\infty,n}$ is injective. Suppose each $\psi'_{\infty,n}$ is injective, then if $a \in \ker \psi_{\infty,n}$, we have $0 = \psi_{\infty,n}(a) = \psi'_{\infty,n}(\varphi_{\infty,n}(a))$ and hence $a \in \ker \varphi_{\infty,n}(a)$. On the other hand if $\ker \varphi_{\infty,n} = \ker \psi_{\infty,n}$, then if $a \in \ker \psi'_{\infty,n}$ there exists $b \in A_n$ with $\varphi_{\infty,n}(b) = a$ such that

$$0 = \psi_{\infty,n}(a) = \psi'_{\infty,n}(\varphi_{\infty,n}(b)) = \psi_{\infty,n}(b)$$

implying that $b \in \ker \psi_{\infty,n}$, i.e., $b \in \ker \varphi_{\infty,n}$ such that $a = 0$, and consequently $\psi'_{\infty,n}$ is injective.

(iv)(b) This is a direct consequence of $\lambda(A) = \overline{\bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(A_n)}$.

This completes the proof. □

Following the above proof with very few modifications, one easily proves the existence of inductive limits in the category of Abelian groups; we thus state the result without proof.

Theorem 2.6. *Let $(G_n, \{\varphi_n\})$ be an inductive sequence in the category of Abelian groups. Then there exists an inductive limit $(G, \{\varphi_{\infty,n}\})$, which satisfies the following:*

(i) $G = \bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(G_n)$;

(ii) *For any $n \in \mathbb{N}$, we have $a \in \ker \varphi_{\infty,n}$ if and only if $a \in \ker \varphi_{m,n}$ for some $m > n$.*

(iii) *Let $(H, \{\psi_{\infty,n}\})$ be a system, where H is an Abelian group and $\psi_{\infty,n} : G_n \to H$ are group homomorphisms satisfying $\psi_{\infty,n+1} \circ \varphi_n = \psi_{\infty,n}$ for all n, and let $\lambda \colon G \to H$ be the unique morphism from Definition 2.2. Then,*

(a) $\ker \varphi_{\infty,n} \subseteq \ker \psi_{\infty,n}$ *for all $n \in \mathbb{N}$ with equality if and only if λ is injective,*

(b) *λ is surjective if and only if $G = \bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(G_n)$.*

In fact, the category of ordered Abelian groups admits inductive limits.

Theorem 2.7. *Let $((G_n, G_n^+), \{\varphi_n\})$ be an inductive sequence in the category of ordered Abelian groups, where φ_n are positive group homomorphisms. Let $(G, \{\varphi_{\infty,n}\})$ be the inductive limit in the category of Abelian groups, which exists by Theorem 2.6, and let $G^+ = \bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(G_n^+)$. Then (G, G^+) is an ordered Abelian group, each $\varphi_{\infty,n}$ is a positive group homomorphism, and $((G, G^+), \{\varphi_{\infty,n}\})$ is the inductive limit of the above sequence in the category of ordered Abelian groups.*

Proof. We first prove that (G, G^+) is an ordered Abelian group. Note that since φ_n are positive maps, we get

$$\varphi_{\infty,n}(G_n^+) = \varphi_{\infty,n+1}(\varphi_n(G_n^+)) \subseteq \varphi_{\infty,n+1}(G_{n+1}^+)$$

such that $\{\varphi_{\infty,n}(G_n^+)\}$ is an increasing sequence of subsets of G. Hence, for any $x, y \in G^+$ there exists $n \in \mathbb{N}$ such that $x, y \in \varphi_{\infty,n}(G_n^+)$ and thus $x + y \in \varphi_{\infty,n}(G_n^+) \subseteq G^+$. If $x \in G^+ \cap (-G^+)$, then $x \in \varphi_{\infty,n}(G^+) \cap (-\varphi_{\infty,n}(G^+)$ for some $n \in \mathbb{N}$, whence there exist $y_1, y_2 \in G_n^+$ such that $x = \varphi_{\infty,n}(y_1) = -\varphi_{\infty,n}(y_2)$. Then $\varphi_{\infty,n}(y_1 + y_2) = 0$, and thus $y_1 + y_2 \in \ker \varphi_{m,n}$ for some $m > n$. However, then $\varphi_{m,n}(y_1) = -\varphi_{m,n}(y_2) \in G_m^+$ such that $\varphi_{m,n}(y_1) = \varphi_{m,n}(y_2) = 0$, and hence $x = \varphi_{\infty,n}(y_1) = \varphi_{\infty,m}(\varphi_{m,n}(y_1)) = 0$. Lastly if $x \in G$ then $x = \varphi_{\infty,n}(y)$ for some $n \in \mathbb{N}$ and $y \in G_n$. Write $y = y^+ - y^-$ for some $y^\pm \in G_n^+$ and thus $x = \varphi_{\infty,n}(y^+) - \varphi_{\infty,n}(y^-) \in G^+ - G^+$. This proves that (G, G^+) is an ordered Abelian group. Moreover, the definition of G^+ clearly implies that $\varphi_{\infty,n}$ is a positive group homomorphism for each $n \in \mathbb{N}$.

Now we prove that $((G, G^+), \{\varphi_{\infty,n}\})$ is the inductive limit of the sequence $((G_n, G_n^+), \{\varphi_n\})$ in the category of ordered Abelian groups. Note that Definition 2.2(i) holds, as $(G, \{\varphi_{\infty,n}\})$ is the inductive limit in the category of Abelian groups, so we only need to show (ii) in this definition. Let $((H, H^+), \{\psi_{\infty,n}\})$ be a system, where (H, H^+) is an ordered Abelian group and $\psi_{\infty,n} : G_n \to H$ are positive group homomorphisms satisfying $\psi_{\infty,n+1} \circ \varphi_n = \psi_{\infty,n}$ for each n. From Theorem 2.6, we know that there exists a unique group homomorphism $\lambda\colon G \to H$ such that $\lambda \circ \varphi_{\infty,n} = \psi_{\infty,n}$, so we only need to show that λ is positive. This is true, since

$$\lambda(G^+) = \lambda\left(\bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(G_n^+)\right) = \bigcup_{n\in\mathbb{N}} \lambda(\varphi_{\infty,n}(G_n^+) = \bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(G_n^+) \subseteq H^+$$

using that $\psi_{\infty,n}$ are positive group homomorphisms. ☐

At this point the only explicit example of an inductive limit has been the union of an increasing sequence of sets, and this was stated without proof. In order to get a better understanding of concrete inductive limits, we shall examine a few in this section, as well as explain the importance of them for later purposes in the project.

Example 2.8. Let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing family of C^*-algebras, i.e. consider the inductive sequence $(A_n, \{\iota_n\})$, where each ι_n is the inclusion map $\iota_n\colon A_n \to A_{n+1}$. We claim that the inductive limit is isomorphic to $A = \overline{\bigcup_{n=1}^\infty A_n}$. Let $\iota_{\infty,n}\colon A_n \to A$ be the inclusion map for each $n \in \mathbb{N}$, and let $(B, \{\psi_{\infty,n}\})$ denote the inductive limit of the sequence. Since $\iota_{\infty,n+1} \circ \iota_n = \iota_{\infty,n}$ holds for every $n \in \mathbb{N}$, there exists a unique *-homomorphism $\lambda\colon B \to A$ such that $\lambda \circ \psi_{\infty,n} = \iota_{\infty,n}$. We claim that λ is an isomorphism. Injectivity is easy, since $\iota_{\infty,n}$ is injective for each $n \in \mathbb{N}$, and surjectivity is similarly easy, since $A = \overline{\bigcup_{n\in\mathbb{N}} \iota_{\infty,n}(A_n)}$ holds by definition.

While the above example is a straightforward one, it is not without importance; when Bratteli first examined AF-algebras in [2], he defined them as unital C^*-algebras being the norm closure of a union of an increasing sequence of finite-dimensional C^*-algebras, which corresponds to an inductive limit by the above example. Our definition in Chapter 3 is more sophisticated and allows for more freedom in the structure of AF-algebras by accepting the lack of a unit, although we only classify the unital ones. If one removes the assumption regarding the existence of a unit, we obtain our definition of AF-algebras.

The next example is a bit more concrete and examines how the rational numbers $\mathbb{Q}$ arise as the inductive limit of an inductive sequence of integers.

Example 2.9. Consider the inductive sequence $(G_n, \{\varphi_n\})$, where $G_n = \mathbb{Z}$ for each $n \in \mathbb{N}$, and $\varphi_n(m) = nm$ for all n, m. Let $(G, \{\varphi_{\infty,n}\})$ be the inductive limit of this sequence; we claim that $G \cong \mathbb{Q}$ as Abelian groups.

Define for each $n \in \mathbb{N}$ the map $\psi_{\infty,n} \colon \mathbb{Z} \to \mathbb{Q}$ by $\psi_{\infty,n}(m) = \frac{m}{(n-1)!}$ and note that it is well-defined and additive. Then,

$$\psi_{\infty,n+1} \circ \varphi_n(m) = \psi_{\infty,n+1}(nm) = \frac{nm}{(n-1+1)!} = \frac{m}{(n-1)!} = \psi_{\infty,n}(m)$$

and hence there exists a unique group homomorphism $\lambda \colon G \to \mathbb{Q}$ such that $\lambda \circ \varphi_{\infty,n} = \psi_{\infty,n}$. We claim that λ is an isomorphism, and we prove it using Theorem 2.6(iii). Injectivity is easy, since each $\psi_{\infty,n}$ is injective. For surjectivity, let $\frac{n}{m} \in \mathbb{Q}$ be arbitrary and assume without loss of generality that $m > 0$. Then,

$$\frac{n}{m} = \frac{n(m-1)!}{m!} = \psi_{\infty,m+1}(n(m-1)!) \in \bigcup_{k \in \mathbb{N}} \psi_{\infty,k}(\mathbb{Z}).$$

In fact, since each φ_n is positive, and it is easily verified that

$$\mathbb{Q}^+ = \bigcup_{n \in \mathbb{N}} \varphi_n(\mathbb{Z}^+) = \mathbb{Q} \cap [0, \infty),$$

we see that $(\mathbb{Q}, \mathbb{Q}^+)$ is the inductive limit of a sequence of ordered Abelian groups, and is consequently an ordered Abelian group.

We shall in Definition 4.1 define the dimension groups to be the inductive limits of simplical groups, i.e., of ordered Abelian groups of the form $(\mathbb{Z}^n, (\mathbb{Z}^+)^n)$. Since it turns out, see Theorem 3.18 and Theorem 4.2, that the dimension groups with order units completely classify unital AF-algebras, we know that there is an AF-algebra with the ordered K_0-group $(\mathbb{Q}, \mathbb{Q}^+, 1)$. In fact, it is a UHF-algebra, and we shall describe it later in Example 3.30.

Having proved existence and provided a few examples of inductive limits in various categories, we return to the category of C^*-algebras. Not surprisingly, it turns out that a lot can be said about the inductive limit given information about each algebra in the inductive sequence. Separability is a routine exercise in $\frac{\varepsilon}{2}$-proofs.

Proposition 2.10. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence of separable C^*-algebras, then the inductive limit $(A, \{\varphi_{\infty,n}\})$ is separable.*

Proof. Let X_n be a countable dense subset of A_n for each $n \in \mathbb{N}$ and let $X = \bigcup_{n \in \mathbb{N}} X_n$. We claim X is dense in A. Let $\varepsilon > 0$ and $a \in A$, then there exists $n \in \mathbb{N}$ with $\|a - \varphi_{\infty,n}(a_n)\| < \frac{\varepsilon}{2}$ for some $a_n \in A_n$. Then there exists $x_n \in X_n$ such that $\|x_n - a_n\| < \frac{\varepsilon}{2}$. The triangle inequality then gives us that $\|a - x_n\| < \varepsilon$. □

We shall establish other such properties throughout in this chapter and the next. First we shall develop the theory of inductive limits of C^*-algebras further. First we show how passing to subsequences does not change the inductive limit.

Lemma 2.11. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence of C^*-algebras and denote the inductive limit by $(A, \{\varphi_{\infty,n}\})$. Let $\{n_i\}_{i \in \mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$, then $(A, \{\varphi_{\infty,n_i}\})$ is the inductive limit of the sequence $(A_{n_i}, \{\varphi_{n_i}\})$.*

Proof. Let $(B, \{\psi_{\infty,n_i}\})$ be the inductive limit of the subsequence. Since $\varphi_{\infty,n_{i+1}} \circ \varphi_{n_{i+1},n_i} = \varphi_{\infty,n_i}$ trivially holds, there exists a unique *-homomorphism $\lambda \colon B \to A$ such that $\lambda \circ \psi_{\infty,n_i} = \varphi_{\infty,n_i}$. We claim that λ is an isomorphism. For injectivity let $a \in \ker \varphi_{\infty,n}$, then $\lim_{m\to\infty} \|\varphi_{m,n_i}(a)\| = 0$ by Theorem 2.5(iii), and hence $\|\varphi_{n_j,n_i}(a)\| \to 0$ as $j \to \infty$, implying that $a \in \ker \psi_{\infty,n_i}$. For surjectivity note that $A = \overline{\bigcup_{n \in \mathbb{N}} \varphi_{\infty,n}(A_n)}$. We thus merely need to prove that $\bigcup_{n \in \mathbb{N}} \varphi_{\infty,n}(A_n) = \bigcup_{i \in \mathbb{N}} \varphi_{\infty,n_i}(A_{n_i})$. The inclusion from the right is trivial, so suppose $a \in A_n$ and pick $n_i \geq n$, then $\varphi_{\infty,n}(a_n) = \varphi_{\infty,n_i} \circ \varphi_{n_i,n}(a)$ proving the other inclusion. □

Lemma 2.12. *If the connecting maps φ_n of an inductive sequence $(A_n, \{\varphi_n\})$ of C^*-algebras are injective, and $(A, \{\varphi_{\infty,n}\})$ is the inductive limit, then the boundary maps $\varphi_{\infty,n}$ are all injective.*

Proof. Immediate from Theorem 2.5(iii). □

It is easily verified that for any inductive sequence of unital C^*-algebras with unit-preserving connecting maps the inductive limit is also unital. The reverse is not exactly true, but it is true in the limit, when the connecting maps are injective.

Lemma 2.13. *Suppose that $(A_n, \{\varphi_n\})$ is an inductive sequence of C^*-algebras where the connecting maps φ_n are all injective. Assume further that the inductive limit $(A, \{\varphi_{\infty,n}\})$ is unital. Then there exists $N \in \mathbb{N}$ such that A_n is unital and $\varphi_n, \varphi_{\infty,n}$ are unit-preserving maps for all $n > N$.*

Proof. Let $(\varphi_{\infty,m_n}(a_n)) \subseteq \bigcup_{n=1}^{\infty} \varphi_{\infty,n}(A_n)$ be a sequence with $\lim_{n\to\infty} \varphi_{\infty,m_n}(a_n) = 1_A$ and $a_n \in A_{m_n}$. Since the set of invertibles is open in A and 1_A is invertible, we have that a_n is invertible for all sufficiently large n. Consider such an n, then the C^*-algebra B generated by $\varphi_{\infty,m_n}(a_n)$ is a unital C^*-subalgebra of A. Write 1 as a power series in $\varphi_{\infty,m_n}(a_n)$ and $\varphi_{\infty,m_n}(a_n)^*$, then as injective *-homomorphisms are isometries, this power series is convergent if and only if the corresponding power series in a_n and a_n^* is convergent; call this limit 1_{m_n}. It is then clear that $\varphi_{\infty,m_n}(1_{m_n}) = 1$, and an easy calculation show that

$$\varphi_{\infty,m_n}(1_{m_n}a) = \varphi_{\infty,m_n}(a1_{m_n})$$

for all $a \in A_{m_n}$. Injectivity of φ_{∞,m_n} then shows that 1_{m_n} is a unit of A_{m_n}. Similarly one can show that $\varphi_{k,m_n}(1_{m_n})$ is a unit for A_k for each $k \geq m_n$, which proves the remainder of the theorem. □

Interestingly, the constraint that the connecting maps are injective in the above lemmas is weak in the sense that we for any inductive sequence can construct another inductive sequence with the same inductive limit, but whose connecting maps are all injective.

Proposition 2.14. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence of C^*-algebras with inductive sequence $(A, \{\varphi_{\infty,n}\})$. Let $B_n = A_n/\ker\varphi_{\infty,n}$ and let $\pi_n \colon A_n \to B_n$ be the quotient map. Then there exist unique injective *-homomorphisms $\psi_n \colon B_n \to B_{n+1}$ and a *-isomorphism $\pi \colon A \to B$ such that the diagram*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & A_3 & \xrightarrow{\varphi_3} & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & & & \downarrow{\scriptstyle \pi} \\
B_1 & \xrightarrow[\psi_1]{} & B_2 & \xrightarrow[\psi_2]{} & B_3 & \xrightarrow[\psi_3]{} & \cdots & \longrightarrow & \varinjlim B_n
\end{array}$$

is commutative.

Proof. Suppose that $\pi_n(a) = \pi_n(b)$, then $a - b \in \ker\varphi_{\infty,n}$ by definition and hence

$$\varphi_{\infty,n+1}(\varphi_n(a-b)) = \varphi_{\infty,n}(a-b) = 0$$

such that $\pi_{n+1} \circ \varphi_n(a) = \pi_{n+1} \circ \varphi_n(b)$. This proves that the map $\psi_n \colon B_n \to B_{n+1}$ by $\psi_n(\pi_n(a)) = \pi_{n+1} \circ \varphi_n(a)$ is well-defined. It is clearly a *-homomorphism. For injectivity, suppose $\pi_n(a) \in \ker\psi_n$, then

$$0 = \psi_n \circ \pi_n(a) = \pi_{n+1} \circ \varphi_n(a)$$

such that $\varphi_n(a) \in \ker\varphi_{\infty,n+1}$. However,

$$0 = \varphi_{\infty,n+1} \circ \varphi_n(a) = \varphi_{\infty,n}(a)$$

implying that $a \in \ker\varphi_{\infty,n}$ and hence $\pi_n(a) = 0$. This proves that the maps ψ_n are injective *-homomorphisms. It follows by construction that $\psi_n \circ \pi_n = \pi_{n+1} \circ \varphi_n$ for all $n \in \mathbb{N}$.

Now let $(B, \{\psi_{\infty,n}\})$ be the inductive limit of the sequence $(B_n, \{\psi_n\})$ and note that each $\psi_{\infty,n}$ is injective by Lemma 2.12. Define for each $n \in \mathbb{N}$ the *-homomorphism $\alpha_{\infty,n} = \psi_{\infty,n} \circ \pi_n$. Then $\alpha_{\infty,n} = \alpha_{\infty,n+1} \circ \varphi_n$ and hence there exists a unique *-homomorphism $\pi \colon A \to B$ such that

$$\pi \circ \varphi_{\infty,n} = \alpha_{\infty,n} = \psi_{\infty,n} \circ \pi_n.$$

We now only need to show that π is an isomorphism. Surjectivity is clear by Theorem 2.5(iv), since

$$B = \overline{\bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(B_n)} = \overline{\bigcup_{n\in\mathbb{N}} \psi_{\infty,n} \circ \pi_n(A_n)} = \overline{\bigcup_{n\in\mathbb{N}} \alpha_{\infty,n}(A_n)}.$$

For injectivity suppose that $a \in \ker \alpha_{\infty,n}$, then

$$0 = \alpha_{\infty,n}(a) = \psi_{\infty,n} \circ \pi_n(a)$$

implying that $a \in \ker \varphi_{\infty,n}$ by injectivity of $\psi_{\infty,n}$ and definition of π_n. □

In order to understand the structure of K_0-groups of inductive limits, we need to understand the structure of the projections in the limit. One thing we need to know is when two projections in the sequence are Murray-von Neumann equivalent in the limit. The following lemma gives a complete description of this.

Lemma 2.15. *Let $(A, \{\varphi_{\infty,n}\})$ be the inductive limit of the inductive sequence $(A_n, \{\varphi_n\})$ of C^*-algebras. Let $p, q \in A_n$ be projections. Then $\varphi_{\infty,n}(p) \sim \varphi_{\infty,n}(q)$ if and only if there exists $m \geq n$ such that $\varphi_{m,n}(p) \sim \varphi_{m,n}(q)$.*

Proof. If $\varphi_{m,n}(p) \sim \varphi_{m,n}(q)$ for some $m \geq n$, it is clear that $\varphi_{\infty,n}(p) \sim \varphi_{\infty,n}(q)$. Conversely, suppose $\varphi_{\infty,n}(p) \sim \varphi_{\infty,n}(q)$ and find a partial isometry $v \in A$ such that $\varphi_{\infty,n}(p) = v^*v$ and $\varphi_{\infty,n}(q) = vv^*$. Find by Theorem 2.5(i) a natural number $\ell \geq n$ and an element $x_\ell \in A_\ell$ such that $\varphi_{\infty,\ell}(x_\ell)$ is sufficiently close to v to ensure that

$$\|\varphi_{\infty,\ell}(x_\ell^* x_\ell) - \varphi_{\infty,n}(p)\| < \frac{1}{2}, \qquad \text{and} \qquad \|\varphi_{\infty,\ell}(x_\ell x_\ell^*) - \varphi_{\infty,n}(q)\| < \frac{1}{2}.$$

Rewriting this and using that $\|\varphi_{\infty,k}(a)\| = \lim_{m\to\infty} \|\varphi_{m,k}(a)\|$ for all $a \in A_k$, Theorem 2.5(ii), we find that

$$\begin{aligned}\frac{1}{2} &> \|\varphi_{\infty,\ell}(x_\ell^* x_\ell) - \varphi_{\infty,n}(p)\| \\ &= \|\varphi_{\infty,\ell}(x_\ell^* x_\ell - \varphi_{\ell,n}(p))\| \\ &= \lim_{m\to\infty} \|\varphi_{m,\ell}(x_\ell^* x_\ell - \varphi_{\ell,n}(p))\| \\ &= \lim_{m\to\infty} \|x_m^* x_m - \varphi_{m,n}(p)\|\end{aligned}$$

where $x_m = \varphi_{m,\ell}(x_\ell)$. In particular, there exists $m \geq \ell$ such that

$$\|\varphi_{m,\ell}(x_\ell^* x_\ell) - \varphi_{m,n}(p)\| < \frac{1}{2}, \qquad \text{and} \qquad \|\varphi_{m,\ell}(x_\ell x_\ell^*) - \varphi_{m,n}(q)\| < \frac{1}{2}.$$

It then follows from Lemma 1.2 that $\varphi_{m,n}(p) \sim \varphi_{m,n}(q)$. □

This lemma can also be used to show that the cancellation property is preserved by inductive limits. The proof uses a fact to be proven later in the chapter, namely that taking matrix algebras preserves inductive limits in the sense that if $(A_n, \{\varphi_n\})$ is an inductive sequence of C^*-algebras with inductive limit $(A, \{\varphi_{\infty,n}\})$, then

$$\lim_{\to}(M_N(A_n), \{\varphi_n^{(N)}\}) = (M_N(A), \{\varphi_{\infty,n}^{(N)}\}).$$

Proposition 2.16. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence of C^*-algebras, where each A_n has the cancellation property. Denote the inductive limit by $(A, \{\varphi_{\infty,n}\})$, then A has the cancellation property.*

Proof. Let $p, q, r \in \mathcal{P}_\infty(A)$ be projections satisfying $p \perp r$ and $q \perp r$ and $(p + r) \sim (q + r)$. We claim that $p \sim q$. Find $N \in \mathbb{N}$ so that $p, q, r \in \mathcal{P}_N(A)$ and note that $p \sim p \oplus 0_N$, $q \sim q \oplus 0_N$, $r \sim 0_N \oplus r$, i.e. embed p and q into a 2×2-matrix over $M_N(A)$ in the upper-left entry, and r into the lower-right entry; we can hence assume that p, q and r are as such in the following.
We shall find $n \in \mathbb{N}$ and projections $p', q', r' \in \mathcal{P}_N(A_n)$ such that $p' \perp r'$ and $q' \perp r'$ along with

$$\left\|\varphi_{\infty,n}^{(N)}(p') - p\right\| < 1$$

and similar inequalities for q and r.
First of all, since taking matrix algebras preserves inductive limits, to be proven later in this project, see Proposition 2.22, we find that $\varphi_{\infty,n}^{(N)}: M_N(A_n) \to M_N(A)$ are the boundary maps in the inductive limit of the inductive sequence $(M_N(A_n), \{\varphi_n^{(N)}\})$. Since $\bigcup_{n=1}^{\infty} \varphi_{\infty,n}^{(N)}(M_N(A_n))$ is dense in $M_N(A)$, see Theorem 2.5(i), there exist $m \in \mathbb{N}$ and $b_m \in M_N(A_m)$ such that $\left\|\varphi_{\infty,m}^{(N)}(b_m) - p\right\| < \frac{1}{5}$. Let $a_m = \frac{b_m + b_m^*}{2}$ be self-adjoint and put $a_n = \varphi_{n,m}^{(N)}(a_n)$ for all $n > m$. It is immediate that a_n is self-adjoint with $\left\|\varphi_{\infty,n}^{(N)}(a_n) - p\right\| < \frac{1}{5}$, for $m \geq n$. In particular, we find that the spectrum of $\varphi_{\infty,n}^{(N)}(a_n)$, being within $\frac{1}{5}$ of a projetion, lies within $[-1/5, 1/5] \cup [4/5, 6/5]$ for all $m \geq n$. Then, $\left\|\varphi_{\infty,m}^{(N)}(a_m^2 - a_m)\right\| < \frac{1}{4}$ and, by Theorem 2.5(ii), there exists $n \geq m$ with $\left\|a_n^2 - a_n\right\| < \frac{1}{4}$. It is now possible using Lemma 1.1 to find a projection p' such that $\|a_n - p'\| < \frac{1}{2}$, and the triangle inequality thus implies that $\left\|\varphi_{\infty,n}^{(N)}(p') - p\right\| < 1$ as desired. Do the same constructions for q and r and note that $p' \perp r'$ and $q' \perp r'$, and that $\varphi_{\infty,n}^{(N)}(p') \sim p$ and similarly for q and r. We hence find that

$$\varphi_{\infty,n}^{(N)}(p' + r') \sim (p + r) \sim (q + r) \sim \varphi_{\infty,n}^{(N)}(q' + r').$$

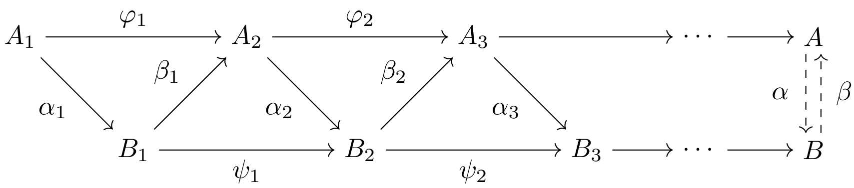
Then Lemma 2.15 implies that there exists $m \geq n$ such that $\varphi_{m,n}^{(N)}(p' + r') \sim \varphi_{m,n}^{(N)}(q' + r')$. Since A_m has the cancellation property, we find that $\varphi_{m,n}^{(N)}(p') \sim \varphi_{m,n}^{(N)}(q')$, and hence

$$p \sim \varphi_{\infty,n}^{(N)}(p') = \varphi_{\infty,m}^{(N)}(\varphi_{m,n}^{(N)}(p')) \sim \varphi_{\infty,m}^{(N)}(\varphi_{m,n}^{(N)}(q')) = \varphi_{\infty,n}^{(N)}(q') \sim q$$

as desired. □

The last lemma states that if two inductive sequences are intertwined, their inductive limits agree.

