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"CLASSIFICATION AND STRUCTURE OF C^*-ALGEBRAS"

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DEF (REPRESENTED C^*-ALG) A C^*-ALG IS A NORM-CLOSED SELF-ADJOINED SUBALGEBRA OF $B(H)$, THE BOUNDED OPERATORS ON A HILBERT SPACE H.

DEF (ABSTRACT C^*-ALG). A C^*-ALG IS AN ALGEBRA OVER $\mathbb{C}$ WITH NORM $a \mapsto \|a\|$ AND AN INVOLUTION $a \mapsto a^*$ SATISFYING SEVERAL AXIOMS, E.G.:

$$\|a^* a\| = \|a\|^2$$

THE TWO DEFINITIONS ABOVE AGREE!

EXAMPLE X ~ LOCALLY CPT. HAUSDORFF, THEN $C_0(X)$ IS A C^*-ALGEBRA.

ALL COMMUTATIVE C^*-ALG'S ARE $\cong C_0(X)$, AND X CAN BE RECOVERED FROM THE C^*-ALG.

STUDY OF C^*-ALG'S = NON-COMMUTATIVE TOPOLOGY!

EXAMPLE (ROTATION C^*-ALG'S). $\theta \in \mathbb{R}$

$$A_\theta := C^*[u, v \mid u, v \text{ unitaries}, \; uv = e^{2\pi i\theta} vu]$$

$$= C(\mathbb{T}) \rtimes_{R_\theta} \mathbb{Z}$$

$\theta = 0 \Rightarrow A_\theta = C(\mathbb{T}^2)$

$\theta \notin \mathbb{Q} \Leftrightarrow A_\theta$ SIMPLE (NO NON-TRIVIAL IDEALS)

EXAMPLE (GLIMM, 1959). UHF-ALGEBRAS

TAKE $n_1, n_2, n_3, \dots \in \mathbb{N}$ s.t. $n_j \mid n_{j+1}$:

$$M_{n_1} \to M_{n_2} \to M_{n_3} \to \cdots \to A$$

[i.e., $A = \overline{\bigcup_{k=1}^{\infty} A_k}$, $A_1 \subseteq A_2 \subseteq \cdots$, $A_k \cong M_{n_k}$]

FACT A SIMPLE C^*-ALG

THERE ARE UNCOUNTABLY MANY ISOMORPHISM CLASSES OF THESE C^*-ALG'S;

CLASSIFIED BY:

$$\prod_{j=0}^{\infty} n_{j+1}/n_j \quad \text{OR BY} \quad \bigcup_{j=1}^{\infty} \frac{1}{n_j}\mathbb{Z} \subseteq \mathbb{Q}$$

EXAMPLE (AF-ALGEBRAS). AS ABOVE, BUT REPLACE M_{n_j} BY ARBITRARY FINITE DIMENSIONAL C^*-ALG.

AF-ALG'S WERE CLASSIFIED BY BRATTELI AND ELLIOTT, THE LATTER BY:

$$(K_0(A), K_0(A)^+)$$

AF-ALG'S ARE SOMETIME, BUT NOT ALWAYS, SIMPLE

EXAMPLE (CUNTZ ALGEBRAS).

FIX $n \in \mathbb{N}$, $n \geq 2$. SET

$$\mathcal{O}_n = C^*[s_1, \dots, s_n \mid s_j^* s_j = 1 = s_1 s_1^* + \cdots + s_n s_n^*]$$

FACT $\mathcal{O}_n$ SIMPLE,

$$K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}, \quad K_1(\mathcal{O}_n) = 0$$

(3) <u>EXAMPLE</u> Γ discrete group.

$\pi : \Gamma \to \mathcal{U}(H)$ universal representation

$\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ left regular representation

$C^*(\Gamma) := C^*(\pi(\Gamma)) \subseteq B(H)$

$C^*_r(\Gamma) := C^*(\lambda(\Gamma)) \subseteq B(\ell^2(\Gamma))$

$C^*(\Gamma) = C^*_r(\Gamma) \iff \Gamma$ amenable

<u>THM</u> (POWERS) $C^*_r(\mathbb{F}_n)$ SIMPLE C^*-ALG.

