INCLUSIONS OF C^*-ALGEBRAS ARISING FROM FIXED-POINT ALGEBRAS

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Abstract. We examine inclusions of C^*-algebras of the form $A^H \subseteq A \rtimes_r G$, where G and H are groups acting on a unital simple C^*-algebra A by outer automorphisms and H is finite. It follows from a theorem of Izumi that $A^H \subseteq A$ is C^*-irreducible, in the sense that all intermediate C^*-algebras are simple. We show that $A^H \subseteq A \rtimes_r G$ is C^*-irreducible for all G and H as above if and only if G and H have trivial intersection in the outer automorphisms of A, and we give a Galois type classification of all intermediate C^*-algebras in the case when H is abelian and the two actions of G and H on A commute.

We illustrate these results with examples of outer group actions on the irrational rotation C^*-algebras. We exhibit, among other examples, C^*-irreducible inclusions of AF-algebras that have intermediate C^*-algebras that are not AF-algebras, in fact, the irrational rotation C^*-algebra appears as an intermediate C^*-algebra.

1. Introduction

Inclusions of unital simple C^*-algebras with the property that all intermediate C^*-algebras are simple were characterized and labelled C^*-irreducible in the recent paper [16] by the second named author. A well-known and classic result of Kishimoto, [13], states that whenever a group G acts by outer automorphisms on a simple C^*-algebra A, then the reduced crossed product $A \rtimes_r G$ is simple as well. It follows easily from the proof of this theorem that the inclusion $A \subseteq A \rtimes_r G$ is C^*-irreducible, when A in addition is unital, cf. [16, Theorem 5.8]. Moreover, Izumi, [12, Corollary 6.6], in the case of finite G, and Cameron and Smith, [4, Theorem 3.5], in the general case established a Galois correspondence between intermediate C^*-algebras $A \subseteq D \subseteq A \rtimes_r G$ and subgroups L of G, via $L \mapsto D = A \rtimes_r L$.

It was observed by Rosenberg, [17], that if H is any finite group acting (outer or not) on any C^*-algebra A, then A^H is isomorphic to a hereditary sub-C^*-algebra of $A \rtimes H$. In particular, if A is simple and the action of H on A is by outer automorphisms, then A^H is simple. A result of Izumi, [12, Corollary 6.6], shows that the inclusion $A^H \subseteq A$ then is C^*-irreducible and that all intermediate algebras are of the form $A^H \subseteq A^L \subseteq A$ for subgroups L of H. This mirrors the situation of crossed products by finite groups, and Izumi indeed directly relates the fixed-point algebra inclusion to the corresponding crossed-product inclusion via a version of Jones basic construction (see [12, Corollary 3.12]).

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Bisch and Haagerup considered in their paper [3] subfactors of the form $P^H \subseteq P \rtimes G$ arising from outer actions of two finite groups H and G on a II_1-factor P. They show that certain properties of the resulting subfactors (finite depth, respectively, amenability) are precisely mirrored by properties of the subgroup of $\mathrm{Out}(P)$ generated by H and G. They also show that the inclusion $P^H \subseteq P \rtimes G$ is irreducible if and only if G and H intersect trivially in $\mathrm{Out}(P)$.

Specifially, as stated in the abstract, we prove in this paper that if α and β are actions of groups G and H on a unital simple C^*-algebra A, and if H is finite, then the inclusion $A^H \subseteq A \rtimes_r G$ is C^*-irreducible if and only if $\alpha_s \circ \beta_t$ is outer for all $(s,t) \in G \times H$ with $(s,t) \neq (e_G, e_H)$. This condition is an exact translation to the realm of C^*-algebras of the Bisch-Haagerup condition ensuring irreducibility in the subfactor case. In the case where H is abelian and the two actions α and β commute, we further establish a Galois correspondence between intermediate C^*-algebras of the inclusion $A^H \subseteq A \rtimes_r G$ and subgroups of $\widehat{H} \times G$, where $\widehat{H}$ denotes the Pontryagin dual of H. Clearly, A itself is an intermediate C^*-algebra of this inclusion.

We apply our results to some well-known outer actions of finite and infinite cyclic groups on the irrational rotation C^*-algebra A_θ. There is a canonical (outer) action of the group $\mathrm{SL}(2,\mathbb{Z})$ on A_θ. It is known that $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are finite cyclic subgroups of $\mathrm{SL}(2,\mathbb{Z})$, and in fact the only ones, up to conjugacy. The corresponding actions of these finite cyclic groups on A_θ were studied in [10], and it was shown therein, that the fixed-point algebra and the crossed product of A_θ by each of these groups gives rise to a simple AF-algebra. We use this, and our main result stated above, to show that when F_1 and F_2 are (certain) combinations of the groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$, then $A_\theta^{F_1} \subseteq A_\theta \rtimes F_2$ is a C^*-irreducible inclusion of simple AF-algebras admitting a non-AF intermediate C^*-algebra, namely A_θ. This answers in the negative Question 6.11 from [16]. We also study several interesting examples of C^*-irreducible inclusions which involve actions of the integer group $\mathbb{Z}$.

The paper is organized as follows. In Section 2 we collect some well-known and some new results about outer actions of groups on C^*-algebras. In Section 3 we prove our main result on C^*-irreducibility of inclusions of the form $A^H \subseteq A \rtimes_r G$, and in Section 4 we establish the Galois correspondence for the intermediate subalgebras of these inclusions (under the assumptions stated above). Finally, in Section 5 we provide examples of our main results relating to actions on the irrational rotation C^*-algebras.

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2. Outer actions on fixed-point algebras

In this section we derive some preliminary results on outer actions of a discrete group G on a C^*-algebra A. The C^*-algebra A may or may not be unital, and if it is not unital we shall consider its multiplier algebra $M(A)$. For a unital C^*-algebra A we let $U(A)$ denote its group of unitary elements.

We shall repeatedly use the classic result by Kishimoto from [13, Theorem 3.1] mentioned in the introduction that if $\alpha \colon G \to \mathrm{Aut}(A)$ is an action of a discrete group

G by outer automorphisms on a simple C^*-algebra A, then the reduced crossed product $A \rtimes_{\alpha,r} G$ is simple as well. We shall often write $A \rtimes_\alpha G$ instead of $A \rtimes_{\alpha,r} G$ if G is known to be amenable (in particular, if G is abelian or finite), since then the full and reduced crossed products coincide. Also, we may write $A \rtimes_r G$ instead of $A \rtimes_{\alpha,r} G$ if the action α is understood. Recall that if G is discrete there is always a canonical inclusion $A \subseteq A \rtimes_{\alpha,r} G$ together with a canonical unitary representation $u\colon G \to UM(A \rtimes_{\alpha,r} G)$ implementing the action α, i.e., $\alpha_g = \operatorname{Ad} u_g$, for $g \in G$. The *algebraic crossed product*

$$A \rtimes_{\alpha,\mathrm{alg}} G := \Big\{ \sum_{g\in G} a_g u_g : a_g \in A, a_g = 0 \text{ for all but finitely many } g \Big\}$$

becomes a dense subalgebra of $A \rtimes_{\alpha,r} G$, and the two algebras coincide if G is finite.

Recall that an action α is *outer* if no α_g is inner, for $g \neq e$, that is $\alpha_g \neq \operatorname{Ad} v$ for all unitaries $v \in M(A)$. On the other extreme, if the action $\alpha\colon G \to \operatorname{Aut}(A)$ is implemented by a unitary representation $v\colon G \to UM(A)$ such that $\alpha_g = \operatorname{Ad} v_g$, for all $g \in G$, we have

$$A \rtimes_{\alpha,r} G \cong A \rtimes_{\mathrm{id},r} G \cong A \otimes C_r^*(G)$$

where the first isomorphism is the extension of the map

$$A \rtimes_{\alpha,\mathrm{alg}} G \to A \rtimes_{\mathrm{id},\mathrm{alg}} G\colon a_g u_g \mapsto (a_g v_g) u_g.$$

We use these results to prove

Lemma 2.1. *Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a discrete group on a simple C^*-algebra A. Then the following are equivalent:*

(i) *The action α is outer.*
(ii) *For all subgroups H of G, the crossed product $A \rtimes_{\alpha,r} H$ is simple.*
(iii) *For all (finite or infinite) cyclic subgroups $C_g := \langle g \rangle$ of G, the crossed product $A \rtimes_\alpha C_g$ is simple.*

Proof. The implication (i) $\Rightarrow$ (ii) is a direct consequence Kishimoto's theorem, since outerness of α implies outerness of the restriction of α to any subgroup of G. The implication (ii) $\Rightarrow$ (iii) is trivial. Thus it suffices to prove (iii) $\Rightarrow$ (i).

