ON THE ORDERED K_0-GROUP OF UNIVERSAL FREE PRODUCT C^*-ALGEBRAS

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Abstract. The ordered K_0-group of the universal, unital free product C^*-algebra $\mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C})$ is calculated in the case where k is prime and not a divisor in l. It is shown that the positive cone of $\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C}))$ is as small as possible in this case. The article also contains results (full and partial) on the ordered K_0-group of more general universal, unital free product C^*-algebras.

1. Introduction

Let $A * B$ denote the universal, unital free product C^*-algebra of the unital C^*-algebras A and B. The purpose of this paper is to calculate the ordered K_0-group $(\mathrm{K}_0(A*B), \mathrm{K}_0(A*B)^+)$, at least for some specific C^*-algebras A and B. Along the same line, we consider the question of what ordered abelian groups (G, G^+) arise as the ordered K_0-groups of some C^*-algebra with particular focus on the case where $G = \mathbb{Z}$.

The K_0-groups — just as abelian groups — of these algebras have been calculated by E. Germain in [G] when A and B are K-nuclear. The ordered K_0-group of the *reduced* free product C^*-algebra $A *_r B$ has been determined in [DR] (under some stronger assumptions on A and B), and it has been shown that these K_0-groups are *weakly unperforated*, ie. $ng > 0$ for some $n \in \mathbb{Z}^+$ and $g \in \mathrm{K}_0(A * B)$ implies $g > 0$. One can rephrase this result by the somewhat imprecise statement, that the positive cone $\mathrm{K}_0(A *_r B)^+$ is as large as one can imagine. The results on the ordered K_0-group of the universal, unital, free product C^*-algebra referred to above can similarly be rephrased by saying that $\mathrm{K}_0(A*B)^+$ is as small as one can imagine.

2. Ordered abelian groups and K_0-groups

Recall that an ordered abelian group is a pair (G, G^+) where G is an abelian group, $G^+ \subseteq G$ and

$$G^+ + G^+ \subseteq G^+, \qquad G^+ - G^+ = G, \qquad G^+ \cap -G^+ = \{0\}.$$

An element $u \in G^+$ is called an *order unit* in G if for every $x \in G$ there exists a $k \in \mathbb{N}$ such that $-ku \le x \le ku$. If every $u \in G^+ \setminus \{0\}$ is an order unit, then

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(G, G^+) is said to be *simple*. A simple ordered group (G, G^+) is said to be *weakly unperforated* if $kx > 0$ for some $x \in G$ and some $k \in \mathbb{N}$ implies $x > 0$.

If A is a stably finite C*-algebra with an approximate unit consisting of projections then $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ is an ordered abelian group. If, in addition, A is simple then so is $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$. George Elliott has proved in [E] that every simple, weakly unperforated, countable abelian ordered group is isomorphic to the ordered K_0-group of a unital, separable, simple, stably finite, nuclear C*-algebra. (This result fits into a program whereby the ordered K_0-group together with other invariants are conjectured — and in specific cases proved — to be a complete invariant for nuclear, simple, unital, separable, stably finite C*-algebras.) The second named author showed in [V] that the ordered K_0-group of a nuclear, simple, unital, separable, stably finite C*-algebra need not be weakly unperforated. This raises the following:

Question 2.1.

a) Which simple, countable ordered groups are isomorphic to the ordered K_0-group of some stably finite C*-algebra?
b) Which such groups are isomorphic to the ordered K_0-group of some *simple,* stably finite, unital C*-algebra?
c) Which such groups are isomorphic to the ordered K_0-group of some *simple,* stably finite, unital and *nuclear* C*-algebra?

We know of no examples of countable ordered groups (simple or not) that do not arise as the K_0-group of a C*-algebra. The example below may illustrate the complexity of ordered groups with perforation.

Example 2.2. Let $n \geq 2$ be an integer, and let $\Gamma_0, \dots, \Gamma_{n-1}$ be arbitrary subsets of $\mathbb{Z}$. Set $G = \mathbb{Z} \oplus \mathbb{Z}$ and

$$G^+ = \{(0,0)\} \cup \{(n+j, x) : 0 \leq j < n,\ x \in \Gamma_j\} \cup \{(k, x) : k \geq 2n, x \in \mathbb{Z}\}.$$

Then (G, G^+) is a simple ordered group (which is *not* weakly unperforated).

In the following we shall primarily study the case where $G = \mathbb{Z}$. In that case G^+ is either a subset of $\mathbb{Z}^+ (= \{0, 1, 2, \dots\})$ or a subset of $-\mathbb{Z}^+$. We may assume the former by replacing G^+ with $-G^+$. If $S \subseteq \mathbb{Z}^+$, then $(\mathbb{Z}, S)$ is an ordered group if and only if S is a subsemigroup of $\mathbb{Z}^+$, $0 \in S$ and $S - S = \mathbb{Z}$.

Let $n_1, \dots, n_r \in \mathbb{Z}^+$ and set

$$S = \langle n_1, \dots, n_r \rangle = \{k_1 n_1 + \cdots + k_r n_r : k_1, \dots, k_r \in \mathbb{Z}^+\}.$$

Then S is a subsemigroup of $\mathbb{Z}^+$ and $0 \in S$. Moreover $S - S = \mathbb{Z}$ if and only if $\gcd(n_1, n_2, \dots, n_r) = 1$.

Lemma 2.3. *Let integers $k, l \geq 2$ with $\gcd(k, l) = 1$ be given. Set $N = kl - k - l$. Then $N \notin \langle k, l \rangle$, but $\{N+1, N+2, N+3, \dots\} \subseteq \langle k, l \rangle$.*

Moreover, if S is a subsemigroup of $\mathbb{Z}^+$, then $S = \langle k, l \rangle$ if (and only if) $k, l \in S$ and $N \notin S$.

Proof. Each $m \in \mathbb{Z}$ can in a unique way be written as $m = ak + bl$ where $a, b \in \mathbb{Z}$ and $0 \leq a < l$. Clearly, $m \in \langle k, l \rangle$ if and only if $b \geq 0$ in this decomposition. Since $N = (l-1)k - l$ it follows that $N \notin \langle k, l \rangle$. Assume that $m > N$. Then $bl = m - ak > N - (l-1)k = (-1)l$, whence $b \geq 0$ and $m \in \langle k, l \rangle$.

