The Stable rank of $C^*_{\mathrm{red}}(F_n)$ is one — A Survey

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1 Introducton

In a recent paper, [1], by Ken Dykema, Uffe Haagerup and the author of this note it was proved that any reduced free product of C^*-algebras with respect to tracial states has stable rank one, provided that the ingoing C^*-algebras satisfy a certain (mild) condition (called the Avitzour condition). Recall that a unital C^*-algebra A has stable rank one if its group $\mathrm{GL}(A)$ of invertible elements is a norm dense subset of A. This definition is due to Marc Rieffel, [4], who associates the stable rank, a number in $\{1, 2, 3, \dots\} \cup \{\infty\}$, to each C^*-algebra, as an analogue of dimension for topological spaces.

Rieffel posed in his paper the problem of calculating the stable rank of some concrete (simple, finite) C^*-algebras of interest, namely the irrational rotation C^*-algebras and the C^*-algebra $C^*_{\mathrm{red}}(F_n)$ arising from the free group F_n of n generators, where $2 \le n \le \infty$. Ian Putnam, [3], settled the first question by proving that all the irrational rotation C^*-algebras have stable rank one. It is proved in [1], as a corollary to its main theorem, that $C^*_{\mathrm{red}}(F_n)$ has stable rank one for all $2 \le n \le \infty$.

The purpose of this note is to give a direct and self contained proof of this corollary. The proof given here does not contain any new ideas, not already contained in [1], and the papers it is based upon ([2] and [5]), but it is shorter, and perhaps also less technical, having the privilege of dealing only with a special case of main theorem of [1].

It is a commonly asked question if every finite, simple C^*-algebra has stable rank one. (A unital C^*-algebra is said to be finite if it contains no non-unitary isometries.) In Section 5 we give an example that shows that the approach taken in [1] cannot be generalized (in any obvious way) to settle this conjecture for all finite, simple C^*-algebras. This example, I believe, has not been published before.

After a first draft of this note was written, Jesper Villadsen has constructed an example of a finite, simple, unital C^*-algebra which is not of stable rank one ([6]).

2 The distance to the invertible elements

We shall in this section give a direct proof of a theorem from [5] that states that if the set of invertible elements in a unital C^*-algebra is not dense, then the C^*-algebra contains an element with the largest possible distance to the invertibles.

Let A be a unital C^*-algebra, and denote, as above, the group of invertible elements in A by $\mathrm{GL}(A)$. Upon representing A faithfully on a Hilbert space H, we may assume that $A \subseteq B(H)$. Each element $a \in A$ has a polar decomposition $a = v|a|$, where v is a partial isometry in $B(H)$ and $|a| = (a^*a)^{1/2} \in A$. For each $\alpha > 0$, define projections

$$p_\alpha = 1_{(\alpha,\infty)}(|a|), \qquad q_\alpha = 1_{(\alpha,\infty)}(|a^*|),$$

on H. Observe that $vp_\alpha v^* = q_\alpha$.

Lemma 2.1 *If $vp_\alpha = yp_\alpha$ for some $\alpha > 0$ and for some $y \in \mathrm{GL}(A)$, then $\mathrm{dist}(a, \mathrm{GL}(A)) \leq \alpha$.*

Proof: Define $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ by $f(t) = \max\{0, t - \alpha\}$. Then $(b =)\ vf(|a|) = yf(|a|)$. Since $y(f(|a|) + \varepsilon \cdot 1) \in \mathrm{GL}(A)$ for all $\varepsilon > 0$, we see that b belongs to the closure of $\mathrm{GL}(A)$. Hence

$$\mathrm{dist}(a, \mathrm{GL}(A)) \leq \|a - b\| = \|v(|a| - f(|a|))\| = \sup_{t \in (|a|)} |t - f(t)| \leq \alpha.$$

□

For each $\alpha > 0$, let $g_\alpha \colon \mathbb{R}^+ \to \mathbb{R}^+$ be the function given by $g_\alpha(t) = \min\{1, \alpha^{-1}t\}$. Set $b_\alpha = vg_\alpha(|a|) \in A$.