Lemma 2.17. *Let $(A_n, \{\varphi_n\})$ and $(B_n, \{\psi_n\})$ be inductive sequences of C^*-algebras with inductive limits $(A, \{\varphi_{\infty,n}\})$ and $(B, \{\psi_{\infty,n}\})$ respectively. Suppose that there exist *-homomorphisms $\alpha_n: A_n \to B_n$ and $\beta_n: B_n \to A_{n+1}$ for each $n \in \mathbb{N}$ such that the diagram*



*is commutative. Then there exist *-isomorphisms $\alpha: A \to B$ and $\beta: B \to A$ as indicated by dashed arrows making the entire diagram commute.*

Proof. Define the *-homomorphisms $\hat{\alpha}_{\infty,n}: A_n \to B$ by $\tilde{\alpha}_{\infty,n} = \psi_{\infty,n} \circ \alpha_n$ and $\hat{\beta}_{\infty,n}: B_n \to A$ by $\hat{\beta}_{\infty,n} = \varphi_{\infty,n+1} \circ \beta_n$. Then,

$$\hat{\alpha}_{\infty,n+1} \circ \varphi_n = \psi_{\infty,n+1} \circ \alpha_{n+1} \circ \varphi_n = \psi_{\infty,n+1} \circ \psi_n \circ \alpha_n = \psi_{\infty,n} \circ \alpha_n = \hat{\alpha}_{\infty,n},$$

and similarly $\hat{\beta}_{\infty,n+1} \circ \psi_n = \hat{\beta}_{\infty,n}$. It then follows from the definition of inductive limits that there exist unique *-homomorphisms $\alpha: A \to B$ and $\beta: B \to A$ such that $\alpha \circ \varphi_{\infty,n} = \hat{\alpha}_{\infty,n}$ and $\beta \circ \psi_{\infty,n} = \hat{\beta}_{\infty,n}$. We claim that α and β are *-isomorphisms with $\alpha = \beta^{-1}$. We prove here that $\beta \circ \alpha = \mathrm{id}_A$; the other way follows analogously. Note that since $A = \overline{\bigcup_{n \in \mathbb{N}} \varphi_{\infty,n}(A_n)}$, it suffices by continuity to prove that $\beta \circ \alpha \circ \varphi_{\infty,n} = \varphi_{\infty,n}$ for all $n \in \mathbb{N}$. This is a simple calculation, since

$$\beta \circ \alpha \circ \varphi_{\infty,n} = \beta \circ \hat{\alpha}_{\infty,n} = \beta \circ \psi_{\infty,n} \circ \alpha_n = \hat{\beta}_{\infty,n} \circ \alpha_n = \varphi_{\infty,n+1} \circ \beta_n \circ \alpha_n = \varphi_{\infty,n+1} \circ \varphi_n = \varphi_{\infty,n}$$

completing the proof. □

2.2 Inverse limits

Given a category $\mathscr{C}$, one can associate the opposite category $\mathscr{C}^{\mathrm{op}}$ with the same objects, but with the arrows reversed. The notion of inverse limits is precisely the same; take the definition of inductive limits and reverse all the arrows. Equivalently, inverse limits in $\mathscr{C}$ are precisely inductive limits in $\mathscr{C}^{\mathrm{op}}$. We are not as interested in specific details on inverse limits in this project, as we are in inductive limits, since we shall not encounter them often; in fact, they only appear in Chapter 5. Once again, we only study *sequential* inverse limits in this project; for the general definition we refer to [13].

Definition 2.18. Let $\mathscr{C}$ be a category. An *inverse sequence* is a sequence $A_1 \xleftarrow{\varphi_1} A_2 \xleftarrow{\varphi_2} \cdots$, where A_n are objects and $\varphi_n \in \mathrm{Hom}(A_{n+1}, A_n)$ are morphisms in $\mathscr{C}$. We shall often write $(A_n, \{\varphi_n\})$ for this inverse sequence.

Note that the notation $(A_n, \{\varphi_n\})$ is the same for inverse sequences and inductive sequences; we shall consequently always explicitly mention what type the sequence is, whenever we use this notation.

Definition 2.19. Let $\mathscr{C}$ be a category and let $A_1 \xleftarrow{\varphi_1} A_2 \xleftarrow{\varphi_2} \cdots$ be an inductive sequence. An *inductive limit* of this sequence is a system $(A, \{\varphi_{n,\infty}\})$, where A is an object in $\mathscr{C}$ and $\varphi_{n,\infty} \in \mathrm{Hom}(A, A_n)$ are morphisms, such that the following properties are satisfied:

(i) For any $n \in \mathbb{N}$, we have $\varphi_n \circ \varphi_{n+1,\infty} = \varphi_{n,\infty}$;

(ii) If $(B, \{\psi_{n,\infty}\})$ is a system, where B is an object in $\mathscr{C}$ and $\psi_{n,\infty} \in \mathrm{Hom}(B, A_n)$ are morphisms satisfying $\varphi_n \circ \psi_{n+1,\infty} = \psi_{n,\infty}$ for each n, then there exists a unique morphism $\lambda \in \mathrm{Hom}(B, A)$ such that $\varphi_{n,\infty} \circ \lambda = \psi_{n,\infty}$ for all $n \in \mathbb{N}$.

We write $(A, \{\varphi_{\infty,n}\}) = \varprojlim(A_n, \{\varphi_n\})$ or, if the maps are understood, just $A = \varprojlim A_n$.

Once again we emphasize that the inverse limits depends strongly on the connecting maps. In the literature, inverse limits are also known as *projective limits* or — again somewhat confusingly — as *limits*. An inverse limit of an inverse sequence is unique up to isomorphism by taking the universal property twice, and we can consequently talk about *the* inverse limit. The notation of the connecting maps φ_n and boundary maps $\varphi_{n,\infty}$ are chosen to mimic the idea of reversing the arrows of inductive limits. For example, we define

$$\varphi_{n,m} = \begin{cases} \varphi_n \circ \varphi_{n+1} \circ \cdots \circ \varphi_{m-1} & \text{if } m > n, \\ \mathrm{id}_{A_n} & \text{if } m = n \\ 0 & \text{if } m < n \end{cases}$$

and get the identities $\varphi_{n,m} \circ \varphi_{m,k} = \varphi_{n,k}$ and $\varphi_{n,m} \circ \varphi_{m,\infty} = \varphi_{n,\infty}$ for all n, m, k.
Several categories admit inverse limits, and there is a general structure to the inverse limits. The following discussion is nonsensical in the sense that the objects examined might not be in the desired category in general, but it gives an intuition for working with inverse limits.
Suppose $(A_n, \{\varphi_n\})$ is an inverse sequence in some category $\mathscr{C}$. Consider the structure

$$A = \left\{ a = (a_1, a_2, \ldots) \in \prod_{n \in \mathbb{N}} A_n \,|\, a_n = \varphi_n(a_{n+1}) \text{ for all } n \in \mathbb{N} \right\} \tag{2.1}$$

and assume that this as well as the projection maps $\varphi_{n,\infty} \colon A \to A_n$ by $\varphi_{n,\infty}(a_1, a_2, \ldots) = a_n$ belong to the category. It is clear that $\varphi_n \circ \varphi_{n+1,\infty} = \varphi_{n,\infty}$. Suppose that $(B, \{\psi_{n,\infty}\})$ is another system, where B is an object and $\psi_{n,\infty} \in \mathrm{Hom}(B, A_n)$ are morphisms in $\mathscr{C}$ satisfying $\varphi_n \circ \psi_{n+1,\infty} = \psi_{n,\infty}$ and construct $\lambda \colon B \to A$ by

$$\lambda(b) = (\psi_{1,\infty}(b), \psi_{2,\infty}(b), \ldots)$$

assuming that it is a morphism in $\mathscr{C}$. It is then immediate that λ is unique with the property $\varphi_{n,\infty} \circ \lambda = \psi_{n,\infty}$ for all $n \in \mathbb{N}$, proving that A — if the above assumptions are true — is the inverse limit.
As an example of how one can use this in a concrete category, consider the category **CptConv** of compact convex subsets of Hausdorff topological vector spaces with morphisms being affine continuous mappings.

Theorem 2.20. *The category* **CptConv** *admits inverse limits.*

Proof. Let $(A_n, \{\varphi_n\})$ be an inverse system in **CptConv** and define the set

$$A = \{(a_1, a_2, \ldots) \in \prod_{n \in \mathbb{N}} A_n \,|\, a_n = \varphi_n(a_{n+1}) \text{ for all } n \in \mathbb{N}\}.$$

Since each φ_n is continuous, $A \subseteq \prod_{n \in \mathbb{N}} A_n$ is a closed subset. By Tychonoff's theorem, $\prod_{n \in \mathbb{N}} A_n$ is compact and, hence, so is A. Convexity holds as each φ_n is affine. Thus A is an object in **CptConv**.

Define the affine continuous projections $\varphi_{n,\infty}\colon A \to A_n$ by $\varphi_{n,\infty}(a_1, a_2, \ldots) = a_n$. These maps clearly satisfies $\varphi_n \circ \varphi_{n+1,\infty} = \varphi_{n,\infty}$. Suppose that $(B, \{\psi_{n,\infty}\})$ is another system such that $\varphi_n \circ \psi_{n+1,\infty} = \psi_{n,\infty}$ and define an affine continuous map $\lambda\colon B \to A$ by $\lambda(b) = (\psi_{1,\infty}(b), \psi_{2,\infty}(b), \ldots)$ for each $b \in B$. It is immediate that this satisfies $\varphi_{n,\infty} \circ \lambda = \psi_{n,\infty}$ for each $n \in \mathbb{N}$, and it is unique by this characterization. □

Other than the existence of inverse limits in **CptConv**, there are not many specific results we need. One result we do need, which is also fairly trivial, is the following.

Lemma 2.21. *Let $(A_n, \{\varphi_n\})$ be an inverse sequence in a category $\mathscr{C}$ and assume it has a canonical inverse limit $(A, \{\varphi_{n,\infty}\})$ as in (2.1). Let $a, a' \in A$ be arbitrary. If $\varphi_{n,\infty}(a) = \varphi_{n,\infty}(a')$ for all sufficiently large $n \in \mathbb{N}$, then $a = a'$.*

Proof. Write $a = (a_1, a_2, \ldots)$ and $a' = (a'_1, a'_2, \ldots)$, then by construction of the canonical inverse limit, we see that $a_n = a'_n$ for all sufficiently large $n \in \mathbb{N}$. The proof then follows from the fact that $a_n = \varphi_n(a_{n+1})$ holds for all $n \in \mathbb{N}$ and the analogue identity for the primed elements. □

2.3 Continuity of K_i for $i = 0, 1$

It is clear that a covariant functor takes inductive/inverse sequences to inductive/inverse sequences, and that a contravariant functor takes inductive/inverse sequences to inverse/inductive sequences. However it is not true in general that the limits are preserved; the minimal tensor product functor $- \otimes_{\min} B$ in the category of C^*-algebras does not in general preserve inductive limits, although the maximal tensor product functor $- \otimes_{\max} B$ does, see [1, II.9.6.5]. We say that a functor F is *continuous* if it preserves inductive limits. This terminology differs from other literature, as one would usually expect that continuous functors preserve limits, that is, inverse limits. We shall, however, follow the terminology of [10].
As we shall see later, AF-algebras are inductive limits of finite-dimensional C^*-algebras, and if we want to calculate their K-theory, it would be significantly easier if K_i is a continuous functor for $i = 0, 1$. Luckily both K_0 and K_1 are continuous functors from $\mathbf{C}^*$ to **Ab** (or **OrdAb**), as we shall see. First we need to show that taking unitization and matrix algebras are continuous functors on $\mathbf{C}^*$.

Proposition 2.22. *Let B be a C^*-algebra and $(A_n, \{\varphi_n\})$ an inductive system of C^*-algebras with inductive limit $(A, \{\varphi_{\infty,n}\})$. Then we have the following isomorphisms of C^*-algebras:*

(i) $\varinjlim(\tilde{A}_n, \{\tilde{\varphi}_n\}) \cong (\tilde{A}, \{\tilde{\varphi}_{\infty,n}\})$;

(ii) $\varinjlim(M_N(A_n), \{\varphi_n^{(k)}\}) \cong (M_N(A), \{\varphi_{\infty,n}^{(N)}\})$.

Proof. The proof of (ii) is very similar to (i), so let us only prove (i). Both proofs essentially just use Theorem 2.5(iv).

Let $(B, \{\psi_{\infty,n}\}) = \varinjlim(\tilde{A}_n, \{\tilde{\varphi}_n\})$ and note that $\tilde{\varphi}_{\infty,n+1} \circ \tilde{\varphi}_n = \tilde{\varphi}_{\infty,n}$ such that there exists a unique *-homomorphism $\lambda\colon B \to \tilde{A}$ with $\lambda \circ \psi_{\infty,n} = \tilde{\varphi}_{\infty,n}$ for all $n \in \mathbb{N}$. We claim that λ is a *-isomorphism. Note that if $x \in \tilde{A}$, then $x = a + \mu \cdot 1$ for $a \in A$ and $\mu \in \mathbb{C}$. Find a sequence such that $\varphi_{\infty,m_n}(a_n) \to a$ as $n \to \infty$, then

$$\tilde{\varphi}_{\infty,m_n}(a_n + \mu \cdot 1) = \varphi_{\infty,m_n}(a_n) + \mu \cdot 1 \to a + \mu \cdot 1$$

as $n \to \infty$, which shows that $\tilde{A} = \overline{\bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(A_n)}$, and hence λ is surjective by Theorem 2.5(iv)(a). Moreover, suppose $x \in \ker \tilde{\varphi}_{\infty,n}$ and write $x = a + \mu \cdot 1$ for $a \in A$ and $\mu \in \mathbb{C}$. Then,

$$0 = \tilde{\varphi}_{\infty,n}(a + \mu \cdot 1) = \varphi_{\infty,n}(a) + \mu \cdot 1$$

such that $x \in \ker \varphi_{\infty,n}$ and $\mu = 0$. Then $\|\varphi_{m,n}(a)\| \to 0$ as $m \to \infty$ by Theorem 2.5(iii), such that

$$\|\tilde{\varphi}_{m,n}(x)\| = \|\varphi_{m,n}(a)\| \to 0, \quad \text{as } m \to \infty$$

The same theorem implies that $\psi_{\infty,n}(x) = 0$, and hence we conclude that λ is injective. □

We are now ready to prove continuity of K_0.

Theorem 2.23 (Continuity of K_0)**.** *Let $(A_n, \{\varphi_n\})$ be an inductive sequence of C^*-algebras. Then $K_0(\varinjlim A_n) \cong \varinjlim K_0(A_n)$ as Abelian groups and, in the case where $(K_0(A_n), K_0(A_n)^+)$ is an ordered Abelian group for each n, then $K_0(\varinjlim A_n) \cong \varinjlim K_0(A_n)$ as ordered Abelian groups.*
In more precise terms, suppose that $(A, \{\varphi_{\infty,n}\})$ is the inductive limit of the inductive sequence, and $(G, \{\psi_{\infty,n}\})$ is the inductive limit of the inductive sequence $(K_0(A_n), \{K_0(\varphi_n)\})$ in $\mathbf{Ab}$, then there exists a unique group isomorphism $\lambda\colon G \to K_0(A)$ such that $\lambda \circ \psi_{\infty,n} = K_0(\varphi_{\infty,n})$. If $(K_0(A_n), K_0(A_n)^+)$ are ordered Abelian groups for each $n \in \mathbb{N}$, then λ is a positive group isomorphism. Moreover, the following properties are satisfied:

(i) $K_0(A) = \bigcup_{n=1}^{\infty} K_0(\varphi_{\infty,n})(K_0(A_n))$*;*

(ii) *For any $n \in \mathbb{N}$, $x \in \ker K_0(\varphi_{\infty,n})$ if and only if $x \in \ker K_0(\varphi_{m,n})$ for some $m > n$.*

(iii) *If $(K_0(A_n), K_0(A_n)^+)$ are ordered Abelian groups for all n, then*

$$K_0(A)^+ = \bigcup_{n=1}^{\infty} K_0(\varphi_{\infty,n})(K_0(A_n)^+).$$

Proof. We first see how (i)-(iii) imply the rest of the theorem.
By functoriality of K_0, we find that $K_0(\varphi_{\infty,n+1}) \circ K_0(\varphi_n) = K_0(\varphi_{\infty,n})$, hence there exists a unique $\lambda\colon G \to K_0(A)$ such that $\lambda \circ \psi_{\infty,n} = K_0(\varphi_{\infty,n})$. If we have proven (i), then λ is surjective by Theorem 2.6. Suppose that we have proven (ii) and let $g \in \ker \lambda$. Since $G = \bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(K_0(A_n))$ we can write $g = \psi_{\infty,n}(h)$ for some $n \in \mathbb{N}$ and $h \in K_0(A_n)$. Then,

$$0 = \lambda(g) = \lambda \circ \psi_{\infty,n}(h) = K_0(\varphi_{\infty,n})(h).$$

Then (ii) implies that there exists $m > n$ such that $K_0(\varphi_{m,n})(h) = 0$, and hence

$$g = \psi_{\infty,n}(h) = \psi_{\infty,m} \circ K_0(\varphi_{m,n})(h) = 0$$

proving injectivity.
Now suppose that each $(K_0(A_n), K_0(A_n)^+)$ is an ordered Abelian group, then the inductive limit (G, G^+) is also an ordered Abelian group by Theorem 2.6 with $G^+ = \bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(K_0(A_n)^+)$. Then, if (iii) is true, we get

$$\lambda(G^+) = \lambda\left(\bigcup_{n\in\mathbb{N}} \psi_{\infty,n}(K_0(A_n)^+)\right) = \bigcup_{n\in\mathbb{N}} \lambda \circ \psi_{\infty,n}(K_0(A_n)^+) = \bigcup_{n\in\mathbb{N}} K_0(\varphi_{\infty,n}(K_0(A_n)^+) = K_0(A)^+$$

proving that λ is a positive group isomorphism.

Having shown how proving (i)-(iii) implies the rest of the theorem, let us turn our attention to proving these three parts. Note that taking unitizations and matrix algebras are continuous functors by Proposition 2.22.
(i): The inclusion from the right is trivial, so assume $g \in K_0(A)$. Then the standard picture of K_0 implies that there exists some projection $p \in M_k(\tilde{A})$ for some $k \in \mathbb{N}$ such that $g = [p]_0 - [s(p)]_0$. The proof of Proposition 2.16 implies that there exist natural numbers $n, N \in \mathbb{N}$ and a projection $p' \in \mathcal{P}_N(A_n)$ such that $\varphi_{\infty,n}^{(N)}(p') \sim p$ are Murray-von Neumann equivalent. Thus,

$$g = [p]_0 - [s(p)]_0 = [\tilde{\varphi}_{\infty,n}^{(N)}(p')]_0 - [s(\tilde{\varphi}_{\infty,n}^{(N)}(p'))]_0 = K_0(\varphi_{\infty,n})([p']_0 - [s(p')]_0)$$

proving the desired inclusion.

(ii): The inclusion from the right is trivial, so suppose $g \in \ker K_0(\varphi_{\infty,n})$. By Proposition 1.7, there exist $k \in \mathbb{N}$ and a projection $p \in M_k(\tilde{A}_n)$ with $g = [p]_0 - [s(p)]_0$ and

$$\tilde{\varphi}_{\infty,n}^{(k)}(p) \sim \tilde{\varphi}_{\infty,n}^{(k)}(s(p)).$$

By Lemma 2.15, there exists $m > n$ such that $\tilde{\varphi}_{m,n}^{(k)}(p) \sim \tilde{\varphi}_{m,n}^{(k)}(s(p))$, and hence

$$K_0(\varphi_{m,n})(g) = [\tilde{\varphi}_{m,n}^{(k)}(p)]_0 - [\tilde{\varphi}_{m,n}^{(k)}(s(p))]_0 = [\tilde{\varphi}_{m,n}^{(k)}(p)]_0 - [s(\tilde{\varphi}_{m,n}^{(k)}(p))]_0 = 0$$

proving (ii).

(iii): The inclusion from the right is immediate, as $K_0(\varphi_{\infty,n})$ are positive group homomorphisms. Suppose that $g \in K_0(A)^+$ and find by definition of the positive cone a natural number $k \in \mathbb{N}$ and a projection $p \in M_k(A)$ with $g = [p]_0$. Using the same strategy as in (i), we can find a natural number $m \in \mathbb{N}$ and a projection $q \in M_k(A_m)$ such that $p \sim \varphi^{(k)}_{\infty,m}(q)$, and consequently

$$g = [p]_0 = [\varphi^{(k)}_{\infty,m}(q)]_0 = K_0(\varphi_{\infty,m})([q]_0).$$

This completes the proof. □

Not surprisingly, K_1 is also a continuous functor. In order to prove it, we need a lemma which effectively gives us a dense subset of the suspension.

Lemma 2.24. *Let X be a locally compact Hausdorff space and A a C^*-algebra. For any $f \in C_0(X)$ and $a \in A$, define $f_a \in C_0(X, A)$ by $f_a(x) = f(x)a$. The set*

$$\mathcal{F} = \{f_a \mid f \in C_0(X),\, a \in A\}$$

has dense span in $C_0(X, A)$.

Proof. Let $f \in C_0(X, A)$ be arbitrary and take an arbitrary tolerance $\varepsilon > 0$. Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X and note that $C_0(X, A) = \{f \in C_0(X^+, A) \mid f(\infty) = 0\}$. For every $x \in X^+$, let U_x denote the open subset of X^+ of elements $y \in X^+$ such that $\|f(x) - f(y)\| < \varepsilon$; then $\{U_x\}_{x\in X^+}$ is clearly an open cover of X^+. By compactness of X^+, we can reduce this open cover to a finite one, say, $U_1, \dots, U_k$. For each $j = 1, \dots, k$ fix a point $x_j \in U_j$; if $\infty \in U_j$, then choose $x_j = \infty$. Take a partition of unity subordinate to this finite open cover $\{h_j\}_{j=1}^k$ and note that $\|f(x)h_j(x) - f(x_j)h_j(x)\| \le \varepsilon h_j(x)$ for all $x \in X$ and $j \in \{1, \dots, k\}$, whence

$$\left\| f(x) - \sum_{j=1}^k f(x_j)h_j(x) \right\| \le \varepsilon, \qquad \text{for all } x \in X.$$

Set $a_j = f(x_j)$ and note that if $x_j = \infty$, then $a_j = 0$ and we can hence disregard these. In the other terms, i.e., for j with $\infty \notin U_j$, we have $h_j \in C_0(X)$ such that $f(x_j)h_j \in \mathcal{F}$, implying the desired density. □

Recall that the suspension S is a functor in the category of C^*-algebras taking a C^*-algebra A to the C^*-algebra $SA = C_0((0,1), A)$, and if $\varphi\colon A \to B$ is a *-homomorphism, then $S\varphi\colon SA \to SB$ is a *-homomorphism given by $S\varphi(f)(t) = \varphi(f(t))$ for all $f \in SA$ and $t \in (0, 1)$.