So assume that (iii) holds for all $g \in G$. If α is not outer, there exists an element $e \neq g \in G$ such that $\alpha_g(a) = \operatorname{Ad} u(a) = uau^*$ for some unitary element $u \in M(A)$. Let C_g be the cyclic subgroup of G generated by g. Suppose first that g has infinite order. Since $\alpha_{g^n} = \operatorname{Ad} u^n$ for all $n \in \mathbb{Z}$, it follows that the restriction of α to $C_g \cong \mathbb{Z}$ is implemented by the unitary representation $n \mapsto u^n \in UM(A)$, and hence we get

$$A \rtimes_\alpha C_g \cong A \otimes C^*(C_g) \cong A \otimes C^*(\mathbb{Z}) \cong A \otimes C(\mathbb{T}),$$

which is certainly not simple.

On the other hand, if C_g is cyclic of order $m \in \mathbb{N}$, then $\operatorname{Ad} u^m = \alpha_e = \mathrm{id}_A$. It follows from simplicty of A that $A' \cap M(A) = \mathbb{C}$, so there must exists $\omega \in \mathbb{T}$ such that $u^m = \omega 1$. Now, if $\eta \in \mathbb{T}$ is an mth root of $\bar{\omega}$, we see that $g^k \mapsto (\eta u)^k \in UM(A)$ implements a homomorphism $\tilde{u}\colon C_g \to UM(A)$ such that $\alpha|_{C_g} = \operatorname{Ad} \tilde{u}$, and hence

$$A \rtimes_\alpha C_g \cong A \otimes C^*(C_g) \cong A \otimes \mathbb{C}^m,$$

which is not simple. □

Remark 2.2. In general, outerness for an action $\alpha\colon G \to \operatorname{Aut}(A)$ on a simple C^*-algebra A (unital or not) is not equivalent to $A \rtimes_{\alpha,r} G$ being simple, even if G is finite and abelian and A is simple and unital. To construct a counterexample, let H be any finite abelian group. Let $G := H \times \widehat{H}$ be the direct product of H with its dual group $\widehat{H}$. For each pair $(g,x) \in H \times \widehat{H}$ let $V_{(g,x)}$ be the unitary operator on $\ell^2(H)$ defined by

$$\big(V_{(g,x)}\xi\big)(h) = \overline{\langle h, x\rangle}\xi(g^{-1}h),$$

where $\langle\,\cdot\,,\,\cdot\,\rangle\colon H \times \widehat{H} \to \mathbb{T}$ denotes the canonical pairing between H and $\widehat{H}$. A short computation then shows that $V\colon H \times \widehat{H} \to U(\ell^2(H))$ is a projective representation such that

$$V_{(g_1,x_1)}V_{(g_2,x_2)} = \langle g_1, x_2\rangle V_{(g_1g_2,x_1x_2)},$$

for all $(g_1,x_1),(g_2,x_2) \in H \times \widehat{H}$. Thus, V is an ω-representation of the Heisenberg-type 2-cocycle $\omega\colon H \times \widehat{H} \to \mathbb{T}$ defined by $\omega\big((g_1,x_1),(g_2,x_2)\big) = \langle g_1, x_2\rangle$. Let $C^*(H \times \widehat{H}, \omega)$ denote the twisted group algebra of $H \times \widehat{H}$ with respect to the cocycle ω (see, e.g., [7, Section 2.8.6] for the construction). Since ω is totally skew in the sense of [1, p. 300] it follows from [1, Theorem 3.3] that V is the unique irreducible ω-representation of $H \times \widehat{H}$, which then implements an isomorphism $C^*(H \times \widehat{H}, \omega) \cong B(\ell^2(H)) \cong M_{|H|}(\mathbb{C})$.

Now let $A := B(\ell^2(H))$ and define $\beta\colon H \times \widehat{H} \to \operatorname{Aut}(A)$ by $\beta_{(g,x)} = \operatorname{Ad} V^*_{(g,x)}$. Then one checks that $A \otimes C^*(H \times \widehat{H}, \omega)$ is isomorphic to $A \rtimes_\beta (H \times \widehat{H})$ via the map $a \otimes \delta_{(g,x)} \mapsto aV_{(g,x)}u_{(g,x)}$ (see, e.g., [7, Remark 2.8.18]). Thus β is an action by inner automorphisms on the simple unital C^*-algebra $A = M_{|H|}(\mathbb{C})$ for which $A \rtimes_\beta (H \times \widehat{H}) \cong M_{|H|}(\mathbb{C}) \otimes M_{|H|}(\mathbb{C})$ is simple.

3. C^*-irreducible inclusions arising from fixed-point algebras into crossed products

We shall here prove our main results regarding C^*-irreducibility of inclusions arising from fixed-point algebras into crossed products. Let H be a finite group and let $\beta\colon H \to \operatorname{Aut}(A)$ be an action of H on the C^*-algebra A. Let

$$A^{H,\beta} := \{a \in A : \beta_h(a) = a \text{ for all } h \in H\}$$

(or simply A^H if confusion seems unlikely) be the fixed-point algebra of β. Consider the projection

$$p^\beta := \frac{1}{|H|}\sum_{h\in H} u_h \in M(A \rtimes_\beta H), \tag{3.1}$$

where $u\colon H \to UM(A \rtimes_\beta H)$ denotes the canonical unitary representation which implements β in the crossed-product. Note that p^β commutes with A^H. Rosenberg observed in [17] that the image of the *-homomorphism $A^H \ni a \mapsto ap^\beta = \frac{1}{|H|}\sum_{h\in H} au_h \in A \rtimes_\beta H$ is equal to $p^\beta(A \rtimes_\beta H)p^\beta$, so that we get an isomorphism

$$A^H \cong p^\beta(A \rtimes_\beta H)p^\beta. \tag{3.2}$$

We say that β is *saturated* if $Ap^\beta A$ (or p^β, if A is unital) is *full* in $A \rtimes_\beta H$, i.e., not contained in any proper closed two-sided ideal in $A \rtimes_\beta H$. Of course, this always holds if the crossed product $A \rtimes_\beta H$ is simple. The following result is then a direct consequence of Izumi's [12, Corollary 6.6].

Theorem 3.1 (Izumi)**.** *Let $\beta\colon H \to \mathrm{Aut}(A)$ be an outer action of a finite group H on a unital C^*-algebra A. Then the inclusion $A^{H,\beta} \subseteq A$ is C^*-irreducible, and the intermediate algebras of the inclusion are precisely the fixed-point algebras $A^{L,\beta}$ for the subgroups $L \subseteq H$.*

The following lemma is a modification of [13, Lemma 3.2] by Kishimoto. We are grateful to Masaki Izumi for pointing out to us a modification of our original argument which assumed, in addition to the assumptions given in the lemma, that α_j commutes with β_t, for all $1 \leq j \leq n$ and $t \in H$.

Lemma 3.2. *Let A be a unital simple C^*-algebra, let $\beta\colon H \to \mathrm{Aut}(A)$ be an action of a finite group H on A. Let $\alpha_1, \dots, \alpha_n$ be automorphisms of A, and let $a_1, \dots, a_n \in A$ and $\varepsilon > 0$ be given. Suppose that $\alpha_j \circ \beta_t$ is outer on A, for all $1 \leq j \leq n$ and for all $t \in H$.*

Then there exists a positive element $h \in A^H$ with $\|h\| = 1$ such that $\|h a_j \alpha_j(h)\| \leq \varepsilon$, for all $j = 1, \dots, n$.

Proof. First observe that $\alpha_j \circ \beta_t$ is outer for all $t \in H$ implies that $\beta_{s^{-1}} \circ \alpha_j \circ \beta_t$ is outer as well, for all $s, t \in H$, which follows from the fact that the conjugate of an outer automorphism by an arbitrary automorphism remains outer.

It follows then from [13, Lemma 3.2] that there exists a positive element $h_0 \in A$ with $\|h_0\| = 1$ and

$$\|h_0 \beta_{s^{-1}}(a_j)(\beta_{s^{-1}} \circ \alpha_j \circ \beta_t)(h_0)\| \leq \varepsilon |H|^{-2}, \quad s, t \in H,\ 1 \leq j \leq n.$$

Applying the automorphism β_s to the inequality above, we obtain that $\|\beta_s(h_0) a_j \alpha_j(\beta_t(h_0))\| \leq \varepsilon |H|^{-2}$, for all $s, t \in H$ and for all $j = 1, 2, \dots, n$. Set $h_1 = |H|^{-1} \sum_{s \in H} \beta_s(h_0)$. Then h_1 is a positive element in A^H, and

$$\|h_1 a_j \alpha_j(h_1)\| \leq |H|^{-2} \sum_{s,t \in H} \|\beta_s(h_0) a_j \alpha_j(\beta_t(h_0))\| \leq \varepsilon |H|^{-2}.$$

Since $\|h_1\| \geq |H|^{-1} \|h_0\| = |H|^{-1}$, it follows that $h := \|h_1\|^{-1} h_1$ has the desired properties. □

We proceed to state our first main result characterizing when inclusions of the form $A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$ are C^*-irreducible. Thanks to some very helpful comments by Izumi we can now state this theorem in a stronger form than in a previous version of this paper, where it was assumed that the actions α and β commute and that the group H is abelian.

