REAL RANK AND SQUARING MAPPINGS FOR UNITAL C^*-ALGEBRAS

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Abstract. It is proved that if X is a compact Hausdorff space of Lebesgue dimension $\dim(X)$, then the squaring mapping $\alpha_m\colon\ (C(X)_{\mathrm{sa}})^m \to C(X)_+$, defined by $\alpha_m(f_1,\dots,f_m) = \sum_{i=1}^m f_i^2$, is open if and only if $m-1 \geq \dim(X)$. Hence the Lebesgue dimension of X can be detected from openness of the squaring maps α_m. In the case $m = 1$ it is proved that the map $x \mapsto x^2$, from the self-adjoint elements of a unital C^*-algebra A into its positive elements, is open if and only if A is isomorphic to $C(X)$ for some compact Hausdorff space X with $\dim(X) = 0$.

1. Introduction

A compact Hausdorff space X is defined to have Lebesgue dimension $\leq m$ if for every closed subset F of X, each continuous map $F \to S^m$ has a continuous extension $X \to S^m$.

Various types of ranks for (unital) C^*-algebras have been inspired by corresponding prototypes in the classical dimension theory of (compact) spaces, such as the one given above. While the Lebesgue dimension of a compact space has numerous equivalent formulations, the extensions of these equivalent formulations to non-commutative C^*-algebras most often differ. Examples of such ranks for C^*-algebras are the *stable rank* defined by Rieffel in [7], the *real rank* defined by Brown and Pedersen in [1], the *analytic rank* defined by Murphy in [4], the *tracial rank* defined by Lin in [3], the *completely positive rank* considered by Winter in [8], and the *bounded rank* defined in [2] (see also [6] for the definition of the *exponential rank*).

It was shown in [2] that a unital C^*-algebra A has real rank at most n if the squaring map $(x_1,\dots,x_{n+1}) \mapsto \sum_{i=1}^{n+1} x_i^2$, from the set of $(n+1)$-tuples of self-adjoint elements to the set of positive elements in A, is open; and it was asked if the reverse also holds, in which case openness of the squaring maps would determine the real rank of the C^*-algebra.

In the present note we answer this question in the affirmative in the commutative case — and in the negative in the general (non-commutative) case.

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The latter — negative — answer follows from our result that the squaring map $x \mapsto x^2$ (from the set of self-adjoint elements to the set of positive elements) is open if and only if the C^*-algebra is commutative and of real rank zero. Hence the squaring map $x \mapsto x^2$ is not open for the C^*-algebra M_2 of 2 by 2 matrices, but this C^*-algebra has real rank zero.

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2. Results

For a C^*-algebra A we use the standard notation A_{sa} and A_+ to denote the set of all self-adjoint and the set of all positive elements of A, respectively. The real rank of a unital C^*-algebra A, denoted by $\mathrm{RR}(A)$, is in [1] defined as follows: For each non-negative integer n, $\mathrm{RR}(A) \leq n$ if for every $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ in A_{sa} and every $\varepsilon > 0$, there exists an $(n+1)$-tuple $(y_1, \ldots, y_{n+1})$ in A_{sa} such that $\sum_{k=1}^{n+1} y_k^2$ is invertible and $\sum_{k=1}^{n+1} \|x_k - y_k\| < \varepsilon$.

Let us say that a unital C^*-algebra A has an open m-squaring map if the map $\alpha_m \colon (A_{sa})^m \to A_+$, defined by $\alpha_m(x_1, \ldots, x_m) = \sum_{k=1}^m x_k^2$, is open. Observe that α_m is open at $(x_1, \ldots, x_m)$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $a \in A_+$ with $\|\sum_{k=1}^m x_k^2 - a\| < \delta$ there is an m-tuple $(y_1, \ldots, y_m)$ in A_{sa} with $\sum_{k=1}^m y_k^2 = a$ and $\sum_{k=1}^m \|x_k - y_k\| < \varepsilon$.

For the reader's convenience we present a shorter proof of [2, Proposition 7.1].

