A simple C^*-algebra with a finite and an infinite projection

Mikael Rørdam

Dedicated to Richard V. Kadison on the occasion of his 75th birthday

Abstract

An example is given of a simple, unital C^*-algebra which contains an infinite and a non-zero finite projection. This C^*-algebra is also an example of an infinite simple C^*-algebra which is not purely infinite. A corner of this C^*-algebra is a finite, simple, unital C^*-algebra which is not stably finite.

Our example shows that the type decomposition for von Neumann factors does not carry over to simple C^*-algebras.

We also give an example of a simple, separable, *nuclear*, C^*-algebra in the UCT class which contains an infinite and a non-zero finite projection. This nuclear C^*-algebra arises as a crossed product $D \rtimes_\alpha \mathbb{Z}$, where D is an inductive limit of type I C^*-algebras.

1 Introduction

The first interesting class of simple C^*-algebras (not counting the simple von Neumann algebras) were the UHF-algebras, also called Glimm algebras, constructed by Glimm in 1959 ([22]). Several other classes of simple C^*-algebras were found over the following 25 years including the (simple) AF-algebras, the irrational rotation C^*-algebras, the free group C^*-algebras $C^*_{\mathrm{red}}(\mathbb{F}_n)$ (and other reduced group C^*-algebras), the Cuntz algebras $\mathcal{O}_n$ and the Cuntz–Krieger algebras $\mathcal{O}_A$, C^*-algebras arising from minimal dynamical systems and from foliations, and certain inductive limit C^*-algebras, among many other examples. Parallel with the appearance of these examples of simple C^*-algebras it was asked if there is a classification for simple C^*-algebras similar to the classification of von Neumann factors into types. Inspired by work of Dixmier in the 1960's, Cuntz studied this and related questions about the structure of simple C^*-algebras in his papers [14], [17], and [15].

A von Neumann algebra is simple precisely when it is either a factor of type I_n for $n < \infty$ (in which case it is isomorphic to $M_n(\mathbb{C})$), a factor of type II_1, or a separable factor of type III. This leads to the question if (non type I) simple C^*-algebras can be divided into two subclasses, one that resembles type II_1 factors and another that resembles type III factors. A II_1 factor is an infinite dimensional factor in which all projections are finite (in the sense of Murray–von Neumann's comparison theory for projections), and II_1 factors have a unique trace. A factor is of type III if all its non-zero projections are infinite, and type III factors admit no traces. Cuntz asked in [17] if each simple C^*-algebra similarly must have the property that its (non-zero) projections either all are finite or all are infinite. Or can a simple C^*-algebra contain both a (non-zero) finite and an infinite projection? We answer the latter question in the affirmative. In other words, we exhibit a simple (non type I) C^*-algebra that neither corresponds to a type II_1 or to a type III factor.

It was shown in the early 1980's that simple C^*-algebras, in contrast to von Neumann factors, can fail to have non-trivial projections. Blackadar ([5]) and Connes ([12]) found examples of unital, simple C^*-algebras with no projections other than 0 and 1—before it was shown that $C^*_{\mathrm{red}}(\mathbb{F}_2)$ is a simple unital C^*-algebra with no non-trivial projections. Simple C^*-algebras can fail to have projections in a more severe way: Blackadar found in [4] an example of a stably projectionless simple C^*-algebra. (A C^*-algebra A is stably projectionless if 0 is the only projection in $A \otimes \mathcal{K}$.) Blackadar and Cuntz proved in [8] that every stably projectionless simple C^*-algebra is finite in the sense of admitting a (densely defined) quasitrace. (Every quasitrace on an exact C^*-algebra extends to a trace as shown by Haagerup [23] (and Kirchberg [27]).) These results lead to the dichotomy for a simple C^*-algebra A: Either A admits a (densely defined) quasitrace (in which case A is stably finite), or A is *stably infinite*, i.e., $A \otimes \mathcal{K}$ contains an infinite projection.

Cuntz defined in [16] a simple C^*-algebra to be *purely infinite* if all its non-zero hereditary sub-C^*-algebras contain an infinite projection. Cuntz showed in [13] that his algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, are simple and purely infinite. The separable, nuclear, simple, purely infinite C^*-algebras are classified up to isomorphism by K- or KK-theory by the spectacular theorem of Kirchberg ([28] and [26]) and Phillips ([35]). This result has made it an important question to decide which simple C^*-algebras are purely infinite. We show here that not all stably infinite simple C^*-algebras A are purely infinite.

Villadsen ([41]) was the first to show that the K_0-group of a simple C^*-algebra need not be weakly unperforated; Villadsen ([42]) also showed that a unital, finite, simple C^*-algebra can have stable rank different from one—thus answering in the negative two longstanding open questions for simple C^*-algebras.

If B is a unital, simple C^*-algebra with an infinite and a non-zero finite projection, then its semigroup of Murray–von Neumann equivalence classes of projections must fail to be weakly unperforated (see Remark 7.8). It is therefore no surprise that Villadsen's ideas play a crucial role in this article. Our article is also a continuation of the work by the author in [37] and [38] where it is shown that one can find a C^*-algebra A such that $M_2(A)$ is stable but A is not stable; and, related to this, one can find a (non-simple) unital C^*-algebra B, such that B is finite and $M_2(B)$ is properly infinite. We show here (Theorem 5.6) that one can make this example simple by passing to a suitable inductive limit.

In Section 6 (added March 2002) an example is given of a crossed product C^*-algebra $D \rtimes_\alpha \mathbb{Z}$, where D is an inductive limit of type I C^*-algebras, such that $D \rtimes_\alpha \mathbb{Z}$ is simple and contains an infinite and a non-zero finite projection. This new example is nuclear and separable. It shows that simple C^*-algebras with this rather pathological behavior can arise from a quite natural setting. It shows that Elliott's classification conjecture (in its present formulation) does not hold (cf. Corollary 7.9); and it also serves as an example of a separable nuclear simple C^*-algebra that is tensorially prime (cf. Corollary 7.5).

I thank Bruce Blackadar, Joachim Cuntz, George Elliott, and Eberhard Kirchberg for valuable discussions and for their comments to earlier versions of this manuscript. I thank Paul M. Cohn and Ken Goodearl for explaining the example included in Remark 7.13. I also thank the referee for suggesting several improvements to this article (including a significant simplification of Proposition 5.2 (ii) and (iii)).

This work was done in the spring of 2001 while the author visited the University of California, Santa Barbara. I thank Dietmar Bisch for inviting me and for his warm hospitality.

The present revised version (with the nuclear example in Section 6 and where the construction in Section 5 is simplified) was completed in March 2002. A part of the work leading to this construction was obtained during a visit in January 2002 to the University of Münster. I thank Joachim Cuntz and Eberhard Kirchberg for their hospitality, and I am indebted to Eberhard Kirchberg for several conversations during the visit that led me to this construction.

2 Finite, infinite, and properly infinite projections

A projection p in a C^*-algebra A is called *infinite* if it is equivalent (in the sense of Murray and von Neumann) to a proper subprojection of itself; and p is said to be *finite* otherwise. If p is non-zero and if there are mutually orthogonal subprojections p_1 and p_2 of p such that $p \sim p_1 \sim p_2$, then p is *properly infinite*. A unital C^*-algebra is said to be *properly infinite* if its unit is a properly infinite projection.

If p and q are projections in A, then let $p \oplus q$ denote the projection $\mathrm{diag}(p, q)$ in $M_2(A)$. Two projections $p \in M_n(A)$ and $q \in M_m(A)$ can be compared as follows: Write $p \sim q$ if there exists v in $M_{m,n}(A)$ such that $v^*v = p$ and $vv^* = q$, and write $p \precsim q$ if p is equivalent (in this sense) to a subprojection of q.

In the proposition below, where some well-known properties of properly infinite projections are recorded, $\mathcal{O}_\infty$ denotes the Cuntz algebra generated by infinitely many isometries with pairwise orthogonal range projections, and $\mathcal{E}_2$ is the Cuntz–Toeplitz algebra generated by two isometries with orthogonal range projections ([13]).

Proposition 2.1 *The following five conditions are equivalent for every non-zero projection p in a C^*-algebra A:*

(i) *p is properly infinite;*

(ii) *$p \oplus p \precsim p$;*

(iii) *there is a unital *-homomorphism $\mathcal{E}_2 \to pAp$;*

(iv) *there is a unital *-homomorphism $\mathcal{O}_\infty \to pAp$;*

(v) *for every closed two-sided ideal I in A, either $p \in I$ or $p + I$ is infinite in A/I.*

The equivalences between (i), (ii), and (iii) are trivial. The equivalence between (iii) and (iv) follows from the fact that there are unital embeddings $\mathcal{E}_2 \to \mathcal{O}_\infty$ and $\mathcal{O}_\infty \to \mathcal{E}_2$. The equivalence between (i) and (v) is proved in [29, Corollary 3.15]; a result that extends Cuntz' important observation from [14] that every infinite projection in a simple C^*-algebra is properly infinite.

We shall use the following two well-known results about properly infinite projections.

Lemma 2.2 *Let p and q be projections in a C^*-algebra A. Suppose that p is properly infinite. Then $q \precsim p$ if and only if q belongs to the closed two-sided ideal in A generated by p.*

Proof: If $q \precsim p$, then, by definition, $q \sim q_0 \leq p$ for some projection q_0 in A. This entails that q belongs to the ideal generated by p. Conversely, if q belongs to the ideal generated by p, then $q \precsim \bigoplus_{j=1}^n p$ for some n (cf. [40, Exercise 4.8]), and $\bigoplus_{j=1}^n p \precsim p$ if p is properly infinite by iterated applications of Proposition 2.1 (ii). $\square$

Proposition 2.3 *Let B be the inductive limit of a sequence $B_1 \to B_2 \to B_3 \to \cdots$ of unital C^*-algebras with unital connecting maps. Then B is properly infinite if and only if B_n is properly infinite for all n larger than some n_0.*

Proof: If B_n is properly infinite for some n, then there are unital *-homomorphisms $\mathcal{E}_2 \to B_n \to B$, and hence B is properly infinite. Conversely, if B is properly infinite, then there is a unital *-homomorphism $\mathcal{E}_2 \to B$. The C^*-algebra $\mathcal{E}_2$ is semiprojective, as shown by Blackadar in [6]. By semiprojectivity (see again [6]), the unital *-homomorphism $\mathcal{E}_2 \to B$ lifts to a unital *-homomorphism $\mathcal{E}_2 \to \prod_{n=n_0}^{\infty} B_n$ for some n_0. This shows that B_n is properly infinite for all $n \geq n_0$. $\square$

3 Vector bundles over products of spheres

We consider here complex vector bundles over the sphere S^2 and over finite products of spheres, $(S^2)^n$.

For each $k \leq n$, let $\pi_k \colon (S^2)^n \to S^2$ denote the kth coordinate mapping, and let $\rho_{m,n} \colon (S^2)^m \to (S^2)^n$ be given by

$$\rho_{m,n}(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_n), \qquad (x_1, x_2, \ldots, x_m) \in (S^2)^m. \tag{3.1}$$

when $m \geq n$.

Whenever $f \colon X \to Y$ is a continuous map and ξ is a k-dimensional complex vector bundle over Y, let $f^*(\xi)$ denote the vector bundle over X induced by f. Let $e(\xi) \in H^{2k}(Y, \mathbb{Z})$ denote the Euler class of ξ. Denote also by f^* the induced map $H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$. By functoriality of the Euler class we have $f^*(e(\xi)) = e(f^*(\xi))$.

For any vector bundle ξ over $(S^2)^n$ and for every $m \geq n$ we have a vector bundle $\xi' = \rho_{m,n}^*(\xi)$ over $(S^2)^m$. It follows from the Künneth Theorem (see [33, Theorem A6]), that the map

$$\rho_{m,n}^* \colon H^*((S^2)^n, \mathbb{Z}) \to H^*((S^2)^m, \mathbb{Z})$$

is injective; so if $e(\xi)$ is non-zero, then so is $e(\xi')$. Our main concern with vector bundles will be whether or not they have non-zero Euler class, and from that point of view it does not matter if we replace the base space $(S^2)^n$ with $(S^2)^m$ for some $m \geq n$.

We remind the reader of some properties of the Euler class for complex vector bundles $\xi_1, \xi_2, \ldots, \xi_n$ over a base space X. First of all we have the product formula (see [33, Property 9.6]):

$$e(\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n) = e(\xi_1) \cdot e(\xi_2) \cdots e(\xi_n). \tag{3.2}$$

Let θ denote the trivial complex line bundle over X. The Euler class of θ is zero; and so it follows from the product formula that $e(\xi) = 0$ whenever ξ is a complex vector bundle that dominates θ in the sense that $\xi \cong \theta \oplus \eta$ for some complex vector bundle η.

Combining the formula

$$\mathrm{ch}(\xi) = 1 + e(\xi) + \frac{1}{2}e(\xi)^2 + \frac{1}{6}e(\xi)^3 + \cdots,$$

that relates the Chern character and the Euler class of a complex line bundle ξ (see [33, Problem 16-B]), with the fact that the Chern character is multiplicative, yields the formula

$$e(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = e(\xi_1) + e(\xi_2) + \cdots + e(\xi_n), \tag{3.3}$$

that holds for all complex *line bundles* $\xi_1, \ldots, \xi_n$ over X.