<u>EXAMPLE</u> Γ discrete group can act on a space X. THIS GIVES US A C^*-ALG:

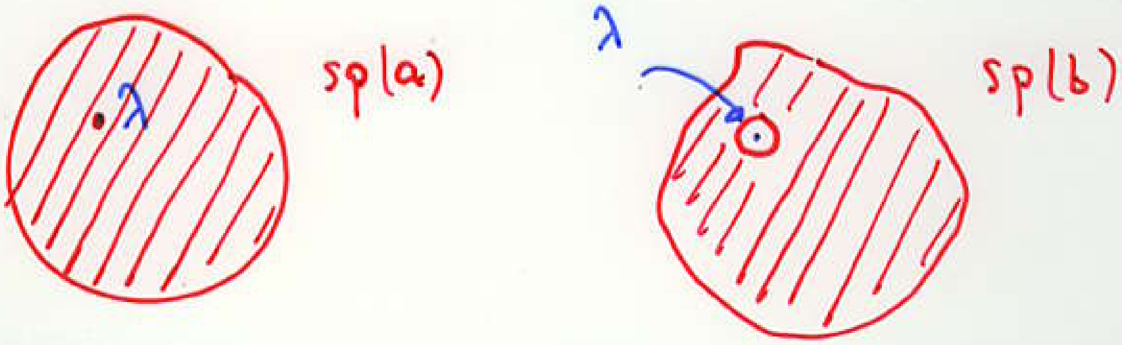
$$C(X) \rtimes \Gamma$$

PROPERTIES OF (SIMPLE) C^*-ALG'S

TWO NOTIONS OF LOW DIMENSION.

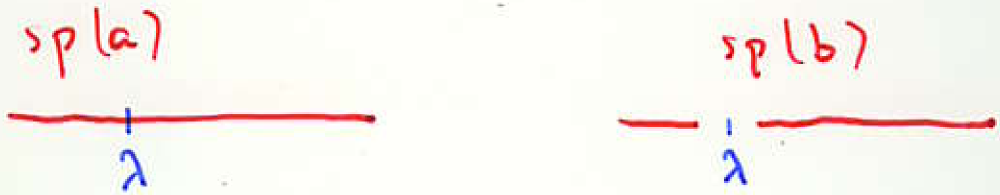
[Stable rank, Rieffel] $A \sim C^*$-ALG.

$$sr(A) = 1 \iff \forall a \in A \ \forall \varepsilon > 0 \ \forall \lambda \in \mathbb{C} \setminus \{0\}$$
$$\exists b \in A : \|a-b\| < \varepsilon, \ \lambda \notin sp(b)$$



[Real rank, Brown-Pedersen] $A \sim C^*$-ALG

$$RR(A) = 0 \iff \forall a = a^* \in A \ \forall \varepsilon > 0 \ \forall \lambda \in \mathbb{R} \setminus \{0\}$$
$$\exists b = b^* \in A : \|a-b\| < \varepsilon, \ \lambda \notin sp(b)$$



COMMUTATIVE CASE

$$sr\, C_0(X) = 1 \iff \dim X \leq 1$$
$$RR\, C_0(X) = 0 \iff \dim X = 0$$

UNITAL CASE : A UNITAL C^*-ALG :

$$sr\, A = 1 \iff \overline{\mathrm{Inv}(A)} = A$$
$$RR\, A = 0 \iff \overline{A_{sa} \cap \mathrm{Inv}(A)} = A_{sa}$$

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FINITE AND INFINITE C^*-ALG's

$A \sim C^*$-ALG, $p \in A$ projection.

p <u>infinite</u> if $\exists\, q \in A$ projection s.t. $p \sim q \lneqq p$. $[p \sim q \Leftrightarrow \exists v \in A : p = v^*v,\ q = vv^*]$

p <u>finite</u> otherwise

A <u>stably infinite</u> if $\exists n : M_n(A)$ contains an infinite projection.

A <u>stably finite</u> otherwise.

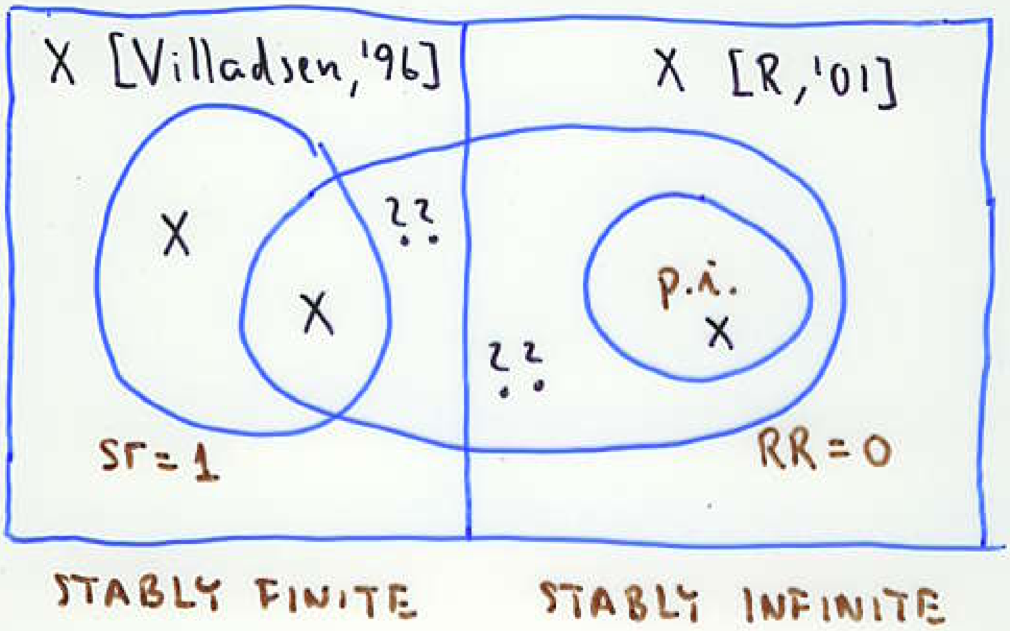
A <u>purely infinite</u> [provided A simple] if $\forall a \in A \setminus \{0\}$