Theorem 3.3. *Let A be a unital, simple C^*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(A)$ be actions of a discrete group G and a finite group H. The following are equivalent:*

(i) *$A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$ is C^*-irreducible,*
(ii) *$(A^{H,\beta})' \cap (A \rtimes_{\alpha,r} G) = \mathbb{C}$,*
(iii) *the automorphisms $\alpha_g \circ \beta_t$ are outer for all $(e_G, e_H) \neq (g, t) \in G \times H$.*

Proof. (i) ⇒ (ii) follows from [16, Remark 3.8].

(ii) ⇒ (iii). Suppose that $\alpha_g \circ \beta_t$ is inner for some $(e_G, e_H) \neq (g, t) \in G \times H$. Then there is a unitary $u \in A$ such that $\beta_t = \alpha_{g^{-1}} \circ \mathrm{Ad}\, u = \mathrm{Ad}\, u_{g-1} u$ (where $g \mapsto u_g \in A \rtimes_{\alpha,r} G$ is the unitary implementation of α). Hence $u_{g^{-1}} u \in (A^H)' \cap (A \rtimes_{\alpha,r} G)$, and $u_{g^{-1}} u \notin \mathbb{C}$ since u belongs to A and $u_{g^{-1}}$ does not.

(iii) ⇒ (i). Let x be a non-zero positive element in $A \rtimes_{\alpha,r} G$. We show that x is full relatively to A^H in the sense of [16, Definition 3.4]. It follows then from [16, Proposition 3.7] that $A^H \subseteq A \rtimes_{\alpha,r} G$ is C^*-irreducible.

Let $E\colon A \rtimes_{\alpha,r} G \to A$ be the canonical conditional expectation. Then $E(x) \in A$ is non-zero and positive. Since $A^H \subseteq A$ is C^*-irreducible by Theorem 3.1 (Izumi), it follows from [16, Proposition 3.7 and Lemma 3.5] that there exist $b_1, \dots, b_n \in A^H$ such that $1_{A^H} \le \sum_{j=1}^n b_j^* E(x) b_j = \sum_{j=1}^n E(b_j^* x b_j)$. Upon replacing x with the non-zero positive element $\sum_{j=1}^n b_j^* x b_j$, we may therefore assume that $E(x) \ge 1_{A^H}$.

Let $0 < \varepsilon < 1$ be given. Choose $y = \sum_{g \in G} a_g u_g \in A \rtimes_{\mathrm{alg}} G$ such that $\|x - y\| < \varepsilon/3$. By Lemma 3.2 we can find a positive element $h \in A^H$ with $\|h\| = 1$ such that $\|h(y - E(y))h\| \le \varepsilon/3$. This implies that $\|h(x - E(x))h\| \le \varepsilon$. Note that

$$hxh \ge hE(x)h - \varepsilon \cdot 1_{A^H} \ge h^2 - \varepsilon \cdot 1_{A^H},$$

so $h^2 x h^2 \ge h^4 - \varepsilon h^2$. Let $\varphi\colon [0,1] \to \mathbb{R}^+$ be a continuous function which vanishes on $[0, \sqrt{\varepsilon}]$ and which is non-zero on $(\sqrt{\varepsilon}, 1]$. Then $d := \varphi(h)(h^4 - \varepsilon h^2)\varphi(h)$ is non-zero and $\varphi(h) h^2 x h^2 \varphi(h) \ge d > 0$. By simplicity of A^H, which follows from outerness of β, cf. the comments below (3.2), there exist $b_1, \dots, b_n \in A^H$ such that $\sum_{j=1}^n b_j^* d b_j = 1_{A^H}$. It follows that

$$\sum_{j=1}^n b_j^* \varphi(h) h^2 x h^2 \varphi(h) b_j \ge \sum_{j=1}^n b_j^* d b_j = 1_{A^H},$$

which proves that x is full relatively to A^H. □

Remark 3.4. It follows from Izumi's [12, Theorem 3.3] that an inclusion $B \subseteq A$ of simple unital C^*-algebras with a conditional expectation $E\colon A \to B$ of finite index is C^*-irreducible if (and only if) it is irreducible (i.e., $A \cap B' = \mathbb{C}$). The inclusions $A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$ considered in Theorem 3.3 do have finite index with respect to the composition of the canonical conditional expectations $E_1\colon A \rtimes_{\alpha,r} G \to A$ and $E_2\colon A \to A^{H,\beta}$ *provided that G is finite.* Hence the implication (ii) ⇒ (i) of Theorem 3.3 is a consequence of Izumi's theorem when G is finite. Note that our proof of Theorem 3.3 does not factor through Izumi's theorem.

Remark 3.5. Condition (iii) of Theorem 3.3 is equivalent to saying that the actions $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(A)$ are outer, so that G and H may be identified with subgroups of $\mathrm{Out}(A)$, the outer automorphisms on A, and that G and H intersect trivially in $\mathrm{Out}(A)$. This condition is identical with the condition in [3, Corollary 4.1 (i)] of Bisch and Haagerup ensuring irreducibility of an inclusion $P^H \subseteq P \rtimes G$ of II_1 factors arising from finite groups G and H acting outerly on a II_1 factor P.

4. A Galois correspondence for the intermediate subalgebras

In this section we shall establish a Galois type classification of the intermediate subalgebras of the inclusions considered in Theorem 3.3 under the additional assumptions that the two actions α and β commute and that H is abelian.

Let us first recall that if $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(A)$ are outer actions on a simple unital C^*-algebra A with G discrete and H finite, then the intermediate algebras of the inclusions $A^{H,\beta} \subseteq A$ and $A \subseteq A \rtimes_{\alpha,r} G$ are in one-to-one correspondence to subgroups $L \subseteq H$ and $K \subseteq G$ by taking the fixed-point algebras

$A^{L,\beta}$ and the crossed products $A \rtimes_{\alpha,r} K$, respectively, as shown by Izumi, [12], and Cameron-Smith, [4].

At present time it is not clear to us how one can describe all intermediate algebras of an inclusion $A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$ in the general setting of Theorem 3.3, but we can give a satisfactory answer in the case where H is abelian and the actions α and β commute. Note that in the abelian case there is a bijection between subgroups L of H and subgroups of the Pontryagin dual $\widehat{H} = \mathrm{Hom}(H, \mathbb{T})$ given by $L \mapsto L^\perp$, where

$$L^\perp := \{x \in \widehat{H} : \langle \ell, x\rangle = 1 \text{ for all } \ell \in L\}. \tag{4.1}$$

Suppose now that $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(A)$ are *commuting* actions of discrete groups G and H on a simple C^*-algebra A. Then we get an action

$$\alpha \times \beta\colon G \times H \to \mathrm{Aut}(A); \qquad (\alpha \times \beta)_{(g,h)} := \alpha_g \circ \beta_h, \quad (g, h) \in G \times H.$$

We shall more than once use the fact that if α and β are commuting actions as above, then β extends naturally to an action $\tilde{\beta}$ on $A \rtimes_{\alpha,r} G$ given, for $h \in H$ and $\sum_{g\in G} a_g u_g \in A \rtimes_{\alpha,\mathrm{alg}} G$, by

$$\tilde{\beta}_h\Big(\sum_{g\in G} a_g u_g\Big) = \sum_{g\in G} \beta_h(a_g) u_g.$$

The following lemma is well-known to experts (e.g., see [9, Lemma 2.9], where a more general result is shown for full crossed products). For completeness we include the easy proof.