Let ζ be a complex line bundle over S^2 such that its Euler class $e(\zeta)$, which is an element in $H^2(S^2, \mathbb{Z})$, is non-zero. (Any such line bundle will do, but the reader may take ζ to be the Hopf bundle over S^2.) For each natural number n and for each non-empty, finite subset $I = \{n_1, n_2, \ldots, n_k\}$ of $\mathbb{N}$ define complex line bundles ζ_n and ζ_I over $(S^2)^m$ (for all $m \geq n$, respectively, $m \geq \max\{n_1, \ldots, n_k\}$) by

$$\zeta_n = \pi_n^*(\zeta), \qquad \zeta_I = \zeta_{n_1} \otimes \zeta_{n_2} \otimes \cdots \otimes \zeta_{n_k}, \tag{3.4}$$

where, as above, $\pi_n \colon (S^2)^m \to S^2$ is the nth coordinate map. The Euler classes (in $H^2((S^2)^m, \mathbb{Z})$) of these line bundles are by functoriality and equation (3.3) given by

$$\begin{aligned} e(\zeta_n) &= \pi_n^*(e(\zeta)), & (3.5)\\ e(\zeta_I) &= \pi_{n_1}^*(e(\zeta)) + \pi_{n_2}^*(e(\zeta)) + \cdots + \pi_{n_k}^*(e(\zeta)). & (3.6) \end{aligned}$$

Lemma 3.1 *For each n and for each $m \geq n$ there is a complex line bundle η_n over $(S^2)^m$ such that $\zeta_n \oplus \zeta_n \cong \theta \oplus \eta_n$.*

Proof: Since

$$\dim(\zeta \oplus \zeta) = 2 > 1 \geq \tfrac{1}{2}(\dim(S^2) - 1),$$

it follows from [24, 9.1.2] that there is a complex vector bundle η over S^2 of dimension $\dim(\eta) = 2 - 1 = 1$ such that $\zeta \oplus \zeta \cong \theta \oplus \eta$. We conclude that

$$\zeta_n \oplus \zeta_n = \pi_n^*(\zeta \oplus \zeta) \cong \pi_n^*(\theta \oplus \eta) = \theta \oplus \pi_n^*(\eta).$$

$\square$

Proposition 3.2 *Let $I_1, I_2, \ldots, I_m$ be non-empty, finite subsets of $\mathbb{N}$. The following three conditions are equivalent:*

(i) $e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \cdots \oplus \zeta_{I_m}) \neq 0$.

(ii) *For all subsets F of $\{1, 2, \ldots, m\}$ we have $\left|\bigcup_{j\in F} I_j\right| \geq |F|$.*

(iii) *There exists a matching $t_1 \in I_1, t_2 \in I_2, \ldots, t_m \in I_m$ (i.e., the elements $t_1, \ldots, t_m$ are pairwise distinct).*

Proof: Choose N large enough so that each ζ_{I_j} is a vector bundle over $(S^2)^N$.

(ii) $\Leftrightarrow$ (iii) is the Marriage Theorem (see any textbook on combinatorics).

(i) $\Rightarrow$ (ii). Assume that $\left|\bigcup_{j\in F} I_j\right| < |F|$ for some (necessarily non-empty) subset $F = \{j_1, j_2, \ldots, j_k\}$ of $\{1, 2, \ldots, m\}$, and write

$$J \stackrel{\text{def}}{=} \bigcup_{j\in F} I_j = \{n_1, n_2, \ldots, n_l\}.$$

Let $\rho\colon (S^2)^N \to (S^2)^l$ be given by $\rho(x) = (\pi_{n_1}(x), \pi_{n_2}(x), \ldots, \pi_{n_l}(x))$. Then

$$\xi \stackrel{\text{def}}{=} \zeta_{I_{j_1}} \oplus \zeta_{I_{j_2}} \oplus \cdots \oplus \zeta_{I_{j_k}} = \rho^*(\eta)$$

for some k-dimensional vector bundle η over $(S^2)^l$. Now, $e(\eta)$ belongs to $H^{2k}((S^2)^l, \mathbb{Z})$, and $H^{2k}((S^2)^l, \mathbb{Z}) = 0$ because $2k > 2l$. Hence $e(\xi) = \rho^*(e(\eta)) = 0$, so by the product formula (3.2) we get

$$e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \cdots \oplus \zeta_{I_m}) = e(\xi) \cdot \prod_{j\notin F} e(\zeta_{I_j}) = 0.$$

(iii) $\Rightarrow$ (i). Put

$$x_j = \pi_j^*(e(\zeta)) \in H^2((S^2)^N, \mathbb{Z}), \qquad j = 1, 2, \ldots, N.$$

The element

$$z = x_1 \cdot x_2 \cdots x_N \in H^{2N}((S^2)^N, \mathbb{Z})$$

is non-zero by the Künneth Theorem ([33, Theorem A6]). Using that $x_i^2 = 0$ and that

$x_i x_j = x_j x_i$ for all i, j it follows that if $i_1, i_2, \dots, i_N$ belong to $\{1, 2, \dots, N\}$, then

$$x_{i_1} \cdot x_{i_2} \cdots x_{i_N} = \begin{cases} z, & \text{if } \ i_1, \dots, i_N \ \text{ are distinct,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Now, by (3.2) and (3.6),

$$\begin{aligned} e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \cdots \oplus \zeta_{I_m}) &= e(\zeta_{I_1}) \cdot e(\zeta_{I_2}) \cdots e(\zeta_{I_m}) \\ &= \Big(\sum_{i \in I_1} \pi_i^*(e(\zeta)) \Big) \cdot \Big(\sum_{i \in I_2} \pi_i^*(e(\zeta)) \Big) \cdots \Big(\sum_{i \in I_m} \pi_i^*(e(\zeta)) \Big) \\ &= \Big(\sum_{i \in I_1} x_i \Big) \cdot \Big(\sum_{i \in I_2} x_i \Big) \cdots \Big(\sum_{i \in I_m} x_i \Big) \\ &= \sum_{(i_1, \dots, i_m) \in I_1 \times \cdots \times I_m} x_{i_1} \cdot x_{i_2} \cdots x_{i_m}. \end{aligned}$$

Assume that (iii) holds, and write

$$\{1, 2, \dots, N\} \setminus \{t_1, t_2, \dots, t_m\} = \{s_1, s_2, \dots, s_{N-m}\}.$$

Let k denote the number of permutations σ on $\{1, 2, \dots, m\}$ such that $t_{\sigma(j)} \in I_j$ for $j = 1, 2, \dots, m$. The identity permutation has this property, so $k \geq 1$. The formula for $e(\zeta_{I_1} \oplus \cdots \oplus \zeta_{I_m})$ above and equation (3.7) yield

$$e(\zeta_{I_1} \oplus \zeta_{I_2} \oplus \cdots \oplus \zeta_{I_m}) \cdot x_{s_1} \cdot x_{s_2} \cdots x_{s_{N-m}} = kz \neq 0.$$

It follows that $e(\zeta_{I_1} \oplus \cdots \oplus \zeta_{I_m}) \neq 0$ as desired. $\square$

4 Projections in a certain multiplier algebra

There is a well-known one-to-one correspondence between isomorphism classes of complex vector bundles over a compact Hausdorff space X and Murray–von Neumann equivalence classes of projections in matrix algebras over $C(X)$ (and in $C(X) \otimes \mathcal{K}$). The vector bundle corresponding to a projection p in $M_n(C(X)) = C(X, M_n(\mathbb{C}))$ is

$$\xi_p = \{(x, v) : x \in X, \ v \in p(x)(\mathbb{C}^n)\},$$

(equipped with the topology given from the natural inclusion $\xi_p \subseteq X \times \mathbb{C}^n$), so that the fibre $(\xi_p)_x$ over $x \in X$ is the range of the projection $p(x)$. If p and q are two projections in $C(X)\otimes\mathcal{K}$, then $\xi_p \cong \xi_q$ if and only if $p \sim q$. It follows from Swan's theorem, which to each complex vector bundle ξ gives a complex vector bundle η such that $\xi \oplus \eta$ is isomorphic to the trivial n-dimensional complex vector bundle over X for some n, that every complex vector bundle is isomorphic to ξ_p for some projection p in $M_n(C(X))$ for some n.

View each matrix algebra $M_n(\mathbb{C})$ as a sub-C^*-algebra of $\mathcal{K}$ via the embeddings

$$\mathbb{C} \hookrightarrow M_2(\mathbb{C}) \hookrightarrow M_3(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow \mathcal{K},$$

where $M_n(\mathbb{C})$ is mapped into the upper left corner of $M_{n+1}(\mathbb{C})$. Identify $C(X,\mathcal{K})$ with $C(X)\otimes\mathcal{K}$ and identify $C(X, M_n(\mathbb{C}))$ with $C(X)\otimes M_n(\mathbb{C})$.

In Section 3 we picked a non-trivial complex line bundle ζ over S^2 (which could be the Hopf bundle). This line bundle ζ corresponds to a projection p in some matrix algebra over $C(S^2)$, and, as is well known, such a projection p can be found in $M_2(C(S^2)) = C(S^2, M_2)$. (The projection $p \in M_2(S^2, M_2)$ corresponding to the Hopf bundle is in operator algebra texts often referred to as the Bott projection.) Put

$$Z = \prod_{n=1}^{\infty} S^2.$$

Let $\pi_n \colon Z \to S^2$ be the nth coordinate map, and let $\rho_{\infty,n} \colon Z \to (S^2)^n$ be given by

$$\rho_{\infty,n}(x_1, x_2, x_3, \dots) = (x_1, x_2, \dots, x_n), \qquad (x_1, x_2, x_3, \dots) \in Z.$$

With $\widehat{\rho}_n \colon C((S^2)^n) \to C((S^2)^{n+1})$ being the *-homomorphism induced by the map $\rho_n = \rho_{n+1,n}$ defined in (3.1) we obtain that $C(Z)$ is the inductive limit

$$C(S^2) \xrightarrow{\widehat{\rho}_1} C((S^2)^2) \xrightarrow{\widehat{\rho}_2} C((S^2)^3) \xrightarrow{\widehat{\rho}_3} \cdots \longrightarrow C(Z)$$

with inductive limit maps $\widehat{\rho}_{\infty,n} \colon C((S^2)^n) \to C(Z)$.

For n in $\mathbb{N}$ and for each non-empty finite subset $I = \{n_1, n_2, \dots, n_k\}$ of $\mathbb{N}$, let p_n and

p_I be the projections in $C(Z) \otimes \mathcal{K} = C(Z, \mathcal{K})$ given by

$$p_n(x) = p(x_n), \tag{4.1}$$

$$\begin{aligned} p_I(x) &= p(x_{n_1}) \otimes p(x_{n_2}) \otimes \cdots \otimes p(x_{n_k}) \\ &= p_{n_1}(x) \otimes p_{n_2}(x) \otimes \cdots \otimes p_{n_k}(x), \end{aligned} \tag{4.2}$$

for all $x = (x_1, x_2, \dots) \in Z$ (identifying M_2, respectively, $M_2 \otimes M_2 \otimes \cdots \otimes M_2$ with sub-C^*-algebras of $\mathcal{K}$).

We shall now make use of the multiplier algebra, $\mathcal{M}(C(Z)\otimes\mathcal{K})$, of $C(Z)\otimes\mathcal{K} = C(Z,\mathcal{K})$. We can identify this multiplier algebra with the set of all bounded functions $f\colon Z \to B(H)$ for which f and f^* are continuous, when $B(H)$, the bounded operators on the Hilbert space H on which $\mathcal{K}$ acts, is given the strong operator topology.

It is convenient to have a convention for adding finitely or infinitely many projections in $\mathcal{M}(C(Z)\otimes\mathcal{K})$, or more generally in $\mathcal{M}(A)$, where A is any stable C^*-algebra—a convention that extends the notion of forming direct sums of projections discussed in Section 2.

Assuming that A is a stable C^*-algebra, so that $A = A_0 \otimes \mathcal{K}$ for some C^*-algebra A_0, then we can take a sequence $\{T_j\}_{j=1}^\infty$ of isometries in $\mathbb{C}\otimes B(H) \subseteq \mathcal{M}(A_0\otimes\mathcal{K}) = \mathcal{M}(A)$ such that $1 = \sum_{j=1}^\infty T_j T_j^*$ in the strict topology. (Notice that 1 is a properly infinite projection in $\mathcal{M}(A)$.) For any sequence $q_1, q_2, \dots$ of projections in A and for any sequence $Q_1, Q_2, \dots$ of projections in $\mathcal{M}(A)$, define

$$q_1 \oplus q_2 \oplus \cdots \oplus q_n = \sum_{j=1}^{n} T_j q_j T_j^* \in A, \tag{4.3}$$

$$\bigoplus_{j=1}^{\infty} q_j = \sum_{j=1}^{\infty} T_j q_j T_j^* \in \mathcal{M}(A), \tag{4.4}$$

$$Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n = \sum_{j=1}^{n} T_j Q_j T_j^* \in \mathcal{M}(A), \tag{4.5}$$

$$\bigoplus_{j=1}^{\infty} Q_j = \sum_{j=1}^{\infty} T_j Q_j T_j^* \in \mathcal{M}(A), \tag{4.6}$$

Observe that $q_j' = T_j q_j T_j^* \sim q_j$, that the projections $q_1', q_2', \dots$ are mutually orthogonal, and that the sum $\sum_{j=1}^\infty q_j'$ is strictly convergent. The projections in (4.3)–(4.6) are, up to unitary equivalence in $\mathcal{M}(A)$, independent of the choice of isometries $\{T_j\}_{j=1}^\infty$. Indeed, if $\{R_j\}_{j=1}^\infty$ is another sequence of isometries in $\mathcal{M}(A)$ with $1 = \sum_{j=1}^\infty R_j R_j^*$, then $U = \sum_{j=1}^\infty R_j T_j^*$ is a unitary element in $\mathcal{M}(A)$ and

$$\sum_{j=1}^{\infty} R_j X_j R_j^* = U\big(\sum_{j=1}^{\infty} T_j X_j T_j^*\big)U^*$$

for any bounded sequence $\{X_j\}_{j=1}^{\infty}$ in $\mathcal{M}(A)$. It follows in particular that

$$\bigoplus_{j=1}^{\infty} q_j \sim \bigoplus_{j=1}^{\infty} q_{\sigma(j)} \tag{4.7}$$

for every permutation σ on $\mathbb{N}$.

In the lemma below the correspondence between projections and vector bundles is given by the mapping $p \mapsto \xi_p$ defined at the beginning of this section. By identifying the projections $p_n, p_I, p_{I_1}, \dots, p_{I_k}$ with projections in $C((S^2)^N) \otimes \mathcal{K}$, where N is any integer large enough to ensure that these projections belong to the image of

$$\widehat{\rho}_{\infty,N} \otimes \mathrm{id}_{\mathcal{K}} \colon C((S^2)^N) \otimes \mathcal{K} \to C(Z) \otimes \mathcal{K},$$

we can take the base space to be $(S^2)^N$.