Assume now that S is a subsemigroup of $\mathbb{Z}^+$, that $k, l \in S$, and that $N \notin S$. Then $\langle k, l \rangle \subseteq S$. Suppose $\langle k, l \rangle \neq S$, and choose $m \in S \setminus \langle k, l \rangle$. Then, by the argument above, we can write $m = ak + bl$ where $a, b \in \mathbb{Z}$, $0 \leq a < l$, and $b < 0$. Hence

$$N - m = ((l-1) - a)k + ((-b) - 1)l \in \langle k, l \rangle \subseteq S.$$

But then $N = m + (N - m) \in S$, in contradiction with our assumptions. □

Proposition 2.4. *Let S be a subsemigroup of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$. Then*

i) *there exits $N \in \mathbb{Z}^+$ such that $\{N+1, N+2, N+3, \dots\} \subseteq S$,*
ii) *there exists a finite set $n_1, \dots, n_r \in \mathbb{Z}^+$ such that $S = \langle n_1, \dots, n_r \rangle$.*

Proof. i). By assumption $1 \in S - S$ which entails that $k, k+1 \in S$ for some k. Set $N = k(k+1) - 2k - 1$. Then by Lemma 2.3,

$$\{N+1, N+2, N+3, \dots\} \subseteq \langle k, k+1 \rangle \subseteq S.$$

ii). Let k and N be as above, and let $n_1, n_2, \dots, n_r$ be the elements of the set $S \cap \{1, 2, \dots, N\}$. Then clearly $\langle n_1, \dots, n_r \rangle \subseteq S$, and $S \cap \{1, 2, \dots, N\} \subseteq \langle n_1, \dots, n_r \rangle$. Since $k, k+1$ belong to $\{n_1, \dots, n_r\}$ it follows from i) that

$$\{N+1, N+2, N+3, \dots\} \subseteq \langle k, k+1 \rangle \subseteq \langle n_1, \dots, n_r \rangle,$$

and therefore $S = \langle n_1, \dots, n_r \rangle$ as desired. □

Proposition 2.5. *Let S be a subsemigroup of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$. Then S has a (unique) smallest generating set $(n_1, \dots, n_r)$, i.e. $S = \langle n_1, \dots, n_r \rangle$, and if $S = \langle m_1, \dots, m_s \rangle$ then $\{n_1, \dots, n_r\} \subseteq \{m_1, \dots, m_s\}$.*

Moreover, if $n_1, \dots, n_r$ are listed increasingly with respect to the usual order on $\mathbb{Z}$, then

$$n_1 = \min S \setminus \{0\}, \qquad n_j = \min S \setminus \langle n_1, \dots, n_{j-1} \rangle, \quad 2 \leq j \leq r. \tag{2.1}$$

Proof. Let S be given, and let $n_1, n_2, \dots$ be the finite or infinite set given recursively by (2.1). (It will follow that this set is actually finite.) By Proposition 2.4 there exists a finite set $(m_1, \dots, m_s) \in \mathbb{Z}^+$ such that $S = \langle m_1, \dots, m_s \rangle$. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_s$, and for the purpose of proving the given statement, we may refine this set in such a way that $m_i \notin \langle m_1, \dots, m_{i-1} \rangle$ for all $i = 2, 3, \dots, s$. To prove the proposition, it suffices to show that $\{n_1, n_2, \dots\} = \{m_1, m_2, \dots, m_s\}$.

Notice first that $m_1 = \min S \setminus \{0\} = n_1$. Assume that $m_1 = n_1, \dots, m_{i-1} = n_{i-1}$, where $2 \leq i \leq s$. Let t be an element in $S \setminus \langle m_1, \dots, m_{i-1} \rangle$. Then

$$t = k_1 m_1 + \cdots + k_{i-1} m_{i-1} + k_i m_i + \cdots + k_s m_s \geq k_i m_i + \cdots + k_s m_s,$$

for some $k_1, \dots, k_s \in \mathbb{Z}^+$ where at least one of $k_i, k_{i+1}, \dots, k_s$ is nonzero. Hence $t \geq m_i$ and so

$$m_i = \min S \setminus \langle m_1, \dots, m_{i-1} \rangle = \min S \setminus \langle n_1, \dots, n_{j-1} \rangle = n_i.$$

It follows by induction that $\{n_1, n_2, \dots\} = \{m_1, m_2, \dots, m_s\}$ as desired. □

Notice that there are countably infinitely many orderings on $\mathbb{Z}$ by the two previous propositions. The proposition below shows (among other things) that only one of these orderings is weakly unperforated.

Recall that an ordered group (G, G^+) is said to have the Riesz Interpolation Property if whenever $x_1, x_2, y_1, y_2 \in G$ satisfy $x_i \leq y_j$, for $i, j = 1, 2$, there exists an element $z \in G$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$.

Proposition 2.6. *Let S a subsemigroup of $\mathbb{Z}^+$ which satisfies $0 \in S$ and $S - S = \mathbb{Z}$. Then the following three conditions are equivalent:*

i) $S = \mathbb{Z}^+$,
ii) $(\mathbb{Z}, S)$ *is weakly unperforated,*
iii) $(\mathbb{Z}, S)$ *has the Riesz Interpolation Property.*

Proof. The implications i) ⇒ ii) and i) ⇒ iii) are trivial.

ii) ⇒ i). Since $S - S = \mathbb{Z}$ there is $k \in S$ with $k > 0$. Accordingly, $k \cdot 1 \in S$, and so if $(\mathbb{Z}, S)$ is weakly unperforated, then $1 \in S$. This entails that $S = \mathbb{Z}^+$.

iii) ⇒ i). Suppose that $S \neq \mathbb{Z}^+$. We show that $(\mathbb{Z}, S)$ does not have the Riesz decomposition property, and hence not the Riesz interpolation property. To do so we must find $a, b, c \geq 0$ such that $a + b \geq c$ and so that there is no pair $a_1, b_1 \in \mathbb{Z}$ with $0 \leq a_1 \leq a$, $0 \leq b_1 \leq b$, and $c = a_1 + b_1$. (All inequalites are with respect to the ordering on $\mathbb{Z}$ given by S.)

Let n be the smallest (with respect to the usual order) number in $S \setminus \{0\}$. Then $n \neq 1$, and therefore S is not contained in the set $\{kn \mid k \in \mathbb{Z}^+\}$. Let m be the smallest (with respect to the usual order) number in $S \setminus \{kn \mid k \in \mathbb{Z}^+\}$. Let k be the smallest (with respect to the usual order) integer such that $kn - m \in S$, and note that $k \in \{2, 3, 4, \dots\}$. Put $a = n$, $b = (k-1)n$, and $c = m$. Assume that there were $a_1, b_1 \in \mathbb{Z}$ with $0 \leq a_1 \leq a$, $0 \leq b_1 \leq b$, and $c = a_1 + b_1$. Then either $a_1 = 0$ or $a_1 = a = n$. In the first case, $m = c = b_1 \leq b = (k-1)n$, in contradiction with the choice of k. In the other case, $m - n = c - a_1 = b_1 \in S$, in contradiction with the choice of m. □

3. The ordered K_0-groups of free products of matrix algebras

Let A and B be unital C*-algebras, and let $\mathcal{A} = A * B$ be the universal free product of A and B (in the category of unital C*-algebras). There are inclusion maps $\iota_A\colon A \to \mathcal{A}$ and $\iota_B\colon B \to \mathcal{A}$; and for each unital C*-algebra D with unital $*$-homomorphisms $\phi_A\colon A \to D$ and $\phi_B\colon B \to D$ there is a unique $*$-homomorphism $\phi\colon \mathcal{A} \to D$ making the diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\iota_A} & & \searrow^{\phi_A} & \\ \mathcal{A} & & \overset{\phi}{\dashrightarrow} & & D \\ & \nwarrow_{\iota_B} & & \nearrow_{\phi_B} & \\ & & B & & \end{array}$$

commutative. Define maps

$$\lambda\colon \mathbb{Z} \to \mathrm{K}_0(A) \oplus \mathrm{K}_0(B), \quad \mu\colon \mathrm{K}_0(A) \oplus \mathrm{K}_0(B) \to \mathrm{K}_0(\mathcal{A})$$

by

$$\lambda(k) = (k[1_A]_0, -k[1_B]_0), \quad \mu(g,h) = \mathrm{K}_0(\iota_A)(g) + \mathrm{K}_0(\iota_B)(h),$$

and consider the sequence:

$$\mathbb{Z} \xrightarrow{\lambda} \mathrm{K}_0(A) \oplus \mathrm{K}_0(B) \xrightarrow{\mu} \mathrm{K}_0(\mathcal{A}) \longrightarrow 0. \tag{3.1}$$