Theorem 2.25. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence with inductive limit $(A, \{\varphi_{\infty,n}\})$. Then it follows that $(SA, \{S\varphi_{\infty,n}\})$ is the inductive limit of the sequence $(SA, \{S\varphi_n\})$.*

Proof. Let $(B, \{\psi_{\infty,n}\}) = (\varinjlim SA_n, \{S\varphi_n\})$. Noting that we for any $f \in SA$ and $t \in (0, 1)$ have

$$S\varphi_{\infty,n+1}(S\varphi_n(f))(t) = \varphi_{\infty,n+1}(S\varphi_n(f)(t)) = \varphi_{\infty,n+1}(\varphi_n(f(t)) = \varphi_{\infty,n}(f(t)) = S\varphi_{\infty,n}(f)(t),$$

we find that there exists a unique *-homomorphism $\lambda\colon B \to SA$ such that $\lambda \circ \psi_{\infty,n} = S\varphi_{\infty,n}$. We claim that λ is an isomorphism, and we prove it using Theorem 2.5(iv). First we prove surjectivity. Let $f \in SA$ be arbitrary and take $\varepsilon > 0$. Find, by Lemma 2.24, functions $g_j \in C_0((0, 1))$ and coefficients $a_j \in A$ for $j = 1, \dots, k$ such that

$$\left\| f - \sum_{j=1}^k a_j g_j \right\| < \frac{\varepsilon}{2}.$$

Find some sufficiently large n such that for each $j = 1, \dots, k$ there exists $b_j \in A_n$ with $\varphi_{\infty,n}(b_j) = a_j$ such that $\left\|a_j - \varphi_{\infty,n_j}(b_{n_j})\right\| < \frac{\varepsilon}{2k\|g_j\|}$. Then for any $t \in (0,1)$, we have

$$\begin{aligned}\left\|f(t) - \sum_{j=1}^{k} S\varphi_{\infty,n}(b_j g_j)(t)\right\| &\leq \left\|f(t) - \sum_{j=1}^{k} a_j g_j(t)\right\| + \left\|\sum_{j=1}^{k} a_j g_j(t) - \sum_{j=1}^{k} S\varphi_{\infty,n}(b_j g_j)(t)\right\| \\ &\leq \frac{\varepsilon}{2} + \sum_{j=1}^{k} \|a_j g_j(t) - \varphi_{\infty,n}(b_j g_j(t))\| \\ &\leq \frac{\varepsilon}{2} + \sum_{j=1}^{k} \frac{\varepsilon \|g_j\|}{2k \|g_j\|} \\ &= \varepsilon\end{aligned}$$

proving that $\bigcup_{n\in\mathbb{N}} S\varphi_{\infty,n}(SA_n)$ is dense in SA, which implies surjectivity of λ by Theorem 2.5(iv)(b). For injectivity suppose that $f \in \ker S\varphi_{\infty,n}$, then for any $t \in (0,1)$ we have

$$0 = S\varphi_{\infty,n}(f)(t) = \varphi_{\infty,n}(f(t))$$

and hence $f(t) \in \ker \varphi_{\infty,n}$. In particular, $\lim_{m\to\infty} \|\varphi_{m,n}(f(t))\| = 0$ for all $t \in (0,1)$, and hence

$$\|S\varphi_{m,n}(f)(t)\| = \|\varphi_{m,n}(f(t))\| \to 0$$

and thus $\|S\varphi_{m,n}(f)\| \to 0$ as $m \to \infty$. Consequently $f \in \ker \psi_{\infty,n}$, proving injectivity and completing the proof. □

Corollary 2.26. *K_1 is a continuous functor, that is,*

$$K_1(\varinjlim A_n) \cong \varinjlim K_1(A_n).$$

Proof. Note that $K_1 = K_0 \circ S$ and use Theorem 2.25 and Theorem 2.23 □

This is for our purposes only a mild curiosity and shall not be used for other than a brief fact about approximately finite-dimensional C^*-algebras later.

3 AF-algebras

3.1 Definition and elementary properties

In this section, we discuss AF-algebras and completely classify unital AF-algebras. We first examine a few properties of AF-algebras, which in some sense justify the name. Then we look at a way of visualising AF-algebras graphically due to Bratteli, which in fact gives rise to a classification. Lastly we invoke several results in the past two chapters in order to prove that unital AF-algebras are completely classified by their ordered K_0-groups. We end the chapter with a discussion on further classification results, as well as a classification of a subclass of AF-algebras called UHF-algebras, where we shall see how our classification of AF-algebras can be used to realize the original classification of UHF-algebras. We use various references in this chapter, but our main ones are [2] and [10].

Definition 3.1. An AF-algebra is a C^*-algebra which is the inductive limit of an inductive sequence of finite-dimensional C^*-algebras.

AF stands for *approximately finite-dimensional*, which the above definition, and Definition 3.3 and Theorem 3.4 below in particular, encompasses.

Since finite-dimensional C^*-algebras are clearly separable, all AF-algebras in the above sense are separable by Proposition 2.10. One could define AF-algebras to be the inductive limit of arbitrary inductive systems, in which case separability might fail; it is not customary to do so and, consequently, we shall only work with AF-algebras in the sense of Definition 3.1.
Another way of describing AF-algebras is that they are precisely the C^*-algebras A, for which there exist nested finite-dimensional C^*-algebras A_n such that the union $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A. The two definitions are equivalent, and we shall use these interchangeably. From both definitions it is immediate that finite-dimensional C^*-algebras are AF.
The original definition by Bratteli [2] is the latter one with the added assumption that A is unital, and that each A_n contains the unit of A. In this project, however, we allow our AF-algebras to be non-unital, which expands the class of AF-algebras as the example below shows.

Example 3.2. Let H be an infinite-dimensional separable Hilbert space, and consider for each $n \in \mathbb{N}$ the *-homomorphism $\varphi_n \colon M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. We aim to prove that $\mathbb{K}(H)$ is the inductive limit of the inductive sequence $(M_n(\mathbb{C}), \{\varphi_n\})$, which would imply that $\mathbb{K}(H)$ is an AF-algebra. We can regard the connecting maps as inclusions $M_n(\mathbb{C}) \subseteq M_{n+1}(\mathbb{C})$, such that the inductive limit above is just the closure of $\bigcup_{n\in\mathbb{N}} M_n(\mathbb{C})$ with these inclusion maps. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H and p_n the projection onto the span of $\{e_1, \dots, e_n\}$ for each $n \in \mathbb{N}$ such that $p_n\mathbb{B}(H)p_n = \mathbb{B}(p_nH) \cong M_n(\mathbb{C})$. Now use the well-known fact that $\mathbb{K}(H)$ is the closure of $\bigcup_{n\in\mathbb{N}} p_n\mathbb{B}(H)p_n = \bigcup_{n\in\mathbb{N}} M_n(\mathbb{C})$ to see that $\mathbb{K}(H)$ is the inductive limit of the inductive sequence $(M_n(\mathbb{C}), \{\varphi_n\})$. Consequently, $\mathbb{K}(H)$ is an AF-algebra.

Since the identity operator is compact if and only if the Hilbert space is finite-dimensional, we have hence constructed non-unital AF-algebras. This added freedom in the structure of AF-algebras allow for some more complex C^*-algebras, although we shall not bother with these too much, as we only classify the unital case.

Given the name approximately finite-dimensional, it should come as no surprise that AF-algebras satisfy an approximation property.

Definition 3.3. A C^*-algebra A is said to be a *local AF-algebra* if for any finite subset $\{a_1, \dots, a_n\} \subseteq A$ and $\varepsilon > 0$ there exist a finite-dimensional C^*-subalgebra B of A and elements $b_1, \dots, b_n \in B$ such that $\|a_j - b_j\| < \varepsilon$ for all j.

These two notions coincide on separable C^*-algebras.

Theorem 3.4. *A separable C^*-algebra is an AF-algebra if and only if it is a local AF-algebra.*

Proof. We only prove sufficiency here. For necessity, we refer to [2, Theorem 2.2], where it is proven in the unital case.
Let $A = \varinjlim A_n$. We can, by Proposition 2.14, assume without loss of generality that the connecting

maps are injective, such that we can regard A_n as a C^*-subalgebra of A_{n+1} etc., and regard A_n as a C^*-subalgebra of A. Let $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$ be arbitrary and use the above identification to find $N \in \mathbb{N}$ along with $b_i \in A_N$ for all $i = 1, \ldots, n$ with $\|a_i - b_i\| < \varepsilon$. Since A_N is finite-dimensional, this proves the desired direction. □

This local characterization gives an intrinsic definition of AF-algebras, and hence one can determine whether something is an AF-algebra without resolving to inductive limits. We can, for instance, use it in the following way to find our first non-AF C^*-algebra.

Example 3.5. The C^*-algebra $C([0,1])$ is not an AF-algebra. Note that since $[0,1]$ is connected, the only projections on $C([0,1])$ are 0 and 1. This means that $C([0,1])$ only contains two finite-dimensional C^*-subalgebras, namely $\{0\}$ and $\mathbb{C}$. In particular, $C([0,1])$ cannot be an AF-algebra by Theorem 3.4.

Another reason for this local characterization is that it is used in proving stably finiteness of AF-algebras.

Proposition 3.6. *Every AF-algebra is stably finite.*

Proof. Let $A = \varinjlim A_n$ be an AF-algebra. Since the matrix C^*-algebra $M_N(B)$ and the unitization $\tilde{B}$ are finite-dimensional C^*-algebras, whenever B is a finite-dimensional C^*-algebra, it follows from Proposition 2.22 that $M_N(A)$ and $\tilde{A}$ are AF-algebras. In particular, $M_N(\tilde{A})$ is a unital AF-algebra. By definition of stable finiteness, it thus suffices to prove that every unital AF-algebra is finite.

Let A be an arbitrary unital AF-algebra. We claim that A is finite. By Proposition 1.4, it suffices to prove that every isometry on A is unitary. Let $s \in A$ be an isometry, then by Theorem 3.4 there exist a C^*-subalgebra $B \subseteq A$ and $x \in B$ such that $\|s - x\| < 1$. A simple calculation shows that

$$\|1 - s^*x\| = \|s^*s - s^*x\| \leq \|s^*\| \, \|s - x\| < 1$$

and hence s^*x is invertible in A. In particular, x is left-invertible, which is equivalent to x^*x being invertible.
Since invertibility is inherited by C^*-subalgebras, we see that x^*x is invertible in B, i.e. x is left-invertible in B. As B is finite-dimensional, this is equivalent to x being invertible, and hence x is invertible in B as well as in A. Then s is invertible, and since $s^*s = 1$, we see that s must be invertible with $s^{-1} = s^*$, i.e., s is unitary. □

Since AF-algebras are stably finite, it follows from Proposition 1.25 that unital AF-algebras have ordered K_0-groups. In fact, all AF-algebras have ordered K_0-groups by continuity of K_0, see Theorem 2.23. What is special for a unital AF-algebra A, however, is that the ordered K_0-group has a natural order unit $[1_A]_0$, which turns $(K_0(A), K_0(A)^+, [1_A]_0)$ into an ordered Abelian group with a distinguished order unit. We show that this actually uniquely determines the underlying AF-algebra, and consequently this is a classification invariant of unital AF-algebras. The K_1-group of AF-algebras, on the other hand, is trivial.

Proposition 3.7. *If A is an AF-algebra, then $K_1(A) = 0$.*

Proof. Combine continuity of K_1, Corollary 2.26 with the fact that the K_1-groups of finite-dimensional C^*-algebras is trivial, Proposition 1.17. □

Let us now examine some permanence properties of AF-algebras. It is not true in general that the AF-property passes to subalgebras. One way of seeing this is by examining the C^*-algebra of continuous functions on the Cantor set. The Cantor set $\mathcal{C}$ can be realized as an inverse limit of finite-dimensional spaces, and since the functor taking compact Hausdorff spaces X to C^*-algebras $C(X)$ takes inverse limits to inductive limits, [1, II.8.2.2(i)], one can realize $C(\mathcal{C})$ as the inductive limit of finite-dimensional C^*-algebras, i.e., $C(\mathcal{C})$ is an AF-algebra. However, every compact metric space is the continuous image of $\mathcal{C}$, so consider the compact metric space $X = [0,1]$ as in Example 3.5. Since X is the continuous image of $\mathcal{C}$, i.e., there exists a surjection $\varphi\colon \mathcal{C} \to X$, which induces an injection $\varphi^* : C(X) \to C(\mathcal{C})$ by $\varphi^*(f)(x) = f(\varphi(x))$ for all $x \in \mathcal{C}$ and $f \in C(X)$. In particular, we have determined a C^*-subalgebra of an AF-algebra, which is *not* approximately finite-dimensional. This raises the question of when we can regard C^*-algebras as subalgebras of AF-algebras, which is a still unanswered question; more details can be found [3, Chapter 8].

While being AF does not pass to general subalgebras, it does pass to ideals and quotients.

Proposition 3.8. *Suppose $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of C^*-algebras, and let $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ be the inductive limit. Let I be a closed two-sided ideal in A, then*

$$I = \overline{\bigcup_{n=1}^{\infty} A_n \cap I}.$$

If each A_n is finite-dimensional, then I is an AF-algebra, and moreover A/I is an AF-algebra, proving that being AF passes to ideals and quotients.

Proof. Define $I_n = A_n \cap I$ for each $n \in \mathbb{N}$ and note that I_n is a closed two-sided ideal in A_n, and that we have the inclusion $\overline{\bigcup_{n=1}^{\infty} I_n} \subseteq I$. For the other inclusion suppose that $x \notin \overline{\bigcup_{n\in\mathbb{N}} I_n}$, then we claim that $x \notin I$.
Find a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $x_k \in A_{n_k}$ for each $k \in \mathbb{N}$ and $x_k \to x$ in norm. Let $\varepsilon > 0$ be defined by

$$2\varepsilon = \inf\Big\{\|x - y\| \mid y \in \bigcup_{n\in\mathbb{N}} I_n\Big\} > 0$$

using that $x \notin \overline{\bigcup_{n\in\mathbb{N}} I_n}$. Find $N \in \mathbb{N}$ such that $\|x_k - x\| < \varepsilon$ for each $k \geq N$. A simple application of the triangle inequality then gives us that

$$\|x_k - y\| \geq \|x - y\| - \|x - x_k\| > \varepsilon$$

for all $y \in I_{n_k}$. Let $\pi\colon A \to A/I$ be the canonical quotient mapping and note that $\ker \pi|_{A_n} = I_n$. By uniqueness of norms of C^*-algebras, we find that

$$\|\pi(x_k)\| = \inf_{y\in I_{n_k}} \|x_k - y\|$$

where we use that the right-hand side is the norm of $\pi(x_k)$ when restricting it to be the quotient mapping $A_{n_k} \to A_{n_k}/I_{n_k}$. In particular, we see that $\|\pi(x_k)\| \geq \varepsilon$ for each $k \in \mathbb{N}$, and by continuity, we have

$$\|\pi(x)\| = \lim_{k\to\infty} \|\pi(x_k)\| \geq \varepsilon.$$

This proves that $x \notin I$, and we conclude that $I = \overline{\bigcup_{n\in\mathbb{N}} I_n}$. If each A_n is a finite-dimensional $C*$-algebra, then so is I_n, and hence I is an AF-algebra. To prove that A/I is an AF-algebra, let $\pi\colon A \to A/I$ be the quotient mapping and note that $A/I = \pi(A) = \overline{\bigcup_{n=1}^{\infty} \pi(A_n)}$. Since any AF-algebra A can be realized as the closure of a union of an increasing sequence of finite-dimensional C^*-algebra, we have shown that being approximately finite-dimensional passes to ideals and quotients. □

Proposition 3.9. *Suppose $(A_n, \{\varphi_n\})$ is an inductive sequence of AF-algebras, then the inductive limit $(A, \{\varphi_{\infty,n}\})$ is again AF.*

Proof. We use the local description of AF-algebras, see Theorem 3.4. Since quotients of AF-algebras are AF by Proposition 3.8, we can use Proposition 2.14 to assume that the connecting maps $\varphi_n\colon A_n \to A_{n+1}$ are injective, and consequently assume $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$. For any finite subset $F = \{a_1, \ldots, a_n\}$ of A and $\varepsilon > 0$, there exist sufficiently large $N \in \mathbb{N}$ along with a finite set $F' = \{a_1', \ldots, a_n'\} \subseteq A_N$ such that $\|a_i - a_i'\| < \frac{\varepsilon}{2}$ for each i. Using that A_N is an AF-algebra, there exist a finite-dimensional C^*-subalgebra $B \subseteq A_N$ along with a finite set $F'' = \{b_1, \ldots, b_n\} \subseteq B$ with $\|a_i' - b_i\| < \frac{\varepsilon}{2}$ for each i. Since $B \subseteq A$ is a C^*-subalgebra, and $\|a_i - b_i\| < \varepsilon$ by the triangle inequality, it follows from Theorem 3.4 that A is an AF-algebra. □

We end this section with a brief discussion regarding another property which encompasses the notion of approximating through finite-dimensional C^*-algebra called nuclearity. We shall only sketch the proof of inductive limits preserving nuclearity, and consequently that AF-algebras are nuclear. Recall that a linear map $\varphi\colon A \to B$ between C^*-algebras is called *contractive* if $\|\varphi\| \leq 1$. Moreover, it is called *completely positive* if all the amplifications $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ are positive. If φ is contractive and completely positive, we write that φ is ccp.

Definition 3.10. A C^*-algebra A is *nuclear* if for any finite set $F \subseteq A$ and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and ccp maps $\varphi\colon A \to M_n(\mathbb{C})$ and $\psi\colon M_n(\mathbb{C}) \to A$ such that $\|\psi \circ \varphi(a) - a\| < \varepsilon$ for all $a \in F$.

Finite-dimensional C^*-algebras are all nuclear. The next proposition states that locally nuclear C^*-algebras are nuclear.

Proposition 3.11. *Suppose that A is a C^*-algebra satisfying the following property: For any finite set $F = \{a_1, \dots, a_n\} \subseteq A$ and $\varepsilon > 0$, there exist a nuclear C^*-subalgebra $B \subseteq A$ and elements $b_1, \dots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$ for all $i = 1, \dots, n$. Then A is nuclear.*

The proof uses Arveson's extension theorem, a proof of which can be found in [3, Theorem 1.6.1].

Theorem 3.12 (Arveson)**.** *Let A be a unital C^*-algebra, H a Hilbert space and $E \subseteq A$ a closed, self-adjoint subspace containing the unit of A (that is, E is an operator subsystem of A). Then every ccp map $\varphi\colon E \to \mathbb{B}(H)$ extends to a ccp map $\overline{\varphi}\colon A \to \mathbb{B}(H)$.*

Proof of Proposition 3.11. Let $F = \{a_1, \dots, a_n\} \subseteq A$ be a finite set and assume $\varepsilon > 0$. Find a nuclear C^*-subalgebra $B \subseteq A$ with elements $b_1, \dots, b_n \in B$ such that $\|b_i - a_i\| < \frac{\varepsilon}{3}$ for each $i = 1, \dots, n$. Find by nuclearity ccp maps $\varphi\colon B \to M_n(\mathbb{C})$ and $\psi\colon M_n(\mathbb{C}) \to B$ such that $\|\psi \circ \varphi(b_i) - b_i\| < \frac{\varepsilon}{3}$ for $i = 1, \dots, n$. Find by Arveson's extension theorem a ccp map $\overline{\varphi}\colon A \to M_n(\mathbb{C})$ extending φ. A simple application of the triangle inequality then gives us that $\|\psi \circ \overline{\varphi}(a_i) - a_i\| < \varepsilon$ for all $i = 1, \dots, n$ proving nuclearity of A. □

Since AF-algebras are clearly locally nuclear by Theorem 3.4, we find that they are nuclear. It is a general fact, however, that inductive limits of nuclear C^*-algebras are once again nuclear; we sketch the proof beneath.

Proposition 3.13. *If $(A_n, \{\varphi_n\})$ is an inductive sequence of nuclear C^*-algebras, then the inductive limit A is also nuclear. In particular, all AF-algebras are nuclear.*

Proof. Suppose first that the connecting maps are injective. We can then without loss of generality assume that $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of nuclear C^*-algebras with inductive limit $A = \overline{\bigcup_{n \in \mathbb{N}} A_n}$. Using Proposition 3.11, it is easily verified that A is nuclear. Since nuclearity passes to quotients, see [1, IV.3.1.13], and we by Proposition 2.14 can obtain an inductive sequence with injective connecting maps by passing to certain quotients without changing the limit, the general case follows. □

For AF-algebras, we would not need the full strength of [1, IV.3.1.13], since passing to quotients in the inductive sequence preserves the finite-dimensional C^*-algebraic structure.

3.2 Bratteli diagrams

One way of visualizing AF-algebras is by *Bratteli diagrams*, named after the Norwegian mathematician Ola Bratteli, who first examined AF-algebras in [2]. With these diagrams one can easily encode the information of the inductive sequence, and from this deduce properties of the corresponding AF-algebra.

Definition 3.14. The *multiplicity* of a *-homomorphism $\varphi\colon M_k(\mathbb{C}) \to M_\ell(\mathbb{C})$ is $\frac{\mathrm{Tr}(\varphi(e))}{\mathrm{Tr}(e)}$, where e is any non-zero projection in A.

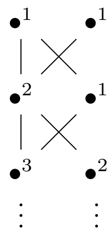
It is easily verified that the above is well-defined, i.e., the multiplicity is independent of the projection. As the name suggests, the multiplicity in a sense encodes the number of copies, the map is embedding. For example, the map $x \mapsto \mathrm{diag}(x, \dots, x, 0, \dots, 0)$ with n copies of x has multiplicity n. If two unital *-homomorphisms between matrix algebras have the same multiplicity, they are unitarily equivalent.

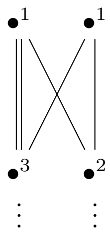
Given an AF-algebra A with corresponding inductive sequence $(A_n, \{\varphi_n\})$, we can construct the Bratteli diagram as follows, and we study some examples later. The nth row corresponds to the finite-dimensional C^*-algebra A_n, and the kth vertex on the nth row corresponds to the kth matrix algebra in some decomposition of A_n as in Theorem 1.15, and the vertex is labelled by its dimension. Given a vertex on the nth row, corresponding to the matrix algebra $M_k(\mathbb{C})$ in A_n and a vertex on the

$(n+1)$th row, corresponding to the matrix algebra $M_{k'}(\mathbb{C})$ in A_{n+1}, the number of edges connecting them is equal to the multiplicity of the *-homomorphism

$$M_k(\mathbb{C}) \hookrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \twoheadrightarrow M_{k'}(\mathbb{C}).$$

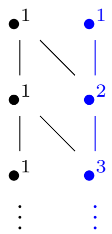
For any vertex, there should be a edge connecting it to a vertex on the subsequent row, and, likewise, a edge connecting it to a vertex on the previous row. See below for examples of Bratteli diagrams.
It is proven in [2, Section 1.8] that any Bratteli diagram corresponds to *exactly* one AF-algebra. It is not true, however, that an AF-algebra has a unique Bratteli diagram; first of all, we can interchange the vertices, together with the number of edges, on any given row in a Bratteli diagram, which does not change the AF-algebra. Moreover, in the same way one can pass to a subsequence of the inductive sequence in Lemma 2.11, one can "telescope" the Bratteli diagram without changing the AF-algebra. For example, the following two Bratteli diagrams corresponds to the same AF-algebra.





Graphically, telescoping the Bratteli diagram works by removing rows and counting the number of possible paths between each vertex in the new diagram. This is precisely the same as passing to subsequences of inductive limits, and the number of edges between each pair of vertices is exactly the multiplicity of the new connecting maps.
However, while the correspondence between AF-algebras and Bratteli diagrams may not be one-to-one on the nose, the diagrams still contain the data of their corresponding AF-algebras, and they are hence both a visual guidance as well as a way of getting information about AF-algebras. In fact, the diagrams combinatorically classify the AF-algebras by the possible diagram structures. We shall not pursue this here.