Lemma 4.1 *Let ζ_n and ζ_I be the complex line bundles defined in* (3.4)*.*

(i) *The vector bundle ζ_n corresponds to p_n for each n in $\mathbb{N}$.*

(ii) *The vector bundle ζ_I corresponds to p_I for each non-empty finite subset I of $\mathbb{N}$.*

(iii) *The vector bundle $\zeta_{I_1} \oplus \zeta_{I_2} \oplus \cdots \oplus \zeta_{I_k}$ corresponds to $p_{I_1} \oplus p_{I_2} \oplus \cdots \oplus p_{I_k}$ whenever $I_1, \dots, I_k$ are non-empty finite subsets of $\mathbb{N}$.*

Proof: (i). Since p corresponds to ζ, $p_n = p \circ \pi_n$ corresponds to $\zeta_n = \pi_n^*(\zeta)$, where $\pi_n \colon (S^2)^N \to S^2$ is the nth coordinate map.

(ii). Write $I = \{n_1, n_2, \dots, n_k\}$. We shall here view p_n as a projection in $C((S^2)^N, M_2)$ and p_I as a projection in $C((S^2)^N, M_2 \otimes \cdots \otimes M_2)$. By (i), ζ_n is the complex line bundle over $(S^2)^N$ whose fibre over $x \in (S^2)^N$ is equal to $p_n(x)(\mathbb{C}^2)$. The fibre of the complex line bundle $\zeta_I = \zeta_{n_1} \otimes \zeta_{n_2} \otimes \cdots \otimes \zeta_{n_k}$ over $x \in (S^2)^N$ is by definition

$$\begin{aligned}
(\zeta_I)_x &= (\zeta_{n_1})_x \otimes (\zeta_{n_2})_x \otimes \cdots \otimes (\zeta_{n_1})_x \\
&= p_{n_1}(x)(\mathbb{C}^2) \otimes p_{n_2}(x)(\mathbb{C}^2) \otimes \cdots \otimes p_{n_k}(x)(\mathbb{C}^2) \\
&= p_I(x)(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2).
\end{aligned}$$

This shows that ζ_I corresponds to p_I.

(iii). This follows from (ii) and additivity of the map $p \mapsto \xi_p$. $\square$

The next three lemmas are formulated for an arbitrary stable C^*-algebra A and its multiplier algebra $\mathcal{M}(A)$, but they shall primarily be used in the case where $A = C(Z) \otimes \mathcal{K}$.

The lemma below is a trivial, but much used, generalization of (4.7):

Lemma 4.2 *Let A be a stable C^*-algebra, and let $q_1, q_2, \dots$ and $r_1, r_2, \dots$ be two sequences of projections in A. Assume that there is a permutation σ on $\mathbb{N}$ such that $q_j \precsim r_{\sigma(j)}$, respectively $q_j \sim r_{\sigma(j)}$, in A for all j in $\mathbb{N}$. Then $\bigoplus_{j=1}^\infty q_j \precsim \bigoplus_{j=1}^\infty r_j$, respectively $\bigoplus_{j=1}^\infty q_j \sim \bigoplus_{j=1}^\infty r_j$, in $\mathcal{M}(A)$.*

An element in a C^*-algebra A is said to be *full in A* if it is not contained in any proper closed two-sided ideal of A.

Lemma 4.3 *Let A be a stable C^*-algebra. The following three conditions are equivalent for all projections Q in $\mathcal{M}(A)$:*

(i) $Q \sim 1$, (ii) *Q is properly infinite and full in $\mathcal{M}(A)$,* (iii) $1 \precsim Q$.

Proof: (i) $\Rightarrow$ (iii) is trivial. Assume that $1 \precsim Q$. Then Q is full in $\mathcal{M}(A)$ (the closed two-sided ideal in $\mathcal{M}(A)$ generated by Q contains 1 and hence all of $\mathcal{M}(A)$). It was noted above (4.3) that 1 is properly infinite in $\mathcal{M}(A)$, and so $Q \oplus Q \le 1 \oplus 1 \precsim 1 \precsim Q$, whence Q is properly infinite; cf. Proposition 2.1. This proves (iii) $\Rightarrow$ (ii). Assume finally that Q is properly infinite and full in $\mathcal{M}(A)$. Since $K_0(\mathcal{M}(A)) = 0$ (see [7, Proposition 12.2.1]) the two projections Q and 1 represent the same element in $K_0(\mathcal{M}(A))$; and since these two projections both are properly infinite and full they must be Murray–von Neumann equivalent (see [16, Section 1] or [40, Exercise 4.9 (iii)]), i.e., $Q \sim 1$. $\square$

Lemma 4.4 *Let A be a stable C^*-algebra and let $q, q_1, q_2, \dots$ be projections in A. If $q \precsim \bigoplus_{j=1}^\infty q_j$ in $\mathcal{M}(A)$, then $q \precsim q_1 \oplus q_2 \oplus \cdots \oplus q_k$ in A for some k.*

Proof: We have $\bigoplus_{j=1}^\infty q_j = \sum_{j=1}^\infty q_j'$ $(= Q)$ for some strictly summable sequence of mutually orthogonal projections $q_1', q_2', \dots$ in A with $q_j' \sim q_j$. By the assumption that $q \precsim Q$ there is a partial isometry v in $\mathcal{M}(A)$ such that $vv^* = q$ and $v^*v \le Q$. As $v = qv$, v belongs to A, and by the strict convergence of the sum $Q = \sum_{j=1}^\infty q_j'$ there is k such that

$$\|v - v\sum_{j=1}^k q_j'\| < 1/2.$$

Put $x = v\sum_{j=1}^{k} q'_j$. Then $xx^* \le q$, $x^*x \le q'_1 + \cdots + q'_k$, and $\|xx^* - q\| < 1$. This shows that xx^* is invertible in qAq with inverse $(xx^*)^{-1}$. Put $u = (xx^*)^{-1/2}x$. Then $uu^* = q$ and $u^*u \le q'_1 + \cdots + q'_k$, whence $q \precsim q_1 \oplus \cdots \oplus q_k$. $\square$

Let g be a constant one-dimensional projection in $C(Z, \mathcal{K}) = C(Z) \otimes \mathcal{K}$ (that corresponds to the trivial complex line bundle θ over X). The (easy-to-prove) statement in part (iii) of the proposition below is not used in this paper, but it may have some independent interest.

Proposition 4.5 *Let $I_1, I_2, \dots$ be a sequence of non-empty, finite subsets of $\mathbb{N}$. Put*

$$Q = \bigoplus_{j=1}^{\infty} p_{I_j} \in \mathcal{M}(C(Z) \otimes \mathcal{K}).$$

(i) *If $\left|\bigcup_{j\in F} I_j\right| \ge |F|$ for all finite subsets F of $\mathbb{N}$, then $g \not\precsim Q$ and Q is not properly infinite.*

(ii) *$g \precsim p_n \oplus p_n$ for every natural number n.*

(iii) *If infinitely many of the sets $I_1, I_2, \dots$ are singletons, then $Q \oplus Q$ is properly infinite and $Q \oplus Q \sim 1$ in $\mathcal{M}(C(Z) \otimes \mathcal{K})$.*

Proof: (i). We show first that $g \not\precsim Q$ in $\mathcal{M}(C(Z) \otimes \mathcal{K})$. Indeed, assume to the contrary that $g \precsim Q$. Then

$$g \precsim p_{I_1} \oplus p_{I_2} \oplus \cdots \oplus p_{I_k} \tag{4.8}$$

in $C(Z) \otimes \mathcal{K}$ for some k by Lemma 4.4. As noted earlier, $C(Z) \otimes \mathcal{K}$ is an inductive limit

$$C(S^2) \otimes \mathcal{K} \xrightarrow{\widehat{\rho}_1 \otimes \mathrm{id}_{\mathcal{K}}} C((S^2)^2) \otimes \mathcal{K} \xrightarrow{\widehat{\rho}_2 \otimes \mathrm{id}_{\mathcal{K}}} C((S^2)^3) \otimes \mathcal{K} \longrightarrow \cdots \longrightarrow C(Z) \otimes \mathcal{K}.$$

Take N such that all projections appearing in (4.8) belong to the image of

$$\widehat{\rho}_{\infty,n} \otimes \mathrm{id}_{\mathcal{K}} \colon C((S^2)^n) \otimes \mathcal{K} \to C(Z) \otimes \mathcal{K}$$

whenever $n \ge N$. Use a standard inductive limit argument to see that (4.8) holds relatively to $C((S^2)^n) \otimes \mathcal{K}$ for some large enough $n \ge N$. In the language of vector bundles over $(S^2)^n$, (4.8) and Lemma 4.1 imply that

$$\theta \oplus \eta \cong \zeta_{I_1} \oplus \zeta_{I_2} \oplus \cdots \oplus \zeta_{I_k} \tag{4.9}$$

for some vector bundle η over $(S^2)^n$. Now, (4.9) and (3.2) imply that $e(\zeta_{I_1} \oplus \cdots \oplus \zeta_{I_k}) = 0$, in contradiction with Proposition 3.2 and the assumption on the sets I_j.

The projection p_{I_1} is a full element in $C(Z) \otimes \mathcal{K}$ and $p_{I_1} \leq Q$. Hence g belongs to the ideal generated by Q. It now follows from Lemma 2.2 and from the fact that $g \not\precsim Q$ that Q cannot be properly infinite.

(ii) follows from Lemma 3.1 and Lemma 4.1.

(iii). The unit 1 of $\mathcal{M}(C(Z) \otimes \mathcal{K})$ can be written as a strictly convergent sum $1 = \sum_{j=1}^{\infty} g_j$, where $g_j \sim g$ for all j. Let Γ denote the infinite subset of $\mathbb{N}$ consisting of those j for which I_j is a singleton. By Lemma 4.2 and (ii) we get

$$1 \sim \bigoplus_{j=1}^{\infty} g \precsim \bigoplus_{j \in \Gamma} (p_{I_j} \oplus p_{I_j}) \precsim \bigoplus_{j=1}^{\infty} (p_{I_j} \oplus p_{I_j}) \sim Q \oplus Q.$$

Lemma 4.3 now tells us that $Q \oplus Q$ is properly infinite and that $Q \oplus Q \sim 1$. □

5 A non-exact example

We construct here a simple, unital C^*-algebra that contains a finite and an infinite projection; thus proving one of our main results: Theorem 5.6 below.

Let again Z denote the infinite product space $\prod_{j=1}^{\infty} S^2$. Set $A = C(Z) \otimes \mathcal{K} = C(Z, \mathcal{K})$; recall from Section 4 that $\mathcal{M}(A)$ denotes the multiplier algebra of A and that it can be identified with the set of bounded *-strongly continuous functions $f \colon Z \to B(H)$.

Choose an injective function $\nu \colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. Choose points $c_{j,i} \in S^2$ for all $j, i \in \mathbb{N}$ with $j \geq i$ such that

$$\overline{\{(c_{j,1}, c_{j,2}, \ldots, c_{j,n}) \mid j \geq n\}} = S^2 \times S^2 \times \cdots \times S^2 \tag{5.1}$$

for every natural number n. Set

$$I_j = \{\nu(j,1), \nu(j,2), \ldots, \nu(j,j)\}, \tag{5.2}$$

for $j \in \mathbb{N}$.

Define *-homomorphisms $\varphi_j \colon A \to A$ for all integers j as follows. For $j \leq 0$, set

$$\varphi_j(f)(x) = f\big(x_{\nu(j,1)}, x_{\nu(j,2)}, x_{\nu(j,3)}, \ldots\big), \qquad f \in A,\ x = (x_1, x_2, \ldots) \in Z. \tag{5.3}$$

Let p_n and p_I be the projections in $A = C(Z, \mathcal{K})$ defined in (4.1) and (4.2). Choose an isomorphism $\tau\colon \mathcal{K}\otimes\mathcal{K}\to\mathcal{K}$. For f in A, $x=(x_1,x_2,\dots)$ in Z, and $j\ge 1$ define

$$\varphi_j(f)(x)=\tau\big(f(c_{j,1},\dots,c_{j,j},x_{\nu(j,j+1)},x_{\nu(j,j+2)},\dots)\otimes p_{I_j}(x)\big). \tag{5.4}$$

Choose a sequence $\{S_j\}_{j=-\infty}^{\infty}$ of isometries in $\mathcal{M}(A)$ such that $\sum_{j=-\infty}^{\infty}S_jS_j^*=1$ with the sum being strictly convergent. Define a *-homomorphism $\psi\colon A\to\mathcal{M}(A)$ by

$$\psi(f)=\sum_{j=-\infty}^{\infty}S_j\varphi_j(f)S_j^*,\qquad f\in A. \tag{5.5}$$

Lemma 5.1 *Let $\{e_n\}_{n=1}^{\infty}$ be an increasing approximate unit for A. Then $\{\psi(e_n)\}_{n=1}^{\infty}$ converges strictly to a projection $F\in\mathcal{M}(A)$, and F is equivalent to the identity 1 in $\mathcal{M}(A)$.*

Proof: If $\psi(e_n)$ converges strictly to $F\in\mathcal{M}(A)$ for *some* approximate unit $\{e_n\}$ for A, then this conclusion will hold for *all* approximate units for A. We can therefore take $\{e_n\}_{n=1}^{\infty}$ to be the approximate unit given by $e_n(x)=\widehat{e}_n$, where $\{\widehat{e}_n\}_{n=1}^{\infty}$ is an increasing approximate unit for $\mathcal{K}$.

We show first that $\{\varphi_j(e_n)\}_{n=1}^{\infty}$ converges strictly to a projection F_j in $\mathcal{M}(A)$ for each $j\in\mathbb{Z}$. Indeed, since $\varphi_j(e_n)=e_n$ when $j\le 0$ it follows that $\varphi_j(e_n)\to 1$ strictly; and so $F_j=1$ when $j\le 0$. Consider next the case $j\ge 1$. Here we have $\varphi_j(e_n)(x)=\tau(\widehat{e}_n\otimes p_{I_j}(x))$. Extend $\tau\colon\mathcal{K}\otimes\mathcal{K}\to\mathcal{K}$ to a strongly continuous unital *-homomorphism $\overline{\tau}\colon B(H\otimes H)\to B(H)$ and define F_j in $\mathcal{M}(A)$ by $F_j(x)=\overline{\tau}(1\otimes p_{I_j}(x))$ for $x\in Z$. Then F_j is a projection and $\{\varphi_j(e_n)\}_{n=1}^{\infty}$ converges strictly to F_j.