Then, obviously, $\mathrm{Im}(\lambda) \subseteq \mathrm{Ker}(\mu)$. E. Germain has proved, in [G], that if A and B are K-nuclear, then we have an exact six-term sequence:

$$\begin{array}{ccccc} \mathrm{K}_0(\mathbb{C}) & \xrightarrow{\lambda} & \mathrm{K}_0(A) \oplus \mathrm{K}_0(B) & \xrightarrow{\mu} & \mathrm{K}_0(\mathcal{A}) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(\mathcal{A}) & \longleftarrow & \mathrm{K}_1(A) \oplus \mathrm{K}_1(B) & \longleftarrow & \mathrm{K}_1(\mathbb{C}) \end{array}$$

In particular (3.1) is exact. Notice that

$$\mu(\mathrm{K}_0(A)^+ \oplus \mathrm{K}_0(B)^+) \subseteq \mathrm{K}_0(\mathcal{A})^+. \tag{3.2}$$

Question 3.1. Is

$$\mathrm{K}_0(\mathcal{A})^+ = \mu(\mathrm{K}_0(A)^+ \oplus \mathrm{K}_0(B)^+)$$

for all (K-nuclear) unital C*-algebras A and B?

Question 3.2. Let $p \in A$ and $q \in B$ be projections, and suppose that $p \neq 0$ and that there exists no subprojection of q which is equivalent to 1_B, i.e. $1 \not\precsim q$ in B. Does it follow that $\iota_A(p) \not\precsim \iota_B(q)$ in $\mathcal{A}$?

In other words, if $p \in A$ and $q \in B$ are projections such that $\iota_A(p) \precsim \iota_B(q)$ in $\mathcal{A}$, does it then follow that either $p = 0$ or $1 \precsim q$?

Remark 3.3. Recall that a projections p in a C*-algebra D is called properly infinite if there exist sub-projections p_1, p_2 of p such that $p_1 \sim p_2 \sim p$ and $p_1 \perp p_2$.

Assume that $e_1, e_2 \in D$ are projections such that $e_1 \sim e_2$ and $e_1 \perp e_2$. It is not known if $e_1 + e_2$ is properly infinite implies that e_1 (and e_2) are properly infinite. If the answer to Question 3.2 is affirmative, then there do exist e_1 and e_2 as above with $e_1 + e_2$ properly infinite and with e_1 and e_2 not properly infinite as shown below:

Let A be the Cuntz algebra $\mathcal{O}_2$, and set $B = \mathrm{M}_2(\mathbb{C})$ with $p = 1$, the unit of $\mathcal{O}_2$ and q a 1-dimensional projection in B. Set $e_1 = \iota_B(q)$ and $e_2 = \iota_B(1-q)$. Then $e_1 \sim e_2$, $e_1 \perp e_2$, and $e_1 + e_2 = 1$ is properly infinite because the unit of $\mathcal{O}_2$ is properly infinite. If e_1 is properly infinite in $\mathcal{A}$, then there exist subprojections $f_1, f_2 \in \mathcal{A}$ of e_1 with $f_1 \perp f_2$, and $f_1 \sim f_2 \sim e_1 \sim e_2$. Hence

$$\iota_A(p) = 1 = e_1 + e_2 \sim f_1 + f_2 \leq e_1 = \iota_B(q),$$

in which case we could answer Question 3.2 in the negative.

Note added in proof: The existence of non properly infinite (actually finite) projections e_1, e_2 in some C*-algebra D with $e_1 \sim e_2$, $e_1 \perp e_2$, and $e_1 + e_2$ properly infinite, has subsequently been found in [R].

For a C*-algebra D let $V(D)$ be the ordered semi-group of Murray-von Neumann equivalence classes of projections in $D \otimes \mathcal{K}$. If $p \in D \otimes \mathcal{K}$ is a projection, then $[p]$ and $[p]_0$ will denote its classes in $V(D)$ and $\mathrm{K}_0(D)$ respectively. Let $\eta\colon V(D) \to \mathrm{K}_0(D)$ be the canonical map. Then $\eta([p]) = [p]_0$.

Theorem 3.4. *Let $k, l \geq 2$ be integers with k prime and l not divisible by k. There exists a C*-algebra D, projections $p, q \in D$, and an element $g \in \mathrm{K}_0(D)$ such that*

i) $[p]_0 = kg, \quad [q]_0 = lg$,
ii) $l[p] = k[q]$,
iii) $mg \notin \mathrm{K}_0(D)^+$ *if* $m \notin \langle k, l \rangle$ *(the semi-group generated by k and l).*

It follows in particular from Theorem 3.4 that

$$\{n \in \mathbb{Z} : ng \in \mathrm{K}_0(D)^+\} = \langle k, l \rangle.$$

Proof. Set $B_k = \mathbb{D}/{\sim}$, where $\mathbb{D} \subseteq \mathbb{C}$ is the unit disc and $z \sim w$ if $z, w \in \mathbb{T}$ and $z^k = w^k$. Then $\mathrm{H}^2(B_k) \cong \mathbb{Z}_k$.

Let ω be a non-trivial complex line bundle over B_k. Then the Euler class of ω is nonzero and $k\omega \cong \theta_k$, i.e. the k-fold direct sum of ω is a trivial vector bundle.

Let B_k^{k-1} denote the $(k-1)$-fold Cartesian product of B_k with itself and let $\rho_i : B_k^{k-1} \to B_k$, $1 = i, \ldots, k-1$ denote the coordinate projections. Once again, let $(B_k^{k-1})^l$ denote the l-fold Cartesian product of B_k^{k-1} and let $\pi_j : (B_k^{k-1})^l \to B_k^{k-1}$, $j = 1, \ldots, l$, be the corresponding coordinate projections. Put

$$\zeta = \rho_1^*(\omega) \otimes \cdots \otimes \rho_{k-1}^*(\omega), \qquad \xi = \zeta^{\times l} \cong \pi_1^*(\zeta) \oplus \cdots \oplus \pi_l^*(\zeta).$$

Then ζ is a line-bundle over B_k^{k-1}, and ξ is a vector bundle over $(B_k^{k-1})^l$ of fiber dimension l. Successive applications of the isomorphism $k\omega \cong \theta_k \cong k\theta_1$ yields $k\zeta \cong \theta_k$, and this in turns implies that $k\xi \cong \theta_{kl}$.