One possible thing to read directly from a Bratteli diagram is the ideal structure of the corresponding AF-algebra. Suppose $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ is an AF-algebra satisfying the conditions of Proposition 3.8 and let I be a closed two-sided ideal in A. Since A_n is finite-dimensional and the ideals of finite-dimensional C^*-algebras are the subsums of its decomposition into matrix algebras, each $I_n = A_n \cap I$ is of this form. This implies that I is an AF-algebra whose Bratteli diagram is a subdiagram of the Bratteli diagram of A, see [2] for the details. In particular, if there are edges connecting each pair of vertices on subsequent rows, then A is simple. For example, the Bratteli diagrams considered above are Bratteli diagrams for a simple AF-algebra. Another example of a simple AF-algebra is the compact operators on a Hilbert space, see Example 3.2. For an example of a non-simple AF-algebra, consider the following diagram where the subdiagram marked in blue, i.e., the right-hand vertical subdiagram, forms a non-trivial ideal, see [2, Section 3.4] where the ideal structure is classified completely.



In Chapter 5, we shall use this sufficient condition for an AF-algebra to be simple in order to pass from a general AF-algebra to a simple one without changing the structure of the tracial state space.

3.3 Classification of unital AF-algebras

Having studied several properties of AF-algebras, in this section we focus our attention towards classifying the unital AF-algebras. For this we need a few lemmas. The first two state how under certain circumstances we can classify maps on K-theory and lift them to maps on the level of C^*-algebras.

Lemma 3.15. *Let A be a finite-dimensional C^*-algebra and B a unital C^*-algebra with the cancellation property. If $\alpha\colon K_0(A) \to K_0(B)$ is a positive group homomorphism with $\alpha([1_A]_0) \le [1_B]_0$, then there exists a *-homomorphism $\varphi\colon A \to B$ such that $K_0(\varphi) = \alpha$. If $\alpha([1_A])_0 = [1_B]_0$, then φ is unit-preserving.*

Proof. Our proof is essentially a construction of a (not necessarily spanning) set of matrix units for B using the group homomorphism α on the level of K_0-groups and using the existence of a spanning set of matrix units for A.

We first claim that if $g_1, \dots, g_N \in K_0(B)^+$ are elements such that $\sum_{j=1}^N g_j \le [1_B]_0$, then there exist pairwise orthogonal projections $p_1, \dots p_n \in \mathcal{P}(B)$ such that $g_j = [p_j]_0$. Suppose we have shown that if $0 \le g \le [1_B]_0 - [p]_0$ for some projection $p \in \mathcal{P}(B)$, then there is a projection $q \in \mathcal{P}(B)$ orthogonal to p such that $g = [q]_0$. Then we can proceed iteratively to find the previous claim. So suppose $p \in \mathcal{P}(B)$ and $g \in K_0(B)^+$ satisfies $g \le [1_B]_0 - [p]_0$. Since $g \in K_0(B)^+$, there exist some $e \in \mathcal{P}_n(B)$ and $f \in \mathcal{P}_m(B)$ such that $g = [e]_0$ and $[f]_0 = [1_B]_0 - [p]_0 - g$. Then

$$[e \oplus f]_0 = [e]_0 + [f]_0 = [1_B]_0 - [p]_0 = [1_B - p]_0$$

and since B has the cancellation property, we find that $e \oplus f \sim_0 1_B - p$. Let $v \in M_{1,n+m}(B)$ be the partial isometry implementing this equivalence, i.e. such that $e \oplus f = v^*v$ and $1_B - p = vv^*$. If we put $q = v(e \oplus 0_m)v^*$, then $q \in \mathcal{P}(B)$ is a projection satisfying $q \le 1_B - p$, such that $q \perp p$. Furthermore $q \sim_0 e$, which proves that $g = [q]_0$, and which completes the proof of the claim.

Let $\{e_{ij}^{(k)}\}$ be the standard matrix units for A, then by positivity of α, and as $\alpha([1_A]_0) \le [1_B]_0$, we can construct pairwise orthogonal projections $f_{ii}^{(k)}$ for B with $\alpha([e_{ii}^{(k)}]_0) = [f_{ii}^{(k)}]_0$ by the above. Note that $e_{ii}^{(k)} \sim e_{jj}^{(k)}$ for all i, j, k, which implies $[e_{ii}^{(k)}]_0 = [e_{jj}^{(k)}]_0$. By construction of $\{f_{ii}^{(k)}\}$, we see that $[f_{ii}^{(k)}]_0 = [f_{jj}^{(k)}]_0$, which by the cancellation property of B implies that $f_{ii}^{(k)} \sim f_{jj}^{(k)}$. Use Lemma 1.12 to extend $\{f_{ii}^{(k)}\}$ to a system of matrix units $\{f_{ij}^{(k)}\}$ in B, and find the unique *-homomorphism $\varphi\colon A \to B$ with $\varphi(e_{ij}^{(k)}) = f_{ij}^{(k)}$. Since $\{[e_{11}^{(k)}]_0\}$ generates the finitely generated group $K_0(A)$ by Proposition 1.30, and as

$$K_0(\varphi)([e_{11}^{(k)}]_0) = [\varphi(e_{11}^{(k)}]_0 = [f_{11}^{(k)}]_0 = \alpha([e_{11}^{(k)}])_0,$$

we conclude that $K_0(\varphi) = \alpha$ as desired.

In the case with $\alpha([1_A]_0) = [1_B]_0$, we define the projection $p = \sum_{k=1}^r \sum_{i=1}^{n_k} f_{ii}^{(k)}$ in B and note that

$$\varphi(1_A) = \varphi\left(\sum_{k=1}^r \sum_{i=1}^{n_k} e_{ii}^{(k)}\right) = \sum_{k=1}^r \sum_{i=1}^{n_k} f_{ii}^{(k)} = p.$$

But then,

$$[1_B - p]_0 = [1_B]_0 - [p]_0 = \alpha([1_A]_0) - K_0(\varphi)([1_A]_0) = 0$$

as $\alpha = K_0(\varphi)$ from before, which implies that $(1_B - p) \sim_0 0$ by the cancellation property of B. However, this is only possible if $1_B - p = 0$, which implies that φ is unital. □

Lemma 3.16. *Let A be a finite-dimensional C^*-algebra and B a unital C^*-algebra with the cancellation property. Suppose $\varphi, \psi\colon A \to B$ are *-homomorphisms. Then $K_0(\varphi) = K_0(\psi)$ if and only if there exists a unitary $u \in B$ such that $\psi = \operatorname{Ad} u \circ \varphi$.*

Proof. Since unitary equivalence of projections implies Murray-von Neumann equivalence, it is clear that if $\psi = \operatorname{Ad} u \circ \varphi$ for some $u \in \mathcal{U}(B)$, then $K_0(\varphi) = K_0(\psi)$. For the other direction, suppose that $K_0(\varphi) = K_0(\psi)$. Let $\{e_{ij}^{(k)}\}$ be the matrix units for A and note that

$$[\varphi(e_{11}^{(k)})]_0 = K_0(\varphi)([e_{11}^{(k)}]_0) = K_0(\psi)([e_{11}^{(k)}]_0) = [\psi(e_{11}^{(k)})]_0$$

for all k. Moreover,

$$[1_B - \varphi(1_A)]_0 = [1_B]_0 - K_0(\varphi)[1_A]_0 = [1_B]_0 - K_0(\psi)[1_A]_0 = [1_B - \psi(1_A)]_0.$$

Since B has the cancellation property, we see that $\varphi(e_{11}^{(k)}) \sim \psi(e_{11}^{(k)})$ and $1_B - \varphi(1_A) \sim 1_B - \psi(1_A)$. Let $v_1, \dots, v_r, w \in B$ be the partial isometries implementing these equivalences, that is,

$$v_k^* v_k = \varphi(e_{11}^{(k)}) \qquad v_k v_k^* = \psi(e_{11}^{(k)})$$
$$w^* w = 1_B - \varphi(1_A) \qquad w w^* = 1_B - \psi(1_A).$$

A straight-forward, but long, computation shows that for each i, k, the element $s_{i,k} = \psi(e_{i1}^{(k)}) v_k \varphi(e_{1i}^{(k)})$ is a partial isometry with

$$s_{i,k}^* s_{i,k} = \varphi(e_{ii}^{(k)}), \qquad \text{and} \qquad s_{i,k} s_{i,k}^* = \psi(e_{ii}^{(k)})$$

and satisfying

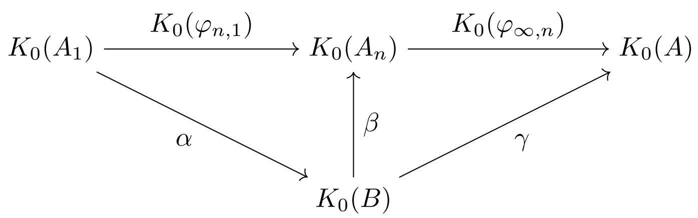
$$\sum_{k=1}^{r} \sum_{i=1}^{n_k} s_{i,k}^* s_{i,k} + w^* w = 1_B = \sum_{k=1}^{r} \sum_{i=1}^{n_k} s_{i,k} s_{i,k}^* + w w^*.$$

From this one calculates that $u = \sum_{k=1}^{r} \sum_{i=1}^{n_k} s_{i,k} + w$ is unitary. Let i, j, k be arbitrary. A quick computation shows that

$$u \varphi(e_{ij}^{(k)}) = \psi(e_{i1}^{(k)}) v_k \varphi(e_{1i}^{(k)}) = \psi(e_{ij}^{(k)}) u$$

implying that $\psi = \operatorname{Ad} u \circ \varphi$ as desired. □

Lemma 3.17. *Let $(A_n, \{\varphi_n\})$ be an inductive sequence of finite-dimensional C^*-algebras with inductive limit $(A, \{\varphi_{\infty,n}\})$. Suppose B is a finite-dimensional C^*-algebra, and that $\alpha\colon K_0(A_1) \to K_0(B)$ and $\gamma\colon K_0(B) \to K_0(A)$ are positive group homomorphisms with $\gamma \circ \alpha = K_0(\varphi_{\infty,1})$. Then there exist a natural number $n \in \mathbb{N}$ and a positive group homomorphism $\beta\colon K_0(B) \to K_0(A_n)$ such that the following diagram is commutative:*



If all φ_n are unital and $\alpha([1_{A_1}]_0) = [1_B]_0$, then $\beta([1_B]_0) = [1_{A_n}]_0$.

Proof. Write $B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ with matrix units $\{e_{ij}^{(k)}\}$ and note that $K_0(B)$ is generated by $[e_{11}^{(k)}]$ for $1 \le k \le r$ by Proposition 1.30. Define for each k the positive element $x_k = \gamma([e_{11}^{(k)}]_0)$. Continuity of K_0 implies that there exist $m \in \mathbb{N}$ and $y_1, \dots, y_r \in K_0(A_m)^+$ with $x_k = K_0(\varphi_{\infty,m})(y_k)$. Construct the unique positive group homomorphism $\beta'\colon K_0(B) \to K_0(A_m)$ satisfying $\beta'([e_{11}^{(k)}]_0) = y_k$ for each k. Then,

$$K_0(\varphi_{\infty,m} \circ \beta')([e_{11}^{(k)}]_0) = x_k = \gamma([e_{11}^{(k)}]_0)$$

proving that $K_0(\varphi_{\infty,m}) \circ \beta' = \gamma$. Moreover,

$$K_0(\varphi_{\infty,m}) \circ (\beta' \circ \alpha - K_0(\varphi_{m,1})) = \gamma \circ \alpha - K_0(\varphi_{\infty,1}) = 0.$$

Since A_1 is a finite-dimensional C^*-algebra, we find that $K_0(A_1) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}$ by Proposition 1.17, and hence $K_0(A_1)$ is finitely generated. By Theorem 2.6(ii), there thus exists $n > m$ such that $\operatorname{im}(\beta' \circ \alpha - \varphi_{m,1}) \subseteq \ker K_0(\varphi_{n,m})$. Define $\beta = \varphi_{n,m} \circ \beta'$, then $\beta \circ \alpha = K_0(\varphi_{n,1})$ and $K_0(\varphi_{\infty,n}) \circ \beta = \gamma$. Moreover, if φ_k is a unit-preserving *-homomorphism for each k, and $\alpha([1_{A_1}]_0) = [1_B]_0$, then

$$\beta([1_B]_0) = \beta \circ \alpha([1_{A_1}]_0) = K_0(\varphi_{n,1})([1_{A_1}]_0) = [1_{A_n}]_0$$

completing the proof. □

We are now in good shape to prove classification of unital AF-algebras.

Theorem 3.18 (Elliott, 1976)**.** *Let A, B be unital AF-algebras. Then A and B are isomorphic if and only if*

$$(K_0(A), K_0(A)^+, [1_A]_0) \cong (K_0(B), K_0(B)^+, [1_B]_0)$$

*as ordered Abelian groups with distinguished order units. More specifically, $A \cong B$ if and only if there exists a unit preserving order isomorphism $\alpha\colon K_0(A) \to K_0(B)$, and if such α exists, then there exists a *-isomorphism $\varphi\colon A \to B$ such that $K_0(\varphi) = \alpha$.*

Proof. Isomorphic stably finite and unital C^*-algebras have the same ordered K_0-groups by Theorem 1.27. The other direction is harder. The idea of the proof is as follows: If A and B are unital AF-algebras with an order preserving isometry on the level of K-theory, we can assume their inductive sequences have unital injective connecting maps. Using Lemma 3.17, one can find an intertwining diagram on the level of K-theory, which by Lemma 3.15 can be lifted to a not necessarily commuting intertwining diagram of finite-dimensional C^*-algebras. This non-commutativity can lastly be fixed using Lemma 3.16, and then Proposition 2.17 gives the desired isomorphism of A and B.

Having the above overview, we prove the theorem. Suppose we have the isomorphism

$$(K_0(A), K_0(A)^+, [1_A]_0) \cong (K_0(B), K_0(B)^+, [1_B]_0)$$

of ordered Abelian groups with distinguished order units via an isomorphism $\alpha\colon K_0(A) \to K_0(B)$. Using Lemma 2.13 and Proposition 2.14, let $(A_n, \{f_n\})$ and $(B_n, \{g_n\})$ be inductive sequences of finite-dimensional C^*-algebras such that the connecting maps are unit preserving and injective, and such that $(A, \{f_{\infty,n}\}) = \varinjlim(A_n, \{f_n\})$ and $(B, \{g_{\infty,n}\}) = \varinjlim(B_n, \{g_n\})$. Let $B_0 = \mathbb{C}$ and $g_0\colon B_0 \to B_1$ and $\psi_0\colon B_0 \to A_1$ be the unique unital *-homomorphisms. Define $\beta_0 = K_0(\psi_0)$, and note that

$$\alpha \circ K_0(f_{\infty,1}) \circ \beta_0 = K_0(g_{\infty,0}),$$

where $g_{\infty,0} = g_{\infty,1} \circ g_0$. This implies by Lemma 3.17 that there exist $m_1 \in \mathbb{N}$ and a group homomorphism $\alpha_1\colon K_0(A_1) \to K_0(B_{m_1})$ such that $\alpha_1 \circ \beta_0 = K_0(g_{m_1,0})$. From this, again using Lemma 3.17, we find $n_2 \in \mathbb{N}$ and a group homomorphism $\beta_1\colon K_0(B_{m_1}) \to K_0(A_{n_2})$ such that $\beta_1 \circ \alpha_1 = K_0(f_{n_2,1})$. Continue in this manner to inductively construct the commutative diagram

$$\begin{array}{ccccccccccc}
 & & K_0(A_1) & \longrightarrow & K_0(A_{n_2}) & \longrightarrow & K_0(A_{n_3}) & \longrightarrow & \cdots & \longrightarrow & K_0(A) \\
 & \beta_0 \nearrow & & \searrow \alpha_1 \quad \beta_1 \nearrow & & \searrow \alpha_2 \quad \beta_2 \nearrow & & \searrow \alpha_3 & & & \alpha \downarrow \uparrow \alpha^{-1} \\
K_0(B_0) & \longrightarrow & K_0(B_{m_1}) & \longrightarrow & K_0(B_{m_2}) & \longrightarrow & K_0(B_{m_3}) & \longrightarrow & \cdots & \longrightarrow & K_0(B)
\end{array}$$

By Lemma 2.11, the inductive sequences $(A_{n_i}, \{f_{n_{i+1},n_i}\})$ and $(B_{m_i}, \{g_{m_{i+1},m_i}\})$ have inductive limits A and B, respectively. We can thus simplify the notation and let $n_1 = 1$, $n_2 = 2$ and $m_0 = 0$, $m_1 = 1$ etc.. We want to lift this diagram to a commutative one on the level of C^*-algebras.

Use Lemma 3.15 to lift each α_j and β_j to unit-preserving *-homomorphisms $\varphi'_j\colon A_j \to B_j$ and $\psi'_j\colon B_j \to A_{j+1}$ such that $K_0(\varphi'_j) = \alpha_j$ and $K_0(\psi'_j) = \beta_j$. Note that as

$$K_0(f_j) = \beta_j \circ \alpha_j = K_0(\psi'_j \circ \varphi'_j) \quad \text{and} \quad K_0(g_j) = \alpha_{j+1} \circ \beta_j = K_0(\varphi'_{j+1} \circ \psi'_j),$$

for all j, the following construction is possible by Lemma 3.16:
By uniqueness of ψ_0, we see that $\psi_0 = \psi'_0$, and uniqueness of g_0 implies that $g_0 = \varphi'_1 \circ \psi'_0$. The following might seem strange, but there is a reason for it: Define $v_1 = 1$ and $\varphi_1 = \mathrm{Ad}(v_1) \circ \varphi'_1$, such that $g_0 = \varphi_1 \circ \psi_0$. Now, as

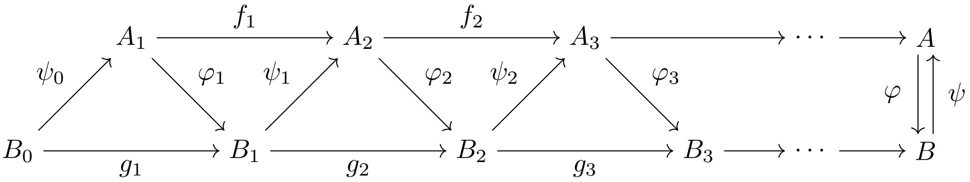
$$K_0(f_1) = K_0(\psi'_1 \circ \varphi'_1) = K_0(\psi'_1 \circ \varphi_1),$$

there exists unitary $u_1 \in B$ such that $f_1 = \mathrm{Ad}\, u_1 \circ \psi'_1 \circ \varphi_1$. Let $\psi_1 = \mathrm{Ad}\, u_1 \circ \psi'_1$, such that $f_1 = \psi_1 \circ \varphi_1$. Continue this construction inductively to find unitaries $u_j \in \mathcal{U}(A_{j+1})$ and $v_j \in \mathcal{U}(B_j)$ with $v_1 = 1$ such that

$$f_j = \psi_j \circ \varphi_j, \quad \text{and} \quad g_j = \varphi_{j+1} \circ \psi_j,$$

for each j, where $\varphi_j = \operatorname{Ad} v_j \circ \varphi'_j$ and $\psi_j = \operatorname{Ad} u_j \circ \psi'_j$. By Lemma 3.16, we also find that $K_0(\varphi_j) = K_0(\varphi'_j)$ and $K_0(\psi_j) = K_0(\psi'_j)$.

Hence we have constructed the commutative diagram



where we know that $\varphi\colon A \to B$ and $\psi\colon B \to A$ exist and are inverses of one another by Lemma 2.17. We conclude that A and B are isomorphic C^*-algebras. Note moreover that $K_0(\varphi_j) = \alpha_j$ and that the following diagrams are commutative

$$\begin{CD} K_0(A_n) @>{K_0(\varphi_{\infty,n})}>> K_0(A) \\ @V{K_0(\varphi_n)}VV @VV{K_0(\varphi)}V \\ K_0(B_n) @>>{K_0(\psi_{\infty,n})}> K_0(B) \end{CD} \qquad\qquad \begin{CD} K_0(A_n) @>{K_0(\varphi_{\infty,n})}>> K_0(A) \\ @V{K_0(\varphi_n)}VV @VV{\alpha}V \\ K_0(B_n) @>>{K_0(\psi_{\infty,n})}> K_0(B) \end{CD}$$

Since $K_0(A) = \bigcup_{n\in\mathbb{N}} K_0(\varphi_{\infty,n})(K_0(A_n))$ by Theorem 2.23(i), we conclude that $\alpha = K_0(\varphi)$, which completes the proof. □

Let us briefly touch on classification of general AF-algebras. If A is a C^*-algebra, one can define the *dimension range* $\mathcal{D}_0(A)$ as

$$\mathcal{D}_0(A) = \{[p]_0 \,|\, p \in \mathcal{P}(A)\} \subseteq K_0(A)^+.$$

Since the classification invariant of Theorem 3.18 obviously is not defined in the non-unital case, it is clear that in order to classify all AF-algebras, one needs to change it. In the general case, the classification invariant is $(K_0(A), \mathcal{D}_0(A))$, see [5]. The structure of the proof is essentially the same, with the only differences being the change from the ordered K-groups to the dimension range. This general classification invariant does not contain more data than the one, we are using, in the unital case, which the following proposition shows.

Proposition 3.19. *If A is a unital, stably finite C^*-algebra with the cancellation property, then*

$$\mathcal{D}_0(A) = \{g \in K_0(A) \,|\, 0 \le g \le [1_A]_0\}.$$

Proof. Since A is unital and stably finite, its ordered K_0-group is an ordered Abelian group with a distinguished order unit $[1_A]_0$, see Proposition 1.25. Suppose $g \in \mathcal{D}_0(A)$, then $g = [p]_0$ for some $p \in \mathcal{P}(A)$, and therefore $p \le 1_A$ implying that $0 \le g = [p]_0 \le [1_A]_0$. If, on the other hand, $0 \le g \le [1_A]_0$ for some $g \in K_0(A)$, then we can use the technique of the proof in Lemma 3.15 to find a projection $p \in A$ with $g = [p]_0$, which completes the proof. □

As stated in the introduction, classifying classes of C^*-algebras through their K-theory is not exclusive to AF-algebras. Let us briefly touch upon how the Elliott invariant can be used to classify a more general class of C^*-algebras, which extends the classification of AF-algebras of Theorem 3.18.

Definition 3.20. A C^*-algebra A is called *approximately homogeneous*, shortened AH, if it is the inductive limit of a sequence of C^*-algebras of the form

$$M_{k_1}(C(X_1)) \oplus \cdots \oplus M_{k_n}(C(X_k)) \tag{3.1}$$

for $n \in \mathbb{N}$ and $k_i \in \mathbb{N}$, where each X_i is a compact Hausdorff space. If each $X_j = \mathbb{T}$ is the unit circle in $\mathbb{C}$, we call A an *$A\mathbb{T}$-algebra*.

AF-algebras form a subclass of AH-algebras with X_j being finite spaces; in fact, we can take them all to be singletons. Since Abelian C^*-algebras are nuclear, and $M_n(A)$ is nuclear whenever A is nuclear, it follows from Proposition 3.13 that AH-algebras are nuclear. While AF-algebras have trivial K_1-groups, this does not hold for general AH-algebras; for example, $K_1(C(\mathbb{T})) = \mathbb{Z}$, see [10]. One would therefore expect that the classification of AH-algebras should in some way include the K_1-group.

Definition 3.21. A C^*-algebra has *real rank zero* if any self-adjoint element can be written as a norm-limit of self-adjoint elements with finite spectrum.

For instance all AF-algebras have real rank zero, see [15, Proposition 1.2.4]. Define for a C^*-algebra A the *graded K-theory* $K_*(A) = K_0(A) \oplus K_1(A)$. We say that a group homomorphism $\alpha\colon K_*(A) \to K_*(B)$ of graded K-theory is *graded* if $\alpha(K_0(A)) \subseteq K_0(B)$ and $\alpha(K_1(A)) \subseteq K_1(B)$. Define moreover the *graded dimension range* $\mathcal{D}_*(A)$ as the set

$$\mathcal{D}_*(A) = \{([p]_0, [u]_1) \,|\, p \in \mathcal{P}(A), u \in \mathcal{U}(pAp)\}.$$

A proof of the following theorem, which we express as in [15, Theorem 3.2.6], may be found in [6, Theorem 7.1].

Theorem 3.22 (Elliott, 1993)**.** *Let A and B be $A\mathbb{T}$-algebras of real rank zero. Then A and B are isomorphic if and only if there is a graded group isomorphism $\alpha\colon K_*(A) \to K_*(B)$ such that $\alpha(\mathcal{D}_*(A)) = \mathcal{D}_*(B)$. In the affirmative case, α has a *-isomorphism $\varphi\colon A \to B$ lifting α, i.e. $\alpha = K_*(\varphi)$.*

The above is clearly an extension of the classification of AF-algebras. The classification structure $(K_*(A), \mathcal{D}_*(A))$ also has a specific structure, which we shall discuss in the next chapter, where we also characterize the structure of the ordered K_0-groups of AF-algebras.

3.4 UHF algebras

A specific type of AF-algebras, which was first studied in the 1960's by Glimm [8] prior to Bratteli's examination of AF-algebras, is the so-called *uniformly hyperfinite algebras*, usually denoted UHF-algebras. Being unital AF-algebras, they are classified by Theorem 3.18, but the UHF-algebras can be further classified by a certain generalization of the natural numbers called the *supernatural numbers*, which in essence are sequences in $\mathbb{N}_0 \cup \{\infty\}$. This classification is due to Glimm [8], and we shall see how the supernatural numbers arise exactly from the ordered K_0-groups.