Now,

$$\psi(e_n)=\sum_{j=-\infty}^{\infty}S_j\varphi_j(e_n)S_j^*\;\underset{n\to\infty}{\overset{\text{strictly}}{\longrightarrow}}\;\sum_{j=-\infty}^{\infty}S_jF_jS_j^*\;\overset{\text{def}}{=}\;F\in\mathcal{M}(A),$$

As $1=F_0\sim S_0F_0S_0^*\le F$ it follows from Lemma 4.3 that $F\sim 1$ in $\mathcal{M}(A)$. $\square$

Take an isometry T in $\mathcal{M}(A)$ with $TT^*=F$ (where F is an in Lemma 5.1). Define

$$\varphi(f)=T^*\psi(f)T=\sum_{j=-\infty}^{\infty}T^*S_j\varphi_j(f)S_j^*T,\qquad f\in A. \tag{5.6}$$

Then $\varphi\colon A\to\mathcal{M}(A)$ is a *-homomorphism that maps an approximate unit for A into a sequence in $\mathcal{M}(A)$ that converges strictly to the identity in $\mathcal{M}(A)$ (by Lemma 5.1 and the choice of T). It follows from [32, Proposition 2.5] that φ extends to a unital *-homomorphism $\overline{\varphi}\colon\mathcal{M}(A)\to\mathcal{M}(A)$.

We collect below some properties of the *-homomorphisms φ and $\overline{\varphi}$. A subset of a C^*-algebra A is called *full in A* if it is not contained in any proper closed two-sided ideal in A.

Proposition 5.2 *Let p_1 be the projection in A defined in* (4.1) *and let g be a constant 1-dimensional projection in $A = C(Z, \mathcal{K})$.*

(i) $\varphi(g) \sim 1$ *in $\mathcal{M}(A)$, and $\varphi(f)$ is full in $\mathcal{M}(A)$ for every full element f in A.*

(ii) *If f is a non-zero element in $\mathcal{M}(A)$, then $\overline{\varphi}(f)$ does not belong to A, and $A\overline{\varphi}(f)$ is full in A.*

(iii) *If f is a non-zero element in $\mathcal{M}(A)$, then $A\overline{\varphi}^k(f)$ is full in A for every $k \in \mathbb{N}$.*

(iv) *None of the projections $\overline{\varphi}^k(p_1)$, $k \in \mathbb{N}$, are properly infinite in $\mathcal{M}(A)$.*

It follows immediately from (ii) that $\overline{\varphi}$ and φ are injective, and that $\overline{\varphi}(\mathcal{M}(A)) \cap A = \{0\}$ and $\varphi(A) \cap A = \{0\}$.

The proof of Proposition 5.2 is divided into a few lemmas, the first of which (included for emphasis) is standard and follows from the fact that any closed two-sided ideal in $C(Z, \mathcal{K})$ is equal to $C_0(U, \mathcal{K})$ for some open subset U of Z.

Lemma 5.3 *Let f be an element in $A = C(Z, \mathcal{K})$. Then f is full in A if and only if $f(x) \neq 0$ for all $x \in Z$.*

Proof of Proposition 5.2 (i): Observe first that $\varphi_j(g) = g$ for every $j \leq 0$. Accordingly,

$$1 \;\sim\; \bigoplus_{j=-\infty}^{0} g \;\sim\; \sum_{j=-\infty}^{0} T^* S_j \varphi_j(g) S_j^* T \;\leq\; \varphi(g) \quad \text{in } \mathcal{M}(A).$$

This and Lemma 4.3 imply that $\varphi(g) \sim 1$ and that $\varphi(g)$ is full in $\mathcal{M}(A)$. If f is any full element in A, then the closed two-sided ideal generated by $\varphi(f)$ contains $\varphi(g)$ and therefore all of $\mathcal{M}(A)$. This proves the second claim in (i). $\square$

Proof of Proposition 5.2 (ii): Take a non-zero element f in $\mathcal{M}(A)$. There is an element a in A such that $af \neq 0$. The two claims in (ii) will clearly follow if we can show that $\overline{\varphi}(af) \notin A$ and that $A\overline{\varphi}(af)$ is full in A, and we can therefore, upon replacing f by af, assume that f is a non-zero element in $A = C(Z, \mathcal{K})$.

There are $\delta > 0$, $r \in \mathbb{N}$, and non-empty open subsets $U_1, \dots, U_r$ of S^2 such that

$$x \in U_1 \times U_2 \times \cdots \times U_r \times S^2 \times S^2 \times \cdots \implies \|f(x)\| \geq \delta. \tag{5.7}$$

Use (5.1) to find an infinite set Λ of integers $j \geq r$ such that

$$(c_{j,1}, c_{j,2}, \dots, c_{j,r}) \in U_1 \times U_2 \times \cdots \times U_r \quad \text{for all } \ j \in \Lambda. \tag{5.8}$$

It follows from Lemma 5.3, (5.4), (5.7), and (5.8) that $\|\varphi_j(f)\| \geq \delta$ and $\varphi_j(f)$ is full in A for every j in the infinite set Λ. This entails that $\varphi(f) = \sum_{j=-\infty}^{\infty} T^* S_j \varphi_j(f) S_j^* T$ does not belong to A. (A strictly convergent sum $\sum_{j=-\infty}^{\infty} a_j$ of pairwise orthogonal elements from A belongs to A if and only if $\lim_{j \to \pm\infty} \|a_j\| = 0$.) The closed two-sided ideal in A generated by $A\varphi(f)$ contains the full element $\varphi_j(f) = S_j^* T \varphi(f) T^* S_j$ and therefore all of A (for each—and hence at least one—j in Λ). □

Proof of Proposition 5.2 (iii): This follows from injectivity of $\overline{\varphi}$ and Proposition 5.2 (ii). □

We proceed to prove Proposition 5.2 (iv).

Lemma 5.4 *Let J be a finite subset of $\mathbb{N}$ and let j be an integer. Then $\varphi_j(p_J) \sim p_{\alpha_j(J)}$, where*

$$\alpha_j(J) = \begin{cases} \nu(j, J), & j \leq 0 \\ \nu(j, J \setminus \{1, 2, \dots, j\}) \cup I_j, & j \geq 1. \end{cases} \tag{5.9}$$

We have in particular that $\nu(j, J) \subseteq \alpha_j(J)$ for all finite subsets J of $\mathbb{N}$ and for all $j \in \mathbb{Z}$.

Proof: Write $J = \{t_1, t_2, \dots, t_k\}$, where $t_1 < t_2 < \cdots < t_k$. We consider first the case where $j \leq 0$. Then

$$\begin{aligned} \varphi_j(p_J)(x) &= p_J(x_{\nu(j,1)}, x_{\nu(j,2)}, x_{\nu(j,3)}, \dots) \\ &= p(x_{\nu(j,t_1)}) \otimes p(x_{\nu(j,t_2)}) \otimes \cdots \otimes p(x_{\nu(j,t_k)}) \\ &= p_{\nu(j,t_1)}(x) \otimes p_{\nu(j,t_2)}(x) \otimes \cdots \otimes p_{\nu(j,t_k)}(x) \ = \ p_{\nu(j,J)}(x), \end{aligned}$$

as desired.

Suppose next that $j \geq 1$, and put $q(x) = p_J(c_{j,1}, \dots, c_{j,j}, x_{\nu(j,j+1)}, x_{\nu(j,j+2)}, \dots)$. Then $\varphi_j(p_J)(x) = \tau(q(x) \otimes p_{I_j}(x))$. Suppose that $1 \leq j < t_k$ and let m be such that $t_{m-1} \leq j < t_m$

(with the convention $t_0 = 0$). Then

$$\begin{aligned} q(x) &= p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_{m-1}}) \otimes p(x_{\nu(j,t_m)}) \otimes \cdots \otimes p(x_{\nu(j,t_k)}) \\ &= p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_{m-1}}) \otimes p_{\nu(j,t_m)}(x) \otimes \cdots \otimes p_{\nu(j,t_k)}(x) \\ &= p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_{m-1}}) \otimes p_{\nu(j,J\setminus\{1,2,\dots,j\})}(x). \end{aligned}$$

Thus $q \sim p_{\nu(j,J\setminus\{1,2,\dots,j\})}$, which shows that $\varphi_j(p_J)$ is equivalent to the projection defined by

$$x \mapsto \tau(p_{\nu(j,J\setminus\{1,2,\dots,j\})}(x) \otimes p_{I_j}(x)),$$

and this projection is equivalent to $p_{\nu(j,J\setminus\{1,2,\dots,j\})\cup I_j}$. If $j \geq t_k$, then $J \setminus \{1,2,\dots,j\} = \emptyset$ and $q(x) = p(c_{j,t_1}) \otimes \cdots \otimes p(c_{j,t_k})$, i.e., q is a constant projection. In this case, $\varphi_j(p_J) \sim p_{I_j}$, thus affirming the first claim of the lemma.

The last claim follows from the definition of the sets I_j in (5.2). □

Lemma 5.5 *Let $J_1, J_2, \dots$ be finite subsets of $\mathbb{N}$. Put $Q = \bigoplus_{i=1}^\infty p_{J_i} \in \mathcal{M}(A)$. Then*

$$\overline{\varphi}(Q) \sim \bigoplus_{i=1}^{\infty} \bigoplus_{j=-\infty}^{\infty} p_{\alpha_j(J_i)},$$

where α_j is as defined in (5.9). Moreover, if $|\bigcup_{i\in F} J_i| \geq |F|$ for all finite subsets F of $\mathbb{N}$, then $|\bigcup_{(j,i)\in G} \alpha_j(J_i)| \geq |G|$ for all finite subsets G of $\mathbb{Z}\times\mathbb{N}$.

Proof: By (4.4), $Q = \sum_{i=1}^\infty T_i p_{J_i} T_i^*$; and because $\overline{\varphi}$ is strictly continuous we get

$$\overline{\varphi}(Q) = \sum_{i=1}^{\infty} \overline{\varphi}(T_i)\varphi(p_{J_i})\overline{\varphi}(T_i)^* \sim \bigoplus_{i=1}^{\infty} \varphi(p_{J_i}) \sim \bigoplus_{i=1}^{\infty} \bigoplus_{j=-\infty}^{\infty} \varphi_j(p_{J_i}) \sim \bigoplus_{i=1}^{\infty} \bigoplus_{j=-\infty}^{\infty} p_{\alpha_j(J_i)},$$

where the first equivalence is proved below (4.3)–(4.6), and the last equivalence follows from Lemma 5.4.

By the Marriage Theorem we can find natural numbers $t_i \in J_i$ such that $\{t_i\}_{i\in\mathbb{N}}$ are mutually distinct. Set $s_{j,i} = \nu(j,t_i)$. Then $s_{j,i}$ belongs to $\alpha_j(J_i)$ by Lemma 5.4, and $\{s_{j,i}\}_{(j,i)\in\mathbb{Z}\times\mathbb{N}}$ are mutually distinct because ν is injective and the t_i's are mutually distinct. This proves the second claim of the lemma. □

Proof of Proposition 5.2 (iv): Put $Q_0 = p_1$ and put $Q_n = \overline{\varphi}^n(Q_0)$. We must show that none of the projections Q_n, $n \geq 0$, are properly infinite. It is clear that Q_0 is finite, and hence not properly infinite.

Use Lemmas 5.4 and 5.5 to see that

$$Q_1 = \sum_{j=-\infty}^{\infty} T^* S_j \varphi_j(p_1) S_j^* T \sim \bigoplus_{j=-\infty}^{\infty} \varphi_j(p_1) \sim \bigoplus_{j=-\infty}^{0} p_{\nu(j,1)} \oplus \bigoplus_{j=1}^{\infty} p_{I_j} = \bigoplus_{j=-\infty}^{\infty} p_{J_j},$$

where $J_j = \{\nu(j,1)\}$ for $j \leq 0$ and $J_j = I_j$ for $j \geq 1$. It is easily seen that the sequence of sets $\{J_j\}_{j=-\infty}^{\infty}$ satisfies the condition $|\bigcup_{j \in F} J_j| \geq |F|$ for all finite subsets F of $\mathbb{Z}$. Hence Q_1 is not properly infinite by Proposition 4.5 (i).

The claim that Q_n is not properly infinite for all n follows by induction using Lemma 5.5 and Proposition 4.5 (i). $\square$

Theorem 5.6 *Consider the inductive limit B of the sequence:*

$$\mathcal{M}(C(Z) \otimes \mathcal{K}) \xrightarrow{\overline{\varphi}} \mathcal{M}(C(Z) \otimes \mathcal{K}) \xrightarrow{\overline{\varphi}} \mathcal{M}(C(Z) \otimes \mathcal{K}) \xrightarrow{\overline{\varphi}} \cdots \longrightarrow B.$$

Then B has the following properties:

(i) *B is unital and simple.*

(ii) *The unit of B is infinite.*

(iii) *B contains a non-zero finite projection.*

(iv) *$K_0(B) = 0$ and $K_1(B) = 0$.*

Proof: (i). B is unital being the inductive limit of a sequence of unital C^*-algebras with unital connecting maps.

Write again A for $C(Z) \otimes \mathcal{K}$, and let $\overline{\varphi}_{\infty,n} \colon \mathcal{M}(A) \to B$ be the inductive limit map from the nth copy of $\mathcal{M}(A)$ into B. Let L be a non-zero closed two-sided ideal in B, and set

$$L_n = \overline{\varphi}_{\infty,n}^{-1}(L) \lhd \mathcal{M}(A).$$

Then L_n is non-zero for some n. Since A is an essential ideal in $\mathcal{M}(A)$, also $A \cap L_n$ is non-zero.

Take a non-zero element e in $A \cap L_n$. Then $\overline{\varphi}(e)$ belongs to L_{n+1}, hence $A\overline{\varphi}(e) \subseteq L_{n+1}$, and so it follows from Proposition 5.2 (ii) that $A \subseteq L_{n+1}$. Take now a full element f in $A \subseteq L_{n+1}$. Then $\overline{\varphi}(f)$ belongs to L_{n+2}. It follows from Proposition 5.2 (i) that $\overline{\varphi}(f)$ is full in $\mathcal{M}(A)$ and therefore $L_{n+2} = \mathcal{M}(A)$. Hence $L = B$, and this shows that B is simple.

(ii). This is clear because the unit of $\mathcal{M}(A)$ is infinite.