Lemma 2.2 *If $0 < \alpha < \beta$ and if $\|b_\alpha - y\| < 1$ for some $y \in \mathrm{GL}(A)$, then there exists $y' \in \mathrm{GL}(A)$ such that $vp_\beta = y'p_\beta$.*

Proof: Observe first that

$$\begin{aligned}\|(1-y^*v)p_\alpha\| &= \|(v^*-y^*)vp_\alpha\| = \|(v^*-y^*)q_\alpha v\| \\ &= \|(v^*g_\alpha(|a^*|)-y^*)q_\alpha\| \leq \|v^*g_\alpha(|a^*|)-y^*\| \\ &= \|b_\alpha^*-y^*\| = \|b_\alpha - y\| < 1.\end{aligned}$$

Let $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying $0 \leq h \leq 1$, h is zero on the interval $[0,\alpha]$, and h is equal to 1 on the interval $[\beta,\infty)$. Put

$$z = (1-y^*v)h(|a|) = (1-y^*v)p_\alpha h(|a|).$$

Then $z \in A$ because $h(0)=0$. Moreover, $\|z\| \leq \|(1-y^*v)p_\alpha\| < 1$ and $zp_\beta = (1-y^*v)p_\beta$. Hence $1-z \in \mathrm{GL}(A)$ and $(1-z)p_\beta = y^*vp_\beta$. It follows that $vp_\beta = y'p_\beta$, when $y' = (y^*)^{-1}(1-z) \in \mathrm{GL}(A)$.

□

Theorem 2.3 ([5, Theorem 2.6]) *If A has stable rank not equal to one, then there exists an element x in A with*

$$\mathrm{dist}(x,\mathrm{GL}(A)) = \|x\| = 1.$$

Proof: Assume that A is a unital C^*-algebra of stable rank different from one. Then there is an element $a \in A$ not in the closure of $\mathrm{GL}(A)$. Choose α,β such that

$$0 < \alpha < \beta < \mathrm{dist}(a,\mathrm{GL}(A)).$$

If $\|b_\alpha - y\| < 1$ for some $y \in \mathrm{GL}(A)$, then $vp_\beta = y'p_\beta$ for some $y' \in \mathrm{GL}(A)$ by Lemma 2.2. However, from Lemma 2.1, this would entail that $\mathrm{dist}(a,\mathrm{GL}(A)) \leq \beta$, in contradiction with the choice of β.

It follows that $\mathrm{dist}(b_\alpha,\mathrm{GL}(A)) \geq 1$. In combination with the obvious fact that $\|b_\alpha\| \leq 1$, this yields that $\|x\| = \mathrm{dist}(x,\mathrm{GL}(A)) = 1$, when $x = b_\alpha$.

□

3 A norm estimate

The two first lemmas of this section are from Uffe Haagerup's paper [2]. Proposition 3.3 is an easy consequence of these two lemmas, and it will, together with Theorem 2.3, go into the proof of Theorem 4.2.

Let e_i denote the generators of F_n. Each element of the free group F_n is a finite word in e_i and their inverses. Such a word in F_n is called *reduced* if no occurrence of e_i follows or is followed by e_i^{-1}. The length of a $g \in F_n$, which is written $l(g)$, is the number of factors e_i or e_i^{-1} appearing in the reduced representation of g.

Let $\lambda\colon F_n \to B(\ell^2(F_n))$ be the left regular representation. Set

$$\mathfrak{A}_n = \operatorname{span}\{\lambda(g) \mid g \in F_n\}, \quad \mathfrak{A}_n^{(j)} = \operatorname{span}\{\lambda(g) \mid g \in F_n,\, l(g) = j\}.$$

Then $C^*_{\mathrm{red}}(F_n)$ is the norm closure of $\mathfrak{A}_n$.

$C^*_{\mathrm{red}}(F_n)$ has a unique trace τ, which gives rise to an inner product on $C^*_{\mathrm{red}}(F_n)$ defined by $\langle a, b\rangle = \tau(b^*a)$ and to the norm $\|a\|_2 = \langle a, a\rangle^{1/2}$. Denote by E_j the orthogonal projection from $\mathfrak{A}_n$ onto $\mathfrak{A}_n^{(j)}$.