Proposition 2.1. *Let A be a unital C^*-algebra. If the $(n+1)$-squaring map on A is open, then $\mathrm{RR}(A) \leq n$.*

Proof. Let $(x_1, \ldots, x_{n+1})$ be an $(n+1)$-tuple of self-adjoint elements in A and let $\varepsilon > 0$. By openness of the $(n+1)$-squaring map there is $\delta > 0$ and an $(n+1)$-tuple $(y_1, \ldots, y_{n+1})$ of self-adjoint elements in A such that $\sum_{k=1}^{n+1} \|x_k - y_k\| < \varepsilon$ and $\sum_{k=1}^{n+1} y_k^2 = \sum_{k=1}^{n+1} x_k^2 + \delta \cdot 1$, and the latter element is invertible (because each x_k^2 is positive). □

Next, we prove the reverse of Proposition 2.1 in the commutative case.

Proposition 2.2. *If X is a compact space such that $\dim X \leq n$, then $C(X)$ has open $(n+1)$-squaring map.*

Proof. Let $f = f_1^2 + \cdots + f_{n+1}^2$. Put $m_i = \sup f_i$ for $i = 1, \ldots, n+1$ and let $m = \max\{m_i\}$. Fix $\varepsilon > 0$ and let

$$\delta = \min\left\{\frac{(\varepsilon/3)^4}{m^2}, \left(\frac{\varepsilon}{3}\right)^2\right\} \text{ and } U = \left\{x \in X \colon f(x) > \left(\frac{\varepsilon}{3}\right)^2\right\}.$$

Let also $A = f^{-1}([0, (\varepsilon/3)^2])$ and $S = f^{-1}((\varepsilon/3)^2)$. Then A and S are closed subsets of X such that $A = X \backslash U$ and $S \subseteq A$.

Now consider the diagonal product

$$F(x) = (f_1(x), \dots, f_{n+1}(x)) : X \to \mathbb{R}^{n+1}$$

and note that

$$A = F^{-1}\left(B^{n+1}\right) \text{ and } S = F^{-1}\left(S^n\right),$$

where

$$B^{n+1} = B^{n+1}\left(\mathbf{0}, \varepsilon/3\right) \text{ and } S^n = \partial B^{n+1}\left(\mathbf{0}, \varepsilon/3\right).$$

Since $\dim A \leq \dim X \leq n$, the map $F|S \colon S \to S^n$ admits an extension $H \colon A \to S^n$ (see, for instance, [5, Ch. 3, Theorem 2.2]). Let $h_i \colon A \to \mathbb{R}$ be the i-th component of the map H. Since $H(A) \subseteq S^n$ it follows that $h_1^2 + \dots + h_{n+1}^2 = (\varepsilon/3)^2$. Note also that since $H|S = F|S$ we have $h_i|S = f_i|S$ for each $i = 1, \dots, n+1$.

The last condition allows us to define for each $i = 1, \dots, n+1$ a continuous map $\widetilde{h}_i$ on X by letting

$$\widetilde{h}_i(x) = \begin{cases} f_i(x), & \text{if } x \in U \\ h_i(x), & \text{if } x \in A \end{cases}$$

Observe that the function $\widetilde{h} = \widetilde{h}_1^{\,2} + \dots + \widetilde{h}_{n+1}^2$ is strictly positive on X. Notice also that $\widetilde{h}|_U = f|_U$ and $\widetilde{h}|_A = (\varepsilon/3)^2$.

Take next a positive function g in $C(X)$ with $\|f - g\|_\infty < \delta$. Define a function λ on X by $\lambda(x) = \left(g(x)/\widetilde{h}(x)\right)^{1/2}$. Note that $\lambda \geq 0$.

Now define g_i for $i = 1, \dots, n+1$, on X by the formula $g_i(x) = \widetilde{h}_i(x) \cdot \lambda(x)$. Clearly

$$g_1^2(x) + \dots + g_{n+1}^2(x) = \left(\widetilde{h}_1(x) \cdot \lambda(x)\right)^2 + \dots + \left(\widetilde{h}_{n+1}(x) \cdot \lambda(x)\right)^2 =$$
$$\lambda^2(x)\left(\widetilde{h}_1^2(x) + \dots + \widetilde{h}_{n+1}^2(x)\right) = \frac{g(x)}{\widetilde{h}(x)} \cdot \left(\widetilde{h}_1^2(x) + \dots + \widetilde{h}_{n+1}^2(x)\right) = g(x).$$