(iii). As in the proof of Proposition 5.2 (iv), set $Q_0 = p_1$ and $Q_n = \overline{\varphi}^n(Q_0)$ for $n \geq 1$. Put $Q = \overline{\varphi}_{\infty,0}(Q_0) \in B$. It is shown in Proposition 5.2 (ii) that $\overline{\varphi}$ is injective, which implies that $\overline{\varphi}_{\infty,0}$ is injective, and hence Q is non-zero. We show next that Q is finite.

Assume that Q were infinite. Then Q is properly infinite by Cuntz' result (see Proposition 2.1) because B is simple. Applying Proposition 2.3 to the sequence

$$Q_0\mathcal{M}(A)Q_0 \xrightarrow{\lambda_0} Q_1\mathcal{M}(A)Q_1 \xrightarrow{\lambda_1} Q_2\mathcal{M}(A)Q_2 \longrightarrow \cdots \longrightarrow QBQ,$$

with the unital connecting maps $\lambda_j = \overline{\varphi}|_{Q_j\mathcal{M}(A)Q_j}$, we obtain that Q_n is properly infinite for all sufficiently large n. But this contradicts Proposition 5.2 (iv).

(iv). This follows from the fact that the multiplier algebra of a stable C^*-algebra has trivial K-theory (see [7, Proposition 12.2.1]). $\square$

It follows from Proposition 4.5 (ii) and Proposition 5.2 (i) that the finite projection Q in B (found in part (iii) above) satisfies

$$Q \oplus Q \;\sim\; \overline{\varphi}_{\infty,0}(Q_0 \oplus Q_0) \;=\; \overline{\varphi}_{\infty,0}(p_1 \oplus p_1) \;\precsim\; \overline{\varphi}_{\infty,0}(g) = \overline{\varphi}_{\infty,1}(\varphi(g)) \;\sim\; 1,$$

whence $Q \oplus Q \sim 1$ by Lemma 4.3. In other words, the corner C^*-algebra QBQ is unital, finite, and simple, and $M_2(QBQ) \cong B$ is infinite.

The C^*-algebra B from Theorem 5.6 is not separable and not exact. To see the latter, note that $B(H)$, the bounded operators on a separable, infinite dimensional Hilbert space H, can be embedded into $\mathcal{M}(A) = \mathcal{M}(C(Z) \otimes \mathcal{K})$ and hence into B. As $B(H)$ is non-exact (see Wasserman [43, 2.5.4]) it follows from Kirchberg's result that exactness passes to sub-C^*-algebras (see [43, 2.5.2]) that B is non-exact. We use the lemma below from [3] to construct a non-exact *separable* example.

Lemma 5.7 (Blackadar) *Let B be a simple C^*-algebra and let X be a countable subset of B. It follows that B has a separable, simple sub-C^*-algebra B_0 that contains X.*

Corollary 5.8 *There exists a unital,* separable*, non-exact, simple C^*-algebra B_0 such that B_0 contains an infinite and a non-zero finite projection.*

Proof: Let B be as in Theorem 5.6. Let s be a non-unitary isometry in B and let q be a non-zero finite projection in B. The universal C^*-algebra, $C^*(\mathbb{F}_2)$, generated by two unitaries is separable and non-exact (see Wassermann [43, Corollary 3.7]). It admits an embedding into $\mathcal{M}(C(Z) \otimes \mathcal{K})$ and hence into B. Let $u, v \in B$ be the images of the two (canonical) unitary generators in $C^*(\mathbb{F}_2)$. Use Lemma 5.7 to find a separable, simple, and unital C^*-algebra B_0 that contains $\{u, v, s, q\}$.

Then B_0 is infinite because it contains the non-unitary isometry s; and it contains the finite projection q. Finally, B_0 is non-exact because it contains the non-exact sub-C^*-algebra $C^*(u, v) \cong C^*(\mathbb{F}_2)$. □

6 A nuclear example

We show here that an elaboration of the construction in Section 5 yields a *nuclear* and separable example of a simple C^*-algebra with a finite and an infinite projection.

The construction requires that we make a specific choice for the injective map $\nu\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ from Section 5.

Let $\{\Lambda_r\}_{r=0}^{\infty}$ be a partition of the set $\mathbb{N}$ such that $\Lambda_0 = \{1\}$ and such that Λ_r is infinite for each $r \geq 1$. For each $r \geq 1$ choose an injective map $\gamma_r\colon \mathbb{Z} \times \Lambda_{r-1} \to \Lambda_r$ and define $\nu\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ by:

$$\nu(j, t) = \gamma_r(j, t), \qquad r \in \mathbb{N},\ t \in \Lambda_{r-1},\ j \in \mathbb{Z}. \tag{6.1}$$

Observe that

$$t \in \Lambda_r \iff \nu(j, t) \in \Lambda_{r+1}, \qquad j \in \mathbb{Z}. \tag{6.2}$$

To see that ν is injective assume that $\nu(j, t) = \nu(i, s)$. Then $\nu(j, t) = \nu(i, s) \in \Lambda_r$ for some $r \geq 1$. Therefore both s and t belong to Λ_{r-1}. Now, $\gamma_r(j, t) = \nu(j, t) = \nu(i, s) = \gamma_r(i, s)$, which entails that $(j, t) = (i, s)$ by injectivity of γ_r.

Let α_j be as defined in Lemma 5.4 (wrt. the new choice of ν). Let $\Gamma_0 \subseteq P(\mathbb{N})$ be the family containing the one set $\{1\}$, and set

$$\Gamma_{n+1} = \{\alpha_j(I) \mid I \in \Gamma_n,\ j \in \mathbb{Z}\} \subseteq P(\mathbb{N}),$$

for $n \geq 0$. Set $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$. Observe that each $I \in \Gamma$ is a finite subset of $\mathbb{N}$.

Put $Q_0 = p_1 \in A$ (cf. (4.1)) and put $Q_n = \overline{\varphi}^n(Q_0) \in \mathcal{M}(A)$ (where $\overline{\varphi}$ is the endomorphism on $\mathcal{M}(A)$ defined in Section 5 above Proposition 5.2). It then follows by induction from Lemma 5.5 that

$$Q_n \sim \bigoplus_{I \in \Gamma_n} p_I, \qquad n \geq 0, \tag{6.3}$$

when $p_I \in A$ is as defined in (4.2).

Lemma 6.1 *There is an injective function $t\colon \Gamma \to \mathbb{N}$ such that $t(I) \in I$ for all $I \in \Gamma$. It follows in particular that*

$$\Big|\bigcup_{I\in F} I\Big| \geq |F|$$

for all finite subsets F of Γ.

Proof: Define t recursively on each Γ_n as follows. For $n = 0$ we set $t(\{1\}) = 1$. Assume that t has been defined on Γ_{n-1} for some $n \geq 1$. Then define t on Γ_n by $t(\alpha_j(I)) = \nu(j, t(I))$ for $I \in \Gamma_{n-1}$ and $j \in \mathbb{Z}$. It follows from Lemma 5.4 that

$$t(I) \in I \implies t(\alpha_j(I)) \in \alpha_j(I), \qquad I \in \Gamma,\ j \in \mathbb{Z}.$$

It therefore follows by induction that $t(I) \in I$ for all $I \in \Gamma$.

We show next that $t(I) \in \Lambda_n$ if $I \in \Gamma_n$. This is clear for $n = 0$. Let $n \geq 1$ and let $I \in \Gamma_n$ be given. Then $I = \alpha_j(I')$ for some $I' \in \Gamma_{n-1}$ and some $j \in \mathbb{Z}$. It follows that $t(I) = t(\alpha_j(I')) = \nu(j, t(I'))$. Hence $t(I) \in \Lambda_n$ if $t(I') \in \Lambda_{n-1}$, cf. (6.2). Now the claim follows by induction on n.

We proceed to show that t is injective. If $I, J \in \Gamma$ are such that $t(I) = t(J)$, then $t(I) = t(J) \in \Lambda_n$ for some n, whence I, J both belong to Γ_n. It therefore suffices to show that $t|_{\Gamma_n}$ is injective for each n. We prove this by induction on n. It is trivial that $t|_{\Gamma_0}$ is injective. Assume that $t|_{\Gamma_{n-1}}$ is injective for some $n \geq 1$. Let $I, J \in \Gamma_n$ be such that $t(I) = t(J)$. Then $I = \alpha_i(I')$ and $J = \alpha_j(J')$ for some $i, j \in \mathbb{Z}$ and some $I', J' \in \Gamma_{n-1}$, and

$$\nu(i, t(I')) = t(\alpha_i(I')) = t(I) = t(J) = t(\alpha_j(J')) = \nu(j, t(J')).$$

Since ν is injective we deduce that $i = j$ and $t(I') = t(J')$. By injectivity of $t|_{\Gamma_{n-1}}$ we obtain $I' = J'$, and this proves that $I = J$. It has now been shown that $t|_{\Gamma_n}$ is injective, and the induction step is complete. $\square$

Let $g \in A = C(Z, \mathcal{K})$ be a constant 1-dimensional projection, and let Q_n be as defined above (6.3).

Lemma 6.2 *For each natural number m we have*

$$g \not\precsim Q_0 \oplus Q_1 \oplus \cdots \oplus Q_m \quad \textit{in } \mathcal{M}(A).$$

Proof: From (6.3) (and Lemma 4.2) we deduce that

$$Q_0 \oplus Q_1 \oplus \cdots \oplus Q_n \sim \bigoplus_{I \in \Gamma_0 \cup \cdots \cup \Gamma_n} p_I.$$

The claim of the lemma now follows from Proposition 4.5 (i) together with Lemma 6.1. $\square$

As in Theorem 5.6 consider the inductive limit

$$\mathcal{M}(A) \xrightarrow{\overline{\varphi}} \mathcal{M}(A) \xrightarrow{\overline{\varphi}} \mathcal{M}(A) \xrightarrow{\overline{\varphi}} \cdots \longrightarrow B, \tag{6.4}$$

where $A = C(Z) \otimes \mathcal{K}$. Let $\mu_{\infty,n} \colon \mathcal{M}(A) \to B$ be the inductive limit map (from the nth copy of $\mathcal{M}(A)$) for $n \geq 0$, and let $\mu_{m,n} \colon \mathcal{M}(A) \to \mathcal{M}(A)$ be the connecting map from the nth copy of of $\mathcal{M}(A)$ to the mth copy of $\mathcal{M}(A)$ for $n < m$, i.e., $\mu_{m,n} = \overline{\varphi}^{(m-n)}$. The endomorphism $\overline{\varphi}$ on $\mathcal{M}(A)$ extends to an automorphism α on B that satisfies $\alpha(\mu_{\infty,n}(x)) = \mu_{\infty,n}(\overline{\varphi}(x))$ for $x \in \mathcal{M}(A)$ and all $n \in \mathbb{N}$. (The inverse of α is on the dense subset $\bigcup_{n=0}^{\infty} \mu_{\infty,n}(\mathcal{M}(A))$ of B given by $\alpha^{-1}(\mu_{\infty,n}(x)) = \mu_{\infty,n+1}(x)$.)

Put $A_0 = \mu_{\infty,0}(A) \subseteq B$, put $A_n = \alpha^n(A_0) \subseteq B$ for all $n \in \mathbb{Z}$, and put

$$D_n = C^*(A_{-n}, A_{-n+1}, \ldots, A_0, \ldots A_{n-1}, A_n), \qquad D = \overline{\bigcup_{n=1}^{\infty} D_n}. \tag{6.5}$$

It is shown in Lemma 6.6 below that each D_n is a type I C^*-algebra, and so the C^*-algebra D is an inductive limit of type I algebras. In particular, D is nuclear and belongs to the UCT class $\mathcal{N}$. Moreover, D is α-invariant (by construction). Observe that $A_{m-n} = \mu_{\infty,n}(\overline{\varphi}^m(A))$ for all non-negative integers m and n.

Put $Q = \mu_{\infty,0}(p_1)$ $(= \mu_{\infty,n}(Q_n))$ in $D \subseteq B$, and, as above, let $g \in A = C(Z, \mathcal{K})$ be a constant 1-dimensional projection.

Lemma 6.3 *The following two relations hold in D and in B:*

(i) $\mu_{\infty,0}(g) \precsim Q \oplus Q$.

(ii) $\mu_{\infty,0}(g) \not\precsim \bigoplus_{j=-N}^{N} \alpha^j(Q)$ *for all natural numbers N.*

Proof: (i) follows immediately from Proposition 4.5 (ii).

(ii). Assume, to reach a contradiction, that $\mu_{\infty,0}(g) \precsim \sum_{j=-N}^{N} \alpha^j(Q)$ in B (or in D) for some $N \in \mathbb{N}$. For $j \geq -N$ we have

$$\alpha^j(Q) = \alpha^j(\mu_{\infty,0}(Q_0)) = \alpha^j(\mu_{\infty,N}(\overline{\varphi}^N(Q_0))) = \mu_{\infty,N}(\overline{\varphi}^{N+j}(Q_0)).$$

The relation $\mu_{\infty,0}(g) \precsim \sum_{j=-N}^{N} \alpha^j(Q)$ can therefore be rewritten as

$$\mu_{\infty,N}(\overline{\varphi}^N(g)) \;\precsim\; \bigoplus_{j=0}^{2N} \mu_{\infty,N}(\overline{\varphi}^j(Q_0)) \quad \text{in } B.$$

By a standard property of inductive limits this entails that

$$\mu_{M,N}(\overline{\varphi}^N(g)) \;\precsim\; \bigoplus_{j=0}^{2N} \mu_{M,N}(\overline{\varphi}^j(Q_0)) \quad \text{in } \mathcal{M}(A),$$

for some $M \geq N$, or, equivalently,

$$\overline{\varphi}^M(g) \;\precsim\; \bigoplus_{j=0}^{2N} \overline{\varphi}^{j+M-N}(Q_0) \;=\; \bigoplus_{j=M-N}^{N+M} \overline{\varphi}^j(Q_0) \;=\; \bigoplus_{j=M-N}^{N+M} Q_j \;\precsim\; \bigoplus_{j=0}^{N+M} Q_j \quad \text{in } \mathcal{M}(A).$$

Use now that $g \precsim \overline{\varphi}^M(g)$ (which holds because $\varphi_j(g) = g$ for $j \leq 0$, cf. (5.3)) to conclude that $g \precsim \bigoplus_{j=0}^{N+M} Q_j$ in $\mathcal{M}(A)$, in contradiction with Lemma 6.2. □

Let C be an arbitrary unital C^*-algebra and let γ be an automorphism on C.