Definition 3.23. A C^*-algebra is called a *uniformly hyperfinite algebra*, or UHF-algebra for short, if it is the inductive limit of matrix algebras with unit-preserving *-homomorphisms, that is, if it is the inductive limit of a sequence $(A_n, \{\varphi_n\})$, where $A_n \cong M_{k_n}(\mathbb{C})$ for some $k_n \in \mathbb{N}$ and φ_n is unit-preserving.

It is clear that UHF-algebras are unital AF-algebras. In fact, they are all simple, as one immediately gathers from their Bratteli diagrams. Not all simple AF-algebras are UHF-algebras, though, since the compact operators on an infinite-dimensional, separable Hilbert space is a simple non-unital AF-algebra.
The fact that the connecting maps are unital gives a somewhat strong condition on the inductive sequence, since there exists a unital *-homomorphism $\varphi\colon M_k(\mathbb{C}) \to M_\ell(\mathbb{C})$ if and only if k divides ℓ. If φ is unital, then if e is a one-dimensional projection in $M_k(\mathbb{C})$, and if we denote by 1_k the unit on $M_k(\mathbb{C})$, we find that

$$\mathrm{Tr}(\varphi(e)) = \frac{\mathrm{Tr}(\varphi(1_k))}{\mathrm{Tr}(1_k)} = \frac{\ell}{k}$$

since $\varphi(e) \in M_\ell(\mathbb{C})$ is a projection. As $\mathrm{Tr}(\varphi(e))$ is an integer in $\{0, \dots, \ell\}$, this implies that k must divide ℓ. Conversely, if k divides ℓ, then we can take $\varphi\colon M_k(\mathbb{C}) \to M_\ell(\mathbb{C})$ to be the unital *-homomorphism $x \mapsto (x, \dots, x)$ with $\frac{\ell}{k}$ copies of x.
Let us now define what will shortly turn out to be the classification invariant of UHF-algebras.

Definition 3.24. By a supernatural number n, we mean a sequence $n = \{n_j\}_{j\in\mathbb{N}}$ where $n_j \in \mathbb{N}_0 \cup \{\infty\}$.

Let $\{p_1, p_2, \ldots\}$ denote the prime numbers in strictly increasing order, then we view a supernatural number $n = \{n_j\}_{j\in\mathbb{N}}$ as $n = \prod_{j\in\mathbb{N}} p_j^{n_j}$, i.e., n_j denotes the multiplicity of the jth prime in the prime factorization of n. Using this point of view, all natural numbers become supernatural numbers, since each natural number has a unique prime factorization, and consequently this is a generalization of the naturals. If $n = \{n_j\}_{j\in\mathbb{N}}$ and $m = \{m_j\}_{j\in\mathbb{N}}$ are two supernatural numbers, we define the product to be $nm = \{n_j + m_j\}_{j\in\mathbb{N}}$, as one would expect with this prime factorization viewpoint.

Definition 3.25. For each supernatural number $n = \{n_j\}_{j\in\mathbb{N}}$, we denote by $Q(n)$ the subgroup of $\mathbb{Q}$ consisting of elements $\frac{x}{y}$, where $x \in \mathbb{Z}$ and $y = \prod_{j\in\mathbb{N}} p_j^{m_j}$ where $m_j \le n_j$ and $m_j > 0$ for only finitely many j.

The K-theoretic invariants of Theorem 3.18 for UHF-algebras are precisely these subgroups. The following proposition establishes this connection.

Proposition 3.26. *Let A be a UHF-algebra with inductive sequence $(M_{k_i}(\mathbb{C}), \{\varphi_i\})$. Write $k_i = \prod_{j=1}^{\infty} p_j^{n_{i,j}}$ for some $n_{i,j} \in \mathbb{N}_0$. Define $n_j = \sup_{i\in\mathbb{N}} n_{i,j}$ and let $n = \{n_j\}_{j\in\mathbb{N}}$ be the corresponding supernatural number. Then $Q(n) = \bigcup_{i=1}^{\infty} k_i^{-1}\mathbb{Z}$ and there exists a group isomorphism $\alpha\colon Q(n) \to K_0(A)$ with $\alpha(1) = [1_A]_0$.*

Proof. Since $n_{i,j} \le n_j$ for all $i \in \mathbb{N}$, we see that $k_j^{-1} \in Q(n)$. Suppose $t = \frac{x}{y} \in Q(n)$, and write $y = \prod_{j=1}^{r} p_j^{m_j}$, where $m_j \le n_j$ for each j. Without loss of generality, we can assume that x and y are relatively prime. By definition of $Q(n)$, we know that $m_j \le n_j$ for all j, and by construction of $\{n_j\}_{j\in\mathbb{N}}$ there exists a sufficiently large i such that $m_j \le n_{i,j}$ for all j. In particular, y is a divisor in k_i for this i, and hence $t \in k_i^{-1}\mathbb{Z}$.
We now construct the desired isomorphism. Denote by $\tau_i = \frac{1}{k_i}\mathrm{Tr}$ the normalized trace on $M_{k_i}(\mathbb{C})$ for each $i \in \mathbb{N}$, and note that since $K_0(\mathrm{Tr})\colon K_0(M_{k_i}(\mathbb{C})) \to \mathbb{Z}$ is an isomorphism, we have that $K_0(\tau_i)\colon K_0(M_{k_i}(\mathbb{C})) \to k_i^{-1}\mathbb{Z}$ is an isomorphism. Since we clearly have that $\tau_{j+1} \circ \varphi_j = \tau_j$ for each j, this passes to morphisms of K_0-groups, i.e., $K_0(\tau_{j+1}) \circ K_0(\varphi_j) = K_0(\tau_j)$ for each j. This implies by definition of inductive limits that there exists a group homomorphism $\alpha\colon K_0(A) \to Q(n)$ such that $\alpha \circ K_0(\varphi_{\infty,j}) = K_0(\tau_j)$ for each j; here we have implicitly used continuity of K_0, Theorem 2.23. Since $K_0(\tau_j)$ is injective for each $j \in \mathbb{N}$, and $Q(n) = \bigcup_{j=1}^{\infty} K_0(\tau_j)(K_0(M_{k_j}(\mathbb{C}))$ by the previous part, it follows from Theorem 2.6(iii) that α is an isomorphism. □

We call the supernatural number n in the above proposition for the supernatural number associated to the UHF-algebra A. At this point, however, the name is not very well justified, as it is not clear that there exists a unique correspondence between supernatural numbers and UHF-algebras. The following proposition shows that there is such a correspondence.

Proposition 3.27. *Any subgroup of $(\mathbb{Q}, +)$ containing 1 is of the form $Q(n)$ for some supernatural number n. Moreover, there exists a group isomorphism $\alpha\colon Q(n) \to Q(m)$ with $\alpha(1) = 1$ if and only if $n = m$.*

Proof. Suppose G is a subgroup of $(\mathbb{Q}, +)$ containing 1. Define for each j the number

$$n_j = \sup\{m \in \mathbb{N}_0 \mid p_j^{-m} \in G\} \in \mathbb{N}_0 \cup \{\infty\} \tag{3.2}$$

and consider the supernatural number $n = \{n_j\}_{j=1}^{\infty}$. Let $t \in \mathbb{Q}$ be a non-zero rational number expressed by $t = \frac{x}{y}$, where x, y are relatively prime. Find, by Bézout's lemma, integers $a, b \in \mathbb{Z}$ such that $1 = ax + by$, and suppose that y has prime factorization $y = \prod_{j=1}^{k} p_j^{m_j}$, where $m_j \in \mathbb{N}_0$. Put $y_j = y p_j^{-m_j}$ and note that

$$p_j^{-m_j} = \frac{y_j}{y} = \frac{y_j(ax + by)}{y} = y_j(at + b \cdot 1). \tag{3.3}$$

Since the numbers $y_1, \ldots, y_k$ are relatively prime, we can find integers $c_1, \ldots, c_k \in \mathbb{Z}$ such that $1 = c_1 y_1 + \cdots + c_k y_k$. Hence,

$$t = \frac{x}{y} = \frac{x(c_1 y_1 + \cdots + c_k y_k)}{y} = \frac{x c_1}{p_1^{m_1}} + \cdots + \frac{x c_k}{p_k^{m_k}}. \tag{3.4}$$

If $t \in G$, then in particular $p_j^{-m_j} \in G$ by (3.3) and consequently $m_j \le n_j$ for each j. Therefore $1/y \in Q(n)$ by definition, and then (3.4) implies that $t \in Q(n)$. If, on the other hand, $t \in Q(n)$, then

$m_j \leq n_j$ and hence $p_j^{-m_j} \in G$ for each j. Then it follows from (3.4) that $t \in G$. Hence, we conclude that $G = Q(n)$.

We now prove uniqueness of the supernatural number. Suppose n, m are supernatural numbers. If $n = m$, then we can just take $\alpha\colon Q(n) \to Q(m)$ to be the identity map. Conversely, if there exists a group isomorphism $\alpha\colon Q(n) \to Q(m)$ such that $\alpha(1) = 1$, then it is easily verified through additivity and $\alpha(1) = 1$ that $\alpha(x/y) = x/y$ for each $x/y \in Q(n)$. This implies that α is, in fact, the identity map, and hence $Q(n) = Q(m)$. Since the supernatural number n can be recovered from $Q(n)$ by (3.2), we find that $n = m$. □

We are almost able to invoke Theorem 3.18 and classify UHF-algebras by their associated supernatural number. There is, however, a catch; the isomorphism in Proposition 3.26 is not of ordered Abelian groups, as in Theorem 3.18. This is not a problem, though, as the positive cone can be derived from the quantity $(K_0(A), [1_A]_0)$ as the following theorem, which is the classification of UHF-algebras, shows.

Theorem 3.28. *Let A and B be UHF-algebras with associated supernatural number n and m, respectively. The following are equivalent:*

(i) $A \cong B$*;*

(ii) $n = n'$*;*

(iii) *There exists a group isomorphism $\alpha\colon K_0(A) \to K_0(B)$ such that $\alpha([1_A]_0) = [1_B]_0$;*

(iv) *There is an isomorphism $(K_0(A), K_0(A)^+, [1_A]_0) \cong (K_0(B), K_0(B)^+, [1_B]_0)$ of ordered Abelian groups with distinguished order units.*

Proof. It follows from Proposition 3.26 and Proposition 3.27 that (ii) and (iii) are equivalent. It is known from classification of unital AF-algebras, Theorem 3.18, that (i) and (iv) are equivalent, and it is trivial that (iv) implies (iii). We hence only need to establish that (iii) implies (iv), i.e. that any group isomorphism between $K_0(A)$ and $K_0(B)$ sending $[1_A]_0$ to $[1_B]_0$ necessarily preserves the positive cone.

Let $g \in K_0(A)$ and identify via. Proposition 3.26 $(K_0(A), [1_A]_0)$ with $(Q(n), 1)$ for some supernatural number n. Then $g = \frac{x}{y}[1_A]_0$, and hence $yg = x[1_A]_0$, for some $x \in \mathbb{Z}$ and $y \in \mathbb{N}$. If $x \geq 0$, then $yg \geq 0$, and as $K_0(A)$ is unperforated, see Definition 4.3 and Theorem 4.8 in the next chapter, we conclude that $g \geq 0$. If $x \leq 0$, then we similarly find that $-g \leq 0$. We thus find that $g \in K_0(A)^+$ if and only if $gy = x[1_A]_0$ for some $x \in \mathbb{N}_0$ and $y \in \mathbb{N}$ with $\frac{x}{y} \in Q(n)$.
Since this description of $K_0(A)^+$ is determined completely by rational multiples of $[1_A]_0$, and as $\alpha\colon K_0(A) \to K_0(B)$ is a group isomorphism with $\alpha([1_A]_0) = [1_B]_0$, one immediately finds that $\alpha(K_0(A)^+) = K_0(B)^+$ proving the theorem. □

Since we have classified the UHF-algebras via their associated supernatural numbers, it is natural to ask: Does there for every supernatural number n exist a UHF-algebra A, whose associated supernatural number is n? The answer is affirmative, and it is almost answered in Proposition 3.26, since it is a simple consequence hereof.

Proposition 3.29. *For any supernatural number n, there exists a UHF-algebra A whose associated supernatural number is n.*

Proof. Let $n = \{n_j\}_{j=1}^{\infty}$ be an arbitrary supernatural number and define for each j the element $k_j = \prod_{i=1}^{j} p_i^{\min\{j, n_i\}}$ in $\mathbb{N}_0 \cup \{\infty\}$. It is easily seen that n is the supernatural number associated to the sequence $\{k_j\}_{j=1}^{\infty}$ as in Proposition 3.26. Since k_{j+1} is divisible by k_j for each j, there exist unital *-homomorphisms $\varphi_j\colon M_{k_j}(\mathbb{C}) \to M_{k_{j+1}}(\mathbb{C})$. Let A be the inductive limit of the sequence $(M_{k_j}, \{\varphi_j\})$, then Proposition 3.26 states that $(K_0(A), [1_A]_0)$ is isomorphic to the pair $(Q(n), 1)$, and hence A has n as its associated supernatural number. □

The original proof of the classification of UHF-algebras by Glimm [8] was precisely by the associated supernatural numbers. It was later realized that this classification was in fact K-theoretic in nature, and the proof in Theorem 3.28 uses this realization to go from the classification of unital AF-algebras via ordered K_0-groups to the supernatural numbers.

Let us end this chapter by examining a specific UHF-algebra, whose ordered K_0-group has been studied previously in this project.

Example 3.30. In Example 2.9, we discovered that the rational numbers $\mathbb{Q}$ could be realized as the inductive limit of an inductive sequence of integers. Combining Proposition 3.27, Theorem 3.28 and Proposition 3.29, there exists a UHF-algebra $\mathcal{Q}$ whose ordered K_0-group is precisely $\mathbb{Q}$ equipped with the positive cone $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$. Following the arguments of Theorem 4.2 in the next chapter, we can realize $\mathcal{Q}$ as the UHF-algebra with Bratteli diagram

i.e. with inductive sequence $A_n = M_{(n-1)!}(\mathbb{C})$ and multiplicity n for the connecting *-homomorphism $\varphi_n \colon A_n \to A_{n+1}$, such that they are unitarily equivalent to the maps $x \mapsto \operatorname{diag}(x, \ldots, x)$. The corresponding supernatural number is $n = \{n_j\}_{j \in \mathbb{N}}$ with $n_j = \infty$ for each j. This UHF-algebra is sometimes known as the *universal UHF-algebra* [15].

4 Dimension groups

In the previous chapter, we saw how we can classify unital AF-algebras through some K-theoretic invariants, and then we saw that the subclass of UHF-algebras actually have a particular structure to their K-theoretic invariants; they are all subgroups of the additive group $\mathbb{Q}$ containing 1, and any such group is the K-theory invariant of a UHF-algebra. We can hence completely characterize the classification invariants. It is natural to ask whether this is also the case for AF-algebras, and the answer is "yes". Given the major difference in complexity between UHF-algebras and AF-algebras, the structure of the K-theoretic invariants for a general AF-algebra can be expected to be more flexible than just associating a supernatural number, as is the case for UHF-algebras. We shall see that the classification invariants for AF-algebras still have some nice properties, and that they have an inductive as well as an intrinsic characterization. For this chapter, we mainly rely on [9] for the background theory on Riesz groups, and the orginal papers of Shen, [16], and Effros, Handelman and Shen, [4], for the intrinsic characterization of dimension groups.

Recall that $\mathbb{Z}^n$ equipped with the positive cone $(\mathbb{Z}^+)^n$ is an ordered Abelian group, and that we call these *simplical groups*.

Definition 4.1. An ordered Abelian group is called a *dimension group* if it is isomorphic to an inductive limit of simplical groups.

It is clear that dimension groups are always countable with our definition of inductive limits, and it turns out that they are precisely the K-theoretic invariants, we studied in Chapter 3.

Theorem 4.2. *The ordered K_0-group of an AF-algebra is a dimension group, and, conversely, any dimension group is the ordered K_0-group of some AF-algebra.*

Proof. If B is a finite-dimensional C^*-algebra, then $(K_0(B), K_0(B)^+) \cong (\mathbb{Z}^k, (\mathbb{Z}^+)^k)$ as ordered Abelian groups for some $k \in \mathbb{N}$. It hence follows by continuity of K_0, that if A is an AF-algebra, then $(K_0(A), K_0(A)^+)$ is a dimension group.
Suppose therefore that (G, G^+) is a dimension group, $G = \varinjlim(\mathbb{Z}^{n_i}, \{\varphi_i\})$. We want to construct an AF-algebra whose ordered K_0-group is precisely (G, G^+). The idea of the proof is simple: For each $i \in \mathbb{N}$, find some finite-dimensional C^*-algebra A_i with $K_0(A_i) = \mathbb{Z}^{n_i}$, and lift the positive group homomorphisms $\mathbb{Z}^{n_i} \to \mathbb{Z}^{n_{i+1}}$ to *-homomorphisms $A_i \to A_{i+1}$.

First, we inductively choose order units $u_j = (k_1^{(j)}, \ldots, k_{n_j}^{(j)}) \in (\mathbb{Z}^+)^j$ such that $\varphi_j(u_j) \leq u_{j+1}$. Note that $k_i^{(j)} \in \mathbb{N}$, as u_j are order units. Define

$$A_j = M_{k_1^{(j)}}(\mathbb{C}) \oplus \cdots \oplus M_{k_{n_j}^{(j)}}(\mathbb{C})$$

and let by Proposition 1.30 $\gamma_j \colon \mathbb{Z}^{n_j} \to K_0(A_j)$ be the canonical order isomorphism satisfying $\gamma_j(u_j) = [1_{A_j}]_0$. Define $\alpha_j \colon K_0(A_j) \to K_0(A_{j+1})$ by $\alpha_j = \gamma_{j+1} \circ \varphi_j \circ \gamma_j^{-1}$ and find by Lemma 3.15 a *-homomorphism $\psi_j \colon A_j \to A_{j+1}$ such that $K_0(\psi_j) = \alpha_j$ for each j. Then $K_0(\psi_j) \circ \gamma_j = \gamma_{j+1} \circ \varphi_j$ by construction. Let $A = \varinjlim(A_n, \{\psi_n\})$, then A is an AF-algebra, and by continuity of K_0 we conclude that $K_0(A) = G$ as desired. $\square$

Let us try to establish some intuition for what the above proof actually says, which may be hidden in the details. Given a dimension group, find an inductive sequence of simplical groups. We can graphically represent this inductive sequence in a Bratteli diagram, where the multiplicity of the connecting maps can be read directly from the corresponding matrices. If we consider the AF-algebra corresponding to the Bratteli diagram, this will precisely correspond to the dimension group we started with. This gives us a way to rephrase the definition of dimension groups to express its importance in the study of AF-algebras, but no new information about the structure of dimension groups can be extracted from this. Our main goal of this chapter is to understand dimension groups by giving an intrinsic characterization.

Definition 4.3. Let (G, G^+) be an ordered Abelian group. We say that

(i) G is *unperforated* if $ng \geq 0$ implies that $g \geq 0$ for all $n \in \mathbb{N}$ and $g \in G$;

(ii) G has *Riesz interpolation* if for every $x_1, x_2, y_1, y_2 \in G$ with $x_i \leq y_j$ for $i, j = 1, 2$ there exists $z \in G$ with $x_i \leq z \leq y_j$ for $i = 1, 2$;

(iii) G satisfies *Riesz decomposition property* if for any $x_1, x_2, y_1, y_2 \in G^+$ with $x_1 + x_2 = y_1 + y_2$ there exist $z_{ij} \in G^+$, $i, j = 1, 2$, such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.

If (G, G^+) satisfies (i) and (ii), we say that G is a *Riesz group*.

It turns out that (ii) and (iii) are actually equivalent, as the following lemma from [9] shows.

Lemma 4.4. *Let (G, G^+) be an ordered Abelian group. Then the following are equivalent:*

(i) *G has Riesz interpolation;*

(ii) *Given $x, y_1, y_2 \in G^+$ with $x \leq y_1 + y_2$, there exist $x_1, x_2 \in G^+$ such that $x = x_1 + x_2$ and $x_j \leq y_j$ for each j;*

(iii) *G satisfies Riesz decomposition property.*

Proof. (i)⇒(ii): Note that if $x, y_1, y_2 \in G^+$ satisfies $x \leq y_1 + y_2$, then $0 \leq x$ and $0 \leq y_2$, as well as $x - y_2 \leq y_1$ and $x - y_2 \leq x$. By Riesz interpolation, there exists $x_1 \in G_1$ such that $0 \leq x_1 \leq x$ and $x - y_2 \leq x_1 \leq y_1$. Put $x_2 = x - x_1$, then $x = x_1 + x_2$, and we see that $x_1 \leq y_1$ and $0 \leq x_2 \leq y_2$.

(ii)⇒(i): Take $x_1, x_2, y_1, y_2 \in G$ such that $x_i \leq y_j$ for all $i, j = 1, 2$. Then $y_j - x_i \in G^+$ for all i, j and hence

$$0 \leq y_2 - x_1 \leq (y_2 - x_1) + (y_1 - x_2) = (y_1 - x_1) + (y_2 - x_2),$$

which by (ii) implies that there exist $z_1, z_2 \in G^+$ such that $z_1 + z_2 = y_2 - x_1$ and $z_j \leq y_j - x_j$ for $j = 1, 2$. Put $z = x_1 + z_1$, then we claim that $x_i \leq z \leq y_j$ for all i, j.
First we note that $x_1 \leq z$, as $z_1 \geq 0$. Since $z_1 \leq y_1 - x_1$, we have $z \leq y_1$, and as $z_1 + z_2 = y_2 - x_1$, we have $z = y_2 - z_2 \leq y_2$. Lastly, as $z_2 \leq y_2 - x_2$, we have $x_2 \leq y_2 - z_2 = z$ using the definition of z_1 and z_2. We conclude that G has Riesz interpolation.

(ii)⇒(iii): Suppose $x_1, x_2, y_1, y_2 \in G^+$ with $x_1 + x_2 = y_1 + y_2$. Writing $x_1 = y_1 + y_2 - x_2$ shows that $x_1 \leq y_1 + y_2$. Then, by (ii), there exist $z_{11}, z_{12} \in G^+$ such that $x_1 = z_{11} + z_{12}$ and $z_{1j} \leq y_j$ for $j = 1, 2$. Now put $z_{2j} = y_j - z_{1j}$ for each j, then $z_{2j} \in G^+$ and $y_j = z_{1j} + z_{2j}$. We hence only need to show that $x_2 = z_{21} + z_{22}$. However, this is easily seen by the following calculations:

$$x_1 + x_2 = y_1 + y_2 = (z_{11} + z_{21}) + (z_{12} + z_{22}) = (z_{11} + z_{12}) + (z_{21} + z_{22}) = x_1 + z_{21} + z_{22}.$$

(iii)⇒(ii): Suppose $x, y_1, y_2 \in G^+$ with $x \leq y_1 + y_2$. Consider the positive elements $w_1 = x$ and $w_2 = y_1 + y_2 - x$, which satisfy $w_1 + w_2 = y_1 + y_2$. By (iii), there exist $z_{ij} \in G^+$ such that $w_i = z_{i1} + z_{i2}$ and $y_j = z_{1j} + z_{2j}$ for all i, j. Put $x_j = z_{1j}$ for $j = 1, 2$ and note that $x = w_1 = x_1 + x_2$. Moreover,

$$x_j = z_{1j} \leq z_{1j} + z_{2j} = y_j$$

showing that G satisfies (ii). This completes the proof. □

The next proposition, again from [9], states that if G satisfies one of the three equivalent properties of Lemma 4.4, then these properties can be extended to any finite number of elements.

Proposition 4.5. *Suppose G is a Riesz group.*

(i) *If $x_1, \ldots, x_n, y_1, \ldots, y_k \in G$ satisfy that $x_i \leq y_j$ for all i, j, then there exists $z \in G$ such that $x_i \leq z \leq y_j$, for all i, j.*

(ii) *If $x, y_1, \ldots, y_k \in G^+$ satisfies that $x \leq y_1 + \cdots + y_k$, then there exist $x_1, \ldots, x_k \in G^+$ with $x = x_1 + \cdots + x_k$ and $x_i \leq y_i$ for all i.*

(iii) *If $x_1, \ldots, x_n, y_1, \ldots, y_k \in G^+$ satisfies that*

$$x_1 + \cdots + x_n = y_1 + \cdots + y_k$$

then there exist $z_{ij} \in G^+$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$ such that

$$x_i = z_{i1} + \cdots + z_{ik}, \qquad \text{and} \qquad y_j = z_{1j} + \cdots + z_{nj}$$

Proof. (i): We prove this by induction. The case $n = 1$ is trivial, and so is $k = 1$. So consider $n, k \geq 2$. If $n = k = 2$, it follows by Lemma 4.4. So we can assume $n + k > 4$ and proceed by complete induction on $n + k$, i.e. on the number of x_i's and y_j's.
We can without loss of generality assume that $n > 2$. Find by induction hypothesis $w \in G$ with $x_i \leq w \leq y_j$ for $i = 1, \dots, n-1$ and $j = 1, \dots, k$. We can then use the induction hypothesis, since $2 + k < n + k$, again on the elements w, x_n and $y_1, \dots, y_k$ to find $z \in G$ with $w, x_n \leq z \leq y_j$ for all j. It is then easily seen that $x_i \leq z \leq y_j$ for all i, j.