$$\overline{a^* A a} \subseteq A \quad \text{[hereditary sub-}C^*\text{-ALG]}$$

contains an infinite projection.

<u>THM</u> [Blackadar-Handelman, Haagerup]
$A \sim$ unital <u>exact</u> C^*-Alg. Then A has a tracial state iff A is stably finite.

MAP OF SIMPLE C*-ALG'S



X = ∃ EXAMPLES ; ?? = WE DON'T KNOW

p.i. = PURELY INFINITE.

FAMOUS OPEN PROBLEM : IS EVERY SIMPLE C*-ALG OF RR = 0 EITHER PURELY INF. OR STABLY FINITE ?

NUCLEARITY

A – C^*-ALG. A IS NUCLEAR IF $\exists$

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "\varphi_n"'] & & A \\ & M_n \arrow[ur, "\psi_n"'] & \end{tikzcd}$$

S.T. $\psi_n \circ \varphi_n(a) \to a$, $\forall a \in A$, WHERE φ_n, ψ_n ARE COMPLETELY POSITIVE CONTRACTIONS.

MANY OTHER EQUIV. COND., E.G. A^{**} INJECTIVE, UNIQUE NORMS ON $A \otimes_{alg} B$ $\forall$ C^*-ALG. B, … .

EXAMPLE : UHF, AF-ALGEBRAS, A_θ, $\mathcal{O}_n$ ARE NUCLEAR.

EXAMPLE : $C^*_\lambda(\Gamma)$ NUCLEAR $\Leftrightarrow$ Γ AMENABLE (WHEN Γ DISCRETE GROUP)

EXAMPLE : $B(H)$ NOT NUCLEAR.

ELLIOTT'S CLASSIFICATION PROGRAM

K-THEORY : (SEP.) C^*-ALG $\overset{K_0}{\underset{K_1}{\rightrightarrows}}$ (CTBL) ABELIAN GROUPS

$$A \mapsto K_0(A),\ K_1(A)$$

IF A UNITAL :

$$K_0(A) = \{[p]-[q] \mid p,q \in M_n(A) \text{ projections}, n \in \mathbb{N}\},$$

$$K_1(A) = \{[u] \mid u \in M_n(A) \text{ unitary}, n \in \mathbb{N}\},$$

$$K_0(A)^+ = \{[p] \mid p \in M_n(A) \text{ proj.}, n \in \mathbb{N}\}.$$

THM [ELLIOTT, '76] $(K_0(A), K_0(A)^+, [1])$ IS A COMPLETE INV. FOR UNITAL AF-ALG'S.

RANGE OF INVARIANT DESCRIBED BY EFFROS-HANDELMAN-SHEN.

THM [ELLIOTT, '89] $(K_0(A), K_0(A)^+, [1], K_1(A))$ IS A COMPLETE INV. FOR UNITAL A$\mathbb{T}$-ALG'S OF RR = 0.

THM [ELLIOTT, GONG, LI]
$(K_0(A), K_0(A)^+, [1], K_1(A), T(A), T(A) \times K_0(A) \to \mathbb{R})$ IS A COMPLETE INV. FOR A SPECIFIC (RATHER LARGE) CLASS OF UNITAL, SIMPLE, SEPARABLE, NUCLEAR, STABLY FINITE C^*-ALG'S.

(9) ELLIOTT CONJECTURE [RR = 0 CASE]

$$(K_0(A), K_0(A)^+, [1], K_1(A))$$

IS A COMPLETE INV. FOR ALL UNITAL, SIMPLE, SEPARABLE, NUCLEAR C^*-ALG'S OF RR = 0.

RANGE OF INVARIANT KNOWN, MODULO POSSIBLE STRONG PERFORATION IN $(K_0(A), K_0(A)^+)$.

REMARK A STABLY INFINITE $\Rightarrow K_0(A)^+ = K_0(A)$ AND $T(A) = \emptyset$

THEOREM (Kirchberg-Phillips) $(K_0(A), [1], K_1(A))$ IS A COMPLETE INV. FOR THE CLASS OF UNITAL, SIMPLE, SEPARABLE, NUCLEAR, PURELY INFINITE, C^*-ALG'S IN THE UCT-CLASS.