We show that the Euler class $\mathrm{e}((k-1)\xi)$ is nonzero. Using the product formula for the Euler class we get

$$\mathrm{e}((k-1)\xi) = \mathrm{e}(\xi)^{k-1} = \prod_{j=1}^{l} \pi_j^*(\mathrm{e}(\zeta)^{k-1}).$$

From the definition of ζ it follows that

$$\mathrm{e}(\zeta) = \sum_{i=1}^{k-1} \rho_i^*(\mathrm{e}(\omega)),$$

and since $\mathrm{e}(\omega)^2 = 0$ we get

$$\mathrm{e}(\zeta)^{k-1} = (k-1)! \prod_{i=1}^{k-1} \rho_i^*(\mathrm{e}(\omega)).$$

Let

$$\mu \colon \bigotimes_{1}^{(k-1)l} \mathrm{H}^2(B_k; \mathbb{Z}) \to \mathrm{H}((B_k^{k-1})^l; \mathbb{Z})$$

be given by

$$\mu(x_{11} \otimes \cdots \otimes x_{k-1,l}) = \prod_{ij} \pi_j^*(\rho_i^*(x_{ij})).$$

By the Künneth Theorem for singular cohomology, μ is injective. Now,

$$\mathrm{e}((k-1)\xi) = \mu\big(((k-1)!)^l \mathrm{e}(\omega) \otimes \mathrm{e}(\omega) \otimes \cdots \otimes \mathrm{e}(\omega)\big)$$

and since k does not divide $((k-1)!)^l$ it follows that $\mathrm{e}((k-1)\xi)$ is nonzero.

Consider the C*-algebra

$$D = C\big(B_k^{(k-1)l}\big) \otimes \mathcal{K}.$$

Let $p \in D$ be a projection corresponding to a trivial bundle of dimension k, and let $q \in D$ be a projection which corresponds to the bundle ξ. Then $k[q] = l[p]$ (these classes both correspond to the trivial bundle of dimension kl). Choose $a, b \in \mathbb{Z}$ such that $1 = ak + bl$, and put $g = a[p]_0 + b[q]_0 \in \mathrm{K}_0(D)$. Then $kg = [p]_0$ and $lg = [q]_0$.

We must show that $mg \notin \mathrm{K}_0(D)^+$ if $m \notin \langle k, l \rangle$. By Lemma 2.3 it is enough to show that $(kl - k - l)g \notin \mathrm{K}_0(D)^+$. Suppose, for a moment, that $(kl - k - l)g$ is positive. Since

$$(kl - k - l)g = (k - 1)lg - kg = (k - 1)[q]_0 - [p]_0$$

it follows that $(k - 1)[\xi] - [\theta_k]$ is positive in $\mathrm{K}^0(B_k^{(k-1)l})$, and so, in particular, $(k - 1)[\xi] - [\theta_1]$ is positive. Consequently, there exists a complex vector bundle ϑ and $d \in \mathbb{N}$ such that $(k - 1)\xi \oplus \theta_d \cong \vartheta \oplus \theta_{d+1}$. Hence,

$$\mathrm{e}((k - 1)\xi) = \mathrm{e}(\vartheta \oplus \theta_1) = \mathrm{e}(\vartheta)\mathrm{e}(\theta_1) = 0$$

in contradiction with the choice of ξ. $\square$

We shall consider the ordered K_0-group of the free product $\mathcal{A} = \mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C})$. It follows from Germain's theorem (quoted earlier in this section), that $\mathrm{K}_0(\mathcal{A}) = \mathbb{Z}$ when k and l are relatively prime. We give below an elementary proof of this fact.

Proposition 3.5. *Let $k, l \in \mathbb{N}$ be relatively prime, set $\mathcal{A} = \mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C})$, let*

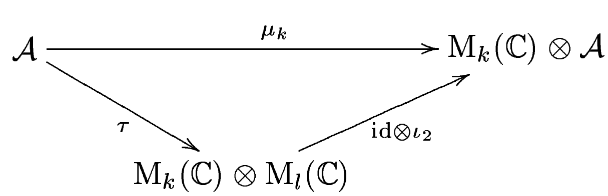
$$\phi_1 \colon \mathrm{M}_k(\mathbb{C}) \to \mathrm{M}_k(\mathbb{C}) \otimes \mathrm{M}_l(\mathbb{C}), \qquad \phi_2 \colon \mathrm{M}_l(\mathbb{C}) \to \mathrm{M}_k(\mathbb{C}) \otimes \mathrm{M}_l(\mathbb{C}),$$

be the natural homomorphisms, and let $\tau \colon \mathcal{A} \to \mathrm{M}_k(\mathbb{C}) \otimes \mathrm{M}_l(\mathbb{C})$ be the homomorphism induced by ϕ_1 and ϕ_2.

Then $\mathrm{K}_0(\tau)$ and $\mathrm{K}_1(\tau)$ are isomorphisms, and in particular we have that $\mathrm{K}_0(\mathcal{A}) \cong \mathbb{Z}$ and $\mathrm{K}_1(\mathcal{A}) = 0$.

Proof. Let $\iota_1 \colon \mathrm{M}_k(\mathbb{C}) \to \mathcal{A}$ and $\iota_2 \colon \mathrm{M}_l(\mathbb{C}) \to \mathcal{A}$ denote the canonical inclusion mappings. Let $e_1 \in \mathrm{M}_k(\mathbb{C})$ and $e_2 \in \mathrm{M}_l(\mathbb{C})$ be one-dimensional projections, and set $f_j = \iota_j(e_j) \in \mathcal{A}$. Then $\tau(f_j) = \phi_j(e_j)$, and hence $\tau(f_1)$ has dimension l and $\tau(f_2)$ has dimension k. Since k and l are relatively prime, this shows that $\mathrm{K}_0(\tau)$ is onto.

The diagram,



where μ_k is given by $a \mapsto 1_k \otimes a$, commutes up to homotopy. Indeed, the diagram commutes exactly on the image of ι_2, and it commutes up to homotopy on the image of ι_1 (since the two canonical homomorphisms $\mathrm{M}_k(\mathbb{C}) \to \mathrm{M}_k(\mathbb{C}) \otimes \mathrm{M}_k(\mathbb{C})$ are homotopic). It follows that the diagram commutes exactly at the level of K-groups, and therefore that

$$\mathrm{Ker}(\mathrm{K}_j(\tau)) \subseteq \mathrm{Ker}(\mathrm{K}_j(\mu_k)) = \mathrm{Ker}(k \cdot \mathrm{id}_{\mathrm{K}_j(\mathcal{A})}).$$

By interchanging the roles of the first and second factor in $\mathcal{A}$, we get

$$\mathrm{Ker}(\mathrm{K}_j(\tau)) \subseteq \mathrm{Ker}(l \cdot \mathrm{id}_{\mathrm{K}_j(\mathcal{A})}).$$

Since k and l are relatively prime,

$$\mathrm{Ker}(k \cdot \mathrm{id}_{\mathrm{K}_j(\mathcal{A})}) \cap \mathrm{Ker}(l \cdot \mathrm{id}_{\mathrm{K}_j(\mathcal{A})}) = 0,$$

and this proves that $\mathrm{K}_j(\tau)$ is injective for $j = 0, 1$. $\square$

Theorem 3.6. *Let $k, l \geq 2$ be integers with k prime and l not divisible by k. Put $\mathcal{A} = \mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C})$. Then*

i) $(\mathrm{K}_0(\mathcal{A}), \mathrm{K}_0(\mathcal{A})^+) \cong (\mathbb{Z}, \langle k, l \rangle)$,
ii) $\mathrm{K}_0(\mathcal{A})^+ = \mu(\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}))^+ \oplus \mathrm{K}_0(\mathrm{M}_l(\mathbb{C}))^+)$.