Lemma 4.1. *Suppose that $\alpha \times \beta\colon G \times H \to \mathrm{Aut}(A)$ is an action of the discrete product group $G\times H$, as above, with H is finite. Suppose further that $\beta\colon H \to \mathrm{Aut}(A)$ is saturated. Then the following hold:*

(i) *The fixed-point algebra $A^{H,\beta}$ is a G-invariant subalgebra of A, and α therefore restricts to a well-defined action $\alpha^H : G \to \mathrm{Aut}(A^{H,\beta})$;*
(ii) *the natural extension of β to $\tilde{\beta}\colon H \to \mathrm{Aut}(A \rtimes_{\alpha,r} G)$ is also saturated;*
(iii) *the canonical inclusion $A^{H,\beta} \rtimes_{\alpha^H,r} G \hookrightarrow A \rtimes_{\alpha,r} G$ co-restricts to an isomorphism*

$$A^{H,\beta} \rtimes_{\alpha^H,r} G \cong (A \rtimes_{\alpha,r} G)^{H,\tilde{\beta}}.$$

Proof. The first assertion is a direct consequence of the fact that α and β commute. For the proof of (ii) we first observe that the canonical inclusion

$$A \rtimes_\beta H \hookrightarrow (A \rtimes_\beta H) \rtimes_{\tilde{\alpha},r} G \cong (A \rtimes_{\alpha,r} G) \rtimes_{\tilde{\beta}} H$$

maps the projection $p^\beta \in M(A \rtimes_\beta H)$ to the projection $p^{\tilde{\beta}}$ in the multiplier algebra $M((A \rtimes_{\alpha,r} G) \rtimes_{\tilde{\beta}} H)$. Since p^β is full in $A \rtimes_\beta H$ it follows that

$$\begin{aligned}(A \rtimes_{\alpha,r} G) \rtimes_{\tilde{\beta}} H &= (A \rtimes_\beta H) \rtimes_{\tilde{\alpha},r} G \\ &\cong \overline{\big((A \rtimes_\beta H)p^\beta(A \rtimes_\beta H)\big) \rtimes_{\tilde{\alpha},r} G} \\ &= \overline{\big((A \rtimes_\beta H) \rtimes_{\tilde{\alpha},r} G\big)p^\beta\big((A \rtimes_\beta H) \rtimes_{\tilde{\alpha},r} G\big)} \\ &= \overline{\big((A \rtimes_{\alpha,r} G) \rtimes_{\tilde{\beta}} H\big)p^{\tilde{\beta}}\big((A \rtimes_{\alpha,r} G) \rtimes_{\tilde{\beta}} H\big)}.\end{aligned}$$

Hence $p^{\tilde\beta}$ is full in $(A\rtimes_{\alpha,r}G)\rtimes_{\tilde\beta}H$ which proves (ii). The proof of (iii) then follows from

$$\begin{aligned}(A\rtimes_{\alpha,r}G)^{H,\tilde\beta}&=p^{\tilde\beta}\big((A\rtimes_{\alpha,r}G)\rtimes_{\tilde\beta}H\big)p^{\tilde\beta}\\&=p^{\beta}\big((A\rtimes_{\beta}H)\rtimes_{\tilde\alpha,r}G\big)p^{\beta}\\&=\big(p^{\beta}(A\rtimes_{\beta}H)p^{\beta}\big)\rtimes_{\tilde\alpha,r}G\\&=A^{H,\beta}\rtimes_{\alpha,r}G,\end{aligned}$$

where the first and the last isomorphism in the above computation follow from Rosenberg's equation (3.2). □

Using the above observation, we can now prove:

Proposition 4.2. *Let α and β be commuting actions of discrete groups G and H on a simple C^*-algebra A, with H finite, as above. Suppose further that $\alpha\times\beta\colon G\times H\to\mathrm{Aut}(A)$ is outer. Then the restricted action $\alpha^H: G\to\mathrm{Aut}(A^{H,\beta})$ on the fixed-point algebra $A^{H,\beta}$ is outer.*

Proof. Let $\alpha\times\beta\colon G\times H\to\mathrm{Aut}(A)$ be as above. Since A is simple and β is outer, it follows from Kishimoto's theorem that $A\rtimes_\beta H$ is simple as well. Hence β is saturated and $A^{H,\beta}$ is a full corner of $A\rtimes_\beta H$ by the full projection p^β. Since full corners of simple C^*-algebras are simple, it follows that $A^{H,\beta}$ is simple.

Thus, by Lemma 2.1, it suffices to show that for every subgroup $M\subseteq G$ the crossed product $A^{H,\beta}\rtimes_{\alpha^H,r}M$ is simple. But it follows from Lemma 4.1 that $A^{H,\beta}\rtimes_{\alpha^H,r}M=(A\rtimes_{\alpha,r}M)^{H,\tilde\beta}$ which is a full corner of $(A\rtimes_{\alpha,r}M)\times_{\tilde\beta}H\cong A\rtimes_{\alpha\times\beta,r}(M\times H)$. But the latter is simple, again by Kishimoto's theorem. □

We shall also need the lemma below. Let $\beta\colon H\to\mathrm{Aut}(A)$ be an action of a *discrete abelian* group H on a C^*-algebra A. The dual action $\widehat\beta\colon\widehat H\to\mathrm{Aut}(A\rtimes_\beta H)$ is for $x\in\widehat H$ and $b=\sum_{h\in H}a_hu_h\in A\rtimes_{\beta,\mathrm{alg}}H$ given by

$$\widehat\beta_x(b)=\sum_{h\in H}\overline{\langle h,x\rangle}\,a_hu_h.$$

Since $\widehat H$ is a compact abelian group, the subgroup $L^\perp$ of $\widehat H$, defined in (4.1), associated with a subgroup L of H, is compact as well.

Lemma 4.3. *Suppose that $\beta\colon H\to\mathrm{Aut}(A)$ is an action of a discrete abelian group on a C^*-algebra A and let L be a subgroup of H. Then*

$$A\rtimes_\beta L=(A\rtimes_\beta H)^{L^\perp,\widehat\beta},$$

when $A\rtimes_\beta L$ is viewed as a subalgebra of $A\rtimes_\beta H$.

Proof. Let $b=\sum_{l\in L}a_lu_l\in A\rtimes_{\mathrm{alg},\beta}L$. Then

$$\widehat\beta_x(b)=\sum_{l\in L}\overline{\langle l,x\rangle}\,a_lu_l=\sum_{l\in L}a_lu_l=b,$$

for all $x\in L^\perp$, so b lies in $(A\rtimes_\beta H)^{L^\perp}$. This proves that $A\rtimes_\beta L\subseteq(A\rtimes_\beta H)^{L^\perp}$.

To prove the converse inclusion we make use of the conditional expectation $E\colon A\rtimes_\beta H\to A\rtimes_\beta L$ given by $E(b)=\int_{L^\perp}\widehat\beta_x(b)\,dx$, where the integral is with respect to the normalized Haar measure. To see that E indeed maps $A \rtimes_\beta H$ onto $A \rtimes_\beta L$, note first that

$$\int_{L^\perp} \langle h, x \rangle \, dx = \begin{cases} 1, & \text{for } h \in L, \\ 0, & \text{for } h \in H \setminus L. \end{cases} \tag{4.2}$$

Hence, for $b = \sum_{h\in H} a_h u_h \in A \rtimes_{\beta,\mathrm{alg}} H$, we have

$$E(b) = \int_{L^\perp} \widehat{\beta}_x(b) \, dx = \int_{L^\perp} \sum_{h\in H} \overline{\langle h, x \rangle} \, a_h u_h \, dx = \sum_{l\in L} a_l u_l \in A \rtimes_\beta L.$$

This shows that the range of E is contained in $A \rtimes_\beta L$ and that E is the identity on $A \rtimes_\beta L$. Now, since $E(b) = b$, whenever $b \in (A \rtimes_\beta H)^{L^\perp}$, we are done. □

We now provide an elaboration of the observation by Rosenberg stated in (3.2) relating the fixed-point algebra to a crossed product. Two inclusions $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ of C^*-algebras are said to be isomorphic if there is a *-isomorphism $\phi\colon A_1 \to A_2$ with $\phi(B_1) = B_2$. Clearly, if $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ are isomorphic, and if one of the inclusions is C^*-irreducible, then so is the other.

Proposition 4.4. *Let β be an action of a finite abelian group H on a C^*-algebra A. Then, with $p^\beta \in M(A \rtimes_\beta H)$ as defined above* (3.2)*, there is an isomorphism $\psi\colon A \to p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p^\beta$ satisfying $\psi(A^{H,\beta}) = p^\beta(A \rtimes_\beta H)p^\beta$, thus implementing an isomorphism between the two inclusions*

$$A^{H,\beta} \subseteq A \quad \textit{and} \quad p^\beta(A \rtimes_\beta H)p^\beta \subseteq p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p^\beta.$$

Moreover, for each subgroup $L \subseteq H$ we have $\psi(A^{L,\beta}) = p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} L^\perp)p^\beta$, where $L^\perp \subseteq \widehat{H}$ is the annihilator defined above Lemma 4.3.