RANGE OF THE INVARIANT: ALL PAIRS OF COUNTABLE ABELIAN GROUPS CAN ARISE AS $(K_0(A), K_1(A))$ WITH A AS IN THE THM.

ARE ALMOST COMMUTING S.A. MATRICES CLOSE TO EXACTLY COMMUTING S.A. MATRICES ?

<u>EXAMPLE</u> :

$$A_n = \begin{bmatrix} 1/n & & & & 0 \\ & 2/n & & & \\ & & 3/n & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 & 1/2 & & & 0 \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & \ddots & \\ & & \ddots & \ddots & 1/2 \\ & 0 & & 1/2 & 0 \end{bmatrix}$$

$\| A_n B_n - B_n A_n \| \leq 1/n \xrightarrow{n \to \infty} 0$; $\|A_n\|, \|B_n\| \leq 1$,

FIND S.A. MATRICES A_n', B_n' s.t.

$$A_n' B_n' = B_n' A_n' \; ; \quad \| A_n - A_n' \| + \| B_n - B_n' \| \to 0$$

<u>THM</u> (H. LIN ; NEW PROOF BY P. FRIIS & R)

$\forall \varepsilon > 0$ $\boxed{\exists \delta > 0 \ \forall n \in \mathbb{N}}$: FOR ALL SA. $n \times n$ MATRICES A, B S.T.

$$\| AB - BA \| < \delta, \quad \|A\|, \|B\| \leq 1,$$

$\exists$ SA. $n \times n$ MATRICES A', B' S.T.

$$A'B' = B'A', \quad \| A - A' \| + \| B - B' \| < \varepsilon.$$

PROOF - Indirect: Assume a counter-example exists:

$$(*)\quad \left| \begin{array}{l} A_n, B_n \in M_{k_n}(\mathbb{C})_{S.A.}, \quad \|A_nB_n - B_nA_n\| \to 0, \\ \|A_n\|, \|B_n\| \leq 1, \quad \text{“FAR AWAY” FROM EXACTLY} \\ \text{COMMUTING S.A. MATRICES.} \end{array} \right.$$

THEN DEFINE THE FOLLOWING:

$$\mathfrak{A} := \prod_{n=1}^{k_n} M_{k_n} \xrightarrow{\pi} \mathcal{B} := \prod_{n=1}^{k_n} M_{k_n} \Big/ \sum_{n=1}^{\infty} M_{k_n}$$

$$T := \{A_n + iB_n\}_{n=1}^{\infty} \in \mathfrak{A}, \quad \pi(T) \in \mathcal{B}$$

CRUCIAL FACT: $\pi(T)$ IS NORMAL.

IF WE CAN LIFT $\pi(T)$ TO A NORMAL ELEMENT $S = \{A_n' + iB_n'\} \in \mathfrak{A}$, THEN WE ARE DONE: $A_n'B_n' = B_n'A_n'$ AND $\|A_n - A_n'\| + \|B_n - B_n'\| \to 0$. [SO $(*)$ DOES NOT HOLD]

HERE IS WHAT WE CAN SHOW:

(i) $\forall \varepsilon > 0 \ \exists R \in \mathcal{B}$ NORMAL S.T. $\|R - \pi(T)\| < \varepsilon$ AND Spectrum(R) FINITE.

(ii) R LIFTS TO A NORMAL ELEMENT IN $\mathfrak{A}$

(i) + (ii) LEAD TO A CONTRADICTION OF $(*)$

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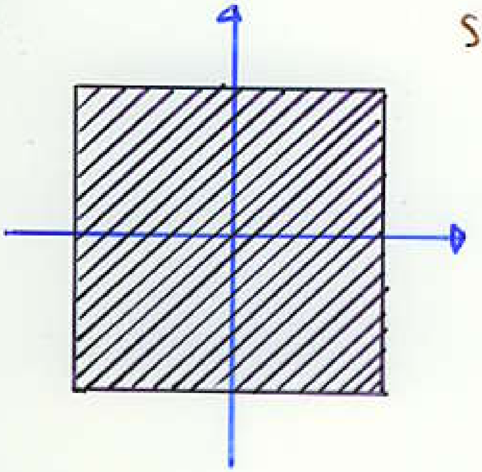
$sp(\pi(T))$