Moreover, if $p \in \mathrm{M}_k(\mathbb{C})$ and $q \in \mathrm{M}_l(\mathbb{C})$ are projections with $p \neq 0$ and $q \neq 1$, then $p \not\precsim q$ in $\mathcal{A}$.

Proof. Let $\tau \colon \mathcal{A} \to \mathrm{M}_k(\mathbb{C}) \otimes \mathrm{M}_l(\mathbb{C})$ be as in Proposition 3.5. Identify $\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}))$, $\mathrm{K}_0(\mathrm{M}_l(\mathbb{C}))$ and $\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}) \otimes \mathrm{M}_l(\mathbb{C}))$ with $\mathbb{Z}$ in the natural way. Let μ be as in (3.1), and set $\mu'(x, y) = lx + ky$. Then by Proposition 3.5 the diagram

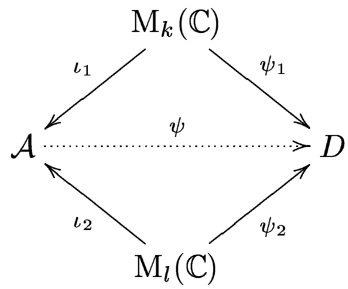
$$\begin{array}{ccc} \mathrm{K}_0(\mathrm{M}_k(\mathbb{C})) \oplus \mathrm{K}_0(\mathrm{M}_l(\mathbb{C})) & \xrightarrow{\ \mu\ } & \mathrm{K}_0(\mathcal{A}) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong\ \mathrm{K}_0(\tau)} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\ \mu'\ ]{} & \mathbb{Z} \end{array}$$

commutes, and hence

$$(3.3) \qquad (\mathrm{K}_0(\tau) \circ \mu)\big(\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}))^+ \oplus \mathrm{K}_0(\mathrm{M}_l(\mathbb{C}))^+\big) = \mu'(\mathbb{Z}^+ \oplus \mathbb{Z}^+) = \langle k, l \rangle.$$

Put $S = \mathrm{K}_0(\tau)(\mathrm{K}_0(\mathcal{A})^+) \subseteq \mathbb{Z}^+$. Then, by (3.1) and (3.3), $\langle k, l \rangle \subseteq S$. We proceed to show that $S \subseteq \langle k, l \rangle$, and this will prove i) and ii).

Let D, $p, q \in D$ and $g \in \mathrm{K}_0(D)$ be as in Theorem 3.4. Upon replacing D with a corner of $D \otimes \mathcal{K}$, we may assume that D is unital and that $l[p] = k[q] = [1_D]$. Let $e_1 \in \mathrm{M}_k(\mathbb{C})$ and $e_2 \in \mathrm{M}_l(\mathbb{C})$ be 1-dimensional projections. By the choice of p and q there exist unital $*$-homomorphisms $\psi_1 \colon \mathrm{M}_k(\mathbb{C}) \to \mathcal{A}$ and $\psi_2 \colon \mathrm{M}_l(\mathbb{C}) \to \mathcal{A}$ such that $\psi_1(e_1) = q$ and $\psi_2(e_2) = p$. By the universal property of $\mathcal{A}$, ψ_1 and ψ_2 factor through a unital $*$-homomorphism $\psi\colon \mathcal{A} \to D$:



At the level of K-theory, we have the following commuting diagram

$$\begin{CD} S @<{V(\tau)}<{\cong}< V(\mathcal{A}) @>{V(\psi)}>> V(D) \\ @V{\subseteq}VV @V{\eta}VV @V{\eta}VV \\ \mathbb{Z} @<{\cong}<{\mathrm{K}_0(\tau)}< \mathrm{K}_0(\mathcal{A}) @>>{\mathrm{K}_0(\psi)}> \mathrm{K}_0(D). \end{CD}$$

Let $h \in \mathrm{K}_0(A)$ be the element satisfying $\mathrm{K}_0(\tau)(h) = 1$. Viewing e_1 and e_2 as elements of $\mathcal{A}$, we have $\psi(e_1) = q$, $\psi(e_2) = p$, $\mathrm{K}_0(\tau)([e_1]_0) = l$, and $\mathrm{K}_0(\tau)([e_2]_0) = k$. Hence $lh = [e_1]_0$ and $kh = [e_2]_0$, and so

$$l \cdot \mathrm{K}_0(\psi)(h) = \mathrm{K}_0(\psi)([e_1]_0) = [q]_0 = lg, \quad k \cdot \mathrm{K}_0(\psi)(h) = \mathrm{K}_0(\psi)([e_2]_0) = [p]_0 = kg,$$

which implies that $\mathrm{K}_0(\psi)(h) = g$.

Assume $m \in S$, and find $x \in V(\mathcal{A})$ which is mapped to m. Then $mh = \eta(x)$, and therefore

$$mg = \mathrm{K}_0(\psi)(mh) = \mathrm{K}_0(\psi) \circ \eta(x) = \eta \circ V(\psi)(x) \in \mathrm{K}_0(D)^+.$$

By Theorem 3.4 this implies that $m \in \langle k, l\rangle$. We have now shown that $S = \langle k, l\rangle$ and the proof of i) and ii) is complete.

Let $p \in \mathrm{M}_k(\mathbb{C})$ and $q \in \mathrm{M}_l(\mathbb{C})$ be given such that $p \neq 0$ and $q \neq 1$. Then $[p]_0 = x[e_1]_0$ and $[q]_0 = y[e_2]_0$ where $x, y \in \mathbb{Z}$ satisfy $1 \leq x \leq k$ and $0 \leq y < l$. Since

$$\mathrm{K}_0(\tau)([q]_0 - [p]_0) = yk - xl \notin \langle k, l\rangle = S,$$

it follows that $[q]_0 - [p]_0 \notin \mathrm{K}_0(\mathcal{A})^+$, and so $p \not\precsim q$ in $\mathcal{A}$ $\square$

Remark 3.7. It would be interesting to know if

$$\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C}))^+ = \mu(\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}))^+ \oplus \mathrm{K}_0(\mathrm{M}_l(\mathbb{C}))^+)$$

for *all* integers $k, l \geq 2$. It should be noted that if $\gcd(k, l) = m > 1$, then $\mathrm{K}_0(\mathrm{M}_k(\mathbb{C}) * \mathrm{M}_l(\mathbb{C})) \cong \mathbb{Z} \oplus \mathbb{Z}/m$.

Remark 3.8. It would also be interesting to know if one to each subsemigroup S of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$ can find a C^*-algebra A with $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+) \cong (\mathbb{Z}, S)$.