(ii): The case $k = 1$ is trivial, and the case $k = 2$ holds by Lemma 4.4. We proceed by complete induction on k. Note that

$$x_1 \leq (y_1 + y_2) + y_3 + \cdots + y_k$$

and use the induction hypothesis to find $x_{12}, x_3, \dots, x_k \in G^+$ such that $x = x_{12} + x_3 + \cdots + x_k$ with $x_{12} \leq y_1 + y_2$ and $x_j \leq y_j$ for $j \geq 3$. By the case $k = 2$, there exist $x_1, x_2 \in G^+$ such that $x_{12} = x_1 + x_2$ and $x_j \leq y_j$ for $j = 1, 2$. Then $x = x_1 + \cdots + x_k$ and $x_j \leq y_j$ for all $j = 1, \dots, k$.

(iii): Lastly, we prove (iii) by induction on n. The case $n = 1$ is trivially true. For $n > 1$ we see that

$$x_1 \leq x - 1 + \cdots + x_n = y_1 + \cdots + y_k$$

and hence by (ii) there exist positive elements $z_{1j} \in G^+$ such that $x_1 = \sum_{j=1}^k z_{1j}$ and $z_{1j} \leq y_j$ for all $j = 1, \dots, k$. In particular, $y_j - z_{1j} \geq 0$ and we have the positive decomposition

$$x_2 + \cdots + x_n = y_1 + \cdots + y_k - x_1 = (y_1 - z_{11}) + \cdots + (y_k - z_{1k}) .$$

By induction hypothesis, there exist elements $z_{ij} \in G^+$ for $i = 2, \dots n$ and $j = 1, \dots k$ such that

$$x_i = z_{i1} + \cdots + z_{ik}, \qquad \text{and} \qquad y_j - z_{1j} = z_{2j} + \cdots + z_{2n}.$$

for all $i = 2, \dots, n$ and $j = 1, \dots, k$. Adjoining these z_{ij}'s with the z_{1j}'s gives us that

$$x_i = z_{i1} + \cdots + z_{ik}, \quad \text{and} \quad y_j = z_{1j} + z_{2j} + \cdots + z_{nj}$$

for all $i = 1, \dots, n$ and $j = 1, \dots, k$. This completes the proof. □

We shall see that dimension groups are precisely countable Riesz groups. We first need a lemma explaining how we can pull back positive elements in an inductive limit to positive elements in the sequence.

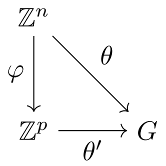
Lemma 4.6. *Let $(G_n, \{\varphi_n\})$ be an inductive sequence of ordered Abelian groups with inductive limit $(G, \{\varphi_{\infty,n}\})$. Then $\varphi_{\infty,n}(x) \geq 0$ if and only if there exists $m \geq n$ with $\varphi_{m,n}(x) \geq 0$.*

Proof. The implication from the right is trivial, since $\varphi_{\infty,n}(x) = \varphi_{\infty,m} \circ \varphi_{m,n}(x)$, and as the boundary maps are positive. Suppose $\varphi_{\infty,n}(x) \geq 0$, then as $G^+ = \bigcup_{n=1}^{\infty} \varphi_{\infty,n}(G_n^+)$ there exists $y \in G_k^+$ such that $\varphi_{\infty,n}(x) = \varphi_{\infty,k}(y)$. Suppose $n \geq k$; the other case follows analogously. Then $\varphi_{\infty,n}(x - \varphi_{n,k}(y)) = 0$ which, by Theorem 2.6(ii), implies there exists $m \geq n$ with $\varphi_{m,n}(x - \varphi_{n,k}(y)) = 0$. Then $\varphi_{m,n}(x) = \varphi_{m,k}(y) \geq 0$ completing the proof. □

Note that this lemma implies that dimension groups are unperforated. If G is a dimension group with inductive sequence $(\mathbb{Z}^{n_k}, \{\varphi_k\})$ and we have $g \in G$ and $n \in \mathbb{N}$ such that $ng \geq 0$, then $g = \varphi_{\infty,k}(h)$ for some $k \in \mathbb{N}$ and $h \in \mathbb{Z}^{n_k}$. Then $\varphi_{\infty,k}(nh) = ng \geq 0$, which by the above lemma implies that there exists $m \in \mathbb{N}$ with $\varphi_{m,k}(nh) \geq 0$. Unperforation of simplical groups implies that $\varphi_{m,k}(h) \geq 0$, and thus $g = \varphi_{\infty,m}(\varphi_{m,k}(h))$ is positive.

The following result gives a local characterization of dimension groups and is due to Shen, see [16].

Theorem 4.7 (Shen, 1979)**.** *An unperforated, countable, ordered Abelian group G is a dimension group if and only if the following condition holds:*
For any simplical group $\mathbb{Z}^n$ and any positive group homomorphism $\theta \colon \mathbb{Z}^n \to G$ and arbitrary $\alpha \in \ker \theta$, there exist a simplical group $\mathbb{Z}^p$ and positive group homomorphisms $\varphi \colon \mathbb{Z}^n \to \mathbb{Z}^p$ and $\theta' \colon \mathbb{Z}^p \to G$ such that $\alpha \in \ker \varphi$ and the following diagram commutes:



Proof. First suppose that G is a dimension group, and let $(G, \{\varphi_{\infty,n}\}) = (\varinjlim(\mathbb{Z}^{n_k}, \varphi_k)$. Suppose $\theta\colon \mathbb{Z}^n \to G$ is a positive group homomorphism, and that $\alpha \in \ker\theta$. Let (e_i) be a basis for $\mathbb{Z}^n$ and write $\alpha = \sum_{i=1}^n \alpha_i e_i$. By construction of the inductive limit, there exists $k \in \mathbb{N}$ such that $\theta(e_i) \in \varphi_{\infty,k}(\mathbb{Z}^{n_k})$ for all $i = 1, \ldots, n$. Find $u_i \in \mathbb{Z}^{n_k}$ such that $\varphi_{\infty,k}(u_i) = \theta(e_i)$. By Lemma 4.6, there exists $m \geq k$ such that $\varphi_{m,k}(u_i) \geq 0$ and thus

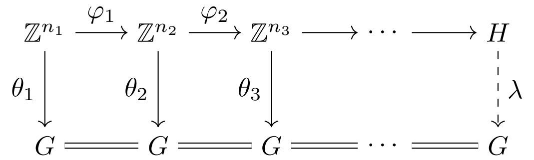
$$\varphi_{\infty,k}(u_i) = \varphi_{\infty,m}(\varphi_{m,k}(u_i))$$

such that we without loss of generality can assume that $u_i \geq 0$. Moreover we find that

$$\varphi_{\infty,k}(\sum_{i=1}^n \alpha_i u_i) = \sum_{i=1}^n \alpha_i \varphi_{\infty,k}(u_i) = \sum_{i=1}^n \alpha_i \theta(e_i) = \theta(\alpha) = 0$$

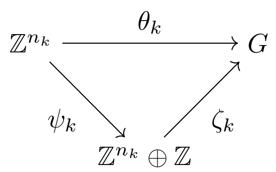
and hence we may assume that $\sum_{i=1}^n \alpha_i u_i = 0$. Now define the positive group homomorphism $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}^{n_k}$ by $\varphi(e_i) = u_i$ and let $\theta' = \varphi_{\infty,k}$, then $\theta' \circ \varphi = \theta$ and $\alpha \in \ker\varphi$ by construction.

Now suppose that G is an unperforated ordered Abelian group with the localization property in the theorem. Suppose we have constructed a diagram of order homomorphisms:



such that $\ker\theta_k \subseteq \ker\varphi_k$ and $G^+ = \bigcup_k \theta_k((\mathbb{Z}^+)^{n_k})$. We prove that the dashed map $\lambda\colon H \to G$ exists and is an isomorphism of ordered groups. Let $(H, \{\varphi_{\infty,k}\}) = \varinjlim(\mathbb{Z}^{n_k}, \{\varphi_k\})$. Then the definition of inductive limits implies that there exists a unique group homomorphism $\lambda\colon H \to G$ such that $\lambda \circ \varphi_{\infty,k} = \theta_k$ for all k. Injectivity of λ is clear by Theorem 2.6(iii), as $\ker\theta_k \subseteq \ker\varphi_k$, and surjectivity is obvious. Moreover it is positive by the proof of Theorem 2.7, and if $g \in G^+$, then there exist $k \in \mathbb{N}$ and $a \in (\mathbb{Z}^+)^{n_k}$ such that $g = \theta_k(g)$, and putting $h = \varphi_{\infty,k}(a) \geq 0$ gives us that $\lambda(h) = g$. We have hence shown that λ is an order isomorphism and, consequently, if we were to construct such a diagram, G would be the inductive limit of a sequence of simplical groups, i.e., a dimension group.

Since G is assumed to be countable, let $G^+ \setminus \{0\} = \{g_1, g_2, \ldots\}$ be an enumeration of the non-zero positive elements of G. Let $n_1 = 1$ and $\theta_1\colon \mathbb{Z} \to G$ be given by $\theta_1(k) = kg_1$ for $k \in \mathbb{Z}$; note that it is a positive group homomorphism. Now construct θ_{k+1} inductively as follows: Consider the diagram



where the direct sum is understood as of ordered Abelian groups, and the maps $\psi_k\colon \mathbb{Z}^{n_k} \to \mathbb{Z}^{n_k} \oplus \mathbb{Z}$ and $\zeta_k\colon \mathbb{Z}^{n_k} \oplus \mathbb{Z} \to G$ are given by

$$\begin{aligned} \psi_k(g) &= (g, 0), && g \in \mathbb{Z}^{n_k}, \\ \zeta_k(g, m) &= \theta_k(g) + m g_{k+1}, && (g, m) \in \mathbb{Z}^{n_k} \oplus \mathbb{Z}. \end{aligned}$$

Note that the diagram is clearly commutative by construction. Since the kernel $\ker\zeta_k$ is a free module of finite rank, we can use the localization property, which G is assumed to satisfy, a finite number of times to find a simplical group $\mathbb{Z}^{n_{k+1}}$ and positive group homomorphisms $\Phi_k\colon \mathbb{Z}^{n_k}\oplus\mathbb{Z}\to\mathbb{Z}^{n_{k+1}}$ and $\theta_{k+1}\colon \mathbb{Z}^{n_{k+1}}\to G$ such that $\ker\zeta_k\subseteq\ker\Phi_k$ and $\theta_{k+1}\circ\Phi_k=\zeta_k$. Define $\varphi_k=\Phi_k\circ\psi_k$, then

$$\theta_{k+1}\circ\varphi_k=\theta_{k+1}\circ\Phi_k\circ\psi_k=\zeta_k\circ\psi_k=\theta_k.$$

Suppose $g\in\ker\theta_k$, then $\zeta_k(\psi_k(g))=0$ implying that $\psi_k(g)\in\ker\zeta_k\subseteq\ker\Phi_k$. Therefore $\varphi_k(g)=\Phi_k\circ\psi_k(g)=0$ proving that $\ker\theta_k\subseteq\varphi_k$.
Finally, having constructed this diagram, note that $g_1=\theta_1(1)$ and for $k>1$ we get

$$g_k=\zeta_{k-1}(0,1)=\theta_k(\Phi_{k-1}(0,1))$$

and since Φ_{k-1} is an order homomorphism, $\Phi_{k-1}(0,1)\in(\mathbb{Z}^+)^{n_k}$, and hence $G^+=\bigcup_{k\in\mathbb{N}}\theta_k\left((\mathbb{Z}^+)^{n_k}\right)$, which completes the proof. □

We are now able to prove the intrinsic characterization of dimension groups [4].

Theorem 4.8 (Effros-Handelmann-Shen, 1980)**.** *Let G be a countable ordered Abelian group. Then G is a dimension group if and only if G is a Riesz group.*

Proof. Suppose that $(G,\{\varphi_{\infty,k}\})=\varinjlim(\mathbb{Z}^{n_k},\{\varphi_k\})$ is a dimension group. We saw as a consequence of Lemma 4.6 that dimension groups are unperforated, since unperforation passes to inductive limits. We hence only need to prove that Riesz' interpolation property is preserved by inductive limits. Let $x_i,y_i\in G$, $i=1,2$, be such that $x_i\le y_j$ for all i,j. Find $k\in\mathbb{N}$ and $a_i,b_i\in\mathbb{Z}^{n_k}$ such that $\varphi_{\infty,k}(a_i)=x_i$ and $\varphi_{\infty,k}(b_i)=y_i$. Since $x_i\le y_j$ for $i,j=1,2$, we find that $\varphi_{\infty,k}(b_j-a_i)\ge 0$, and hence by Lemma 4.6 there exists $m\ge k$ such that $\varphi_{m,k}(b_j-a_i)\ge 0$ for all $i,j=1,2$. Then $\varphi_{m,k}(a_i)\le\varphi_{m,k}(b_j)$ for all $i,j=1,2$, and since $\mathbb{Z}^{n_m}$ has Riesz interpolation, there exists some $c\in\mathbb{Z}^{n_m}$ with

$$\varphi_{m,k}(a_i)\le c\le\varphi_{m,k}(b_j),\qquad i,j=1,2.$$

Setting $z=\varphi_{\infty,m}(c)$ and using that the boundary maps are positive group homomorphisms and hence preserves the ordering, we conclude that $x_i\le z\le y_j$ for all i,j, and hence G has Riesz interpolation. We conclude that if G is a dimension group, then G is a Riesz group.

Now assume that G is a Riesz group, then we want to construct G as an inductive limit of simplical groups. We shall use the characterization given in Theorem 4.7. Suppose $\theta\colon\mathbb{Z}^n\to G$ is an arbitrary positive group homomorphism and let $\alpha\in\ker\theta$. We define for each element $g\in\mathbb{Z}^n$ the *degree* of g by the following: We can decompose g into its positive and negative entries, i.e.: Find an orthonormal basis (e_i,f_j,g_k) of $\mathbb{Z}^n$ such that

$$g=\sum_{i=1}^{r}m_ie_i-\sum_{j=1}^{s}n_jf_j$$

where $m_i,n_j>0$ are strictly positive integers, and note that the basis is unique up to relabelling. Then $\deg g=(p,d)$, where p is the maximal coefficient, i.e. $p=\max_{i,j}\{m_i,n_j\}$, and d is the number of times this coefficient appears, i.e.,

$$d=\#\{i\,|\,m_i=p\}+\#\{j\,|\,n_j=p\}.$$

Define moreover $\deg 0=(0,0)$. Then the lexicographical order, i.e., $(p,d)\le(p',d')$ if $p\le p'$ or if

$$p=p'\quad\text{and}\quad d\le d',$$

defines a total ordering on the degrees. Our goal is to construct positive group homomorphisms $\tilde\varphi\colon\mathbb{Z}^n\to\mathbb{Z}^m$ and $\theta'\colon\mathbb{Z}^m\to G$ for some m such that $\deg\tilde\varphi(\alpha)<\deg\alpha$ and $\theta'\circ\tilde\varphi=\theta$. If this is possible, we can just continue this process finitely many times to get a map $\varphi\colon\mathbb{Z}^n\to\mathbb{Z}^m$ for which $\deg\varphi(\alpha)=(0,0)$, and consequently $\varphi(\alpha)=0$.

Find, again up to relabelling, the unique orthonormal basis (e_i,f_j,g_k) of $\mathbb{Z}^n$ such that

$$\alpha=\sum_{i=1}^{r}m_ie_i-\sum_{j=1}^{s}n_jf_j$$

and define $(p, d) = \deg \alpha$. Set $a_i = \theta(e_i)$ and $b_j = \theta(f_j)$, then $a_i, b_j \geq 0$ by positivity of θ. Moreover, since $\alpha \in \ker \theta$, we find that

$$\sum_{i=1}^{r} m_i a_i = \sum_{j=1}^{s} n_j b_j.$$

If $s = 0$, then $a_i = 0$ for all $i = 1, \ldots, r$ and if we put $m = n$ and define $\tilde{\varphi}\colon \mathbb{Z}^n \to \mathbb{Z}^n$ by $\tilde{\varphi}(e_i) = 0$ and $\tilde{\varphi}(g_k) = g_k$ and $\theta' = \theta$, then $\theta \circ \tilde{\varphi} = \theta$ trivially with $\tilde{\varphi}(\alpha) = 0$. We get a similar result if $r = 0$. We can hence assume that $r, s \neq 0$. Assume moreover that $m_1 = \max_{i,j}\{m_i, n_j\}$, such that $\deg \alpha = (m_1, d)$; if $\max_{i,j}\{m_i, n_j\} = m_{i_0}$, just permute the indices, and if $\max_{i,j}\{m_i, n_j\} = n_{j_0}$ an argument similar to the following will work. With this assumption, we get the inequalities

$$m_1 a_1 \leq \sum_{i=1}^{r} m_i a_i = \sum_{j=1}^{s} n_j b_j \leq m_1 \sum_{j=1}^{s} b_j,$$

and since G is unperforated, we see that $a_1 \leq \sum_{j=1}^{s} b_j$. By the generalized Riesz decomposition, Proposition 4.5, there exist $a_{1j} \in G^+$ for $j = 1, \ldots, s$ such that $a_{1j} \leq b_j$ for each j and $\sum_{j=1}^{s} a_{1j} = a_1$.

Let $m = 2s + t + (r - 1)$ and let

$$(e'_{11}, \ldots, e'_{1s}, e'_2, \ldots, e'_r, f'_1, \ldots, f'_s, g'_1, \ldots, g'_t)$$

be a basis for $\mathbb{Z}^m$. Define the positive group homomorphism $\tilde{\varphi}\colon \mathbb{Z}^n \to \mathbb{Z}^m$ by

$$e_1 \mapsto \sum_{j=1}^{s} e'_{1j}, \quad e_i \mapsto e'_i \text{ for } 2 \leq i \leq r, \quad f_j \mapsto f'_j + e'_{ij}, \quad g_k \mapsto g'_k$$

and the positive group homomorphism $\theta'\colon \mathbb{Z}^m \to G$ by

$$e'_{1j} \mapsto a_{1j}, \quad e'_i \mapsto a_i, \quad f'_j \mapsto b_j - a_{1j}, \quad g'_k \mapsto \theta(g_k).$$

It is straight-forward to verify that $\theta' \circ \tilde{\varphi} = \theta$. Moreover,

$$\tilde{\varphi}(\alpha) = \sum_{j=1}^{s} (m_1 - n_j) e'_{1j} + \sum_{j=2}^{r} m_j e'_j - \sum_{j=1}^{s} n_j f'_j.$$

If we put $\deg \tilde{\varphi}(\alpha) = (p', d')$, then we either get $p' < p$ or, if $p' = p$, we get $d' < d$. This completes the proof. □

We have hence proven that countable dimension groups are precisely the ordered Abelian groups, which are unperforated with Riesz interpolation. Consequently, we have found an intrinsic characterization of the dimension groups and thus the ordered K_0-groups of AF-algebras.

Recall from Theorem 3.22 that A$\mathbb{T}$-algebras of real rank zero can be classified completely by their graded K-theory and graded dimension range. In [15], it is stated that if A is a simple unital A$\mathbb{T}$-algebra of real rank zero, the classification invariant — much like the case for unital AF-algebras — can be reduced to $(K_0(A), K_0(A)^+, [1_A]_0, K_1(A))$, i.e., the ordered K_0-group, as for AF-algebras, along with the K_1-group structure. In this case, the classification invariant has the structure that $(K_0(A), K_0(A)^+)$ is a simple dimension group and that $K_1(A)$ is a torsion-free Abelian group. Moreover, for any tuple (G_0, G_0^+, g_0, G_1), where (G_0, G_0^+) is a simple dimension group, g_0 some order unit for G_0 and G_1 is a countable torsion-free Abelian group, there exists a simple unital A$\mathbb{T}$-algebra A whose classification invariant is precisely this tuple. More generally, any pair (G_0, G_1) with G_0 a countable dimension group and G_1 an arbitrary countable Abelian group can be realized as the K-theory of an AH-algebra [3, Remark 2.4.6].

5 Tracial states on AF-algebras

This last chapter of this project examines the tracial simplices of unital AF-algebras. The first important result is that taking tracial simplices of unital C^*-algebras takes inductive limits to inverse limits, and therefore we can understand the tracial simplex structure of AF-algebra by understanding the structure of the tracial simplices of finite-dimensional algebras, as well as their inverse limits. Using a result of Lazar and Lindenstrauss, [11], as well as the above mentioned, we show that any metrizable Choquet simplex can be realized as the tracial simplex of a simple AF-algebra. Lastly, we discuss some specific examples of such metrizable Choquet simplices in this framework. The main reference for this chapter is [14].

5.1 Tracial simplices of AF-algebras and metrizable Choquet simplices

Recall that an n-simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$ is the closed convex hull of the standard unit vectors $\{e_k \mid k = 0, \dots, n\}$ for $\mathbb{R}^{n+1}$. Since these are compact and convex sets, it makes sense by Theorem 2.20 to consider the inverse limits of such simplices.

Definition 5.1. A *Choquet simplex* is a compact, convex set Δ which is realized as the inverse limit of n-simplices. If the extreme boundary $\partial_e\Delta$ is closed, we call Δ a *Bauer simplex*. We say that a Choquet simplex Δ is *finite-dimensional*, if it is an n-simplex for some $n \in \mathbb{N}_0$; otherwise we call it *infinite-dimensional*.

The above is not the usual definition of a Choquet simplex, but we take this as our definition, as we then do not need to develop too much theory regarding simplices. For a more thorough exposition on Choquet simplices, we refer to [9]; note in particular Theorem 11.6 herein, which is exactly our definition above.

A state ρ on a unital C^*-algebra A is a positive linear functional $\rho\colon A \to \mathbb{C}$ such that $\rho(1_A) = 1$. The collection of states, also called the *state space*, is denoted by $S(A)$ and is a compact and convex subset of the dual A^* of A. A *trace* on A is a linear functional $\tau\colon A \to \mathbb{C}$ such that τ satisfies the trace property:

$$\tau(ab) = \tau(ba), \qquad \text{for all } a, b \in A.$$

If A is unital, and τ is a positive trace with $\tau(1_A) = 1$, then we call τ a *tracial state*. We denote by $T(A)$ the set of tracial states on A. It is clear that $T(A)$ is a convex subset of $S(A)$. In fact, we have, see [1, II.6.8.11]:

Proposition 5.2. *For a unital C^*-algebra A, the tracial state space $T(A)$ is a Choquet simplex.*

It is natural to ask what structure the tracial state space of an AF-algebra may have. To answer this question we need to examine how tracial simplices of inductive limits behave.

If $\varphi\colon A \to B$ is a unit-preserving *-homomorphism, then it induces an affine map $T(\varphi)\colon T(B) \to T(A)$ by $T(\varphi)(\tau) = \tau \circ \varphi$. Note that T is functorial in the sense that if $\varphi\colon A \to B$ and $\psi\colon B \to C$, then $T(\psi \circ \varphi) = T(\varphi) \circ T(\psi)$. We can express this as follows, where $\mathbf{C}^*_\mathbf{1}$ is the category of unital C^*-algebras with unit-preserving *-homomorphisms, and **CptConv** is the category of compact convex spaces with affine continuous maps as morphisms:

Proposition 5.3. *The functor T is contravariant from $\mathbf{C}^*_\mathbf{1}$ to* **CptConv**.

As T is a contravariant functor, it takes inverse sequences to inductive sequences, and hence one might ask whether it preserves these limits. This is true as the following theorem proves. Recall that a continuous bijection $f\colon X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.