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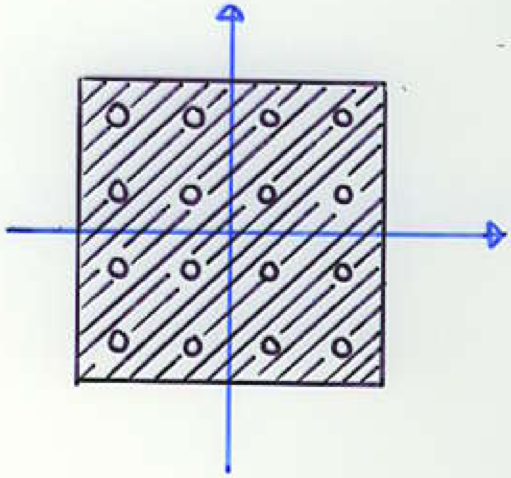

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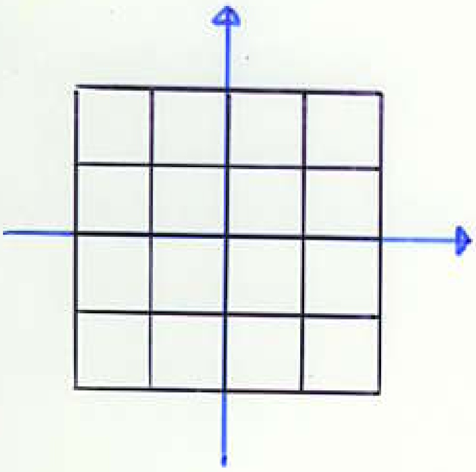

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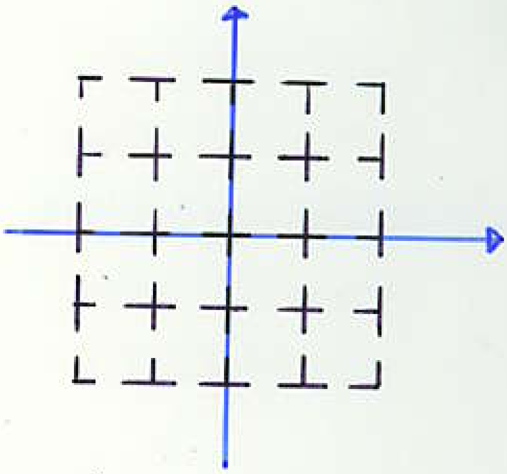

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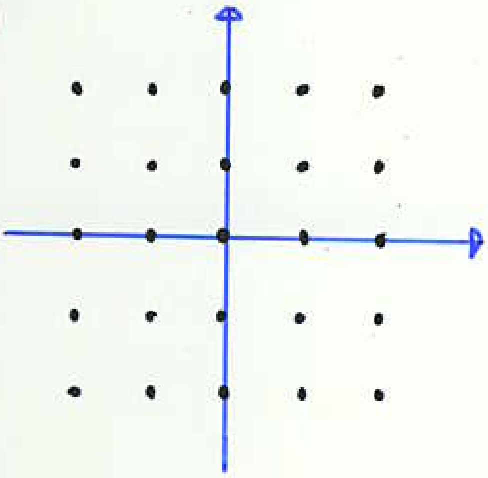


$sp(R)$

(13) ON THE PROOF OF (i):

(1) → (2) Use

- $sr(\mathcal{B}) = 1$, i.e. $\overline{\mathrm{INV}(\mathcal{B})} = \mathcal{B}$ [EASY FACT]
- [R,'88]: $\overline{\mathrm{INV}(\mathcal{B})} \cap \mathrm{NORMAL}(\mathcal{B})$

$$= \overline{\mathrm{INV}(\mathcal{B}) \cap \mathrm{NORMAL}(\mathcal{B})}$$

→ $\forall F \overset{\text{finite}}{\subseteq} \mathbb{C}$ $\exists R_0 \in \mathcal{B}$ NORMAL s.t. $\|R_0\| \leq 1$,

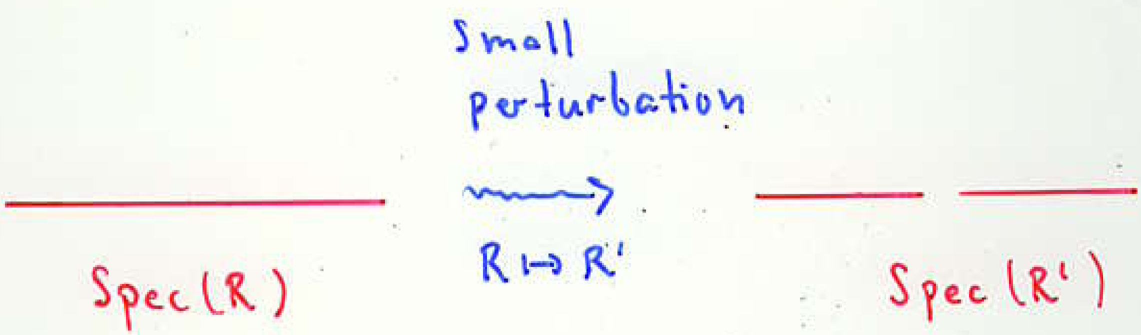
$\pi(T)\,\|\cancel{R} - R_0\|$ SMALL, $\mathrm{Spec}(R_0) \cap F = \emptyset$

(2) → (3) AND (4) → (5)

$R_0 \rightsquigarrow f(R_0)$ FOR SUITABLE CTS. RETRACT f

(3) → (4) A DELICATE ARGUMENT – DUE TO H. LIN – THAT USES $RR(\mathcal{B}) = 0$, $K_1(\mathcal{B}) = 0$.

