Divisibility properties for C^*-algebras

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Abstract

We consider three notions of divisibility in the Cuntz semigroup of a C^*-algebra, and show how they reflect properties of the C^*-algebra. We develop methods to construct (simple and non-simple) C^*-algebras with specific divisibility behaviour. As a byproduct of our investigations, we show that there exists a sequence (A_n) of simple unital infinite dimensional C^*-algebras such that the product $\prod_{n=1}^{\infty} A_n$ has a character.

1 Introduction

A unital embedding of a matrix algebra $M_m(\mathbb{C})$ into a unital C^*-algebra A can exist only if the equation $mx = [1_A]$ has a solution $x \in K_0(A)$. Thus, only C^*-algebras in which the class of the unit in K_0 is m-divisible admit a unital embedding of $M_m(\mathbb{C})$. Whereas all von Neumann algebras (with no central summand of type I_n for n finite) have this divisibility property for all m, the same is not true for C^*-algebras, even for the simple ones. C^*-algebras can fail to have non-trivial projections. Even if they have many projections, as in the real rank zero case, one cannot expect to solve the equation $mx = [1_A]$ exactly in $K_0(A)$. This paper is concerned with different weaker notions of divisibility, phrased in terms of the Cuntz semigroup of the C^*-algebra, and with how they relate to embeddability properties of the C^*-algebra. Instead of solving the equation $mx = [1_A]$ for $x \in K_0(A)$, one should look for less restrictive notions of divisibility. One can try, for example, to solve the inequalities $mx \le \langle 1_A \rangle \le nx$ in the Cuntz semigroup of A for fixed positive integers m and n (typically with $m < n$). We say that A is (m,n)-divisible if one can solve this inequality. This is one of three divisibility properties we shall consider in this paper. We show that there is a full *-homomorphism from $CM_m(\mathbb{C})$, the cone over $M_m(\mathbb{C})$, into A if and only if A is (m,n)-divisible for some n.

Let us mention three embedding problems that served as motivation for this paper. Let A be a unital C^*-algebra with no non-zero finite dimensional representations. Can one always find an embedding of some unital simple infinite dimensional C^*-algebra into A? Can one always find an embedding of $CM_2(\mathbb{C})$ into A whose image is full in A? Can one

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always find two positive mutually orthogonal full elements in A? An affirmative answer to the former problem will imply an affirmative answer to the second problem, which is known as the "Global Glimm Halving problem". An affirmative answer to the Global Glimm Halving problem will imply an affirmative answer to the last mentioned problem. We suspect that all three problems may have negative answers.

The second and the third problem led us to consider two new notions of divisibility properties. In more detail, we say that A is weakly (m, n)-divisible if there are elements $x_1, \dots, x_n$ in $\mathrm{Cu}(A)$ such that $mx_j \leq \langle 1_A \rangle \leq x_1 + \cdots + x_n$. Weak divisibility measures the rank of A in the sense that A is weakly (m, n)-divisible for some n if and only if A has no non-zero representations of dimension $< m$. In particular, A has no non-zero finite dimensional representations if and only if for every m there is n such that A is weakly (m, n)-divisible. We say that A is (m, n)-decomposable if there are elements $y_1, \dots, y_m$ in $\mathrm{Cu}(A)$ such that $y_1 + \cdots + y_m \leq \langle 1_A \rangle \leq ny_j$. For a given m, A is (m, n)-decomposable for some n if and only if A contains m pairwise orthogonal, pairwise equivalent full positive elements.

It was shown in [DHTW09] that there exists a simple unital infinite dimensional C^*-algebra which does not admit a unital embedding of the Jiang-Su algebra $\mathcal{Z}$. This answered in the negative a question posed by the second named author. It is implicit in [DHTW09] that this simple C^*-algebra has bad divisibility properties, cf. Remark 3.14. This leads us to a useful observation, which loosely can be formulated as follows: if A and B are unital C^*-algebras, and if there is a unital *-homomorphism from A to B, then the divisibility properties of B are no worse than those of A. In other words, if A has better divisibility properties than B, then you can not unitally embed A into B.