One obvious generalization of Theorem 3.6 is as follows: Let $k_1, k_2, \dots, k_r \geq 2$ be integers and consider the unital C*-algebra

$$\mathcal{A} = \mathrm{M}_{k_1}(\mathbb{C}) * \mathrm{M}_{k_2}(\mathbb{C}) * \cdots * \mathrm{M}_{k_r}(\mathbb{C}),$$

where the free product, $*$, is the universal free product in the category of unital C^*-algebras. By Germain s theorem (see below (3.1)), $\mathrm{K}_0(\mathcal{A}) \cong \mathbb{Z}$ if and only if $\gcd(k_i, k_j) = 1$ whenever $i \neq j$.

Assume now that $\gcd(k_i, k_j) = 1$ for all $i \neq j$. Set

$$n_j = \prod_{i \neq j} k_i. \tag{3.4}$$

Set $S = \langle n_1, n_2, \dots, n_r \rangle$. One can show that $(\mathrm{K}_0(\mathcal{A}), \mathrm{K}_0(\mathcal{A})^+) \cong (\mathbb{Z}, S)$ provided that S satisfies the following condition:

(S) For each $m \in \mathbb{Z}^+ \setminus S$ there exist a C*-algebra B and homomorphisms $\beta \colon S \to V(B)$ and $\gamma \colon \mathbb{Z} \to \mathrm{K}_0(B)$ such that

$$\begin{array}{ccc} S & \xrightarrow{\ \beta\ } & V(B) \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \eta} \\ \mathbb{Z} & \xrightarrow{\ \gamma\ } & \mathrm{K}_0(B) \end{array}$$

commutes, and such that $\gamma(m) \notin \mathrm{K}_0(B)^+$.

Only rather special semi-groups S arise from such a set $k_1, k_2, \dots, k_r$. For example, if S has minimal generating set $(n_1, n_2, \dots, n_r)$, where $r \geq 3$ and $\gcd(n_i, n_j) = 1$ for some $i \neq j$, then $(n_1, n_2, \dots, n_r)$ cannot be obtained from any set $k_1, k_2, \dots, k_r$ as in (3.4).

We sketch below another construction which to an arbitrary subsemigroup S of $\mathbb{Z}^+$ with $0 \in S$ and $S - S = \mathbb{Z}$ associates a C^*-algebra $\mathcal{F}(S)$, whose ordered K_0-group is likely to be isomorphic to $(\mathbb{Z}, S)$.

Let $(n_1, n_2, \dots, n_r)$ be the minimal generating set for S (cf. Proposition 2.5). For each pair of indices i, j with $1 \leq i, j \leq r$, $i \neq j$, set $m_j^i = n_i / \gcd(n_i, n_j)$ and find projections $e_j^i \in \mathcal{K}$ with $\dim(e_j^i) = m_j^i$ and such that the projections

$$e_j^1, e_j^2, \dots, e_j^{j-1}, e_j^{j+1}, e_j^{j+2}, \dots, e_j^r$$

are mutually orthogonal for every $j = 1, 2, \ldots, r$.

Put $\mathcal{F}_r = \mathcal{K} * \mathcal{K} * \cdots * \mathcal{K}$ (with r copies of $\mathcal{K}$, and where $*$ denotes universal free product with no amalgamation). Let $\phi'_1, \phi'_2, \ldots, \phi'_r \colon \mathcal{K} \to \mathcal{F}_r$ be the canonical inclusions. Let I be the closed two-sided ideal in $\mathcal{F}_r$ generated by the set

$$\{\phi'_i(e_i^j) - \phi'_j(e_j^i) : 1 \le i, j \le r,\, i \neq j\}.$$

Set $\mathcal{F}(S) = \mathcal{F}_r/I$, let $\pi \colon \mathcal{F}_r \to \mathcal{F}(S)$ be the quotient mapping, and set $\phi_j = \pi \circ \phi'_j \colon \mathcal{K} \to \mathcal{F}(S)$.

Then $\mathcal{F}(S) = C^*(\phi_1(\mathcal{K}), \phi_2(\mathcal{K}), \ldots, \phi_r(\mathcal{K}))$, ie., $\mathcal{F}(S)$ is generated by r copies of $\mathcal{K}$. Moreover, it can be shown that

$$(\mathrm{K}_0(\mathcal{F}(S)), \mathrm{K}_0(\mathcal{F}(S))^+) \cong (\mathbb{Z}, S)$$

provided that the semi-group S satisfies the condition (S) described above, and provided that

$$\mathrm{K}_0(\mathcal{F}(S)) = \mathrm{K}_0(\phi_1)(\mathrm{K}_0(\mathcal{K})) + \cdots + \mathrm{K}_0(\phi_r)(\mathrm{K}_0(\mathcal{K})).$$

4. Comparison of projections in free products

Combining Theorem 3.6 with Propositions 4.1 and 4.2 below we shall show that one can answer Questions 3.1 and 3.2 in the affirmative for a rather large class of pairs of C^*-algebras. To systematize our treatment, let $\mathcal{C}_1$ denote the class of all pairs of K-nuclear C^*-algebras (A, B) such that

$$\mathrm{K}_0(A * B)^+ = \mu(\mathrm{K}_0(A)^+ \oplus \mathrm{K}_0(B)^+),$$

(cf. (3.1) and (3.2)). Let $\mathcal{C}_2$ denote the class of pairs of C^*-algebras (A, B) such that for every pair of projections $p \in A$ and $q \in B$, if $p \neq 0$, $q \neq 0$, $1 \not\precsim p$, and $1 \not\precsim q$, then p and q are incomparable in $A * B$, ie., $p \not\precsim q$ and $q \not\precsim p$ in $A * B$.

If follows from Theorem 3.6 that $(\mathrm{M}_k(\mathbb{C}), \mathrm{M}_l(\mathbb{C})) \in \mathcal{C}_j$ for $j = 1, 2$ when k is prime and $\gcd(k, l) = 1$ (or vice versa).

Recall that a (unital) C^*-algebra A is said to be *finite* if for every pair of projections $p, q \in A$, $p \precsim q \le p$ implies $p = q$. If A and B are finite C^*-algebras, then $(A, B) \in \mathcal{C}_2$ if and only if for every pair of projections $p \in A$ and $q \in B$, $p \precsim q$ implies that either $p = 0$ or $q = 1$ (and vice versa).

Proposition 4.1.

i) *If $(A, B) \in \mathcal{C}_1$, then $(B, A) \in \mathcal{C}_1$.*

ii) *If $(A_1, B), (A_2, B) \in \mathcal{C}_1$, then $(A_1 \oplus A_2, B) \in \mathcal{C}_1$.*

iii) *If A is the inductive limit of a sequence of unital C^*-algebras*

$$A_1 \to A_2 \to A_3 \to \cdots$$

with unital connecting maps, and if $(A_n, B) \in \mathcal{C}_1$ for all n, then $(A, B) \in \mathcal{C}_1$.