Lemma 3.1 ([2, Lemma 1.3]) *Let $a \in \mathfrak{A}_n^{(k)}$ and $b \in \mathfrak{A}_n^{(l)}$ be given, and let $j \in \mathbb{N}$. Then*

$$\|E_j(ab)\|_2 \leq \|a\|_2\|b\|_2.$$

Moreover, if $j < |k-l|$, if $j > k+l$, or if $k+l-j$ is odd, then $E_j(ab) = 0$.

Proof: The last claims (about $E_j(ab)$ being zero) follow from the fact that if $g, h \in F_n$, then $l(gh) = l(g) + l(h) - 2m$ for some $0 \leq m \leq \min\{l(g), l(h)\}$.

Suppose now that $|k-l| \leq j \leq k+l$ and that $k+l-j = 2m$ for some $m \in \mathbb{N}$. Let $g, h \in F_n$ with $l(g) = k$ and $l(h) = l$. Write $g = g_1g_2$ and $h = h_2h_1$ as reduced words with $l(g_2) = l(h_2) = m$ (and consequently, $l(g_1) = k-m$ and $l(h_1) = l-m$). Then $l(gh) = j$ if and only if $g_2 = h_2^{-1}$ and g_1h_1 is reduced. Hence

$$\begin{aligned} E_j(\lambda(gh)) &= \langle \lambda(g_2), \lambda(h_2^{-1})\rangle E_j(\lambda(g_1h_1)) \\ &= \begin{cases} \langle \lambda(g_2), \lambda(h_2^{-1})\rangle \lambda(g_1h_1), & \text{if } g_1h_1 \text{ is reduced,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Write

$$a = \sum \alpha_{g_1 g_2} \lambda(g_1 g_2), \qquad b = \sum \beta_{h_2 h_1} \lambda(h_2 h_1),$$

summing over all g_1, g_2, respectively h_1, h_2, such that $l(g_1) = k - m$, $l(h_1) = l - m$, $l(g_2) = l(h_2) = m$ and such that $g_1 g_2$ and $h_2 h_1$ are reduced. Then,

$$\begin{aligned}
\|E_j(ab)\|_2^2 &= \|\sum_{g_1,h_1} \big(\sum_{g_2,h_2} \alpha_{g_1g_2}\beta_{h_2h_1}\langle\lambda(g_2),\lambda(h_2^{-1})\rangle\big)E_j(\lambda(g_1h_1))\|_2^2 \\
&\leq \sum_{g_1,h_1} \big|\sum_{g_2,h_2} \alpha_{g_1g_2}\beta_{h_2h_1}\langle\lambda(g_2),\lambda(h_2^{-1})\rangle\big|^2 \\
&= \sum_{g_1,h_1} \big|\big\langle \sum_{g_2} \alpha_{g_1g_2}\lambda(g_2), \big(\sum_{h_2} \beta_{h_2h_1}\lambda(h_2)\big)^*\big\rangle\big|^2 \\
&\leq \sum_{g_1,h_1} \big\|\sum_{g_2} \alpha_{g_1g_2}\lambda(g_2)\big\|_2^2 \cdot \big\|\sum_{h_2} \beta_{h_2h_1}\lambda(h_2)\big\|_2^2 \\
&= \sum_{g_1,h_1} (\sum_{g_2} |\alpha_{g_1g_2}|^2)(\sum_{h_2} |\beta_{h_2h_1}|^2) \\
&= \sum_{g_1,g_2}\sum_{h_1,h_2} |\alpha_{g_1g_2}|^2|\beta_{h_2h_1}|^2 \\
&= \|a\|_2^2\|b\|_2^2.
\end{aligned}$$

□

Lemma 3.2 ([2, Lemma 1.4]) *For each* $k \in \mathbb{N}$ *and for each* $a \in \mathrm{span}\mathfrak{A}_n^{(k)}$,

$$\|a\| \leq (2k+1)\|a\|_2.$$

Proof: It suffices to show that

$$\|ab\|_2 \leq (2k+1)\|a\|_2\|b\|_2$$

for all $b \in \mathfrak{A}_n$. Put $b_l = E_l(b)$. It follows from Lemma 3.1 that

$$\begin{aligned} \|E_j(ab)\|_2 &= \Big\| \sum_{l=|j-k|}^{j+k} E_j(ab_l) \Big\|_2 \le \sum_{l=|j-k|}^{j+k} \|E_j(ab_l)\|_2 \\ &\le \sum_{l=|j-k|}^{j+k} \|a\|_2 \|b_l\|_2 \le (2k+1)^{1/2} \|a\|_2 \Big(\sum_{l=|j-k|}^{j+k} \|b_l\|_2^2 \Big)^{1/2}. \end{aligned}$$