Next let us show that g_i is sufficiently close to f_i for each $i = 1, \dots, n+1$. Indeed, since $\|f - g\|_\infty < \delta$ we conclude that for each $x \in A$ we have

$$g(x) < f(x) + \delta < \left(\frac{\varepsilon}{3}\right)^2 + \left(\frac{\varepsilon}{3}\right)^2.$$

Since $g_i^2 \leq g$ and $f_i^2 \leq f$, the last inequality implies that

$$|g_i(x)| < \sqrt{2\left(\frac{\varepsilon}{3}\right)^2} < 2\frac{\varepsilon}{3} \text{ and } |f_i(x)| \leq \frac{\varepsilon}{3}$$

for all $x \in A$. Hence

$$|f_i(x) - g_i(x)| \leq |f_i(x)| + |g_i(x)| < \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} = \varepsilon$$

as required.

Further, if $x \in U$, then $\widetilde{h}(x) = f(x)$ and consequently

$$\widetilde{h}(x) - \delta < g(x) < \widetilde{h}(x) + \delta.$$

Hence

$$1 - \frac{\delta}{\widetilde{h}(x)} < \frac{g(x)}{\widetilde{h}(x)} < 1 + \frac{\delta}{\widetilde{h}(x)}$$

for $x \in U$. Since

$$\widetilde{h}(x) = f(x) > \left(\frac{\varepsilon}{3}\right)^2 \text{ for } x \in U \text{ and } \delta \leq \frac{(\varepsilon/3)^4}{m^2}$$

we have (for $x \in U$)

$$1 - \left(\frac{\varepsilon}{3m}\right)^2 = 1 - \frac{\frac{1}{m^2} \cdot \left(\frac{\varepsilon}{3}\right)^4}{\left(\frac{\varepsilon}{3}\right)^2} < \frac{g(x)}{\widetilde{h}(x)} < 1 + \frac{\frac{1}{m^2} \cdot \left(\frac{\varepsilon}{3}\right)^4}{\left(\frac{\varepsilon}{3}\right)^2} = 1 + \left(\frac{\varepsilon}{3m}\right)^2$$

and

$$1 - \left(\frac{\varepsilon}{3m}\right)^2 < \lambda^2(x) < 1 + \left(\frac{\varepsilon}{3m}\right)^2.$$

Consequently,

$$\left|1 - \lambda^2(x)\right| < \left(\frac{\varepsilon}{3m}\right)^2.$$

Since $\lambda(x) \geq 0$, this implies

$$[1 - \lambda(x)]^2 \leq |1 - \lambda(x)| \cdot |1 + \lambda(x)| = |1 - \lambda^2(x)| < \left(\frac{\varepsilon}{3m}\right)^2.$$

Therefore

$$|1 - \lambda(x)| \leq \frac{\varepsilon}{3m} \ \text{ for any } \ x \in U.$$

Finally we have

$$|f_i(x) - g_i(x)| = |1 - \lambda(x)| \cdot |f_i(x)| < \frac{\varepsilon}{3m} \cdot m < \frac{\varepsilon}{3} \text{ for any } x \in U.$$

This completes the verification of the fact that $|f_i(x) - g_i(x)| < \varepsilon$ for each $x \in X$ and any $i = 1, \dots, n+1$. □

Corollary 2.3. *Let A be a unital C^*-algebra. Then the following conditions are equivalent:*

(i) *The squaring map $x \mapsto x^2$ from A_{sa} to A_+ is open.*
(ii) *A is commutative and $\mathrm{RR}(A) = 0$.*
(iii) *A is isomorphic to a C^*-algebra of the form $C(X)$ for a compact Hausdorff space X with $\dim X = 0$.*

Proof. The equivalence of (ii) and (iii) follows from Gelfand's duality and [1, Proposition 1.1].

The implication (iii) $\Rightarrow$ (i) follows from Proposition 2.2.

(i) $\Rightarrow$ (ii). Assume that (i) holds. Then $\mathrm{RR}(A) = 0$ by Proposition 2.1. It remains to show that A is commutative. Since A is of real rank zero it suffices to show that any two projections p, q in A commute.