Theorem 5.4. *If $(A_n, \{\varphi_n\})$ is an inductive sequence of unital C^*-algebras with inductive limit $(A, \{\varphi_{\infty,n}\})$, then $(T(A), \{T(\varphi_{\infty,n})\})$ is the inverse limit of the inverse sequence $(T(A_n), \{T(\varphi_n\})$.*

Proof. Recall that

$$\varprojlim T(A_n) = \left\{ \tau = (\tau_1, \tau_2, \dots) \in \prod_{n \in \mathbb{N}} T(A_n) \,\middle|\, \tau_n = T(\varphi_n)(\tau_{n+1}) \text{ for all } n \right\}.$$

Recall moreover that $T(\varphi_{\infty,n})(\tau) = \tau \circ \varphi_{\infty,n}$. We claim that the map $\Phi\colon T(A) \to \varprojlim T(A_n)$ by $\Phi(\tau) = (\tau \circ \varphi_{\infty,1}, \tau \circ \varphi_{\infty,2}, \ldots)$, for $\tau \in T(A)$, is a well-defined isomorphism of convex compact sets. Firstly, it is well-defined since if $\tau \in T(A)$, then $\tau \circ \varphi_{\infty,n} \in T(A_n)$, and moreover

$$(\tau \circ \varphi_{\infty,n+1}) \circ \varphi_n = \tau \circ \varphi_{\infty,n}$$

proving that $(\tau\circ\varphi_{\infty,1}, \tau\circ\varphi_{\infty,2}, \ldots) \in \varprojlim T(A_n)$. It is easily verified that Φ is an affine homomorphism, so we only need to verify that is a continuous bijection. Φ is continuous, as

$$\|\Phi(\tau)\| = \sup_{n\in\mathbb{N}} \|\tau(\varphi_{n,\infty})\| \leq \|\tau\|.$$

For injectivity suppose that $\Phi(\tau) = \Phi(\tau')$, then $\tau \circ \varphi_{\infty,n} = \tau' \circ \varphi_{\infty,n}$ for all $n \in \mathbb{N}$. In particular, the traces coincide on $\varphi_{\infty,n}(A_n)$. Since the union of these is dense in A by Theorem 2.5(i), continuity of traces implies that $\tau = \tau'$. For surjectivity suppose $(\tau_1, \tau_2, \ldots) \in \varprojlim T(A_n)$, then we wish to construct $\tau \in T(A)$ with $\Phi(\tau) = (\tau_1, \tau_2, \ldots)$. Construct the function $\tilde{\tau}\colon \bigcup_{n\in\mathbb{N}} \varphi_{\infty,n}(A_n) \to \mathbb{C}$ by $\tilde{\tau}(\varphi_{\infty,n}(a)) = \tau_n(a)$. The map is well-defined by the following argument: Suppose that $\varphi_{\infty,n}(a) = \varphi_{\infty,m}(a')$ and assume without loss of generality that $n \geq m$. Then,

$$\tau_n(a) = \tau_n \circ \varphi_{\infty,m}(a') = \tau_m(a').$$

It is clear that $\tilde{\tau}$ is a positive linear function with the trace property. Noting that

$$\|\tilde{\tau}(\varphi_{\infty,n}(a))\| = \|\tau_n(a)\| \leq \|a\|$$

for all $n \in \mathbb{N}$ and $a \in A_n$, since τ_n is a tracial state on A_n, we see that we can extend $\tilde{\tau}$ by uniform continuity to a positive linear function $\tau\colon A \to \mathbb{C}$ with the trace property. Moreover it is obviously unital, which proves that it is a trace. □

It is easily verified that $M_n(\mathbb{C})$ has a unique tracial state $\tau \in T(M_n(\mathbb{C}))$ given by $\tau = \frac{1}{n}\mathrm{Tr}$, where Tr denotes the usual trace. Combining this fact with Theorem 5.4 gives us the following result immediately.

Corollary 5.5. *If A is a unital UHF-algebra, there exists a unique tracial state on A.*

There is, consequently, nothing exotic about the structure of tracial simplices of UHF-algebras as they always admit unique traces. This does not hold for general AF-algebras; in fact, not even finite-dimensional C^*-algebras admit unique tracial states in general. Suppose $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ is a finite-dimensional C^*-algebra and define $\tau_1, \ldots, \tau_r\colon A \to \mathbb{C}$ by

$$\tau_i((a_1, \ldots, a_r)) = \frac{1}{n_i}\mathrm{Tr}(a_i), \qquad (a_1, \ldots, a_r) \in A.$$

It is easily verified that each τ_i is a tracial state. In particular, any convex combination of the τ_k is a tracial state. In fact, any trace on A can be realized as such.

Proposition 5.6. *If $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ is a finite-dimensional C^*-algebra, and $\tau \in T(A)$ is a tracial state on A, then $\tau = \sum_{i=1}^r \lambda_i\tau_i$, where $\sum_{i=1}^r \lambda_i = 1$ and $\lambda_i \in [0,1]$ for all i.*

Proof. Define for each $i = 1, \ldots, r$ the scalar $\lambda_i = \tau((0, \ldots, 0, 1_{n_i}, 0, \ldots, 0))$, where 1_{n_i} denotes the identity matrix on $M_{n_i}(\mathbb{C})$. Note that as τ is unital and positive, we get $\sum_{i=1}^r \lambda_i = 1$ and $\lambda_i \in [0,1]$ for all i. The map $\tilde{\tau}_i\colon M_{n_i}(\mathbb{C}) \to \mathbb{C}$ given by

$$\tilde{\tau}_i(B) = \frac{1}{\lambda_i}\tau((0, \ldots, 0, B, 0, \ldots, 0))$$

for all $B \in M_{n_i}(\mathbb{C})$ defines a tracial state on $M_{n_i}(\mathbb{C})$, and hence $\tilde{\tau}_i = \frac{1}{n_i}\mathrm{Tr}$. Therefore,

$$\tau((0, \ldots, 0, B, 0, \ldots, 0)) = \frac{\lambda_i}{n_i}\mathrm{Tr}(B) = \lambda_i\tau_i(B)$$

for all $B \in M_{n_i}(\mathbb{C})$ and all $i = 1, \ldots, r$, whence

$$\tau((B_1, \ldots, B_r)) = \tau((B_1, 0, \ldots, 0)) + \cdots + \tau((0, \ldots, 0, B_r)) = \sum_{i=1}^r \lambda_i\tau_i(B_1, \ldots, B_r)$$

for all $(B_1, \ldots, B_r) \in A$ as desired. □

The following theorem is immediate.

Theorem 5.7. *Let $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ be a finite-dimensional C^*-algebra and define τ_k as before. Then $\partial_e T(A) = \{\tau_1, \ldots, \tau_r\}$ and $T(A)$ is an $(r-1)$-simplex. In particular, any finite-dimensional Choquet simplex can be realized as the tracial simplex $T(A)$ for some finite-dimensional C^*-algebra A.*

By Theorem 5.7, we know the tracial structure of finite-dimensional C^*-algebras. If $\varphi\colon A \to B$ is a *-homomorphism between finite-dimensional C^*-algebras and τ is a tracial state on B, what does $T(\varphi)(\tau)$ look like? This is answered in the following lemma.

Lemma 5.8. *Let $A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$ and $B = M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_s}(\mathbb{C})$ be finite-dimensional C^*-algebras, and let $\varphi\colon A \to B$ be a *-homomorphism. Let $A(i,j)$ be the multiplicity of the *-homomorphism*

$$M_{n_i}(\mathbb{C}) \hookrightarrow A \xrightarrow{\varphi} B \twoheadrightarrow M_{m_j}(\mathbb{C}).$$

Denote by $\{\tau_{A,i}\}_{i=1}^r$ and $\{\tau_{B,j}\}_{j=1}^s$ the extremal tracial states on A and B supported on the $M_{n_i}(\mathbb{C})$ of A and $M_{m_j}(\mathbb{C})$ of B. Then,

$$T(\varphi)(\tau_{B,j}) = \sum_{i=1}^r \frac{A(i,j)n_i}{m_j}\tau_{A,i}, \qquad j = 1, \ldots, s.$$

Proof. Since A is a finite-dimensional C^*-algebra, such that $T(A) = \operatorname{conv}(\{\tau_{A,1}, \ldots, \tau_{A,r}\})$ by Theorem 5.7, we must have $T(\varphi)(\tau_{B,j}) = \sum_{i=1}^r \ell_{i,j}\tau_{A,i}$ for some coefficients $\ell_{i,j} \in [0,1]$, where $\sum_{i=1}^r \ell_{i,j} = 1$. Denote for each $i = 1, \ldots, r$ the element $e^{(i)} \in A$ given by some one-dimensional projection on $M_{n_i}(\mathbb{C})$, which is zero on every other factor in A. Then if $\mathrm{Tr}_{B,j}$ denotes the trace on the factor $M_{m_j}(\mathbb{C})$ of B, and $\mathrm{Tr}_{A,i}$ the trace on the factor $M_{n_i}(\mathbb{C})$ of A, we find — regarding $e^{(i)}$ naturally as a projection in $M_{n_i}(\mathbb{C})$ — that

$$A(i,j) = \frac{\mathrm{Tr}_{B,j}(\varphi(e^{(i)}))}{\mathrm{Tr}_{A,i}(e^{(i)})} = m_j\tau_{B,j}(\varphi(e^{(i)})) = m_j \sum_{k=1}^r \ell_{k,j}\tau_{A,k}(e^{(i)}) = \frac{\ell_{i,j}m_j}{n_i}$$

as desired. □

As AF-algebras are inductive limits of finite-dimensional C^*-algebras, and taking tracial simplices is a contravariant functor taking inductive limits to inverse limits, it follows immediately that the tracial simplices of unital AF-algebras are inverse limits of n-simplices and are hence Choquet simplices. As stated in Proposition 5.2, the tracial state space of any unital C^*-algebra is a Choquet simplex, so it is not particularly remarkable. A more relevant question is what structure tracial simplices of AF-algebras moreover may have, and whether or not they may be arbitrarily exotic. Our goal is to prove that any metrizable Choquet simplex is the tracial state space of some simple AF-algebra. For this we shall without proof use the following result due to Lazar and Lindenstrauss, see the Corollary to Theorem 5.2 in [11].

Theorem 5.9 (Lazar-Lindenstrauss, 1971)**.** *Let Δ be an infinite-dimensional, metrizable Choquet simplex. Then there exists a sequence of affine surjective continuous maps $f_n\colon \Delta_{n+1} \to \Delta_n$, where Δ_n is an n-simplex, such that Δ is the inverse limit of the sequence*

$$\Delta_0 \xleftarrow{f_0} \Delta_1 \xleftarrow{f_1} \Delta_2 \xleftarrow{f_2} \cdots.$$

Since a finite-dimensional C^*-algebra $A = M_{n_1} \oplus \cdots \oplus M_{n_r}$ has a tracial simplex $T(A)$ affinely homeomorphic to an $(r-1)$-simplex by Theorem 5.7, it is natural to ask whether or not we can use the above description of infinite-dimensional metrizable Choquet simplices to write such structures as tracial simplices of AF-algebras. We shall see in this section that it is possible, and we can even describe a corresponding AF-algebra explicitly. We follow the exposition of [14].

Given an infinite-dimensional metrizable Choquet simplex Δ and, by Theorem 5.9, a corresponding inverse sequence $(\Delta_n, \{f_n\})$, we shall write

$$\partial_e\Delta_n = \{e_0^{(n)}, e_1^{(n)}, \ldots, e_n^{(n)}\}.$$

The connecting maps in this construction actually have a nice structure, as extreme points are lifted to extreme points under affine surjections.

Lemma 5.10. *Let K, K' be non-empty compact convex subsets of a locally convex Hausdorff topological space. Let $f\colon K \to K'$ be a surjective affine continuous map, then extreme points of K' lift to extreme points of K.*

Proof. Let $e' \in \partial_e K'$ be an arbitrary extreme point. Consider the preimage of e', i.e.,

$$F = \{x \in K \mid f(x) = e'\}.$$

By surjectivity of f, we have that $F \neq \emptyset$. It is clear from affinity of f that F is convex; in fact, it is a face in K. Suppose $x, y \in K$ and $\lambda \in (0,1)$ satisfy that $\lambda x + (1-\lambda)y \in F$. Then affinity of f implies that

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y) = \lambda e' + (1-\lambda)e' = e'.$$

This proves that F is a face in K. In fact, F is compact by continuity of f, and the Krein-Milman theorem, Theorem 1.9, ensures the existence of an extreme point $e \in \partial_e F$. Assume that $e = \lambda x + (1-\lambda)y$ for some $x, y \in K$ and $0 < \lambda < 1$. Since F is a face and $e \in F$, this implies that $x, y \in F$, and as $e \in \partial_e F$, we see that $x = y = e$. This proves that e is an extreme point of K, and since $e \in F$, we have that $f(e) = e'$. This completes the proof. ☐

This lemma implies that if $f_n\colon \Delta_{n+1} \to \Delta_n$ is a surjective affine continuous map, then we can — up to some relabelling — assume that

$$f_n(e_j^{(n+1)}) = \begin{cases} e_j^{(n)} & \text{if } j = 0, \dots, n \\ \xi^{(n)} & \text{if } j = n+1 \end{cases} \tag{5.1}$$

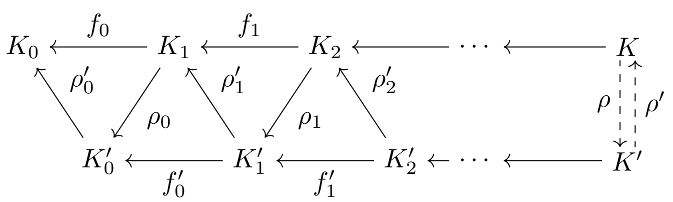
for some $\xi^{(n)} \in \Delta_n$. We thus find that the maps in Theorem 5.9, and consequently the infinite-dimensional metrizable Choquet simplex Δ, are determined by the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}}$ of elements $\xi^{(n)} \in \Delta_n$.

Recall from Lemma 2.17 that if two inductive sequences are intertwined, their inductive limits agree. The same holds true for inverse limits as is easily verified by following the same ideas as in the proof in the case of inductive limits. The commutative assumption can, however, be relaxed; we shall only look at this for inverse limits. Let K and K' be compact metric spaces with metrics d and d' respectively. We equip, as usual, the set of functions from K to K' with the uniform metric d_∞ defined by

$$d_\infty(f, g) = \sup_{x \in K} d'(f(x), g(x))$$

for arbitrary $f, g\colon K \to K'$; this metric space is complete. Recall that a function $f\colon K \to K'$ between metric spaces is called a *contraction* if $d'(f(x), f(y)) \le d(x, y)$ for all $x, y \in K$.

Lemma 5.11. *Let $K_0, K_1, \dots$ and $K_0', K_1', \dots$ be compact and convex metric spaces, let $f_n\colon K_{n+1} \to K_n$ and $f_n'\colon K_{n+1}' \to K_n'$ be affine continuous contractions, and let K and K' be the respective inverse limits. Suppose there exist sequences of affine continuous contractions $\rho_n\colon K_{n+1} \to K_n'$ and $\rho_n'\colon K_n' \to K_n$ such that the diagram*



approximately commutes in the sense that

$$\sum_{j=0}^{\infty} d_\infty(\rho_j' \circ \rho_j, f_j) < \infty, \qquad \text{and} \qquad \sum_{j=0}^{\infty} d_\infty(\rho_j \circ \rho_{j+1}', f_j') < \infty. \tag{5.2}$$

Then there exists an affine continuous homeomorphism $\rho\colon K \to K'$ with inverse ρ'. In particular, K and K' are isomorphic as compact convex sets.

Proof. Let $f_{n,\infty}\colon K \to K_n$ and $f'_{n,\infty}\colon K \to K'_n$ be the canonical boundary maps of the inverse limits. Define for $0 \le i < j$ the functions $\sigma^{(j)}_{i,\infty}\colon K \to K'_i$ and ${\sigma'}^{(j)}_{i,\infty}\colon K' \to K_i$ by

$$\sigma^{(j)}_{i,\infty} = f'_{i,j} \circ \rho_j \circ f_{j+1,\infty}, \qquad \text{and} \qquad {\sigma'}^{(j)}_{i,\infty} = f_{i,j} \circ \rho'_j \circ f'_{j,\infty}.$$

We claim that the sequences $\{\sigma^{(j)}_{i,\infty}\}_{j=i+1}^{\infty}$ and $\{{\sigma'}^{(j)}_{i,\infty}\}_{j=i+1}^{\infty}$ are Cauchy sequences. We prove it for the non-primed sequence; the same method applies to the primed sequence.
We prove that $\sum_{j=i+1}^{\infty} d(\sigma^{(j+1)}_{i,\infty}, \sigma^{(j)}_{i,\infty})$ is convergent, which implies that $\{\sigma^{(j)}_{i,\infty}\}_{j=i+1}^{\infty}$ is a Cauchy sequence. Using the triangle inequality, and the fact that all the maps in the diagram are contractions, we see that for any $j > i$,

$$\begin{aligned} d_\infty(\sigma^{(j)}_{i,\infty}, \sigma^{(j+1)}_{i,\infty}) &= d_\infty(f'_{i,j+1} \circ \rho_{j+1} \circ f_{j+1,\infty}, f'_{i,j} \circ \rho_j \circ f_{j+1,\infty}) \\ &= d_\infty(f'_{i,j} \circ f'_j \circ \rho_{j+1}, f'_{i,j} \circ \rho_j \circ f_{j+1}) \\ &\le d_\infty(f'_j \circ \rho_{j+1}, \rho_j \circ f_{j+1}) \\ &\le d_\infty(f'_j \circ \rho_{j+1}, \rho_j \circ \rho'_{j+1} \circ \rho_{j+1}) + d_\infty(\rho_j \circ \rho'_{j+1} \circ \rho_{j+1}, \rho_j \circ f_{j+1}) \\ &\le d_\infty(f'_j, \rho_j \circ \rho'_{j+1}) + d_\infty(\rho'_{j+1} \circ \rho_{j+1}, f_{j+1}) \end{aligned}$$

which by (5.2) implies that the series $\sum_{j=i+1}^{\infty} d(\sigma^{(j+1)}_{i,\infty}, \sigma^{(j)}_{i,\infty})$ is convergent, and consequently we find that $\{\sigma^{(j)}_{i,\infty}\}_{j=i+1}^{\infty}$ is a Cauchy sequence.
Since the uniform metric induces a complete metric space, these sequences are convergent. Denote by $\sigma_{i,\infty}\colon K \to K'_i$ and $\sigma'_{i,\infty}\colon K' \to K_i$ the limits of these two sequences. Since $f'_i \circ \sigma^{(j)}_{i+1,\infty} = \sigma^{(j)}_{i,\infty}$ and similarly $f_i \circ {\sigma'}^{(j)}_{i+1,\infty} = {\sigma'}^{(j)}_{i,\infty}$ hold for all $i < j$, a simple application of the triangle inequality proves that

$$d_\infty(\sigma_{i,\infty}, f'_i \circ \sigma_{i+1,\infty}) \le d_\infty(\sigma_{i,\infty}, \sigma^{(j)}_{i,\infty}) + d_\infty(\sigma^{(j)}_{i,\infty}, f'_i \circ \sigma^{(j)}_{i+1,\infty}) + d_\infty(f'_i \circ \sigma^{(j)}_{i+1,\infty}, f'_i \circ \sigma_{i+1,\infty}) \to 0$$

in the limit as $j \to \infty$. This implies that $f'_i \circ \sigma_{i+1,\infty} = \sigma_{i,\infty}$ and $f_i \circ \sigma'_{i+1,\infty} = \sigma'_{i,\infty}$ holds for all i. By the universal property of the inverse limit, there exist unique affine continuous maps $\rho\colon K \to K'$ and $\rho'\colon K' \to K$ such that $f'_{i,\infty} \circ \rho = \sigma_{i,\infty}$ and $f_{i,\infty} \circ \rho' = \sigma_{i,\infty}$. For any $x \in K$, we have

$$\begin{aligned} f_{i,\infty} \circ \rho' \circ \rho(x) &= \sigma'_{i,\infty} \circ \rho(x) \\ &= \lim_{j\to\infty} {\sigma'}^{(j)}_{i,\infty} \circ \rho(x) \\ &= \lim_{j\to\infty} (f_{i,j} \circ \rho'_j \circ f'_{j+1,\infty} \circ \rho(x)) \\ &= \lim_{j\to\infty} (f_{i,j} \circ \rho'_j \circ \sigma_{j+1,\infty}(x)) \\ &= \lim_{j\to\infty} \lim_{k\to\infty} (f_{i,j} \circ \rho'_j \circ \sigma^{(k)}_{j+1,\infty}(x)) \\ &= \lim_{j\to\infty} \lim_{k\to\infty} \left(f_{i,j} \circ \rho'_j \circ f'_{j+1,k} \circ \rho_k \circ f_{k+1,\infty}(x)\right). \end{aligned}$$

In a similar manner as before, applying the triangle inequality to consider each triangle in the diagram and using the fact that the maps in the diagram are contractions, we get that

$$d_\infty(\rho'_j \circ f'_{j,k} \circ \rho_k, f_{j,k}) \le \sum_{\ell=j}^{k-1} d_\infty(\rho'_\ell \circ \rho_\ell, f_\ell) + \sum_{\ell=j}^{k-2} d_\infty(\rho_\ell \circ \rho'_{\ell+1}, f'_\ell).$$

In particular, the approximating intertwining property implies that

$$\lim_{j\to\infty} \lim_{k\to\infty} d_\infty(\rho'_j \circ f'_{j,k} \circ \rho_k, f_{j,k}) = 0.$$

Therefore, for arbitrary $x \in K$,

$$\begin{aligned} f_{i,\infty} \circ \rho' \circ \rho(x) &= \sigma'_{i,\infty} \circ \rho(x) \\ &= \lim_{j\to\infty} \lim_{k\to\infty} \left(f_{i,j} \circ \rho'_j \circ f'_{j+1,k} \circ \rho_k \circ f_{k+1,\infty}(x)\right) \\ &= \lim_{j\to\infty} \lim_{k\to\infty} (f_{i,j} \circ f_{j,k} \circ f_{k+1,\infty}(x)) \\ &= f_{i,\infty}(x) \end{aligned}$$

proving that $\rho' \circ \rho = \mathrm{id}_K$. Analogously it is shown that $\rho \circ \rho' = \mathrm{id}_{K'}$ completing the proof. $\square$

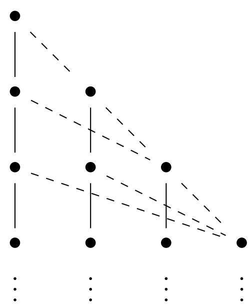
The following corollary follows immediately from Lemma 5.11.

Corollary 5.12. *Let $K_0, K_1, \dots$ be compact convex metric spaces and let $f_n, f_n' \colon K_{n+1} \to K_n$ be affine continuous maps such that the following diagram*

$$\begin{array}{ccccccccc}
K_0 & \xleftarrow{f_0} & K_1 & \xleftarrow{f_1} & K_2 & \xleftarrow{f_2} & \cdots & \longleftarrow & K \\
\| & & \| & & \| & & & & \rho \, \downarrow \uparrow \, \rho' \\
K_0 & \xleftarrow{f_0'} & K_1 & \xleftarrow{f_1'} & K_2 & \xleftarrow{f_2'} & \cdots & \longleftarrow & K'
\end{array}$$

approximately commutes in the sense that $\sum_{j=0}^{\infty} d_\infty(f_j, f_j') < \infty$. Then there exists an affine homeomorphism $\rho \colon K \to K'$ with inverse ρ'.

Now consider the unital AF-algebra A defined by the Bratteli diagram



$$A_0 = M_{k_0}(\mathbb{C})$$

$$A_1 = M_{k_0}(\mathbb{C}) \oplus M_{k_1}(\mathbb{C})$$

$$A_2 = M_{k_0}(\mathbb{C}) \oplus M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$$

$$A_3 = M_{k_0}(\mathbb{C}) \oplus M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C}) \oplus M_{k_3}(\mathbb{C})$$

with the following properties: The connecting maps $\varphi_n \colon A_n \to A_{n+1}$ are unital and hence unique up to unitary equivalence. The multiplicity of the dashed line connecting the (n, j) vertex to the $(n+1, n+1)$ vertex is denoted $m_j^{(n)}$, with the multiplicity of the non-broken lines being 1, and the integers $k_0, k_1, \dots$ are chosen such that $k_0 \geq 1$ is arbitrary and

$$k_{n+1} = \sum_{j=0}^{n} m_j^{(n)} k_j$$

for each $n > 0$. These conditions ensure that A is a well-defined unital AF-algebra, and that it is uniquely determined by the integers $\{k_n\}_{n \in \mathbb{N}_0}$ and the multiplicity vectors

$$m^{(n)} = (m_0^{(n)}, \dots, m_n^{(n)}), \qquad n \geq 0.$$

Now consider the inverse system

$$T(A_0) \leftarrow T(A_1) \leftarrow T(A_2) \leftarrow \cdots \leftarrow T(A) \tag{5.3}$$

with the inverse limit $T(A)$ from Theorem 5.4. If $\tau_j^{(n)}$ denotes the extremal tracial state on A_n supported in M_{k_j} for $j = 0, \dots, n$, then it is easily verified that

$$T(\varphi_n)(\tau_j^{(n+1)}) = \tau_j^{(n)}, \text{ for } j = 0, \dots, n, \quad \text{and} \quad T(\varphi_n)(\tau_{n+1}^{(n+1)}) = \sum_{j=0}^{n} \frac{m_j^{(n)} k_j}{k_{n+1}}.$$

In particular, the connecting maps of (5.3) are surjective. By Theorem 5.7, we can identify $T(A_n)$ with an n-simplex Δ_n through an affine homeomorphism $\chi_n \colon T(A_n) \to \Delta_n$. Define the affine continuous map $f_n' \colon \Delta_{n+1} \to \Delta_n$ by $f_n' = \chi_n \circ T(\varphi_n) \circ \chi_{n+1}^{-1}$, then

$$f_n'(e_j^{(n+1)}) = \begin{cases} e_j^{(n)} & j = 0, \dots, n \\ \left(\frac{m_0^{(n)} k_0}{k_{n+1}}, \dots, \frac{m_n^{(n)} k_n}{k_{n+1}} \right), & j = n+1. \end{cases} \tag{5.4}$$

The maps $f_n, f_n' \colon \Delta_{n+1} \to \Delta_n$ are contractions when equipping the simplices Δ_n with the taxi cab metric d_1 inherited as a subspace of $\mathbb{R}^{n+1}$. We shall only prove it for f_n; the same argument works for f_n'. Note that if $x, y \in \Delta_{n+1}$, then we can write $x = \sum_{j=0}^{n} t_j e_j$ and $y = \sum_{j=0}^{n+1} s_j e_j$, where $\sum_{j=0}^{n+1} t_j = \sum_{j=0}^{n} s_j = 1$. Then,

$$f(x) = \sum_{j=0}^{n} t_j e_j + t_{n+1}\xi^{(n)}, \qquad \text{and} \qquad f(y) = \sum_{j=0}^{n} s_j e_j + s_{n+1}\xi^{(n)}.$$

Then

$$\|f(x) - f(y)\|_1 \leq \sum_{j=0}^{n} |t_j - s_j| + |t_{n+1} - s_{n+1}| = \|x - y\|_1$$

using the triangle inequality, and that $\left\|\xi^{(n)}\right\| = 1$.
We shall immediately use these maps in the following theorem.