Here is what RRO does:



To cope with loops in $\mathrm{Spec}(R)$ we also must have $K_1(\mathcal{B}) = 0$.

(14) TOWARDS HIGH-DIMENSIONAL C^*-ALGs

DEF A C^*-ALG A has strong perforation (in its monoid of Murray-v.Neumann eq. classes of projections) if $\exists\, p, q \in A \otimes \mathcal{K}$ projections and $n \in \mathbb{N}$ s.t.

$$\underbrace{p \oplus p \oplus \cdots \oplus p}_{n+1} \precsim \underbrace{q \oplus q \oplus \cdots \oplus q}_{n}$$

AND S.T. $p \not\precsim q$.

DEF $(K_0(A), K_0(A)^+)$ has strong perforation if $\exists\, g \in K_0(A)$ $\exists\, n \in \mathbb{N}$ s.t.

$$ng,\ (n+1)g \in K_0(A)^+ \ , \quad g \notin K_0(A)^+$$

EXAMPLE WITH $G = \mathbb{Z}$, $G^+ = \{0, 2, 3, 4, 5, \cdots\}$, (G, G^+) HAS STRONG PERFORATION:

$$2 \cdot 1,\ 3 \cdot 1 \in G^+ \ , \quad 1 \notin G^+.$$

WHY IS STRONG PERFORATION TROUBLESOME?

- Known examples of simple C^*-alg's with strong perforation defy classification
- It is difficult (impossible?) to show that Elliott invariant is onto, when we allow strong perforation
- For a long time no examples of simple C^*-alg's w/ strong perforation were known

(15) <u>CUNTZ'S QUESTION</u> (~1980). DOES THERE EXIST A SIMPLE C^*-ALG WITH A NON-ZERO FINITE AND AN INFINITE PROJECTION?

IF YES, THEN ANY SUCH C^*-ALG HAS STRONG PERFORATION:

TAKE: p finite projection, q infinite proj. Can assume $p \oplus p$ infinite. THEN:

$$\underbrace{q \oplus q \oplus \cdots \oplus q}_{n} \precsim p \oplus p \; ; \; q \not\precsim p$$

FOR <u>ALL</u> $n \in \mathbb{N}$.

VILLADSEN'S EXAMPLE #1

THM (VILLADSEN, '95) ∃ SIMPLE, SEPARABLE, NUCLEAR UNITAL C^*-ALG THAT HAS STRONG PERFORATION.

HERE IS WHAT THE EXAMPLE LOOKS LIKE:

$$M_{n_1}(C(\mathbb{T}^{k_1})) \xrightarrow{\varphi_1} M_{n_2}(C(\mathbb{T}^{k_2})) \xrightarrow{\varphi_2} \cdots \to A$$

where: $n_{j+1} = (d_j+1)n_j$; $k_{j+1} = d_j k_j$, $\Sigma 1/d_j < \infty$, AND

$$\varphi_j(f) = \begin{bmatrix} f\circ\pi_1 & & & & 0 \\ & f\circ\pi_2 & & & \\ & & \ddots & & \\ & & & f\circ\pi_{d_j} & \\ 0 & & & & f(z_j) \end{bmatrix},$$

$\pi_i : \mathbb{T}^{k_{j+1}} = (\mathbb{T}^{k_j})^{d_j} \to \mathbb{T}^{k_j}$ [ith coordinate]

$z_j \in \mathbb{T}^{k_j}$ (some point)

DIFFICULTY IT IS WELL KNOWN THAT $C(\mathbb{T}^k)$ HAS STRONG PERFORATION; BUT HOW DOES ONE KEEP THIS PROPERTY WHILE MAKING SURE THAT A BECOMES SIMPLE.

WARNING $(K_0(C(\mathbb{T}^k)), K_0(C(\mathbb{T}^k))^+)$ IS VERY COMPLICATED.

⑰ VILLADSEN USED THE EULER CLASS TO KEEP TRACK OF PERFORATION:

$X \sim$ finite dimensional CW complex

$p \in M_n(C(X))$ projection; $\dim(p) = k$.

$$\mathrm{Euler}(p) = e(p) \in H^{2k}(X, \mathbb{Z})$$

θ_k = trivial (= constant) projection of dimension k.