Hence

$$\begin{aligned} \|ab\|_2^2 &= \sum_{j=0}^{\infty} \|E_j(ab)\|_2^2 \le (2k+1)\|a\|_2^2 \sum_{j=0}^{\infty} \sum_{l=|j-k|}^{j+k} \|b_l\|_2^2 \\ &\le (2k+1)^2 \|a\|_2^2 \sum_{j=0}^{\infty} \|b_j\|_2^2 = (2k+1)^2 \|a\|_2^2 \|b\|_2^2. \end{aligned}$$

□

Proposition 3.3 (c.f. [1, Lemma 3.5]) *For each* $k \in \mathbb{N}$ *and each* $a \in \mathrm{span}_{j \le k} \mathfrak{A}_n^{(j)}$,

$$\|a\| \le (2k+1)^{3/2} \|a\|_2.$$

Proof: Put $a_j = E_j(a)$. It follows from Lemma 3.2 that

$$\begin{aligned} \|a\| &= \Big\| \sum_{j=0}^{k} a_j \Big\| \le \sum_{j=0}^{k} \|a_j\| \\ &\le \sum_{j=0}^{k} (2j+1) \|a_j\|_2 \le (2k+1) \sum_{j=0}^{k} \|a_j\|_2 \\ &\le (2k+1)(k+1)^{1/2} \sum_{j=0}^{k} \|a_j\|_2^2 = (2k+1)(k+1)^{1/2} \|a\|_2 \\ &\le (2k+1)^{3/2} \|a\|_2. \end{aligned}$$

□

One can replace the constant $2k+1$ in Lemma 3.2 and in Proposition 3.3 with $k+1$ by using that $E_j(ab) = 0$ whenever $k+l-j$ is odd (c.f. Lemma 3.1).

4 The stable rank of $C^*_{\mathrm{red}}(F_n)$

As in Section 3 let $\mathfrak{A}_n$ denote the dense subalgebra of $C^*_{\mathrm{red}}(F_n)$ spanned by $\lambda(g)$, $g \in F_n$, where $2 \le n \le \infty$.

Lemma 4.1 (c.f. [1, Lemma 3.7]) *For each $a \in \mathfrak{A}_n$ there exist unitaries $u, v \in \mathfrak{A}_n$ such that $\|(uav)^m\|_2 = \|a\|_2^m$ for all $m \in \mathbb{N}$.*

Proof: As before we let e_i denote the generators of F_n. It follows by the property of the free groups that if $g_1, g_2, \ldots, g_m$ and $h_1, h_2, \ldots, h_m$ are elements in F_n all of length $\le k$, and if

$$e_1^{2k+1} g_1 e_2^{k+1} e_1^{2k+1} g_2 e_2^{k+1} \cdots e_1^{2k+1} g_m e_2^{k+1} = e_1^{2k+1} h_1 e_2^{k+1} e_1^{2k+1} h_2 e_2^{k+1} \cdots e_1^{2k+1} h_m e_2^{k+1},$$

then $g_1 = h_1,\ g_2 = h_2,\ \ldots\ g_m = h_m$.

We can find $k \in \mathbb{N}$ such that $a \in \mathrm{span}_{j \le k} \mathfrak{A}_n^{(j)}$. Put $u = \lambda(e_1^{2k+1})$ and put $v = \lambda(e_2^{k+1})$. Write $a = \sum_{l(g) \le k} \alpha_g \lambda(g)$. Then

$$uav = \sum_{l(g) \le k} \alpha_g \lambda(e_1^{2k+1} g e_2^{k+1}),$$

and consequently

$$(uav)^m = \sum_{g_1} \sum_{g_2} \cdots \sum_{g_m} \alpha_{g_1} \alpha_{g_2} \cdots \alpha_{g_m} \lambda(e_1^{2k+1} g_1 e_2^{k+1} e_1^{2k+1} g_2 e_2^{k+1} \cdots e_1^{2k+1} g_m e_2^{k+1}).$$