Comparability in the Cuntz semigroup is concerned with the extent to which one can conclude that $x \leq y$ if the "size" of x (e.g., measured in terms of states) is (much) smaller than the "size" of y. Comparability and divisibility are probably the two most fundamental properties of the Cuntz semigroup. Good comparability and divisibility properties are necessary and sufficient conditions in Winter's theorem, [Win], to conclude that a simple, separable, unital C^*-algebra with locally finite nuclear dimension tensorially absorbs the Jiang-Su algebra. Also, good comparability and divisibility properties are both necessary and sufficient conditions to ensure that the Cuntz semigroup of a simple, separable, unital, exact C^*-algebra A is (naturally) isomorphic to $\mathrm{Aff}(T(A)) \sqcup V(A)$, cf. [PT07], [BPT08], and [ERS].

The existence of simple C^*-algebras with bad comparability properties was discovered by Villadsen, [Vil98], in the mid 1990's. This discovery was the first indication that the Elliott conjecture could be false (in general), and it was also the first example of a simple C^*-algebra exhibiting "infinite dimensional" behaviour. Villadsen's example in [Vil98] has been generalized extensively by several authors (including Villadsen himself) to exhibit simple C^*-algebras with various kinds of unexpected behaviour, including many ways of failing to have good comparability properties. However, little work has been done to construct simple C^*-algebras with bad divisibility behaviour, and the literature does not contain systematic ways of producing such examples. In this paper we show that there is a duality between comparability and divisibility (Lemma 6.1), and we use this duality to construct examples of simple and non-simple C^*-algebras with bad divisibility behaviour.

We use Lemma 6.1 to obtain a result that concerns the structure of C^*-algebras that arise as the tensor product of a sequence of unital (simple non-elementary) C^*-algebras. Each such C^*-algebra will of course have non-trivial central sequences. Dadarlat and Toms proved in [DT09] that if the infinite tensor power $\bigotimes_{n=1}^{\infty} A$ of a fixed unital C^*-algebra A contains a unital copy of an AHS-algebra without characters, then it automatically absorbs the Jiang-Su algebra. It is not known if this condition always is satisfied, even when A is simple and non-elementary. We show in Section 6 that $\bigotimes_{n=1}^{\infty} A$ has the Corona Factorization Property for every unital A without characters (and in particular for every unital simple C^*-algebra $A \neq \mathbb{C}$). In the other direction we give, in Section 7, an example of a sequence of simple unital infinite dimensional C^*-algebras whose tensor power, $\bigotimes_{n=1}^{\infty} A_n$, does not absorb (or admit an embedding of) the Jiang-Su algebra.

Non-divisibility of a C^*-algebra can be interpreted as a degree of inhomogeneity (or "lumpiness") of the C^*-algebra. Simple C^*-algebras are sometimes thought of as being very homogeneous, as for example in [KOS03]. From this point of view it may at first be surprising that a simple infinite dimensional C^*-algebra can fail to have good divisibility properties. We show that there exists a sequence (A_n) of simple, unital, infinite dimensional C^*-algebras such that $\prod_{n=1}^{\infty} A_n$ (and also the associated ultrapowers of (A_n)) has a character. None of the C^*-algebras A_n can have a character (being simple and not equal to $\mathbb{C}$), however we can show that they posses "almost characters" as defined in Section 8.

In Section 9 we consider what one might call "super-divisibility", which leads to a (new) notion of infiniteness of positive elements (and which implies that a multiple of the given element is properly infinite). We use this to reformulate the Corona Factorization Property of semigroups considered in [OPR]. We study variations of examples, originally due to Dixmier and Douady, and answer in this way two questions from [KR00] in the negative: The sum of two properly infinite positive elements need not be properly infinite, and the multiplier algebra of a C^*-algebra which has a properly infinite strictly positive element need not be properly infinite.