Take the symmetry $s = p - (1 - p)$. Then s is self-adjoint and $s^2 = 1$. By openness of the squaring map there are self-adjoint elements s_n in A such that $\|s_n - s\| \to 0$ and $s_n^2 = 1 + n^{-1}q$. Define $\varphi \colon \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \max\{0, t\}$. For each n, the element $\varphi(s_n)$ commutes with s_n, hence with s_n^2, and hence with q. Since $\varphi(s) = p$ we obtain

$$pq - qp = \lim_{n\to\infty} (\varphi(s_n)q - q\varphi(s_n)) = 0,$$

as desired. □

3. Related comments and open problems

Existence of square roots: Suppose that A is a unital C^*-algebra and that x is a self-adjoint element in A. Does there exist a continuous square root $\rho_x = \rho \colon \Omega \to A_{\mathrm{sa}}$ (i.e., $\rho(a)^2 = a$ for all $a \in \Omega$) defined on an open neighborhood $\Omega \subseteq A_+$ of x^2 such that $\rho(x^2) = x$? If this is true for all self-adjoint elements a in A, then the equivalent conditions of Corollary 2.3 are satisfied.

Suppose that $A = C(X)$ for some 0-dimensional compact Hausdorff space X (i.e., that the conditions of Corollary 2.3 are satisfied). Take a self-adjoint (i.e., real valued) $f \in C(X)$, and suppose that there is a clopen set U such that $f(x) \geq 0$ for all $x \in U$ and $f(x) \leq 0$ for all $x \in X \backslash U$. Then the function $\rho_U \colon C(X)_+ \to C(X)_{\mathrm{sa}}$ defined by

$$\rho_U(g) = \begin{cases} \sqrt{g(x)}, & x \in U, \\ -\sqrt{g(x)}, & x \in X \backslash U, \end{cases}$$

is a continuous square root with $\rho_U(f^2) = f$. It is not clear to the authors if there are continuous square roots at arbitrary real valued functions f in $C(X)$.

In the case where $A = M_n$, the C^*-algebra of n by n matrices, if x is a self-adjoint element and if x^2 has n distinct eigenvalues, then there is a continuous square root ρ with $\rho(x^2) = x$ defined on some neighborhood of x^2.

In the case where $A = M_2$, it follows from Corollary 2.3 (and its proof) that there is no continuous square root ρ defined on a neighborhood of I such that $\rho(I) = \mathrm{diag}(1, -1)$. It is easily checked explicitly that if r is a (small) non-zero real number, then any square root of $\left(\begin{smallmatrix} 1 & r \\ r & 1 \end{smallmatrix}\right)$ is of the form $\left(\begin{smallmatrix} a & s \\ s & a \end{smallmatrix}\right)$, where a and s are real numbers satisfying $a^2 + s^2 = 1$ and $2as = r$, and any such square root has distance at least 1 to $\mathrm{diag}(1, -1)$.

We end this note by listing some open problems related to openness of the squaring maps:

Question 1. Let A be a unital C^*-algebra, let m be a positive integer, and suppose that the squaring map α_m (defined above Proposition 2.1) is open. Does it follow that α_n is open for all $n \geq m$?

The answer to Question 1 is affirmative for commutative C^*-algebras by Propositions 2.1 and 2.2. The difficulty in this question lies in the fact that if Ω is an open subset of A_+ and if $a \in A_+$, then $a + \Omega$ need not be open in A_+. (For instance, $1 + A_+$ is not open in A_+.)

Question 2. Are Propositions 2.1 and 2.2 valid also in the *non-unital* case? (For Proposition 2.2, this means that we will be talking about locally compact Hausdorff spaces rather than compact Hausdorff spaces.) What is the relationship between openness of α_n on a non-unital C^*-algebra A and openness of α_n on its unitization?

Question 3. Are the squaring maps α_m open for all $m \geq 2$ when A is a unital C^*-algebra of real rank zero?

Question 4. Does the class of C^*-algebras, for which the squaring map α_2 is open, have any nice properties? More generally, are there any justifications for considering the rank of a C^*-algebra defined by openness of the squaring maps; and will this rank reflect any "dimension like" properties of the C^*-algebra?

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