FORMULAE / FACTS

- $e(p \oplus q) = e(p) \cdot e(q)$
- $e(p) \neq 0$ AND $\theta_k \precsim p \oplus \theta_\ell \Rightarrow k \leq \ell$
- $\exists\, p \in M_2(C(S^2))$ ST $\dim(p) = 1$, $e(p) \neq 0$

Künneth $\pi_i : X^k \to X$ (ith coordinate), $p \in M_n(C(X))$ proj s.t. $e(p) \neq 0$

PUT $p_i = p \circ \pi_i \in M_n(C(X^k))$, $i = 1, 2, \dots, k$.

THEN $e(p_1 \oplus p_2 \oplus \cdots \oplus p_k) \neq 0$.

FACT $X \sim$ CONNECTED, FINITE DIM. THEN

$C(X)$ HAS NOT STRONG PERFORATION

$\Updownarrow$

$\forall p, q \in M_n(C(X))$: $\dim(p) < \dim(q) \Rightarrow p \precsim q$

COR IF $\exists\, p \in M_n(C(X))$ s.t. $e(p) \neq 0$, $\dim(p) \geq 2$, THEN $C(X)$ HAS STRONG PERFORATION

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THE ANSWER TO CUNTZ QUESTION IS: YES

THM (R, '01) $\exists$ SIMPLE, SEPARABLE, NUCLEAR C^*-ALG WHICH CONTAINS A NON-ZERO FINITE PROJECTION AND AN INFINITE PROJECTION.

COR $\exists$ SIMPLE UNITAL C^*-ALG. A S.T. A IS FINITE AND $M_2(A)$ INFINITE.

ABOVE HAS BEEN KNOWN TO RING THEORISTS FOR RINGS FOR A LONG TIME (P.M. COHN)

RECALL: ELLIOTT INV. FOR STABLY INFINITE, UNITAL SIMPLE C^*-ALG'S IS:

$$(K_0(A), [1], K_1(A)) =: \mathrm{Inv}(A)$$

COR $\exists$ SIMPLE, SEPARABLE, NUCLEAR, UNITAL STABLY INF. C^*-ALG'S A AND B S.T.

$$\mathrm{Inv}(A) \cong \mathrm{Inv}(B), \quad A \not\cong B$$

PROOF BY THM $\exists$ UNITAL, SIMPLE, SEPARABLE, NUCLEAR, STABLY INFINITE C^*-ALG. A S.T. A CONTAINS FINITE PROJECTION. I.E., A NOT PURELY INF.

$\exists$ PURELY INF., SEP., SIMPLE, UNITAL, NUCLEAR C^*-ALG B IN THE UCT CLASS S.T.

$$\mathrm{Inv}(B) \cong \mathrm{Inv}(A).$$

WE ARE DONE, BECAUSE $A \not\cong B$. $\blacksquare$

CONSTRUCTION OF THE SIMPLE C^*-ALG

PUT $\quad \mathbb{X} = \prod_{n=1}^{\infty} S^2$

the compacts

$$A = C(\mathbb{X}, \mathcal{K}) \cong C(\mathbb{X}) \otimes \mathcal{K}$$

SET $\quad \mathcal{M}(A)$ = MULTIPLIER ALG. OF A

$= \{ f : \mathbb{X} \to B(H) \mid f \; {}^*\text{-STRONG CTS} \}$

ONE HAS: $\quad$ A ESSENTIAL IDEAL IN $\mathcal{M}(A)$.

WE CONSTRUCT BELOW A SPECIAL HOMOMORPHISM:

$$\rho : A \to \mathcal{M}(A)$$

FROM THIS WE GET OUR C^*-ALG:

$(A, \rho) \rightsquigarrow$ CUNTZ-PIMSNER ALGEBRA

$= C^*(A, \rho)$

$= A_\infty \rtimes_\alpha \mathbb{Z}$

WHERE $\quad A_\infty = C^*(\cdots, A_{-2}, A_{-1}, \underset{A}{\underset{\parallel}{A_0}}, A_1, A_2, \cdots)$,

$\forall n : \quad A_n \cong A \;, \quad C^*(A_n, A_{n+1}) \cong C^*(A, \rho(A))$

THE AUTOMORPHISM α = SHIFT, AND IT BEHAVES LIKE ρ.

NOTE: ρ EXTENDS TO $\rho : \mathcal{M}(A) \to \mathcal{M}(A)$.

CAN TAKE:

$$A_0 = A, \; A_1 = \rho(A), \; A_2 = \rho^2(A), \; \cdots$$

AND $\alpha = \rho$ ON $C^*(A_0, A_1, A_2, \cdots)$.