Proof. Let $u\colon H \to UM(A \rtimes_\beta H)$ and $\widehat{u}\colon \widehat{H} \to UM(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})$ denote the canonical representations implementing β and $\widehat{\beta}$, respectively. Let $\langle \cdot\,, \cdot \rangle\colon H \times \widehat{H} \to \mathbb{T}$ denote the natural pairing between H and $\widehat{H}$ as in Remark 2.2.

By the definition of the dual action, $\widehat{u}_x \in A' \cap M(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})$, for all $x \in \widehat{H}$, and $\widehat{u}_x u_g \widehat{u}_x^* = \overline{\langle g, x \rangle} \, u_g$, for all $g \in H$ and $x \in \widehat{H}$.

For each $g \in H$ and $x \in \widehat{H}$ set

$$p_x = \frac{1}{|H|} \sum_{g\in H} \overline{\langle g, x \rangle} u_g, \qquad q_g = \frac{1}{|H|} \sum_{x\in \widehat{H}} \langle g, x \rangle \widehat{u}_x.$$

(Note that $|H| = |\widehat{H}|$.) In the notation used above (3.2), $p_e = p^\beta$ and $q_e = p^{\widehat{\beta}}$ (where e denotes the neutral element in both groups). By definition of the dual action and the fact that $\widehat{u}$ implements $\widehat{\beta}$, it follows that

$$\widehat{u}_x u_g \widehat{u}_x^* = \widehat{\beta}_x(u_g) = \overline{\langle g, x \rangle} u_g, \qquad u_g \widehat{u}_x u_g^* = u_g \widehat{u}_x u_{g^{-1}} \widehat{u}_x^* \widehat{u}_x = \langle g, x \rangle \, \widehat{u}_x,$$

for all $g \in H, x \in \widehat{H}$. Together with a variant of equation (4.2) it is then straightforward to verify that

$$1 = \sum_{g\in H} q_g = \sum_{x\in \widehat{H}} p_x, \qquad \widehat{u}_x p_e \widehat{u}_x^* = p_x, \qquad u_g q_e u_g^* = q_g, \tag{4.3}$$

for all $g \in H$ and $x \in \widehat{H}$.

Recall from Lemma 4.3 that $A = (A \rtimes_\beta H)^{\widehat{H}}$. By Rosenberg's result, cf. (3.2), we have *-isomorphisms

$$\varphi\colon A^H \to p_e(A \rtimes_\beta H)p_e, \qquad \psi_0\colon A \to q_e(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})q_e,$$

given by $\varphi(b) = bp_e = |H|^{-1}\sum_{g\in H} bu_g$ and $\psi_0(a) = aq_e = |H|^{-1}\sum_{x\in\widehat{H}} a\widehat{u}_x$, for $b \in A^H$ and $a \in A$.

Now, by Takai duality, the two projections p_e and q_e are equivalent in the C^*-algebra generated by $\{u_g\}_{g\in H} \cup \{\widehat{u}_x\}_{x\in\widehat{H}}$ (since this C^*-algebra is isomorphic to $M_{|H|}(\mathbb{C})$ and p_e and q_e are minimal projections herein). We can also see this directly as follows: For $x \in \widehat{H}$ we have $p_e\widehat{u}_x p_e = p_e p_x \widehat{u}_x = \delta_{e,x}\, p_e$, so $p_e q_e p_e = |H|^{-1}p_e$. Similarly, $q_e p_e q_e = |H|^{-1}q_e$. Set $z = |H|^{1/2}p_e q_e$. Then $z^*z = q_e$ and $zz^* = p_e$. Note that z commutes with A^H. Define a *-isomorphism

$$\psi\colon A \to p_e(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p_e, \quad \psi(a) = z\psi_0(a)z^* \ (= |H|\, p_e a q_e p_e), \ \ a \in A. \tag{4.4}$$

For $b \in A^H$ we have $\psi(b) = z(bq_e)z^* = bzq_e z^* = bp_e = \varphi(b)$. Hence $\psi(A^H) = \varphi(A^H) = p_e(A \rtimes_\beta H)p_e$, as desired.

Let $L \subseteq H$ be a subgroup. We check that $\psi(A^L) = p_e(A \rtimes_\beta H \rtimes_{\widehat{\beta}} L^\perp)p_e$, where we view $A \rtimes_\beta H \rtimes_{\widehat{\beta}} L^\perp$ as a subalgebra of $A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H}$ in the canonical way. Recall from Lemma 4.3, applied to $\widehat{\beta}$ via the isomorphism $H \cong \widehat{\widehat{H}}$, which sends $g \in H$ to $\big(x \mapsto \langle g, x\rangle\big) \in \widehat{\widehat{H}}$, that

$$A \rtimes_\beta H \rtimes_{\widehat{\beta}} L^\perp = (A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})^{L,\widehat{\widehat{\beta}}}.$$

Since $p_e \in A \rtimes_\beta H$ is fixed by $\widehat{\widehat{\beta}}$, we see that $\widehat{\widehat{\beta}}$ restricts to an action on $p_e(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p_e$. So the result will follow if we can show that the isomorphism $\psi\colon A \to p_e(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p_e$ is β-$\widehat{\widehat{\beta}}$ equivariant. To this end observe first that for all $g \in H$ we have

$$\widehat{\widehat{\beta}}_g(q_e) = \frac{1}{|H|}\sum_{x\in\widehat{H}} \overline{\langle g, x\rangle}\, \widehat{u}_x = q_{g^{-1}} = u_g^* q_e u_g.$$

Using this, and the fact that p_e is fixed by $\widehat{\widehat{\beta}}$ we get for all $a \in A$ and $g \in H$:

$$\begin{aligned}\widehat{\widehat{\beta}}_g(\psi(a)) &\overset{(4.4)}{=} |H|\, \widehat{\widehat{\beta}}_g(p_e a q_e p_e) = |H|\, p_e a \widehat{\widehat{\beta}}_g(q_e) p_e = |H|\, p_e a u_g^* q_e u_g p_e \\ &= |H|\, p_e u_g^* \beta_g(a) q_e u_g p_e \overset{(*)}{=} |H|\, p_e \beta_g(a) q_e p_e = \psi(\beta_g(a)),\end{aligned}$$

where at $(*)$ we have used the fact that $p_e u_g^* = u_g p_e = p_e$ for all $g \in H$, which follows easily from the definition of p_e. This finishes the proof. □

Lemma 4.5. *Let $B \subseteq A$ be a unital inclusion of C^*-algebras and let $p \in B$ be a projection. If $B \subseteq A$ is C^*-irreducible, then so is $pBp \subseteq pAp$. Conversely, if p is full in B and if $pBp \subseteq pAp$ is C^*-irreducible, then $B \subseteq A$ is C^*-irreducible as well. Moreover, in this case the assignment $D \mapsto pDp$ gives a bijective correspondence between the intermediate C^*-algebras of $B \subseteq A$ and those of $pBp \subseteq pAp$.*

Proof. Assume first that $B \subseteq A$ is C^*-irreducible. Let $pBp \subseteq C \subseteq pAp$ be an intermediate C^*-algebra, and set $D = C^*(B \cup C)$. Then $B \subseteq D \subseteq A$, so D is simple. Moreover, $C = pDp$, so C is a corner of the simple C^*-algebra D, and is hence simple as well.

Suppose now that p is full and that $pBp \subseteq pAp$ is C^*-irreducible. If $B \subseteq D \subseteq A$ is any intermediate C^*-algebra, then $pBp \subseteq pDp \subseteq pAp$, and hence pDp is simple. Since p is full in B, it follows that p is also full in D, and this implies that D is simple.

As for the last claim, we remarked above that the assignment $C \mapsto C^*(B \cup C)$ gives a map from intermediate C^*-algebras of the inclusion $pBp \subseteq pAp$ to intermediate C^*-algebras of the inclusion $B \subseteq A$, which is a left-inverse of the assignment $D \mapsto pDp$, i.e., $pC^*(B \cup C)p = C$, for any $pBp \subseteq C \subseteq pAp$. If p is full in B, then it is also a right-inverse, i.e., $D = C^*(B \cup pDp)$, for any $B \subseteq D \subseteq A$. Indeed, $1 = 1_B = \sum_{j=1}^n b_j^* p b_j$, for some $b_1, \dots, b_n \in B$ by fullness of p in B. Hence, for each $d \in D$, we have $d = 1 \cdot d \cdot 1 = \sum_{i,j=1}^n b_i^* p b_i d b_j p b_j^*$, which belongs to $C^*(B \cup pDp)$, since $pb_i d b_j p \in pDp$, for all i, j. □

We are now ready to give a Galois type classification of the intermediate subalgebras of (some of) the inclusion $A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$ considered in Theorem 3.3.