Let $\mathcal{K}$ denote the compact operators on $\ell^2(\mathbb{Z})$ and let $\{e_{i,j}\}_{i,j\in\mathbb{Z}}$ be a set of matrix units for $\mathcal{K}$. Define a unital injective *-homomorphism $\psi \colon C \to \mathcal{M}(C \otimes \mathcal{K})$ and a unitary $U \in \mathcal{M}(C \otimes \mathcal{K})$ by

$$\psi(c) = \sum_{n\in\mathbb{Z}} \gamma^n(c) \otimes e_{n,n}, \qquad U = \sum_{n\in\mathbb{Z}} 1 \otimes e_{n,n+1}, \qquad c \in C,$$

(the sums converge strictly in $\mathcal{M}(C \otimes \mathcal{K})$). It is easily seen that

$$U\psi(c)U^* = \psi(\gamma(c)), \qquad c \in C,$$

so that ψ extends to a representation $\widetilde{\psi} \colon C \rtimes_\gamma \mathbb{Z} \to \mathcal{M}(C \otimes \mathcal{K})$. The following standard argument shows that the representation $\widetilde{\psi}$ is faithful.

Put $V_t = \sum_{n\in\mathbb{Z}} 1 \otimes t^{-n} e_{n,n} \in \mathcal{M}(C \otimes \mathcal{K})$ for $t \in \mathbb{T}$, and check that V_t is a unitary element that satisfies $V_t\psi(c)V_t^* = \psi(c)$ and $V_tUV_t^* = tU$ for all $t \in \mathbb{T}$. Let $E \colon C \rtimes_\gamma \mathbb{Z} \to$

C be the canonical faithful conditional expectation, and define $F\colon \mathrm{Im}(\widetilde{\psi}) \to \mathrm{Im}(\widetilde{\psi})$ by $F(x) = \int_{\mathbb{T}} V_t x V_t^* \, dt$. Then $F(\widetilde{\psi}(x)) = \psi(E(x))$ for all $x \in C \rtimes_\gamma \mathbb{Z}$. Now, if $\widetilde{\psi}(x) = 0$ for some positive element x in $C \rtimes_\gamma \mathbb{Z}$, then $\psi(E(x)) = F(\widetilde{\psi}(x)) = 0$, whence $E(x) = 0$ (by injectivity of ψ), and $x = 0$ (because E is faithful).

Lemma 6.4 *Let C be a unital C^*-algebra and let γ be an automorphism on C. Suppose that p, q are projections in C such that*

(i) *$p \precsim \bigoplus_{j=1}^m q$ in C for some natural number m, and*

(ii) *$p \not\precsim \bigoplus_{j=-N}^N \gamma^j(q)$ for all natural numbers N.*

Then q is not properly infinite in $C \rtimes_\gamma \mathbb{Z}$.

Proof: It suffices to show that $\psi(q)$ is not properly infinite in $\mathcal{M}(C \otimes \mathcal{K})$. Assume, to reach a contradiction, that $\psi(q)$ is properly infinite in $\mathcal{M}(C \otimes \mathcal{K})$. Then $\bigoplus_{j=1}^m \psi(q) \precsim \psi(q)$ by Proposition 2.1. As $q \otimes e_{0,0} \le \psi(q)$ we can use (i) to obtain

$$p \otimes e_{0,0} \;\precsim\; \bigoplus_{j=1}^m q \otimes e_{0,0} \;\le\; \bigoplus_{j=1}^m \psi(q) \;\precsim\; \psi(q) \;=\; \sum_{j=-\infty}^{\infty} \gamma^j(q) \otimes e_{j,j}$$

in $\mathcal{M}(C \otimes \mathcal{K})$. By Lemma 4.4 this entails that

$$p \otimes e_{0,0} \;\precsim\; \sum_{j=-N}^{N} \gamma^j(q) \otimes e_{j,j} \quad \text{in } C \otimes \mathcal{K},$$

for some $N \in \mathbb{N}$, or, equivalently, that $p \precsim \bigoplus_{j=-N}^N \gamma^j(q)$ in C, in contradiction with assumption (ii). $\square$

Returning now to our specific C^*-algebra B from (6.4), Lemmas 6.3 and 6.4 imply that:

Lemma 6.5 *The projection $Q = \mu_{\infty,0}(p_1)$ is not properly infinite in $B \rtimes_\alpha \mathbb{Z}$.*

Lemma 6.6 *The C^*-algebra $D_n = C^*(A_{-n}, A_{-n+1}, \dots, A_0, \dots, A_n)$ is of type I for each $n \in \mathbb{N}$.*

Proof: Note first that

$$A_n A_m \subseteq A_{\min\{n,m\}}, \qquad n, m \in \mathbb{Z}. \tag{6.6}$$

Indeed, we can assume without loss of generality that $n \le m$, and then deduce

$$A_n A_m = \alpha^n(\mu_{\infty,0}(A\overline{\varphi}^{m-n}(A))) \subseteq \alpha^n(\mu_{\infty,0}(A)) = A_n.$$

Since $A \cap \overline{\varphi}^{m-n}(A) = \{0\}$ when $n < m$, cf. Proposition 5.2 (ii), it follows also that

$$A_n \cap A_m = \{0\}, \qquad n \neq m. \tag{6.7}$$

Use (6.6) to see that the C^*-algebra $D_{m,n}$ generated by $A_m, A_{m+1}, \dots, A_n$, for $m \le n$, is equal to

$$D_{m,n} = A_m + A_{m+1} + \cdots + A_{n-1} + A_n. \tag{6.8}$$

(To see that the right-hand side of (6.8) is norm closed, use successively the fact that if E is a C^*-algebra, I is a closed two-sided ideal in E, and F is a sub-C^*-algebra of E, then $I + F$ is a sub-C^*-algebra of E.) It follows from (6.6), (6.7), and (6.8) that we have a decomposition series

$$0 \ \lhd \ A_{-n} \ \lhd \ D_{-n,-n+1} \ \lhd \ D_{-n,-n+2} \ \lhd \ \cdots \ \lhd \ D_{-n,n-1} \lhd \ D_{-n,n} = D_n$$

for D_n and that each successive quotient is isomorphic to $A = C(Z) \otimes \mathcal{K}$. This proves that D_n is a type I C^*-algebra. $\square$

Lemma 6.7 *The crossed product C^*-algebra $D \rtimes_\alpha \mathbb{Z}$ contains an infinite projection and a non-zero projection which is not properly infinite. The C^*-algebra D has no non-trivial α^n-invariant closed two-sided ideal for any non-zero integer n.*

Proof: The projection $Q = \mu_{\infty,0}(p_1)$ belongs to $A_0 = \mu_{\infty,0}(A) \subseteq D$, and it is non-zero because $\mu_{\infty,0}$ is injective (which again is because $\overline{\varphi}$ is injective). We have $D \subseteq B$ and hence

$$Q \in D \rtimes_\alpha \mathbb{Z} \subseteq B \rtimes_\alpha \mathbb{Z}.$$

Since Q is not properly infinite in $B \rtimes_\alpha \mathbb{Z}$ (by Lemma 6.5) it follows that Q is not properly infinite in $D \rtimes_\alpha \mathbb{Z}$.

Put $P = \mu_{\infty,0}(g) \in A_0 \subseteq D$, where g is a constant 1-dimensional projection in $A = C(Z, \mathcal{K})$. We have

$$g = \varphi_0(g) \sim S_0\varphi_0(g)S_0^* < \sum_{j=-\infty}^{\infty} S_j\varphi_j(g)S_j^* = \overline{\varphi}(g),$$

cf. (5.3). Hence $P = \mu_{\infty,0}(g)$ is equivalent to a proper subprojection of $\mu_{\infty,0}(\overline{\varphi}(g))$. As $\mu_{\infty,0}(\overline{\varphi}(g)) = \alpha(\mu_{\infty,0}(g)) \sim P$ in $D \rtimes_\alpha \mathbb{Z}$ we conclude that P is an infinite projection in $D \rtimes_\alpha \mathbb{Z}$.

Suppose that n is a non-zero integer (that we can take to be positive) and that I is a non-zero closed two-sided α^n-invariant ideal in D. Then $I \cap D_{kn}$ is non-zero for some natural number k, cf. (6.5). As I is α^n-invariant, $I \cap \alpha^{kn}(D_{kn})$ is non-zero, and

$$\alpha^{kn}(D_{kn}) = C^*(A_0, A_1, \dots, A_{2kn}) = \mu_{\infty,0}\big(C^*(A, \overline{\varphi}(A), \dots, \overline{\varphi}^{2kn}(A))\big).$$

Because $A_0 = \mu_{\infty,0}(A)$ is an essential ideal in $\alpha^{kn}(D_{kn})$ it follows that $I \cap A_0$ is non-zero. Take a non-zero element f in $I \cap A_0$, and write $f = \mu_{\infty,0}(f_0)$ for some non-zero element f_0 in A. Use Proposition 5.2 (iii) to conclude that

$$A_{-m} f = \mu_{\infty,m}\big(A\overline{\varphi}^m(f_0)\big)$$

is full in $\mu_{\infty,m}(A) = A_{-m}$, and hence that $A_{-m} \subseteq I$, for every natural number m. Since I is α^n-invariant, $A_{-m+rn} = \alpha^{rn}(A_{-m}) \subseteq I$ for all $m \in \mathbb{N}$ and all $r \in \mathbb{Z}$. This shows that $A_m \subseteq I$ for all m, which finally entails that $I = D$. $\square$

We remind the reader of the notion of properly outer automorphism introduced by Elliott in [19]:

Definition 6.8 An automorphism γ on a C^*-algebra E is called *properly outer* if for every non-zero γ-invariant closed two-sided ideal I of E and for every unitary u in $\mathcal{M}(I)$ one has $\|\gamma|_I - \operatorname{Ad} u\| = 2$ (the norm is the operator norm).

Olesen and Pedersen list in [34, Theorem 6.6] eleven conditions on an automorphism γ that all are equivalent to γ being properly outer. We shall use the following sufficient (but not necessary) condition for being properly outer: If E has no non-trivial γ-invariant ideals and if $\gamma(p) \not\sim p$ for some projection p in E, then γ is properly outer. To see this, note first that $p \sim upu^* = (\operatorname{Ad} u)(p)$ for every unitary u in $\mathcal{M}(E)$ (the equivalence holds relatively to E). We therefore have $\gamma(p) \not\sim (\operatorname{Ad} u)(p)$, whence $\|\gamma(p) - (\operatorname{Ad} u)(p)\| = 1$. This shows that $\|\gamma - \operatorname{Ad} u\| \geq 1$ for all unitaries u in $\mathcal{M}(E)$, whence γ is properly outer (by (ii) $\Leftrightarrow$ (iii) of [34, Theorem 6.6]).

(One can argue along another line by taking an approximate unit $\{e_\lambda\}$ for E, such that $e_\lambda \geq p$ for all λ, and set $x_\lambda = 2p - e_\lambda$. Then x_λ is a contraction in E for all λ, and one can check that $\lim_{\lambda\to\infty} \|\gamma(x_\lambda) - (\operatorname{Ad} u)(x_\lambda)\| = 2$, thus showing directly that $\|\gamma - \operatorname{Ad} u\| = 2$ for all unitaries u in $\mathcal{M}(E)$ whenever $\gamma(p) \not\sim p$ for some projection p in E.)

More generally, γ is properly outer if for each non-zero γ-invariant ideal I of E there is a projection p in I such that $\gamma(p) \not\sim p$.

Lemma 6.9 *The automorphism α^n on D is properly outer for every non-zero integer n.*

Proof: We know from Lemma 6.7 that D has no α^n-invariant ideals (when $n \neq 0$), so the lemma will follow from the claim (verified below) that $\alpha^n(Q) \not\sim Q$ for all $n \neq 0$ (where Q is as in Lemma 6.3).

Assume, to reach a contradiction, that $\alpha^n(Q) \sim Q$ for some non-zero integer n (that we can take to be positive). Then, by Lemma 6.3 (i), $\mu_{\infty,0}(g) \precsim Q \oplus Q \sim Q \oplus \alpha^n(Q) \precsim \bigoplus_{j=0}^n \alpha^j(Q)$ in D, in contradiction with Lemma 6.3 (ii). $\square$

We now have all ingredients to prove our main result:

Theorem 6.10 *There is a separable C^*-algebra D and an automorphism α on D such that:*

(i) *D is an inductive limit of type I C^*-algebras.*

(ii) *$D \rtimes_\alpha \mathbb{Z}$ is simple and contains an infinite and a non-zero finite projection.*

(iii) *$D \rtimes_\alpha \mathbb{Z}$ is nuclear and belongs to the UCT class $\mathcal{N}$.*

Proof: Let D be the C^*-algebra and let α the automorphism on D defined in (and above) (6.5). Since D is the union of an increasing sequence of sub-C^*-algebras D_n (cf. (6.5)) and each D_n is of type I (by Lemma 6.6), we conclude that D is an inductive limit of type I C^*-algebras, and hence that the crossed product $D \rtimes_\alpha \mathbb{Z}$ is nuclear, separable, and belongs to the UCT class $\mathcal{N}$.

Since D has no non-trivial α-invariant ideals (by Lemma 6.7) and α^n is properly outer for all $n \neq 0$ (by Lemma 6.9), it follows from Olesen and Pedersen, [34, Theorem 7.2], (a result that extends results from Elliott, [19], and Kishimoto, [31]) that $D \rtimes_\alpha \mathbb{Z}$ is simple. By simplicity of $D \rtimes_\alpha \mathbb{Z}$, the (non-zero) projection Q, which in Lemma 6.7 is proved to be not properly infinite, must be finite in $D \rtimes_\alpha \mathbb{Z}$, cf. Proposition 2.1. The existence of an infinite projection in $D \rtimes_\alpha \mathbb{Z}$ follows from Lemma 6.7, and this completes the proof. $\square$

7 Applications of the main results

We begin by listing some corollaries to Theorems 5.6 and 6.10.