Theorem 5.13. *Let Δ be an infinite-dimensional metrizable Choquet simplex, then there exists an AF-algebra A with tracial simplex $T(A)$ isomorphic to Δ as compact convex sets.*

Proof. Denote as usual the n-simplex by Δ_n and express Δ by the inverse limit

$$\Delta_0 \xleftarrow{f_0} \Delta_1 \xleftarrow{f_1} \Delta_2 \xleftarrow{f_2} \cdots \longleftarrow \Delta$$

using Theorem 5.9. This is by (5.1) determined by the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0} \subseteq \prod_{n\in\mathbb{N}_0} \Delta_n$. We aim to construct an AF-algebra A with Bratteli diagram of the form as described previously for which the inverse system of the tracial simplices is approximately intertwined with the inverse system corresponding to Δ above.
Given a sequence of multiplicity vectors $m^{(n)}$ and a sequence of integers $\{k_n\}_{n\in\mathbb{N}_0} \in \mathbb{N}$, we can define for $n \geq 0$ the affine continuous map $f_n' \colon \Delta_{n+1} \to \Delta_n$ given by (5.4). Using the method in the diagram above, we can use these sequences of integers to construct finite-dimensional C^*-algebras A_n with $\Delta_n = T(A_n)$. Denote by A the inductive limit $\lim_{\to} A_n$ with connecting maps $f_n' \colon A_{n+1} \to A_n$. Then $T(A)$ is the inverse limit of the inverse sequence

$$\Delta_0 \xleftarrow{f_0'} \Delta_1 \xleftarrow{f_1'} \Delta_2 \xleftarrow{f_2'} \cdots \longleftarrow T(A).$$

With these maps we have a diagram similar to the one in Corollary 5.12

$$\begin{array}{ccccccccc}
\Delta_0 & \xleftarrow{\quad f_0 \quad} & \Delta_1 & \xleftarrow{\quad f_1 \quad} & \Delta_2 & \xleftarrow{\quad f_2 \quad} & \cdots & \longleftarrow & \Delta \\
\| & & \| & & \| & & & & \\
T(A_0) & \xleftarrow{\quad f_0' \quad} & T(A_1) & \xleftarrow{\quad f_1' \quad} & T(A_2) & \xleftarrow{\quad f_2' \quad} & \cdots & \longleftarrow & T(A)
\end{array}$$

with the caveat that it is not immediate that it approximately commutes such that we can invoke Corollary 5.12 and achieve that $T(A)$ is affinely homeomorphic to Δ. Since the maps f_n are fixed, as the Choquet simplex Δ is given, our only freedom is in choosing f_n' in a sufficiently smart way.

For notational purposes define

$$\zeta^{(n)} = \left(\frac{m_0^{(n)} k_0}{k_{n+1}}, \ldots, \frac{m_n^{(n)} k_n}{k_{n+1}}\right) \in \Delta_n$$

and $\zeta_j^{(n)} = \frac{m_j^{(n)} k_j}{k_{n+1}}$ the jth coordinate of $\zeta^{(n)}$. It is clear that $d_\infty(f_n, f_n') = d_1(\xi^{(n)}, \zeta^{(n)})$, and we previously showed that f_n and f_n' are contractions. If we were to construct the sequence $\zeta^{(n)}$ such that $d_1(\xi^{(n)}, \zeta^{(n)}) < 2^{-n}$, then, by Corollary 5.12, Δ would be isomorphic to $T(A)$ as compact convex sets. Note that $\zeta^{(n)}$ is determined by the multiplicity vectors $\{m^{(n)}\}_{n\in\mathbb{N}_0}$ and integers $\{k_n\}_{n\in\mathbb{N}_0}$.
Let $m_0^{(0)}$ be arbitrary and note that $\xi^{(0)} = \zeta^{(0)} = 1$. Suppose we have constructed $m_j^{(r)}$ for all $0 \leq j \leq r < n$, then we aim to construct $m_j^{(n)}$ for $j = 0, \ldots, n+1$. Note that we can define $k_0, k_1, \ldots, k_n$ by the identity $k_{r+1} = \sum_{j=0}^{r} m_j^{(r)} k_j$ with $k_0 = 1$. Find integers $\ell_0, \ell_1, \ldots, \ell_n \in \mathbb{N}$ with

$$\left|\frac{\ell_j}{\sum_{i=0}^{n} \ell_i} - \xi_j^{(n)}\right| < \frac{1}{2^n n}.$$

Define $K = \prod_{j=0}^{n} k_j$ and set $m_j^{(n)} = K\frac{\ell_j}{k_j}$ for $j = 0, \ldots, n$ and let $k_{n+1} = \sum_{j=0}^{n} m_j^{(n)} k_j = K \sum_{j=0}^{n} \ell_j$. Then,

$$\zeta_j^{(n)} = \frac{m_j^{(n)} k_j}{k_{n+1}} = \frac{\ell_j}{\sum_{i=0}^{n} \ell_i}.$$

In particular, $\left|\zeta_j^{(n)} - \xi_j^{(n)}\right| < \frac{1}{2^n n}$ for each $j = 0, \ldots, n$ and $n \in \mathbb{N}$. Hence,

$$d_1(\xi^{(n)}, \zeta^{(n)}) = \sum_{j=0}^{n} \left|\xi_j^{(n)} - \zeta_j^{(n)}\right| = \sum_{j=0}^{n} \left|\xi_j^{(n)} - \frac{\ell_j}{\sum_{i=0}^{n} \ell_i}\right| < \sum_{j=0}^{n} \frac{1}{2^n n} = 2^{-n}$$

completing the proof. □

Note that the proof states that if Δ is an infinite-dimensional metrizable Choquet simplex, then there are infinitely many AF-algebras A with $T(A) = \Delta$; the easiest way to see this is to note that we had an arbitrary choice when letting $k_0 = 1$, which gives rise to an infinite number of different AF-algebras with the same tracial simplex, but in fact there are *uncountably* many such distinct AF-algebras. Thus there is nothing unique with the above construction; this is not surprising as any AF-algebra can be made into a simple AF-algebra without disturbing the structure of the tracial simplex. We shall only sketch the idea here.

Let A be an AF-algebra with inductive sequence $(A_n, \{\varphi_n\})$ and consider its Bratteli diagram. Let $N_n \in \mathbb{N}$ be arbitrary for each $n \in \mathbb{N}$, and disturb the Bratteli diagram as follows: Multiply the dimension of the matrix algebras on the $(n+1)$st row by N_n for all n, and likewise multiply the number of edges connecting the nth row with the $(n+1)$st row by N_n to preserve unitality of the connecting maps. It follows from Lemma 5.8 that the tracial simplex of the new AF-algebra is exactly the same as the one for the original AF-algebra, i.e., the above procedure does not disturb the tracial simplex. One can then through an intertwining argument similar to Lemma 5.11 show that if we perform this process for some sequence of sufficiently large numbers $\{N_n\}_{n\in\mathbb{N}}$, we can add edges between vertices on subsequent rows with no edges connecting them, and consequently get a simple AF-algebra, without changing the tracial simplex. Using this method, we get the following corollary to Theorem 5.13 and Theorem 5.7.

Corollary 5.14. *Let Δ be a metrizable Choquet simplex, then there exists a* simple *AF-algebra A with tracial simplex $T(A)$ isomorphic to Δ as compact convex sets.*

This, together with Proposition 5.2, shows that AF-algebras can have arbitrary tracial simplices, up to the tracial simplices being metrizable, even in the simple case.

5.2 Examples of metrizable Choquet simplices

Let Δ be an infinite-dimensional metrizable Choquet simplex, and let $f_n \colon \Delta_{n+1} \to \Delta_n$ be the surjective affine continuous connecting maps of Theorem 5.9. By (5.1), Δ is determined by the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$ with $\xi^{(n)} \in \Delta_n$ for each $n \in \mathbb{N}_0$. Since we have no restrictions on the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$, one obvious case to examine is the one with $f_n^(\xi^{(n+1)}) = \xi^{(n)}$ for all $n \in \mathbb{N}$; we call such sequences *stationary sequences*. We shall see that there are two possibilities; either Δ is affinely homeomorphic to a specific infinite-dimensional Bauer simplex, or Δ is not a Bauer simplex.
First we have a lemma regarding the extreme boundary on Choquet simplices as in Theorem 5.9.

Lemma 5.15. *Suppose Δ is a metrizable infinite-dimensional Choquet simplex with inverse sequence $(\Delta_n, \{f_n\})$ of n-simplices. For each $n \in \mathbb{N}_0$, there exists a unique $e_n \in \Delta$ such that $f_{m,\infty}(e_n) = e_n^{(m)}$ for each $m \geq n$. Moreover, $e_n \in \partial_e\Delta$ for each $n \in \mathbb{N}_0$, and the sequence $\{e_n\}_{n\in\mathbb{N}_0}$ is dense in $\partial_e\Delta$.*

Proof. Existence of such e_n follows immediately from the definition of inverse limits, and uniqueness follows from Lemma 2.21. Let $n \in \mathbb{N}_0$ and suppose $e_n = \lambda x + (1-\lambda)y$ for some $x, y \in \Delta$ and $\lambda \in [0, 1]$. Then for any $m \geq n$, we have

$$e_n^{(m)} = f_{\infty,m}(e_n) = f_{\infty,m}(\lambda x + (1-\lambda)y) = \lambda f_{\infty,m}(x) + (1-\lambda) f_{\infty,m}(y).$$

Since $e_n^{(m)}$ is an extreme point of Δ_m, we find that $e_n^{(m)} = f_{\infty,m}(x) = f_{\infty,m}(y)$ for all $m \geq n$. By Lemma 2.21, we conclude that e_n is an extreme point of Δ.
It only remains to be seen that $\{e_n\}_{n\in\mathbb{N}_0}$ is dense in $\partial_e\Delta$. By the partial converse to Krein-Milman, Theorem 1.10, it suffices to prove that $\text{conv}\{e_n\}_{n\in\mathbb{N}_0}$ is dense in Δ. Define $C = \text{conv}\{e_n\}_{n\in\mathbb{N}_0}$ for notational ease. Recall that Δ is given the topology as a subspace of $\prod_{n\in\mathbb{N}_0}\Delta_n$, so an arbitrary open set in Δ can be written as $U \cap \Delta$, where $U \subseteq \prod_{n\in\mathbb{N}_0}\Delta_n$ is open. Suppose $W = U \cap \Delta \neq \emptyset$ is a non-empty open set in Δ. There exists a natural number $N \in \mathbb{N}$ and a topological basis element $V = V_0 \times V_1 \times \cdots \times V_N \times \Delta_{N+1} \times \cdots \subseteq U$ such that $V \cap \Delta \neq \emptyset$ and $V \subseteq W$. There consequently exists an element $(x_1, x_2, x_3, \ldots) \in \Delta \cap V$ such that $x_j \in V_j$ for each $j = 1, \ldots, N$. Then there is $y \in C$ with the property that $f_{\infty,N}(y) = (x_0, x_1, \ldots, x_N)$. and one finds that y is of the form $y = (x_0, x_1, \ldots, x_N, y_{N+1}, \ldots)$ such that $y \in V \cap C$, proving that C is dense in Δ. □

We now determine a necessary and sufficient condition for the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$ to be stationary.

Lemma 5.16. *Let $\{t_n\}_{n\in\mathbb{N}_0}$ be a non-zero sequence of non-negative integers, and let n_0 be the smallest integer such that $t_{n_0} \neq 0$. Define*

$$\xi^{(n)} = \frac{1}{\sum_{j=0}^{n} t_j} \sum_{i=0}^{n} t_i e_i \in \Delta_n \tag{5.5}$$

for each $n \geq n_0$, and let $\xi^{(n)} \in \Delta_n$ be arbitrary for $n < n_0$. Then $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$ is a stationary sequence, and any stationary sequence arises in this manner.

Proof. The expression (5.5) makes sense, as it is a convex combination of the extreme points of Δ_n. Let us first show that such sequences $\{t_n\}_{n\in\mathbb{N}_0}$ give rise to stationary sequences. Let $n \in \mathbb{N}_0$ be arbitrary, then we want to show that $f_n(\xi^{(n+1)}) = \xi^{(n)}$. For notational simplicity we define $\alpha = \sum_{j=0}^{n} t_j$ and $\beta = \sum_{j=0}^{n+1} t_j$. Note in particular that $\beta = \alpha + t_{n+1}$, and that $\xi^{(n)} = \alpha^{-1}\sum_{i=0}^{n} t_i e_i^{(n)}$. Since f_n is affine, we find that

$$f_n(\xi^{(n+1)}) = \beta^{-1}\sum_{i=0}^{n+1} t_i f_n(e_i^{(n+1)}) = \beta^{-1}\sum_{i=0}^{n} t_i e_i^{(n)} + \beta^{-1}t_{n+1}\xi^{(n)} = \beta^{-1}\alpha\xi^{(n)} + \beta^{-1}t_{n+1}\xi^{(n)} = \xi^{(n)}.$$

Now we show the converse claim. Suppose that $f_n(\xi^{(n+1)}) = \xi^{(n)}$ for each $n \in \mathbb{N}_0$. Write $\xi^{(n)} = \sum_{i=0}^{n} t_i^{(n)} e_i^{(n)}$ and note that

$$\begin{aligned}\sum_{i=0}^{n} t_i^{(n)} e_i^{(n)} = \xi^{(n)} = f_n(\xi^{(n+1)}) &= \sum_{i=0}^{n} t_i^{(n+1)} e_i^{(n)} + t_{n+1}^{(n+1)}\xi^{(n)} = \sum_{i=0}^{n}\left(t_i^{(n+1)} + t_{n+1}^{(n+1)} t_i^{(n)}\right) e_i^{(n)} \\ &= \sum_{i=0}^{n}\left[t_i^{(n+1)} + \left(1 - \sum_{j=0}^{n} t_j^{(n+1)}\right) t_i^{(n)}\right] e_i^{(n)}.\end{aligned}$$

Since the $e_i^{(n)}$'s are affinely independent, for each $i = 0, \ldots, n$ we have

$$t_i^{(n)} = t_i^{(n+1)} + \left(1 - \sum_{j=0}^{n} t_j^{(n+1)}\right) t_i^{(n)}$$

and consequently

$$0 = t_i^{(n+1)} - t_i^{(n)} \sum_{j=0}^{n} t_j^{(n+1)}$$

which immediately implies that the elements $\xi^{(n)}$ are of the form (5.5). □

It is easily seen that two such sequences $\{t_n\}_{n\in\mathbb{N}_0}$ and $\{t'_n\}_{n\in\mathbb{N}_0}$ give rise to the same stationary sequence if and only if the sequences are proportional. If $\sum_{j=0}^{\infty} t_j < \infty$, we can thus without loss of generality assume that $\sum_{j=0}^{\infty} t_j = 1$. The aforementioned two different cases for the simplex Δ, whose corresponding sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$ is stationary, are exactly determined by whether the sum of the sequence $\{t_n\}_{n\in\mathbb{N}_0}$ is convergent or not. Before we consider these two cases, we need a brief example.

Example 5.17. Consider the infinite-dimensional Choquet simplex Δ_∞ with extreme boundary $\partial_e\Delta_\infty = \{e_j^{(\infty)} \,|\, 0 \leq j \leq \infty\}$ equipped with the topology such that $\partial_e\Delta_\infty$ is homeomorphic to the one-point compactification $\mathbb{N}_0 \cup \{\infty\}$ of $\mathbb{N}_0$. In particular, Δ_∞ is a Bauer simplex with $e_j^{(\infty)} \to e_\infty^{(\infty)}$ as $j \to \infty$. By Krein-Milman, each $x \in \Delta_\infty$ is uniquely realized as an infinite convex combination $x = \sum_{0\leq j\leq\infty} \lambda_j e_j^{(\infty)}$, where $\lambda_j \geq 0$ with $\sum_{0\leq j\leq\infty} \lambda_j = 1$.

Note that we have not proven the existence of such a simplex — one way to prove existence would be to show that the closed convex hull of the one-point compactification $\mathbb{N}_0 \cup \{\infty\}$ of $\mathbb{N}_0$ is a Choquet simplex. We shall not do this here.

We are now able to state the proposition about the Choquet simplices with corresponding stationary sequences.

Proposition 5.18. *Let Δ be an infinite-dimensional metrizable Choquet simplex, whose corresponding sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$ is stationary. Let $\{t_n\}_{n\in\mathbb{N}_0}$ be a sequence of non-negative integers generating the sequence $\{\xi^{(n)}\}_{n\in\mathbb{N}_0}$ as in Lemma 5.16 with $n_0 \geq 0$ the smallest integer such that $t_{n_0} \neq 0$. Consider the element $e_\infty \in \Delta$ determined by $f_{n,\infty}(e_\infty) = \xi^{(n)}$ for all $n \geq n_0$, and let $\{e_n\}_{n\in\mathbb{N}_0}$ be the dense subset of $\partial_e\Delta$ as constructed in Lemma 5.15. Then there are two possibilities.*

(i) *If $\sum_{j\in\mathbb{N}_0} t_j = \infty$, then Δ is affinely homeomorphic to the Bauer simplex Δ_∞ of Example 5.17, and the extremal boundary is $\partial_e\Delta = \{e_n\}_{n\in\mathbb{N}_0}$. Moreover, $e_n \to e_\infty$ as $n \to \infty$.*

(ii) *If $\sum_{j\in\mathbb{N}_0} t_j = 1$, then $\partial_e\Delta = \{e_n \,|\, 0 \leq n < \infty\}$ and*

$$\lim_{n\to\infty} e_n = e_\infty = \sum_{j\in\mathbb{N}_0} t_j e_j.$$

In particular, Δ is not a Bauer simplex.

Proof. We start by showing that, independently of the sequence $\{t_j\}_{j=0}^\infty$, there exists a continuous, affine surjection $g\colon \Delta_\infty \to \Delta$ such that $g(e_j^{(\infty)}) = e_j$ for $0 \leq j \leq \infty$. For each $n \geq n_0$ define the function $g_n\colon \Delta_\infty \to \Delta_n$ given by

$$g_n(e_j^{(\infty)}) = \begin{cases} e_j^{(n)} & \text{if } 0 \leq j \leq n \\ \xi^{(n)} & \text{if } n < j \leq \infty \end{cases}.$$

Extend this by continuity and affinity. An easy calculation shows that $f_n \circ g_{n+1} = g_n$, and hence universality of inverse limits implies that there exists a continuous affine map $g\colon \Delta_\infty \to \Delta$ satisfying $f_{n,\infty} \circ g = f_n$ for each $n \geq n_0$. Using Lemma 5.15, we see that as

$$f_{n,\infty}(e_j) = e_j^{(n)} = g_n(e_j^{(\infty)}) = f_{n,\infty}(g(e_j^{(\infty)})),$$

for each $n \geq n_0$, we have $e_j = g(e_j^{(\infty)})$ for each $j \in \mathbb{N}_0$. Moreover, as

$$f_{n,\infty}(e_\infty) = \xi^{(n)} = g_n(e_\infty^{(\infty)}) = f_{n,\infty}(g(e_\infty^{(\infty)})),$$

for all $n \geq n_0$, we also have $g(e_\infty^{(\infty)}) = e_\infty$. Lastly, g is surjective by continuity and the Krein-Milman theorem. Continuity further implies that $e_n \to e_\infty$ as $n \to \infty$, since

$$e_n = g(e_j^{(\infty)}) \to g(e_\infty^{(\infty)}) = e_\infty.$$

Having established such a g, we proceed by proving (i). Suppose that $\sum_{j=0}^\infty t_j = \infty$. We first show that Δ is affinely homeomorphic to the Bauer simplex Δ_∞ of Example 5.17. For this we only need to prove that g is injective, so assume $x, y \in \Delta_\infty$ satisfy that $g(x) = g(y)$. Write

$$x = \sum_{0\leq j\leq\infty} x_j e_j^{(\infty)} \quad \text{and} \quad y = \sum_{0\leq j\leq\infty} y_j e_j^{(\infty)}. \tag{5.6}$$

For any $n \geq n_0$, we have by the factorization identity $f_{n,\infty} \circ g = g_n$ that $g_n(x) = g_n(y)$. Writing this out, one finds that

$$g_n(x) = (x_0, \ldots, x_n) + \left(1 - \sum_{j=0}^n x_j\right)\xi^{(n)} = (y_0, \ldots, y_n) + \left(1 - \sum_{j=0}^n y_j\right)\xi^{(n)} = g_n(y) \tag{5.7}$$

for each $n \geq n_0$. Note that the ith coordinate $\xi_i^{(n)}$ of $\xi^{(n)}$ is

$$\xi_i^{(n)} = \frac{t_i}{\sum_{j=0}^n t_j},$$

and as the series $\sum_{j=0}^\infty t_j$ diverges to ∞, the ith coordinate of $\xi^{(n)}$ must converge to 0 as n tends to ∞. Fixing a coordinate i and taking the limit of n in (5.7) then implies that $x_i = y_i$ for each $i \in \mathbb{N}_0$, and as the sums in (5.6) are convex combinations, i.e.,

$$\sum_{0\leq i\leq\infty} x_i = 1 = \sum_{0\leq i\leq\infty} y_i$$

we conclude that $x_i = y_i$ for each $0 \leq i \leq \infty$. This proves injectivity of g, and hence Δ is affinely homeomorphic to Δ_∞.
We now prove that $\partial_e\Delta = \{e_j \,|\, 0 \leq j \leq \infty\}$. By Lemma 5.15, we find that $e_j \in \partial_e\Delta$ for each $j \in \mathbb{N}$. Moreover, as g is surjective, extreme points lift to extreme points by Lemma 5.10 and hence $\partial_e\Delta \subseteq g(\partial_e\Delta_\infty) = \{e_j \,|\, 0 \leq j \leq \infty\}$. We thus only need to show that $e_\infty \in \partial_e\Delta$. Suppose that $e_\infty = \lambda z + (1-\lambda)w$ for some $\lambda \in (0,1)$ and $z, w \in \Delta$. By surjectivity of g, find $x, y \in \Delta_\infty$ such that $g(x) = z$ and $g(y) = w$, and then

$$g(e_\infty^{(\infty)}) = e_\infty = g(\lambda x + (1-\lambda)y)$$

such that $e_\infty^{(\infty)} = \lambda x + (1-\lambda)y$ by injectivity. Since $e_\infty^{(\infty)}$ belongs to the extreme boundary of Δ_∞, we conclude that $x = y = e_\infty^{(\infty)}$, and hence that e_∞ is an extreme point.

We now prove (ii), which provides an example of a non-Bauer simplex. Suppose that $\sum_{j=0}^\infty t_j = 1$. We already know that $\partial_e\Delta$ is contained in $\{e_j \,|\, 0 \leq j < \infty\}$ and that $e_n \to \infty$ as $n \to \infty$, so we only need to prove that e_∞ is not an extreme point of Δ. Noting that $\sum_{j=0}^\infty t_j e_j \in \Delta$, we see that for any $n \geq n_0$,

$$f_{n,\infty}\left(\sum_{j=0}^\infty t_j e_j\right) = \sum_{j=0}^\infty t_j f(e_j) = \sum_{j=0}^n t_j e_j^{(n)} + \left(1 - \sum_{j=0}^n t_j\right)\xi^{(n)} = \xi^{(n)} = f_{n,\infty}(e_\infty)$$

and consequently $e_\infty = \sum_{j=0}^\infty t_j e_j$ by Lemma 2.21. In particular, $e_\infty \notin \partial_e\Delta$, completing the proof. □

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