Theorem 4.6. *Suppose that $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(A)$ are commuting actions of a discrete group G and a finite abelian group H on a unital simple C^*-algebra A.*

(i) *The inclusion $A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$ is isomorphic to the inclusion*

$$p^\beta(A \rtimes_\beta H)p^\beta \subseteq p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H} \rtimes_{\tilde{\alpha},r} G)p^\beta \tag{4.5}$$

where p^β is as defined in (3.1), *and where $\tilde{\alpha}\colon G \to \mathrm{Aut}(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})$ is the extension of α, cf. the explanation above Lemma 4.1.*

(ii) *There is a one-to-one correspondence between subgroups $L \subseteq \widehat{H} \times G$ and intermediate algebras of the inclusion in* (4.5) *given by sending L to $p^\beta(A \rtimes_\beta H)p^\beta \rtimes_{\widehat{\beta} \times \widetilde{\alpha},r} L = p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta} \times \widetilde{\alpha},r} L)p^\beta$.*

(iii) *There is a one-to-one correspondence between subgroups of $\widehat{H} \times G$ and intermediate algebras of the inclusion $A^{H,\beta} \subseteq A \rtimes_{\alpha,r} G$.*

In particular, if $L = L_1 \times L_2$ is a product of subgroups $L_1 \subseteq \widehat{H}$ and $L_2 \subseteq G$, then the corresponding intermediate algebra $A^{H,\beta} \subseteq D \subseteq A \rtimes_{\alpha,r} G$ is $D = A^{L_1^\perp,\beta} \rtimes_{\alpha,r} L_2$, with $L_1^\perp$ the annihilator of L_1 in H, cf. (4.1).

Proof. (i). It was shown in Proposition 4.4 that the inclusion $A^H \subseteq A$ is isomorphic to the inclusion $p^\beta(A \rtimes_\beta H)p^\beta \subseteq p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p^\beta$ via the *-isomorphism

$$\psi\colon A \to p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p^\beta,$$

defined in (4.4), that maps A^H onto $p^\beta(A \rtimes_\beta H)p^\beta$. The isomorphism ψ is easily seen to be α-$\tilde{\alpha}$ equivariant. Hence it extends naturally to a *-isomorphism $\bar{\psi}\colon A \rtimes_{\alpha,r} G \to p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p^\beta \rtimes_{\tilde{\alpha},r} G$. The algebra $p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H})p^\beta \rtimes_{\tilde{\alpha},r} G$ is equal to $p^\beta(A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H} \rtimes_{\tilde{\alpha},r} G)p^\beta$ because $\tilde{\alpha}_g(p^\beta) = p^\beta$ for all $g \in G$ by the definition of $\tilde{\alpha}$. The *-isomorphism $\bar{\psi}$ therefore implements the desired isomorphism of the two inclusions.

(ii). Since $A^H \subseteq A \rtimes_{\alpha,r} G$ is C^*-irreducible by Theorem 3.3, so is the inclusion in (4.5), and hence so is the inclusion

$$A \rtimes_\beta H \subseteq A \rtimes_\beta H \rtimes_{\widehat{\beta}} \widehat{H} \rtimes_{\tilde{\alpha},r} G = A \rtimes_\beta H \rtimes_{\widehat{\beta} \times \tilde{\alpha},r} (\widehat{H} \times G), \tag{4.6}$$

by Lemma 4.5. It follows from [16, Theorem 5.8] that $\widehat{\beta} \times \tilde{\alpha} \colon \widehat{H} \times G \to \mathrm{Aut}(A \rtimes_\beta H)$ is outer.

By Lemma 4.5 there is a bijective correspondence between intermediate C^*-algebras of the inclusion in (4.6) and intermediate C^*-algebras of the inclusion in (4.5) given by compression with p^β. Finally, by the Cameron–Smith theorem, [4, Theorem 3.5], which applies because $\widehat{\beta} \times \tilde{\alpha}$ is outer, each intermediate C^*-algebra of the inclusion in (4.6) is of the form

$$(A \rtimes_\beta H) \rtimes_{\widehat{\beta} \times \tilde{\alpha}, r} L,$$

for some subgroup L of $\widehat{H} \times G$. This proves (ii).

(iii) follows from (i) and (ii) and, for the last claim, inspection of the isomorphism ψ which implements the isomorphism of the two inclusions in (i). □

5. Examples

In this section we want to discuss some interesting examples of the theory as developed in the previous sections arising from group actions on the irrational rotation algebra A_θ, for $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Recall that A_θ is the universal C^*-algebra generated by two unitaries u, v subject to the relation

$$vu = e^{2\pi i \theta} uv.$$

There is an outer action $\alpha \colon \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{Aut}(A_\theta)$ for which

$$n = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

acts on the generators u, v of A_θ by

$$\alpha_n(u) = e^{2\pi i n_{11} n_{21} \theta} u^{n_{11}} v^{n_{21}}, \qquad \alpha_n(v) = e^{2\pi i n_{12} n_{22} \theta} u^{n_{12}} v^{n_{22}}.$$

Up to conjugacy, there are exactly four different finite cyclic subgroups of $\mathrm{SL}(2, \mathbb{Z})$ isomorphic to the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, and $\mathbb{Z}_6$, generated, in that order, by the elements:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \tag{5.1}$$

The resulting crossed products $A_\theta \rtimes_\alpha \mathbb{Z}_k$, $k = 2, 3, 4, 6$, have been studied in detail in [10], where it has been shown that they as well as the fixed-point algebras $A_\theta^{\mathbb{Z}_k}$, $k = 2, 3, 4, 6$, are simple AF-algebras. By [16, Theorem 5.8], all inclusions $A_\theta \subseteq A_\theta \rtimes_\alpha \mathbb{Z}_k$ are C^*-irreducible, and it follows from Theorem 3.1 (Izumi) that the inclusions $A_\theta^{\mathbb{Z}_k} \subseteq A_\theta$ are C^*-irreducible as well. Thus we see that every A_θ, with θ irrational, has a unital C^*-irreducible inclusion into some simple AF-algebra, and that, on the other hand, there always exist simple AF-algebras which admit a unital C^*-irreducible embedding into A_θ. But note that the composition $A_\theta^{\mathbb{Z}_k} \subseteq A_\theta \rtimes_\alpha \mathbb{Z}_k$ of these inclusions is not C^*-irreducible, since $(A_\theta^G)' \cap (A_\theta \rtimes_\alpha G) \neq \mathbb{C}$, as observed earlier for general actions $\alpha \colon G \to \mathrm{Aut}(A)$ of a finite group G. On the other hand, since the entire group $\mathrm{SL}(2, \mathbb{Z})$ acts by outer automorphisms on A_θ, condition (iii) of Theorem 3.3 is satisfied for the actions of two subgroups $F_1, F_2 \subseteq \mathrm{SL}(2, \mathbb{Z})$ on A_θ if and only if their intersection $F_1 \cap F_2$ is trivial in $\mathrm{SL}(2, \mathbb{Z})$. We therefore get:

Proposition 5.1. *Suppose that (F_1, F_2) is either one of the pairs*

$$(\mathbb{Z}_2, \mathbb{Z}_3), \qquad (\mathbb{Z}_3, \mathbb{Z}_4), \qquad (\mathbb{Z}_3, \widetilde{\mathbb{Z}}_3),$$

where $\widetilde{\mathbb{Z}}_3 := \langle R\rangle$ *for some matrix* $R \in \mathrm{SL}(2,\mathbb{Z})$ *which is a conjugate of the matrix* $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ *inside* $\mathrm{SL}(2,\mathbb{Z})$ *and for which* $\mathbb{Z}_3 \cap \widetilde{\mathbb{Z}}_3 = 1$.[1] *Then*

$$A_\theta^{F_1} \subseteq A_\theta \rtimes F_2, \qquad A_\theta^{F_2} \subseteq A_\theta \rtimes F_1$$

are C^**-irreducible inclusions of AF-algebras.*

Proof. In all these cases we have $F_1 \cap F_2 = 1$ in $\mathrm{SL}(2,\mathbb{Z})$, so the result follows from Theorem 3.3. □

Among the finite subgroups of $\mathrm{SL}(2,\mathbb{Z})$ listed in and above (5.1), the pairs (F_1, F_2) listed in the proposition above are the only ones which satisfy item (iii) of Theorem 3.3, so any other combination of subgroups (F_1, F_2) will not provide C^*-irreducible inclusions.