Since all g_j have length $\le k$ the argument in the first paragraph shows that the group elements appearing in the expression above for $(uav)^m$ are mutually distinct. It therefore follows that

$$\begin{aligned} \|(uav)^m\|_2^2 &= \sum_{g_1} \sum_{g_2} \cdots \sum_{g_m} \left| \alpha_{g_1} \alpha_{g_2} \cdots \alpha_{g_m} \right|^2 \\ &= \Big(\sum_{g_1} |\alpha_{g_1}|^2 \Big) \cdot \Big(\sum_{g_2} |\alpha_{g_2}|^2 \Big) \cdots \Big(\sum_{g_m} |\alpha_{g_m}|^2 \Big) \;=\; \|a\|_2^{2m}. \end{aligned}$$

□

The spectral radius of an element x in a C^*-algebra will be denoted by $r(x)$. If $t > r(wx)$ for some unitary w in A, then $x - tw^* = w^*(wx - t \cdot 1)$ is invertible, and so

$$\mathrm{dist}(x, \mathrm{GL}(A)) \leq \|x - (x - tw^*)\| = t.$$

This proves that

$$\mathrm{dist}(x, \mathrm{GL}(A)) \leq \inf_{w \in U(A)} r(wx).$$

Theorem 4.2 ([1, Corollary 3.9]) *The C^*-algebras $C^*_{\mathrm{red}}(F_n)$ have stable rank one for all $2 \leq n \leq \infty$,*

Proof: We begin by proving that

$$\mathrm{dist}(a, \mathrm{GL}(C^*_{\mathrm{red}}(F_n))) \leq \|a\|_2$$

for all $a \in C^*_{\mathrm{red}}(F_n)$. By continuity it suffices to prove this for $a \in \mathfrak{A}_n$. Let $u, v \in \mathfrak{A}_n$ be as in Lemma 4.1. Then $uav \in \mathrm{span}_{j \leq k} \mathfrak{A}_n^{(j)}$ for some k, and $(uav)^m \in \mathrm{span}_{j \leq mk} \mathfrak{A}_n^{(j)}$. By Proposition 3.3 and Lemma 4.1 we get

$$\begin{aligned} d(a, \mathrm{GL}(C^*_{\mathrm{red}}(F_n))) &\leq r(vua) = r(uav) \\ &= \liminf_{m \to \infty} \|(uav)^m\|^{1/m} \\ &\leq \liminf_{m \to \infty} (2mk+1)^{3/2m} \|(uav)^m\|_2^{1/m} = \|a\|_2. \end{aligned}$$

Now, if $C^*_{\mathrm{red}}(F_n)$ had stable rank different from one, then by Theorem 2.3 there would exist an element x in $C^*_{\mathrm{red}}(F_n)$ of norm one and distance one from the invertibles. That would imply

$$1 = \|x\| = \mathrm{dist}(x, \mathrm{GL}(C^*_{\mathrm{red}}(F_n))) \leq \|x\|_2 \leq \|x\|,$$

and hence $\|x\| = \|x\|_2 = 1$. Consequently, $\tau(1 - xx^*) = \tau(1 - x^*x) = 1 - \|x\|_2^2 = 0$, and also $1 - xx^* \geq 0$, $1 - x^*x \geq 0$. Since τ is faithful, this shows that x is unitary. But unitary elements are invertible and do not have distance one to the invertibles. □

5 An example

In the proof of Theorem 4.2 it was shown that

$$\inf_{u \in U(C^*_{\mathrm{red}}(F_n))} r(ux) \leq \|x\|_2$$

for all x in a dense sub-*-algebra of $C^*_{\mathrm{red}}(F_n)$, where $r(\cdot)$ is the spectral radius. Once this is established, density of the invertibles in $C^*_{\mathrm{red}}(F_n)$ follows easily from Theorem 2.3. One might proceed to establish this estimate on the spectral radius for general C^*-algebras (with a unique trace), or appropriate generalizations thereof. For example, it is plausible that if A is any simple, unital C^*-algebra, and if $x \in A$ is a non-zero element such that $ax = 0 = xa$ for some non-zero positive $a \in A$, then

$$\inf_{u \in U(A)} r(ux) < \|x\|.$$

One could moreover hope that this holds for general (non-simple) unital C^*-algebras provided that the element a above is assumed to be full. This is *not* the case, however, as shown in Theorem 5.1 below.