A BIT MORE ABOUT $\rho : A \to M(A)$:

ONE TAKES A SEQUENCE OF ENDOMORPHISMS ON A

$$\rho_j : C(X, K) \to C(X, K), \quad j \in \mathbb{N},$$

THAT "SHUFFLE" THE COORDINATES OF

$$X = \prod_{j=1}^{\infty} S^2$$

AND ~~THROW~~ HAVE BUILT IN SOME "BOTT PROJECTIONS".

THEN:

$$\rho(a) := \sum_{j=1}^{\infty} S_j \rho_j(a) S_j^* = \bigoplus_{j=1}^{\infty} \rho_j(a),$$

WHERE $S_1, S_2, \dots \in M(A)$ ISOMETRIES S.T. $\sum S_j S_j^* = 1$

PROPERTIES OF $C^*(A, \rho)$

PROP: $C^*(A, \rho)$ SIMPLE IF ρ MINIMAL AND ρ^n PROPERLY OUTER $\forall n \in \mathbb{N}$.

DEF ρ MINIMAL IF THERE ARE NO NON-TRIVIAL IDEALS $I \lhd A$ S.T. $A\rho(I)A \subseteq I$.

PROP: A NUCLEAR $\Rightarrow$ $C^*(A, \rho)$ NUCLEAR

PROP: ASSUME $C^*(A, \rho)$ SIMPLE. $p, q \in A$ PROJECTIONS. ASSUME:

(*) $\forall N$: $q \not\precsim p \oplus \rho(p) \oplus \cdots \oplus \rho^N(p)$ IN $M(A)$.

THEN $p \in C^*(A, \rho)$ IS FINITE.

SPECIAL PROJECTIONS IN $A = C(\mathbb{X}, \mathcal{K})$

$p \in M_2(C(S^2))$, $\dim(p) = 1$, $\mathrm{Euler}(p) \neq 0$

If $j \in \mathbb{N}$, $I = \{j_1, j_2, \cdots, j_k\} \overset{\text{finite}}{\subseteq} \mathbb{N}$, SET

$$p_j(x_1, x_2, x_3, \cdots) = p(x_j), \quad p_j \in A = C(\mathbb{X}, \mathcal{K})$$

$$p_I(x) = p_{j_1}(x) \otimes p_{j_2}(x) \otimes \cdots \otimes p_{j_k}(x), \quad p_I \in A$$

<u>PROPOSITION</u> $I_1, I_2, \cdots, I_N \overset{\text{finite}}{\subseteq} \mathbb{N}$. THEN

$$\mathrm{Euler}(p_{I_1} \oplus p_{I_2} \oplus \cdots \oplus p_{I_N}) \neq 0$$

$$\Updownarrow$$

$\exists$ match $t_1 \in I_1, t_2 \in I_2, \cdots, t_N \in I_N$ (i.e., $t_i \neq t_j$ when $i \neq j$).

TO VERIFY $(*)$ TAKE $q \in A$ TO BE TRIVIAL, I.E., WITH $\mathrm{Euler}(q) = 0$.

TAKE $p = p_1$.

USE SPECIAL PROPERTY OF ρ TO SEE THAT $(*)$ HOLDS.

HENCE $p_1 \in A \subseteq C^*(A, \rho)$ IS FINITE IN $C^*(A, \rho)$.

CONCLUDING REMARKS

A. TOMS HAS LATER GIVEN SEVERAL AND SPECTACULAR EXAMPLES – IN THE STABLY FINITE CASE – SHOWING THAT THE ELLIOTT INVARIANT (AS DEFINED HERE) IS NOT COMPLETE.

FUTURE OF THE ELLIOTT PROGRAM:

- SETTLE THE CONJECTURE IN THE SIMPLE RR=0 CASE.
- BECOME FRIENDS WITH THE "HIGH-DIMENSIONAL" EXAMPLES CONSTUCTED BY VILLADSEN, R., TOMS.
- REVISE THE CONJECTURE IN THE SIMPLE, RR>0 CASE: EXPAND THE INVARIANT OR RESTRICT THE CLASS OF C^*-ALG'S TO BE CLASSIFIED.

  E.G., CONSIDER THE CLASS OF C^*-ALG'S A S.T.

$$A \cong A \otimes \mathcal{Z},$$

  WHERE $\mathcal{Z}$ = JIANG-SU ALG: A SIMPLE UNITAL INF. DIM. C^*-ALG'S WITH $\mathrm{Inv}(\mathcal{Z}) = \mathrm{Inv}(\mathbb{C})$.

THE NON-SIMPLE CASE

- KIRCHBERG HAS INTERSTING CLASSIFICATION RESULT IN THE PURELY INFINITE CASE
- DADARLAT – GONG CLASSIFIED A LARGE SPECIFIC CLASS OF STABLY FINITE C^*-ALG'S OF RR=0
- KIRCHBERG, BLANCHARD, R INVESTIGATE THE PURELY INF. C^*-ALG'S.