Since A_θ is not an AF-algebra, Proposition 5.1 leads (as expected) to a negative answer to [16, Question 6.11]:

Corollary 5.2. *There exist* C^**-irreducible inclusions of AF-algebras with intermediate* C^**-algebras that are not AF-algebras.*

Of the three pairs of groups (F_1, F_2) in Proposition 5.1 above, only the pair $(\mathbb{Z}_2, \mathbb{Z}_3)$ satisfies the additional assumptions of Theorem 4.6 which gives a classification of the intermediate C^*-algebras. This pair also satisfies the conditions of the following:

Proposition 5.3. *Suppose that* H *and* G *are finite cyclic groups of prime order* p *and* q*, respectively, such that* $p \neq q$*. Let* $\alpha \times \beta : G \times H \to \mathrm{Aut}(A)$ *be an outer action on the simple unital* C^**-algebra* A*.*

Then $A^{H,\beta} \subseteq A \rtimes_\alpha G$ *is a* C^**-irreducible inclusion, and* A *and* $A^{H,\beta} \rtimes_\alpha G$ *are the only (strict) intermediate* C^**-algebras for this inclusion.*

Proof. Since finite cyclic groups are self-dual, it follows from the assumption on the pair p, q that $\widehat{H} \cong \widehat{H} \times \{e\}$ and $G \cong \{e\} \times G$ are the only non-trivial subgroups of $\widehat{H} \times G$. Thus it follows from Theorem 4.6 that $A = A^{\widehat{H}^\perp,\beta}$ and $A^{H,\beta} \rtimes_\alpha G = A^{\{e\}^\perp,\beta} \rtimes_\alpha G$ are the only strict intermediate C^*-algebras for the inclusion $A^{H,\beta} \subseteq A \rtimes_\alpha G$. □

Corollary 5.4. *Let* $\theta \in \mathbb{R} \setminus \mathbb{Q}$*. The only strict intermediate* C^**-algebras for the* C^**-irreducible inclusion* $A_\theta^{\mathbb{Z}_2,\alpha} \subseteq A_\theta \rtimes_\beta \mathbb{Z}_3$ *are* A_θ *and* $A_\theta^{\mathbb{Z}_2,\alpha} \rtimes_\beta \mathbb{Z}_3$*.*

Similarly, the only strict intermediate C^**-algebras for the* C^**-irreducible inclusion* $A_\theta^{\mathbb{Z}_3,\beta} \subseteq A_\theta \rtimes_\alpha \mathbb{Z}_2$ *are* A_θ *and* $A_\theta^{\mathbb{Z}_3,\beta} \rtimes_\alpha \mathbb{Z}_2$*.*

Note that the intermediate algebras $A_\theta^{\mathbb{Z}_2,\alpha} \rtimes_\beta \mathbb{Z}_3$ and $A_\theta^{\mathbb{Z}_3,\beta} \rtimes_\alpha \mathbb{Z}_2$ are AF algebras. Indeed, it is shown in [10] that $A_\theta \rtimes_\gamma \mathbb{Z}_6 = A_\theta \rtimes_{\alpha\times\beta} (\mathbb{Z}_2 \times \mathbb{Z}_3)$ is an AF-algebra. By Lemma 4.1 together with Rosenberg's isomorphism (3.2) it follows that

$$A_\theta^{\mathbb{Z}_2,\alpha} \rtimes_\beta \mathbb{Z}_3 = (A_\theta \rtimes_\beta \mathbb{Z}_3)^{\mathbb{Z}_2,\alpha}$$

is a (full) corner of $A_\theta \rtimes_\beta \mathbb{Z}_3 \rtimes_{\tilde\alpha} \mathbb{Z}_2 \cong A_\theta \rtimes_\gamma \mathbb{Z}_6$, and similarly for $A_\theta^{\mathbb{Z}_3,\beta} \rtimes_\alpha \mathbb{Z}_2$. Since corners of AF-algebras are AF, it follows that $A_\theta^{\mathbb{Z}_2,\alpha} \rtimes_\beta \mathbb{Z}_3$ and $A_\theta^{\mathbb{Z}_3,\beta} \rtimes_\alpha \mathbb{Z}_2$ are AF-algebras.

[1] One can for example take $R = \left(\begin{smallmatrix} -2 & 1 \\ -3 & 1 \end{smallmatrix}\right) = S \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right) S^{-1}$, with $S = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

Remark 5.5. It would be very interesting also to understand the intermediate C^*-algebras of the inclusions appearing in Proposition 5.1, other than the ones arising from the pair $(\mathbb{Z}_2, \mathbb{Z}_3)$.

Perhaps, the most interesting case is given by the inclusion $A_\theta^{\mathbb{Z}_3} \subseteq A_\theta \rtimes \widetilde{\mathbb{Z}}_3$. The only obvious intermediate C^*-algebra here is A_θ itself, and it might well be that it is the only one. (By an "obvious" intermediate C^*-algebra of an inclusion $A^H \subseteq A \rtimes_r G$, we think here of one of the form $D \rtimes_{r,\alpha} L$, where L is a subgroup of G and D is an L-invariant intermediate algebra $A^H \subseteq D \subseteq A$.) If that would be true it would give us an example of a C^*-irreducible inclusion of two AF algebras with A_θ as the unique intermediate C^*-algebra.

Since $\widetilde{\mathbb{Z}}_3$ is a conjugate of $\mathbb{Z}_3$ by an element of $\mathrm{SL}(2,\mathbb{Z})$, the crossed product $A_\theta \rtimes \widetilde{\mathbb{Z}}_3$ is canonically isomorphic to the crossed product $A_\theta \rtimes \mathbb{Z}_3$ in which $A_\theta^{\mathbb{Z}_3}$ sits as a full corner. In particular, $A_\theta^{\mathbb{Z}_3}$ and $A_\theta \rtimes \widetilde{\mathbb{Z}}_3$ are Morita equivalent AF-algebras.

Actions by infinite cyclic groups. Actions on A_θ can provide further examples of C^*-irreducible inclusions with interesting properties. For this let us consider actions of $\mathbb{Z}$ on A_θ which are given by restrictions of the action of $\mathrm{SL}(2,\mathbb{Z})$ to infinite cyclic subgroups. These are generated by matrices $S \in \mathrm{SL}(2,\mathbb{Z})$ of infinite order. Let us then write α^S for the corresponding action of $\mathbb{Z}$ an A_θ. The crossed products $A_\theta \rtimes_{\alpha^S} \mathbb{Z}$ have been studied and classified in [2]. A particularly interesting example occurs if $\mathrm{Tr}(S) = 3$, e.g., for $S = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. In this case, the classification results of [2] imply that $A_\theta \rtimes_{\alpha^S} \mathbb{Z}$ is actually isomorphic to A_θ itself. Thus by [16, Theorem 5.8] and [4] we obtain a proper C^*-irreducible inclusion

$$A_\theta \subseteq A_\theta \rtimes_{\alpha^S} \mathbb{Z} \cong A_\theta.$$

By the results of Cameron and Smith in [4, Theorem 3.5], all (strict) intermediate C^*-algebras are of the form

$$A_\theta \rtimes_{\alpha^S} (n\mathbb{Z}) = A_\theta \rtimes_{\alpha^{S^n}} \mathbb{Z}, \quad n = 2, 3, 4, \dots.$$

Using the results of [2, Theorem 3.5], all these intermediate algebras can be classified by their Elliott invariants, and it turns out that they are never AF (since by [2, Theorem 3.5] their K_1-groups never vanish) and they are usually not isomorphic to A_θ.

Example 5.6. Let us look again at the matrix $S = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Then S is self-adjoint with $\mathrm{Tr}(S) = 3$. The entries of the powers of S are Fibonnaci numbers:

$$S^n = \begin{pmatrix} f_{2n-1} & f_{2n} \\ f_{2n} & f_{2n+1} \end{pmatrix}, \qquad n \geq 1.$$

In particular, it follows that $\mathrm{Tr}(S^n) > 3$, for all $n \geq 2$, and hence it follows from [2, Theorems 3.5 and 3.9] that the intermediate algebras $A_\theta \rtimes_{\alpha^{S^n}} \mathbb{Z}$ of the inclusion $A_\theta \subseteq A_\theta \rtimes_{\alpha^S} \mathbb{Z} \cong A_\theta$ are never isomorphic to A_θ and are not even irrational rotation algebras.

Indeed, using [2, Remark 3.12], we can conclude that $A_\theta \rtimes_{\alpha^{S^n}} \mathbb{Z}$ and $A_\theta \rtimes_{\alpha^{S^m}} \mathbb{Z}$ are never isomorphic if $n \neq m$, since we have $|2 - \mathrm{Tr}(S^n)| \neq |2 - \mathrm{Tr}(S^m)|$, whenever $n, m \in \mathbb{N}$ with $n \neq m$.