Proposition 4.2.

i) *If* $(A,B)\in\mathcal{C}_2$*, then* $(B,A)\in\mathcal{C}_2$*.*

ii) *If* $(A_1,B),(A_2,B)\in\mathcal{C}_2$*, then* $(A_1\oplus A_2,B)\in\mathcal{C}_2$*.*

iii) *If* A *is the inductive limit of a sequence of unital* C^**-algebras*

$$A_1\to A_2\to A_3\to\cdots$$

with unital connecting maps, and if $(A_n,B)\in\mathcal{C}_2$ *for all* n*, then* $(A,B)\in\mathcal{C}_2$*.*

iv) *Assume* A *and* B *are finite* C^**-algebras with* $(A,B)\in\mathcal{C}_2$*. Then* $(A_0,B_0)\in\mathcal{C}_2$ *for all sub-*C^**-algebras* A_0 *of* A *and* B_0 *of* B *with* $1_A\in A_0$ *and* $1_B\in B_0$*.*

We prove Propositions 4.1 and 4.2 simultaneously, and we note that part i) of these propositions is trivial. To prove part ii) we shall need the following:

Lemma 4.3. *Assume* $(A,B)\in\mathcal{C}_1$*, let* $x\in\mathrm{K}_0(A)$*,* $y\in\mathrm{K}_0(B)$*, and set*

$$g=\mathrm{K}_0(\iota_A)(x)+\mathrm{K}_0(\iota_B)(y)\in\mathrm{K}_0(A*B).$$

Then $g\in\mathrm{K}_0(A*B)^+$ *if and only if* $x+n[1_A]_0\in\mathrm{K}_0(A)^+$ *and* $y-n[1_B]_0\in\mathrm{K}_0(B)^+$ *for some integer* n*.*

Proof. The "if" part follows from (3.2) and the exactness of (3.1). Assume $g\in\mathrm{K}_0(A*B)^+$. Then, by the assumption that $(A,B)\in\mathcal{C}_1$, there exist $x'\in\mathrm{K}_0(A)^+$ and $y'\in\mathrm{K}_0(B)^+$ such that $g=\mathrm{K}_0(\iota_A)(x')+\mathrm{K}_0(\iota_B)(y')$. Hence

$$\mathrm{K}_0(\iota_A)(x'-x)+\mathrm{K}_0(\iota_B)(y'-y)=0.$$

By exactness of the sequence (3.1) there is an integer n such that $x'-x=n[1_A]_0$ and $y'-y=-n[1_B]_0$. This completes the proof. □

Proof of part ii) of Propositions 4.1 and 4.2. From the universal property of the free product, the maps $A_1\oplus A_2\to A_j\to A_j*B$ factor through *-homomorphisms $\phi_j\colon(A_1\oplus A_2)*B\to A_j*B$.

To show that $(A_1\oplus A_2,B)\in\mathcal{C}_1$ it suffices to show that

$$\mathrm{K}_0((A_1\oplus A_2)*B)^+\subseteq\mu(\mathrm{K}_0(A_1\oplus A_2)^+\oplus\mathrm{K}_0(B)^+).$$

Let $g\in\mathrm{K}_0((A_1\oplus A_2)*B)^+$. By exactness of the sequence (3.1), there exist $x_1\in\mathrm{K}_0(A_1)$, $x_2\in\mathrm{K}_0(A_2)$, and $y\in\mathrm{K}_0(B)$, such that

$$g=\mathrm{K}_0(\iota_{A_1\oplus A_2})(x_1,x_2)+\mathrm{K}_0(\iota_B)(y).$$

Now,

$$\mathrm{K}_0(\iota_{A_j})(x_j)+\mathrm{K}_0(\iota_B)(y)=\mathrm{K}_0(\phi_j)(g)\in\mathrm{K}_0(A_j*B)^+.$$

By Lemma 4.3 this entails that $x_j + n_j[1_{A_j}]_0 \geq 0$ and $y - n_j[1_B]_0 \geq 0$ for some integers n_1 and n_2. Set $n = \max\{n_1, n_2\}$. Then

$$x' = (x_1 + n[1_{A_1}]_0, x_2 + n[1_{A_2}]_0) \in \mathrm{K}_0(A_1 \oplus A_2)^+, \quad y' = y - n[1_B]_0 \in \mathrm{K}_0(B)^+,$$

$$g = \mathrm{K}_0(\iota_{A_1 \oplus A_2})(x') + \mathrm{K}_0(\iota_B)(y') \in \mu(\mathrm{K}_0(A_1 \oplus A_2)^+ \oplus \mathrm{K}_0(B)^+)$$

as desired.

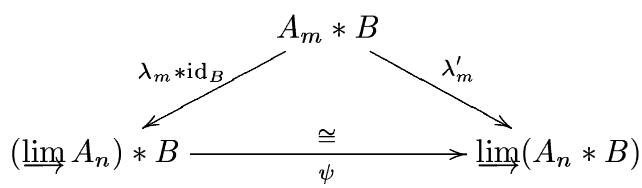
We proceed to show that $(A_1 \oplus A_2, B) \in \mathcal{C}_2$ if $(A_1, B), (A_2, B) \in \mathcal{C}_2$. Let $p = (p_1, p_2)$ be a projection in $A_1 \oplus A_2$, let q be a projection in B, and assume that $p \precsim q$ in $(A_1 \oplus A_2) * B$. Then $p_j = \phi_j(p) \precsim \phi_j(q) = q$ in $A_j * B$. Hence either $p_j = 0$ or $1 \precsim q$ for $j = 1, 2$. This entails that either $p = 0$ or $1 \precsim q$. $\square$

Part iii) of Propositions 4.1 and 4.2 are easy consequences of the following:

Lemma 4.4. *Let*

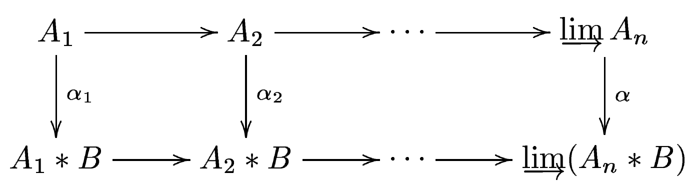
$$A_1 \to A_2 \to A_3 \to \cdots$$

*be a sequence of unital C^*-algebras with unital connecting *-homomorphisms, and let B be a unital C^*-algebra. Then there exists an isomorphism ψ making the diagram*

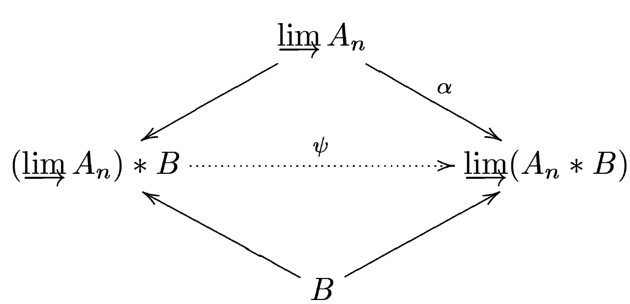


commutative. (The maps $\lambda_m * \mathrm{id}_B$ and λ'_m are the natural maps arising from the functoriality of the free products and the inductive limits.)