Let A be any unital C^*-algebra, let $n, k, l \in \mathbb{N}$ be such that $k + l \leq n$, and let x be any element in $M_n(A)$ such that at most k rows and l columns in the $n \times n$ matrix of x are non-zero. Then for some (unitary) permutation matrices u and v, uxv is strictly upper triangular. It follows that uxv and vux are nilpotent and hence that $r((vu)x) = 0$.

Let $k, l \in \mathbb{N}$, and let $X_{k,l}$ be the space of all complex $k \times l$-matrices of (operator) norm ≤ 1. For each $n \geq \max\{k, l\}$ let $z^{(n)}_{k,l} \in M_n(C(X_{k,l})) = C(X_{k,l}, M_n(\mathbb{C}))$ be given by

$$z^{(n)}_{k,l}(x) = \left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array} \right), \quad x \in X_{k,l}.$$

Clearly $\|z^{(n)}_{k,l}\| = 1$, and the argument above shows that if $k + l \leq n$, then $uz^{(n)}_{k,l}$ is nilpotent for some unitary $u \in M_n(C(X_{k,l}))$, and $z^{(n)}_{k,l}$ belongs to the closure of the invertible elements in $M_n(C(X_{k,l}))$.

Theorem 5.1 *If $k + l > n$, then*

$$\mathrm{dist}(z^{(n)}_{k,l}, \mathrm{GL}(M_n(C(X_{k,l})))) = \|z^{(n)}_{k,l}\| = 1.$$

In particular, $r(uz_{k,l}^{(n)}) = 1$ *for all unitaries* $u \in M_n(C(X_{k,l}))$.

Proof: We need only prove that $\|z_{k,l}^{(n)} - a\| \geq 1$ for all invertible $a \in M_n(C(X_{k,l}))$. (The formula for the spectral radius will then follow from the inequality above Theorem 4.2.)

Suppose that $a \in M_n(C(X_{k,l}))$ and that $\|z_{k,l}^{(n)} - a\| < 1$. We show that a is not invertible. Let $a_0 \in M_{k,l}(C(X_{k,l})) = C(X_{k,l}, M_{k,l}(\mathbb{C}))$ be the upper left $k \times l$ block of the matrix of a. We begin by proving that $a_0(x_0) = 0$ for some $x_0 \in X_{k,l}$.

Assume, to reach a contradiction, that $a_0(x)$ is non-zero for all $x \in X_{k,l}$. Let $Y_{k,l} \subseteq X_{k,l}$ be the set of all $k \times l$ matrices of norm equal to 1. Observe that $Y_{k,l}$ is homeomorphic to the sphere S^{2kl-1}, and that $Y_{k,l}$ therefore is not contractible. Observe also, that

$$\|x - a_0(x)\| \leq \|z_{k,l}^{(n)}(x) - a(x)\| < 1, \quad x \in X_{k,l}.$$

Define a function $f\colon Y_{k,l} \times [0,2] \to M_{k,l}(\mathbb{C})$ by

$$f(y,t) = \begin{cases} a_0(ty), & 0 \leq t \leq 1 \\ (t-1)y + (2-t)a_0(y), & 1 \leq t \leq 2 \end{cases}$$

Then f is continuous, the function $y \mapsto f(y,0)$ is constant, $f(y,2) = y$, and $f(y,t) \neq 0$ for all (y,t). The function $h\colon Y_{k,l} \times [0,2] \to Y_{k,l}$ given by $h(y,t) = f(y,t)/\|f(y,t)\|$ is therefore continuous, $y \mapsto h(y,0)$ is constant, and $h(y,2) = y$. However, no such function h exists because $Y_{k,l}$ is not contractible.

To prove that a is non-invertible, it suffices to show that $a(x_0)$ is non-invertible. Let $v_1, v_2, \ldots, v_n \in \mathbb{C}^n$ be the column vectors of $a(x_0)$. Because $a_0(x_0) = 0$, it follows that $v_1, v_2, \ldots, v_l$ all lie in an $(n-k)$-dimensional subspace of $\mathbb{C}^n$. Since $l > n - k$, the set $(v_1, v_2, \ldots, v_n)$ cannot be linearly independent, and therefore $a(x_0)$ is not invertible.

□

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