Remark 5.7. For any element $S \in \mathrm{SL}(2,\mathbb{Z})$ of infinite order, the intersection $\langle S \rangle \cap F$ is trivial for any finite subgroup $F \subseteq \mathrm{SL}(2,\mathbb{Z})$. Therefore, with $S = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$

as above, we get C^*-irreducible inclusions

$$A_\theta^F \subseteq A_\theta \rtimes_{\alpha^S} \mathbb{Z} \cong A_\theta$$

for every such subgroup F. In the case where $F = \mathbb{Z}_2$, which is generated by the central element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the actions of F and $\mathbb{Z}$ commute and Theorem 4.6 gives a description of all intermediate algebras for this inclusion.

Another interesting consequence of this type of examples is the existence of outer actions β^n of the cyclic groups $\mathbb{Z}_n$ on A_θ, for all $n \in \mathbb{N}$ with $n \geq 2$, such that the crossed products $A_\theta \rtimes_{\beta^n} \mathbb{Z}_n$ as well as the fixed-point algebras $A_\theta^{\mathbb{Z}_n,\beta^n}$ are not AF, quite contrary to the case of the actions of the finite subgroups of $\mathrm{SL}(2,\mathbb{Z})$ considered before. For this we need

Lemma 5.8. *Suppose that $\beta\colon H \to \mathrm{Aut}(A)$ is an outer action of the discrete abelian group H on a simple C^*-algebra A. Then, for each finite subgroup $M \subseteq \widehat{H}$, the restriction of the dual action $\widehat{\beta}\colon \widehat{H} \to \mathrm{Aut}(A \rtimes_\beta H)$ to M is outer as well.*

If $\widehat{H}$ is finite, or more generally if $\widehat{H}$ has no element of infinite order, then the lemma simply says that $\widehat{\beta}$ itself also is outer, cf. Lemma 2.1.

Proof. Let $L \subseteq M \subseteq \widehat{H}$ be any subgroup of M and let $L^\perp$ be the annihilator of L in H. Then it follows from [8, Proposition 2.1] that $(A \rtimes_\beta H) \rtimes_{\widehat{\beta}} L$ is Morita equivalent to $A \rtimes_\beta L^\perp$, which is simple by Lemma 2.1. Thus, since Morita equivalence preserves simplicity, the crossed product $(A \rtimes_\beta H) \rtimes_{\widehat{\beta}} L$ is simple as well. Thus, it follows from Lemma 2.1 that the restriction of $\widehat{\beta}$ to M is by outer automorphisms. □

Example 5.9. Let $S = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ as above (for most of what we do here, one could take any $S \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{Tr}(S) = 3$). Consider the dual action $\widehat{\alpha}^S\colon \mathbb{T} \to \mathrm{Aut}(A_\theta \rtimes_{\alpha^S} \mathbb{Z})$ of α^S. The isomorphism $A_\theta \rtimes_{\alpha^S} \mathbb{Z} \cong A_\theta$ carries this to an action, say $\beta\colon \mathbb{T} \to \mathrm{Aut}(A_\theta)$. For each $n \in \mathbb{N}$, let us identify the cyclic group $\mathbb{Z}_n$ of order n with the group of all nth roots of unity in $\mathbb{T}$, which is the annihilator of $n\mathbb{Z} \subseteq \mathbb{Z}$ under the identification $\mathbb{T} \cong \widehat{\mathbb{Z}}$. Thus $\mathbb{Z}_n$ can be identified with $(n\mathbb{Z})^\perp \subseteq \mathbb{T}$. It follows from Lemma 5.8 that the restriction of β to $\mathbb{Z}_n$ gives an outer action, called β^n below, of $\mathbb{Z}_n$ on A_θ. Thus, using [16, Theorem 5.8] and Theorem 3.3, we obtain C^*-irreducible inclusions

$$A_\theta^{\mathbb{Z}_n,\beta^n} \subseteq A_\theta \quad \text{and} \quad A_\theta \subseteq A_\theta \rtimes_{\beta^n} \mathbb{Z}_n$$

with intermediate algebras given by $A_\theta^{\mathbb{Z}_m,\beta^m}$ and $A_\theta \rtimes_{\beta^m} \mathbb{Z}_m$, respectively, for all $m \in \mathbb{N}$ which divide n. It follows then from Lemma 4.3 that

$$A_\theta^{\mathbb{Z}_m,\beta^m} \cong A_\theta \rtimes_{\alpha^{S^m}} \mathbb{Z}.$$

So at least for $S = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, it follows from Example 5.6 that the sequence of C^*-algebras above are pairwise non-isomorphic, and that none of them are AF-algebras.

Note, if $n, m \in \mathbb{N}$ have no common divisors, then $\mathbb{Z}_n \cap \mathbb{Z}_m = \{0\}$ and Theorem 3.3 implies that the inclusion

$$A_\theta^{\mathbb{Z}_n,\beta^n} \subseteq A_\theta \rtimes_{\beta^m} \mathbb{Z}_m$$

is also C^*-irreducible. Again, in this case Theorem 4.6 allows us to compute all intermediate algebras of this inclusion.

Question 5.10. Let $A_\theta \subseteq A_\theta \rtimes_{\alpha^S} \mathbb{Z} \cong A_\theta$ be the C^*-irreducible inclusion considered in Example 5.9 above. By iteration we get a chain of inclusions

$$A_\theta \subseteq A_\theta \subseteq \cdots \subseteq A_\theta \subseteq \cdots .$$

Are all compositions in this sequence C^*-irreducible?

It has been shown in [2, Remark 3.11] that the direct limit of this sequence is the AF-algebra constructed by Effros and Shen in [11], and into which A_θ embeds with the same ordered K_0-groups, as shown by Pimsner and Voiculescu in [14].

References

[1] L. Baggett and A. Kleppner, Multiplier representations of abelian groups, *J. Functional Analysis* **14** (1973), 299–324.

[2] C. Bönicke, S. Chakraborty, Z. He, and H.-C. Liao, Isomorphism and Morita equivalence classes for crossed products of irrational rotation algebras by cyclic subgroups of $SL_2(\mathbb{Z})$, *J. Functional Analysis* **275** (2018), 3208–3243.

[3] D. Bisch and U. Haagerup, Composition of subfactors: new examples of infinite depth subfactors, *Ann. Sci. École Norm. Sup. (4)* **29** (1996), no. 3, 329–383.

[4] J. Cameron and R. R. Smith, A Galois correspondence for reduced crossed products of simple C*-algebras by discrete groups, *Canad. J. Math.* **71** (2019), no. 5, 1103–1125.

[5] J. Cameron and R. R. Smith, Corrigendum to: A Galois correspondence for reduced crossed products of simple C*-algebras by discrete groups, *Canad. J. Math.* **72** (2020), no. 2, 557–562.

[6] F. Combes, Crossed products and Morita equivalence, *Proc. London Math. Soc. (3)* **49** (1984), no. 2, 289–306.

[7] J. Cuntz, S. Echterhoff, X. Li, and G. Yu, *K-theory for group C^*-algebras and semigroup C^*-algebras*, Oberwolfach Seminars, 47, Birkäuser/Springer, Cham, 2017.

[8] S. Echterhoff, Duality of induction and restriction for abelian twisted covariant systems, *Math. Proc. Cambridge Philos. Soc.* **116** (1994), no. 2, 301–315.

[9] S. Echterhoff and H. Emerson, Structure and K-theory of crossed products by proper actions, *Expo. Math.*, **29** (2011), no. (3), 300–344.

[10] S. Echterhoff, W. Lück, N. C. Phillips, and S. Walters, The structure of crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$, *J. Reine Angew. Math.* **639** (2010), 173–221.

[11] E. G. Effros and C. L. Shen, Approximately finite C^*-algebras and continued fractions, *Indiana Univ. Math. J.*. **29** (1980), no. 2, 191–204.

[12] M. Izumi, Inclusions of simple C^*-algebras, *J. Reine Angew. Math.*, **547** (2002), 97–138.

[13] A. Kishimoto, Outer automorphisms and reduced crossed products of simple C^*-algebras, *Comm. Math. Phys.* **81** (1981), no. 3, 429–435.

[14] M. Pimsner and D. Voiculescu, Imbedding the irrational rotation C^*-algebra into an AF-algebra, *J. Operator Theory* **4** (1980), no. 2, 201–210.

[15] M. A. Rieffel, Actions of finite groups on C^*-algebras, *Math. Scand.* **47** (1980), no. 1, 157–176.

[16] M. Rørdam, Irreducible inclusions of simple C^*-algebras, Preprint 2021. arXiv 2105.11899

[17] J. Rosenberg, Appendix to: "Crossed products of UHF algebras by product type actions" by O. Bratteli, *Duke Math. J.* **46** (1979), no. 1, 25–26.

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