Proof. We have *-homomorphisms α_n and α and the commuting diagram

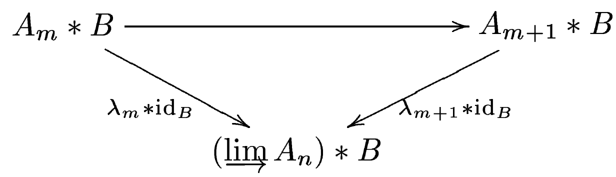


which yields a *-homomorphism ψ:

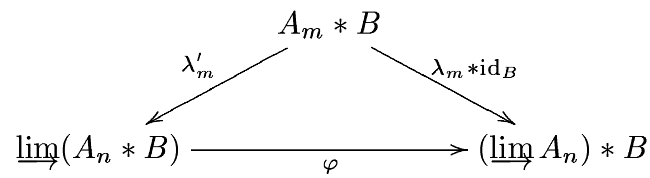


Moreover, ψ makes the diagram in the lemma commutative. (One can check this for example by inspecting elements in A_m and in B separately.)

By commutativity of the diagram:



we get a *-homomorphism $\varphi\colon \varinjlim(A_n * B) \to (\varinjlim A_n) * B$, which makes



commutative. Finally, $\psi \circ \varphi$ is the identity on the image of λ'_m, and $\varphi \circ \psi$ is the identity on the image of $\lambda_m * \mathrm{id}_B$ for each m. This shows that ψ and φ are each others inverses. Hence ψ is an isomorphism. $\square$

Proof of Proposition 4.2 iv). Let $A_0 \subseteq A$ and $B_0 \subseteq B$ with $1_A \in A_0$ and $1_B \in B_0$ be given, and let $\varphi\colon A_0 * B_0 \to A * B$ be the canonical *-homomorphism arising from these inclusions. Let $p \in A_0$ and $q \in B_0$ be projections, and assume that $p \precsim q$ in $A_0 * B_0$. Then $p = \varphi(p) \precsim \varphi(q) = q$ in $A * B$. Since B is assumed to be finite, and since $(A, B) \in \mathcal{C}_2$, this implies that $p = 0$ or $q = 1$. Reverting the roles of A and B yields the other case. $\square$

If we knew that $(\mathrm{M}_k(\mathbb{C}), \mathrm{M}_l(\mathbb{C})) \in \mathcal{C}_1$ for all positive integers k and l, then we could conclude from Proposition 4.1 that $(A, B) \in \mathcal{C}_1$ for all unital AF-algebras A and B. With the present restrictions on k and l in Theorem 3.6 we can still use Proposition 4.1 to reach conclusions about the ordered K_0-group at least for some non-trivial AF-algebras:

Corollary 4.5. *Let $\tilde{\mathcal{K}}$ denote the C^*-algebra of compact operators on a separable Hilbert space with a unit adjoined. Then $(\tilde{\mathcal{K}}, \tilde{\mathcal{K}}) \in \mathcal{C}_1$, ie.,*

$$\mathrm{K}_0(\tilde{\mathcal{K}} * \tilde{\mathcal{K}})^+ = \mu(\mathrm{K}_0(\tilde{\mathcal{K}})^+ \oplus \mathrm{K}_0(\tilde{\mathcal{K}})^+).$$

Proof. Let $\{p_n\}$ and $\{q_n\}$ be two increasing, disjoint, sequences of primes, set

$$A_n = \mathrm{M}_{p_n}(\mathbb{C}) \oplus \mathbb{C}, \qquad B_n = \mathrm{M}_{q_n}(\mathbb{C}) \oplus \mathbb{C}.$$

Then $\tilde{\mathcal{K}} \cong \varinjlim A_n \cong \varinjlim B_n$ (with appropriate choices of unital connecting *-homomorphisms $A_n \to A_{n+1}$ and $B_n \to B_{n+1}$). By Proposition 4.1 (ii) and Theorem 3.6

we have that $(A_n, B_m) \in \mathcal{C}_1$ for all n and m. Two applications of Proposition 4.1 (iii) yield $(\tilde{\mathcal{K}}, B_m) \in \mathcal{C}_1$ for all m, and $(\tilde{\mathcal{K}}, \tilde{\mathcal{K}}) \in \mathcal{C}_1$. $\square$

We have $\mathrm{K}_0(\tilde{\mathcal{K}}) = \mathbb{Z}[1]_0 + \mathbb{Z}[e]_0$, where e is a 1-dimensional projection in $\mathcal{K}$. Hence we can identify $(\mathrm{K}_0(\tilde{\mathcal{K}}), \mathrm{K}_0(\tilde{\mathcal{K}})^+)$ with $(\mathbb{Z} \oplus \mathbb{Z}, G)$, where

$$G = \{(0, y) : y \geq 0\} \cup \{(x, y) : x \geq 1\}.$$

By Corollary 4.5 we get that $(\mathrm{K}_0(\tilde{\mathcal{K}} * \tilde{\mathcal{K}}), \mathrm{K}_0(\tilde{\mathcal{K}} * \tilde{\mathcal{K}})^+)$ equals $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, H)$, where

$$H = \{(0, y, z) : y \geq 0,\ z \geq 0\} \cup \{(1, y, z) : y \geq 0 \text{ or } z \geq 0\} \cup \{(x, y, z) : x \geq 2\}.$$

We call a unital C^*-algebra A *non-divisible* if there for no integer $n \geq 2$ exists $g \in \mathrm{K}_0(A)$ such that $[1_A]_0 = ng$.

Corollary 4.6. *Every pair of unital C^*-algebras, each of which can be unitally embedded into unital, simple, non-divisible AF-algebras, belong to $\mathcal{C}_2$ (cf. the first paragraph of this section).*

Proof. Find four mutually disjoint sequences of primes $\{p_n\}$, $\{p'_n\}$, $\{q_n\}$ and $\{q'_n\}$ such that if

$$A_n = \mathrm{M}_{p_n}(\mathbb{C}) \oplus \mathrm{M}_{p'_n}(\mathbb{C}), \qquad B_n = \mathrm{M}_{q_n}(\mathbb{C}) \oplus \mathrm{M}_{q'_n}(\mathbb{C}),$$

then there exist unital connecting maps $A_n \to A_{n+1}$ and $B_n \to B_{n+1}$ which map each non-zero element of A_n, respectively, B_n, to a full element of A_{n+1}, respectively, B_{n+1}. Set $A = \varinjlim A_n$ and $B = \varinjlim B_n$. Arguing as in the proof of Corollary 4.5 we see that $(A, B) \in \mathcal{C}_2$.

The AF-algebras A and B are unital, simple and infinite-dimensional. The ordered K_0-groups of a unital, simple and infinite-dimensional AF-algebra has the property that for each non-zero positive element g and for each set of positive integers $d_1, d_2, \ldots, d_r$, with greatest common divisor equal to 1, there exist non-zero positive elements $g_1, g_2, \ldots, g_r$ such that $g = d_1 g_1 + d_2 g_2 + \cdots + d_r g_r$.

Using this property it can be shown that every unital, simple, non-divisible AF-algebra can be unitally embedded into A and B (first at the level of K-theory, and then, by the classification theorem for AF-algebras, at the level of algebras). Hence any pair (A', B') of C^*-algebras that can be unitally embedded into unital, non-divisible AF-algebras can be unitally embedded into A and B. Therefore $(A', B') \in \mathcal{C}_2$ by Proposition 4.2 (iv). $\square$

Remark 4.7. Whereas the conclusions of Corollary 4.6 may apply to a very large class of unital, separable, exact C^*-algebras, it does not give us information about infinite C^*-algebras, cf. Remark 3.3.

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