Corollary 7.1 *There is a nuclear, unital, separable, infinite, simple C^*-algebra A in the UCT class $\mathcal{N}$ such that A is not purely infinite.*

Proof: Take the C^*-algebra $D \rtimes_\alpha \mathbb{Z}$ from Theorem 6.10, and take a properly infinite projection p and a non-zero finite projection q in that C^*-algebra. Then $q \sim q_0 \leq p$ for some projection q_0 in $D \rtimes_\alpha \mathbb{Z}$ by Lemma 2.2. Hence $A = p(D \rtimes_\alpha \mathbb{Z})p$ is infinite; and A is not purely infinite because it contains the non-zero finite projection q_0. $\square$

Corollary 7.2 *There is a nuclear, unital, separable, finite, simple C^*-algebra A that is not stably finite, and hence does not admit a tracial state (nor a non-zero quasitrace).*

Proof: Take the C^*-algebra $E = D \rtimes_\alpha \mathbb{Z}$ from Theorem 6.10 and a non-zero finite projection q in E. Put $A = qEq$. Then A is finite, simple, and unital. Since $A \otimes \mathcal{K} \cong E \otimes \mathcal{K}$ we conclude that $A \otimes \mathcal{K}$ (and hence $M_n(A)$ for some large enough n) contains an infinite projection, so A is not stably finite.

Every simple, infinite C^*-algebra is properly infinite, so $M_n(A)$ is properly infinite. No properly infinite C^*-algebra can admit a non-zero trace (or a quasitrace), so $M_n(A)$, and hence A, do not admit a tracial state (nor a non-zero quasitrace). $\square$

A C^*-algebra A is said to have the *cancellation property* if the implication

$$p \oplus r \sim q \oplus r \implies p \sim q \tag{7.1}$$

holds for all projections p, q, r in $A \otimes \mathcal{K}$. It is known that all C^*-algebras of stable rank one have the cancellation property and that no infinite C^*-algebra has the cancellation property. There is no example of a stably finite, simple C^*-algebra which is known not to have the cancellation property (but Villadsen's C^*-algebras from [42] are candidates). A C^*-algebra A is said to have the *weak cancellation property* if (7.1) holds for those projections p, q, r in $A \otimes \mathcal{K}$ where p and q generate the same ideal of A.

Corollary 7.3 *There is a nuclear, unital, separable, simple C^*-algebra A that does not have the weak cancellation property.*

Proof: Take A as in Corollary 7.1, and take a non-zero finite projection q in A. Since A is properly infinite, we can find isometries s_1, s_2 in A with orthogonal range projections; cf. Proposition 2.1. Put $p = s_1 q s_1^* + (1 - s_1 s_1^*)$. Then p is infinite because $s_2 s_2^* \leq p$, and so $p \not\sim q$ (because q is finite). On the other hand, q and p generate the same ideal of A—namely A itself—and

$$p \oplus 1 \;=\; \big(s_1 q s_1^* + (1 - s_1 s_1^*)\big) \oplus 1 \;\sim\; s_1 q s_1^* \oplus (1 - s_1 s_1^*) \oplus s_1 s_1^* \;\sim\; q \oplus 1.$$

□

It was shown in [30, Theorem 9.1] that the following implications hold for any separable C^*-algebra A and for any free filter ω on $\mathbb{N}$:

$$\begin{aligned} A \text{ is purely infinite} \quad &\Longrightarrow \quad A \text{ is weakly purely infinite} \\ &\Longleftrightarrow \quad A_\omega \text{ is traceless} \\ &\Longrightarrow \quad A \text{ is traceless,} \end{aligned}$$

and the first three properties are equivalent for all *simple* C^*-algebras A. (A C^*-algebra is here said to be *traceless* if no algebraic ideal in A admits a non-zero quasitrace. See [30] for the definition of being weakly purely infinite.) It was not known in [30] if the reverse of the third implication holds (for simple or for non-simple C^*-algebras), but we can now answer this in the negative:

Corollary 7.4 *Let ω be any free filter on $\mathbb{N}$. There is a nuclear, unital, separable, simple C^*-algebra A which is traceless, but where $\ell^\infty(A)$ and A_ω admit non-zero quasitraces defined on some (possibly non-dense) algebraic ideal.*

Proof: Take A as in Corollary 7.2. Then A is algebraically simple and A admits no (everywhere defined) non-zero quasitrace. Hence A is traceless in the sense of [30]. Because A is simple and not purely infinite, A_ω cannot be traceless. Since A_ω is a quotient of $\ell^\infty(A)$, the latter C^*-algebra cannot be traceless either. □

Kirchberg has shown in [26] (see also [39, Theorem 4.1.10]) that every exact simple C^*-algebra which is *tensorially non-prime* (i.e., is isomorphic to a tensor product $D_1 \otimes D_2$, where D_1 and D_2 both are simple non-type I C^*-algebras) is either stably finite or purely infinite. Liming Ge has proved in [21] that the II_1-factor $\mathcal{L}(\mathbb{F}_2)$ is (tensorially) prime (in the von Neumann algebra sense), and it follows easily from this result that the C^*-algebra $C^*_{\mathrm{red}}(\mathbb{F}_2)$ is tensorially prime. We can now exhibit a simple, *nuclear* C^*-algebra that is tensorially prime:

Corollary 7.5 *The C^*-algebra $D \rtimes_\alpha \mathbb{Z}$ from Theorem 6.10 is simple, separable, nuclear, and tensorially prime, and so is $p(D \rtimes_\alpha \mathbb{Z})p$ for every non-zero projection p in $D \rtimes_\alpha \mathbb{Z}$.*

Proof: The C^*-algebra $D \rtimes_\alpha \mathbb{Z}$ is simple, separable, nuclear; cf. Theorem 6.10. It is not stably finite because it contains an infinite projection, and it is not purely infinite because it contains a non-zero finite projection. The (unital) C^*-algebra $p(D \rtimes_\alpha \mathbb{Z})p$ is stably isomorphic to $D \rtimes_\alpha \mathbb{Z}$ and is hence also simple, separable, nuclear, and neither stably finite nor purely infinite. It therefore follows from Kirchberg's theorem (quoted above) that these C^*-algebras must be tensorially prime. $\square$

Villadsen's C^*-algebras from [41] and [42] are, besides being simple and nuclear, probably also tensorially prime (although to the knowledge of the author this has not yet been proven). Jiang and Su have in [25] found a non-type I, unital, simple C^*-algebra $\mathcal{Z}$ for which $A \cong A \otimes \mathcal{Z}$ is known to hold for a large class of well-behaved simple C^*-algebras A, such as for example the irrational rotation C^*-algebras and more generally all C^*-algebras that are covered by a classification theorem (cf. [20] or [39]). Such C^*-algebras A are therefore not tensorially prime.

The real rank of the C^*-algebras found in Theorems 5.6 and 6.10 have not been determined, but we guess that they have real rank ≥ 1. That leaves open the following question:

Question 7.6 Does there exist a (separable) unital, simple C^*-algebra A such that A contains an infinite and a non-zero finite projection, and such that:

(i) A is of real rank zero?

(ii) A is both nuclear and of real rank zero?

It appears to be difficult (if not impossible) to construct simple C^*-algebras *of real rank zero* that exhibit bad comparison properties; cf. Remark 7.8 below.

George Elliott suggested the following:

Question 7.7 Does there exist a (separable), (nuclear), unital, simple C^*-algebra A such that all non-zero projections in A are infinite but A is not purely infinite?

If Question 7.7 has affirmative answer, and A is a unital, simple C^*-algebra whose non-zero projections are infinite and A is not purely infinite, then the real rank of A cannot be zero. Indeed, a simple C^*-algebra is purely infinite if and only if it has real rank zero and all its non-zero projections are infinite.

Remark 7.8 (Comparison and dimension ranges) Suppose that A is a unital, simple, infinite C^*-algebra with a non-zero finite projection e. By simplicity of A there is a natural number k such that $1 \precsim e \oplus e \oplus \cdots \oplus e$ (with k copies of e). Let $s_1, s_2, \ldots$ be a sequence of isometries in A with orthogonal range projections; cf. Proposition 2.1. Letting $[p]$ denote the Murray–von Neumann equivalence class of the projection p, we have

$$n[1] = [s_1 s_1^* + s_2 s_2^* + \cdots + s_n s_n^*] \leq [1] \leq k[e]$$

for every natural number n. But $[1] \not\leq [e]$ because e is finite and 1 is infinite.

This shows that if A is a simple C^*-algebra with a finite and an infinite projection, then the semigroup $\mathcal{D}(A)$ of Murray–von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$ is not weakly unperforated.

(An ordered abelian semigroup $(S, +, \leq)$ is said to be *weakly unperforated* if

$$\forall\, g, h \in S \;\forall\, n \in \mathbb{N} : ng < nh \implies g \leq h.$$

The order structure on $\mathcal{D}(A)$ is the algebraic order given by $g \leq h$ if and only if $h = g + f$ for some f in $\mathcal{D}(A)$.)

Villadsen showed in [41] that $K_0(A)$, and also the semigroup $\mathcal{D}(A)$, of a *simple, stably finite* C^*-algebra A can fail to be weakly unperforated. The present article is a natural continuation of Villadsen's work to the stably infinite case.

Let $(S, +)$ be an abelian semigroup with a zero-element 0. An element $g \in S$ is called *infinite* if $g + x = g$ for some non-zero $x \in S$, and g is called *finite* otherwise. The sets of finite, respectively, infinite elements in S are denoted by S_{fin} and S_{inf}. One has $S = S_{\text{fin}} \amalg S_{\text{inf}}$ and $S + S_{\text{inf}} \subseteq S_{\text{inf}}$, but the sum of two finite elements can be infinite.

It is standard and easy to see that the finite and infinite elements in the semigroup $\mathcal{D}(A)$ are given by

$$\begin{aligned} \mathcal{D}_{\text{fin}}(A) &= \{[f] : f \text{ is a finite projection in } A \otimes \mathcal{K}\}, \\ \mathcal{D}_{\text{inf}}(A) &= \{[f] : f \text{ is an infinite projection in } A \otimes \mathcal{K}\}. \end{aligned}$$

If A is a simple C^*-algebra that contains an infinite projection, then the Grothendieck map $\gamma\colon \mathcal{D}(A) \to K_0(A)$ restricts to an isomorphism $\mathcal{D}_{\text{inf}}(A) \to K_0(A)$ as shown by Cuntz in [16, Section 1]. We can therefore identify $\mathcal{D}_{\text{inf}}(A)$ with $K_0(A)$, in which case we can write

$$\mathcal{D}(A) = \mathcal{D}_{\text{fin}}(A) \amalg K_0(A).$$

Note that $[0]$ belongs to $\mathcal{D}_{\text{fin}}(A)$, and that $\mathcal{D}_{\text{fin}}(A) = \{[0]\}$ if and only if all non-zero projections in $A \otimes \mathcal{K}$ are infinite. One can therefore detect the existence of non-zero finite elements in $A \otimes \mathcal{K}$ from the semigroup $\mathcal{D}(A)$; and $K_0(A)$ contains all information about $\mathcal{D}(A)$ if and only if all non-zero projections in $A \otimes \mathcal{K}$ are infinite.

In general, when A is simple and contains both infinite and non-zero finite projections, then $\mathcal{D}_{\mathrm{fin}}(A)$ can be very complicated and large. One can show that $\mathcal{D}_{\mathrm{fin}}(B)$ is uncountable, when B is as in Theorem 5.6. We have no description of $\mathcal{D}(A)$, when $A = D \rtimes_\alpha \mathbb{Z}$ from Theorem 6.10.

We remark finally, that if A is simple and if g is a non-zero element in $\mathcal{D}_{\mathrm{fin}}(A)$, then $ng \in \mathcal{D}_{\mathrm{inf}}(A)$ for some $n \in \mathbb{N}$. In other words, $\mathcal{D}_{\mathrm{inf}}(A)$ eventually absorbs all non-zero elements in $\mathcal{D}(A)$.

The example found in Theorem 6.10 provides a counterexample to Elliott's classification conjecture (see for example [20]) as it is formulated (by the author) in [39, Section 2.2]. The conjecture asserts that

$$\big(K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A \colon T(A) \to S(K_0(A))\big) \tag{7.2}$$

is a complete invariant for unital, separable, nuclear, simple C^*-algebras. If A is stably infinite (i.e., if $A \otimes \mathcal{K}$ contains an infinite projection), then $K_0(A)^+ = K_0(A)$ and $T(A) = \emptyset$. The Elliott invariant for unital, simple, stably infinite C^*-algebras therefore degenerates to the triple $(K_0(A), [1_A]_0, K_1(A))$. (We say that $(K_0(A), [1_A]_0, K_1(A)) \cong (G_0, g_0, G_1)$ if there are group isomorphisms $\alpha_0 \colon K_0(A) \to G_0$ and $\alpha_1 \colon K_1(A) \to G_1$ such that $\alpha_0([1_A]_0) = g_0$.)

Corollary 7.9 *There are two non-isomorphic nuclear, unital, separable, simple, stably infinite C^*-algebras A and B (both in the UCT class $\mathcal{N}$) such that*

$$(K_0(A), [1_A]_0, K_1(A)) \;\cong\; (K_0(B), [1_B]_0, K_1(B)).$$

Proof: Take the C^*-algebra A from Corollary 7.1. It follows from [36, Theorem 3.6] that there is a nuclear, unital, separable, simple, purely infinite C^*-algebra B in the UCT class $\mathcal{N}$ such that

$$(K_0(A), [1_A]_0, K_1(A)) \;\cong\; (K_0(B), [1_B]_0, K_1(B)).$$

Since B is purely infinite and A is not purely infinite, we have $A \not\cong B$. $\square$

One can amend the Elliott invariant by replacing the triple $(K_0(A), K_0(A)^+, [1_A]_0)$ (for a unital C^*-algebra A) with the pair $(\mathcal{D}(A), [1_A])$, cf. Remark 7.8 above, where $\mathcal{D}(A)$ carries the structure of a semigroup. In the unital, stably infinite case, the amended invariant will then become $(\mathcal{D}(A), [1_A], K_1(A))$. (Since $K_0(A)$ is the Grothendieck group of $\mathcal{D}(A)$, and $K_0(A)^+$, respectively, $[1_A]_0$, are the images of $\mathcal{D}(A)$, respectively, $[1_A]$, under the Grothendieck map $\gamma\colon \mathcal{D}(A) \to K_0(A)$, one can recover $(K_0(A), K_0(A)^+, [1_A]_0)$ from $(\mathcal{D}(A), [1_A])$.)

The invariant $(\mathcal{D}(A), [1_A])$ can detect if A has a non-zero finite projection, cf. Remark 7.8; and the triples $(\mathcal{D}(A), [1_A], K_1(A))$ and $(\mathcal{D}(B), [1_B], K_1(B))$ are therefore non-isomorphic, when A and B are as in Corollary 7.9. We have no example to show that $(\mathcal{D}(A), [1_A], K_1(A))$ is not a complete invariant for nuclear, unital, simple, separable, stably infinite C^*-algebras. On the other hand, there is no evidence to suggests that $(\mathcal{D}(A), [1_A], K_1(A))$ indeed is a complete invariant for this class of C^*-algebras.

The Elliott conjecture can also be amended by restricting the class of C^*-algebras that are to be classified. One possibility is to consider only those unital, separable, nuclear, simple C^*-algebras A for which $A \cong A \otimes \mathcal{Z}$ where $\mathcal{Z}$ is the Jiang–Su algebra (see the comment below Corollary 7.5). It seems plausible that the Elliott invariant (7.2) actually is a complete invariant for this class of C^*-algebras; and one could hope that the condition $A \cong A \otimes \mathcal{Z}$ has an alternative intrinsic equivalent formulation, for example in terms of the existence of sufficiently many central sequences.

Remark 7.10 (A non-simple example) Examples of non-simple unital C^*-algebras A, such that A is finite and $M_2(A)$ is infinite, have been known for a long time. Such examples were independently discovered by Clarke in [9] and by Blackadar (see Blackadar [7, Exercise 6.10.1]): One such example is obtained by taking a unital extension

$$0 \longrightarrow \mathcal{K} \longrightarrow A \longrightarrow C(S^3) \longrightarrow 0$$

with non-zero index map $\delta\colon K_1(C(S^3)) \to K_0(\mathcal{K})$. Then A is finite and $M_2(A)$ is infinite.

The proof uses that any isometry or co-isometry s in A (or in a matrix algebra over A) is mapped to a unitary element u in (a matrix algebra over) $C(S^3)$; and every unitary u in $M_n(C(S^3))$ lifts to an isometry or a co-isometry s in $M_n(A)$. Moreover, the isometry or co-isometry s is non-unitary if and only if the unitary element u has non-zero index. The unitary group of $C(S^3)$ is connected, so all unitaries here have zero index. Hence A contains no non-unitary isometry, so A is finite. By construction of the extension, the generator of $K_1(C(S^3))$, which is a unitary element in $M_2(C(S^3))$, has non-zero index, and so it lifts to a non-unitary isometry or co-isometry in $M_2(A)$, whence $M_2(A)$ is infinite.

The C^*-algebra $M_2(A)$ is not properly infinite since the quotient, $M_2(A)/M_2(\mathcal{K}) \cong M_2(C(S^3))$, is finite.

An example of a unital, finite, (non-simple) C^*-algebra A such that $M_2(A)$ is properly infinite was found in [38].

Remark 7.11 (Inductive limits) Suppose that

$$B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow \cdots \longrightarrow B$$

is an inductive limit with unital connecting maps, and that B is a simple C^*-algebra such that B is finite and $M_2(B)$ is infinite. Then $M_2(B)$ is properly infinite, and it follows from Proposition 2.3 that B_n is finite and $M_2(B_n)$ is properly infinite for all sufficiently large n. It is therefore not possible to construct an example of a simple C^*-algebra, which is finite, but not stably finite, by taking an inductive limit of C^*-algebras arising as in the example described in Remark 7.10.

Remark 7.12 (Free products) Let B be a simple, unital C^*-algebra such that B is finite and $M_2(B)$ is infinite. Then we have unital *-homomorphisms

$$\varphi_1 \colon M_2(\mathbb{C}) \to M_2(B), \qquad \varphi_2 \colon \mathcal{O}_\infty \to M_2(B),$$

such that $\varphi_1(e)$ is a finite projection in $M_2(B)$ whenever e is a one-dimensional projection in $M_2(\mathbb{C})$.

The existence of B (already obtained in the non-simple case in [38]) shows that the image of e in the universal unital free product C^*-algebra $M_2(\mathbb{C}) * \mathcal{O}_\infty$ is not properly infinite.

It is tempting to turn this around and seek a simple C^*-algebra A with a finite and an infinite projection by defining A to be a suitable free product of $M_2(\mathbb{C})$ and $\mathcal{O}_\infty$. However, the universal unital free product $M_2(\mathbb{C}) * \mathcal{O}_\infty$ is not simple. The reduced free product C^*-algebra

$$(A, \rho) = (M_2(\mathbb{C}), \rho_1) * (\mathcal{O}_\infty, \rho_2),$$

with respect to faithful states ρ_1 and ρ_2, is simple (at least for many choices of the states ρ_1 and ρ_2, see for example [2]) and properly infinite, but no non-zero projection e in $M_2(\mathbb{C})$ is finite in A. The Cuntz algebra $\mathcal{O}_\infty$ contains a sequence of non-zero mutually orthogonal projections, and it therefore contains a projection f with $\rho_2(f) < \rho_1(e)$. Now, e and f are free with respect to the state ρ and $\rho(f) < \rho(e)$. This implies that $f \precsim e$ (see [1]), and therefore e must be infinite.

It is shown in [18] that reduced free product C^*-algebras often have weakly unperforated K_0-groups, which is another reason why this class of C^*-algebras is unlikely to provide an example of a simple C^*-algebra with finite and infinite projections; cf. Remark 7.8.

We conclude this article by remarking that ring theorists for a long time have known about finite simple *rings* that are not stably finite:

Remark 7.13 (An example from ring theory) A unital ring R is called *weakly finite* if $xy = 1$ implies $yx = 1$ for all x, y in R, and R is called *weakly n-finite* if $M_n(R)$ is weakly finite. (A finite ring is a ring with finitely many elements!) A (unital) non-weakly finite simple ring R is properly infinite in the sense that there are idempotents e, f in R such that $1 \sim e \sim f$ and $ef = fe = 0$. (Equivalence of idempotents is given by $e \sim f$ if and only if $e = xy$ and $f = yx$ for some x, y in R.)

An example of a unital, simple ring which is weakly finite but not weakly 2-finite was constructed by P. M. Cohn as follows:

Take natural numbers $2 \leq m < n$ and consider the universal ring $V_{m,n}$ generated by $2mn$ elements $\{x_{ij}\}$ and $\{y_{ji}\}$, $i = 1, \dots, m$ and $j = 1, \dots, n$, satisfying the relations $XY = I_m$ and $YX = I_n$, where $X = (x_{ij}) \in M_{m,n}(R)$, $Y = (y_{ij}) \in M_{n,m}(R)$, and I_m and I_n are the units of the matrix rings $M_m(R)$ and $M_n(R)$. The rings $M_m(V_{m,n})$ and $M_n(V_{m,n})$ are isomorphic and $M_n(V_{m,n})$ is not weakly finite. Therefore $M_m(V_{m,n})$ is not weakly finite. In other words, $V_{m,n}$ is not weakly m-finite.

It is shown by Cohn in [11, Theorem 2.11.1] (see also the remarks at the end of Section 2.11 of that book) that $V_{m,n}$ is a so-called $(m-1)$-*fir*, and hence a 1-fir; and a ring is a 1-fir if and only if it is an integral domain (i.e., if it has no non-zero zero-divisors). Cohn proved in [10] that every integral domain embeds into a simple integral domain. In particular, $V_{m,n}$ is a subring of a simple integral domain $R_{m,n}$ whenever $2 \leq m < n$. Now, $R_{m,n}$ is weakly finite (an integral domain has no idempotents other than 0 and 1 and must hence be weakly finite), and $R_{m,n}$ is not weakly m-finite (because it contains $V_{m,n}$).

This example cannot in any obvious way be carried over to C^*-algebras, first of all because no C^*-algebra other than $\mathbb{C}$ is an integral domain.

References

[1] J. Anderson, B. Blackadar, and U. Haagerup, *Minimal projections in the reduced group C^*-algebra of $\mathbb{Z}_n * \mathbb{Z}_m$*, J. Operator Theory **26** (1991), 3–23.

[2] D. Avitzour, *Free products of C^*-algebras*, Trans. Amer. Math. Soc. (1982), 423–465.

[3] B. Blackadar, *Weak expectations and nuclear C^*-algebras*, Indiana Univ. J. Math. **27** (1978), 1021–1026.

[4] ———, *A simple C^*-algebra with no non-trivial projections*, Proc. Amer. Math. Soc. **78** (1980), 504–508.

[5] ———, *A simple unital projectionless C^*-algebra*, J. Operator Theory **5** (1981), 63–71.

[6] ———, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), no. 2, 249–275.

[7] ———, *K-Theory for Operator Algebras*, MSRI Monographs, vol. 5, Springer Verlag, Berlin and New York, 1986.

[8] B. Blackadar and J. Cuntz, *The stucture of stable algebraically simple C^*-algebras*, American J. Math. **104** (1982), 813–822.

[9] N. Clarke, *A finite but not stably finite C^*-algebra*, Proc. Amer. Math. Soc. **96** (1966), 85–88.

[10] P. M. Cohn, *Simple rings without zero divisors, and Lie division rings*, Mathematika **6** (1959), 14–18.

[11] ———, *Free rings and their relations*, second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1985.

[12] A. Connes, *An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of* $\mathbf{R}$, Advances in Math. **39** (1981), 31–55.

[13] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[14] ———, *The structure of multiplication and addition in simple C^*-algebras*, Math. Scand. **40** (1977), 215–233.

[15] ———, *Dimension functions on simple C^*-algebras*, Math. Ann. **233** (1978), 145–153.

[16] ———, *K-theory for certain C^*-algebras*, Ann. of Math. **113** (1981), 181–197.

[17] ———, *The internal structure of simple C^*-algebras*, Operator algebras and applications (Providence) (R. V. Kadison, ed.), Proc. Symp. Pure Math., vol. 38, Amer. Math. Soc., 1981, pp. 85–115.

[18] K. J. Dykema and M. Rørdam, *Projections in free product C^*-algebras*, Geom. Funct. Anal. **8** (1998), 1–16.

[19] G. A. Elliott, *Some simple C^*-algebras constructed as crossed products with discrete outer automorphism groups*, Publ. Res. Inst. Math. Sci. **16** (1980), no. 1, 299–311, Kyoto University. Research Institute for Mathematical Sciences. Publications.

[20] ———, *Are amenable C^*-algebras classifiable?*, Representation Theory of Groups and Algebras, vol. 145, Amer. Math. Soc, Providence RI., 1992.

[21] L. Ge, *Application of free entropy to finite von Neumann algebras, II*, Ann. Math. (2) **147** (1998), no. 1, 143–157.

[22] J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.

[23] U. Haagerup, *Every quasi-trace on an exact C^*-algebra is a trace*, preprint, 1991.

[24] D. Husemoller, *Fibre Bundles*, 3rd. ed., Graduate Texts in Mathematics, no. 20, Springer Verlag, New York, 1966, 1994.

[25] X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, American J. Math. **121** (1999), no. 2, 359–413.

[26] E. Kirchberg, *The classification of Purely Infinite C^*-algebras using Kasparov's Theory*, in preparation.

[27] ———, *On the existence of traces on exact stably projectionless simple C^*-algebras*, Operator Algebras and their Applications (P. A. Fillmore and J. A. Mingo, eds.), Fields Institute Communications, vol. 13, Amer. Math. Soc., 1995, pp. 171–172.

[28] ———, *Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren*, C^*-algebras (Berlin), Springer, 2000, (Münster, 1999), pp. 92–141.

[29] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, American J. Math. **122** (2000), 637–666.

[30] ———, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Advances in Math. **167** (2002), no. 2, 195–264.

[31] A. Kishimoto, *Outer Automorphisms and Reduced Crossed Products of Simple C^*-algebras*, Comm. Math. Phys. **81** (1981), 429–435.

[32] E. C. Lance, *Hilbert C^*-modules—A toolkit for operator algebraists*, London Math. Soc.—Lecture notes Series, no. 210, Cambridge University Press, 1995.

[33] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, no. 76, Princeton University Press, Princeton, New Jersey, 1974.

[34] D. Olesen and G. K. Pedersen, *Applications of the Connes Spectrum to C^*-dynamical Systems, III*, J. Funct. Anal. **45** (1981), no. 3, 357–390.

[35] N. C. Phillips, *A Classification Theorem for Nuclear Purely Infinite Simple C^*-Algebras*, Documenta Math. (2000), no. 5, 49–114.

[36] M. Rørdam, *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal. **131** (1995), 415–458.

[37] ______, *Stability of C^*-algebras is not a stable property*, Documenta Math. **2** (1997), 375–386.

[38] ______, *On sums of finite projections*, Operator algebras and operator theory (Shanghai, 1997), Amer. Math. Soc., Providence, RI, 1998, pp. 327–340.

[39] ______, *Classification of Nuclear, Simple C^*-algebras*, Classification of Nuclear C^*-Algebras. Entropy in Operator Algebras (J. Cuntz and V. Jones, eds.), vol. 126, Encyclopaedia of Mathematical Sciences. Subseries: Operator Algebras and Non-commutative Geometry, no. VII, Springer Verlag, Berlin, Heidelberg, 2001, pp. 1–145.

[40] M. Rørdam, F. Larsen, and N. J. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society — Student Texts, vol. 49, Cambridge University Press, Cambridge, 2000.

[41] J. Villadsen, *Simple C^*-algebras with perforation*, J. Funct. Anal. **154** (1998), no. 1, 110–116.

[42] ______, *On the stable rank of simple C^*-algebras*, J. Amer. Math. Soc. **12** (1999), no. 4, 1091–1102.

[43] S. Wassermann, *Exact C^*-algebras and related topics*, Lecture Notes Series, vol. 19, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, Korea, 1994.

Department of Mathematics, University of Southern Denmark, Odense, Campusvej 55, 5230 Odense M, Denmark

E-mail address: `mikael@imada.sdu.dk`

Internet home page: `www.imada.sdu.dk/~mikael/`