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2 Preliminaries

Let A be a C^*-algebra, and let $\mathrm{Cu}(A)$ denote the Cuntz semigroup of A. We remind the reader of the basic ingredients of the Cuntz semigroup: If a and b are positive elements in $A \otimes \mathcal{K}$, then a is *Cuntz smaller* than b, denoted $a \precsim b$, if $d_n^* b d_n \to a$ for some sequence (d_n) in A. If $a \precsim b$ and $b \precsim a$, then a and b are *Cuntz equivalent*. The Cuntz equivalence class containing $a \in A \otimes \mathcal{K}$ is denoted by $\langle a \rangle$. The Cuntz semigroup is the set of Cuntz equivalence classes endowed with the sum arising from orthogonal addition of positive elements and the order arising from the Cuntz relation $\precsim$. In [CEI08], Coward, Elliott and Ivanescu give an alternative picture of the Cuntz semigroup where $\mathrm{Cu}(A)$ consists of suitable equivalence classes of countably generated Hilbert C^*-modules over A. When using the Hilbert modules picture of $\mathrm{Cu}(A)$, we shall denote the equivalence class of a countably generated Hilbert module H by $[H]$.

We present here some well-known definitions and facts about the Cuntz semigroup. The reader is referred to [ABG$^+$] for a general account of the theory of the Cuntz semigroup. First of all, we shall frequently use the axioms of the category **Cu**, of which $\mathrm{Cu}(A)$ is always an object (see [CEI08]). An ordered abelian semigroup S is an object in the category **Cu** if

(A1) every increasing sequence in S has a supremum,

(A2) for every $u \in S$ there exists a sequence (u_i) in S such that $u_i \ll u_{i+1}$ and $\sup_i u_i = u$,

(A3) if $u' \ll u$ and $v' \ll v$, then $u' + v' \ll u + v$,

(A4) if (u_i) and (v_i) are increasing sequences then $\sup_i u_i + \sup_i v_i = \sup_i (u_i + v_i)$.

Recall that $u \ll v$ in S if whenever $v = \sup_i v_i$ for some increasing sequence (v_i) in S, then $u \leq v_i$ for some i. An element $u \in S$ is called *compact* if $u \ll u$.

We also note the following two additional properties of the Cuntz semigroup of a C^*-algebra which are not listed among the axioms of **Cu**. The first of them asserts that the Cuntz semigroup of a C^*-algebra almost has the Riesz Decomposition Property, and the second states that its order relation is almost the algebraic order.

(P1) if $u' \ll u \leq v + w$, then there exist v' and w', with $v' \leq u, v$ and $w' \leq u, w$, and such that $u' \ll v' + w'$.

(P2) if $u' \ll u \leq v$, then there exists w such that $u' + w \leq v \leq u + w$.

For the proofs of these facts, see [Rob11, Proposition 5.1.1] for the first and [RW10, Lemma 7.1 (i)] for the second.

We will also make use of the sequential continuity with respect to inductive limits of the functor $\mathrm{Cu}(\cdot)$ proved in [CEI08] (see also the proof of [ERS, Theorem 4.8]). It can be stated as follows:

Proposition 2.1 ([CEI08])**.** *Let $A = \varinjlim(A_i, \varphi_{i,j})$ be a sequential inductive limit of C^*-algebras.*

(i) *For every $u \in \mathrm{Cu}(A)$ there exists an increasing sequence $(u_i)_{i=1}^\infty$ with supremum u and such that each u_i belongs to $\bigcup_j \mathrm{Im}(\mathrm{Cu}(\varphi_{j,\infty}))$.*

(ii) *If $u, v \in \mathrm{Cu}(A_i)$ are such that $\mathrm{Cu}(\varphi_{i,\infty})(u) \leq \mathrm{Cu}(\varphi_{i,\infty})(v)$, then for every $u' \ll u$ there exists j such that $\mathrm{Cu}(\varphi_{i,j})(u') \leq \mathrm{Cu}(\varphi_{i,j})(v)$.*

Remark 2.2 (Equivalence of positive elements)**.** Two positive elements a and b in a C^*-algebra A are said to be *equivalent*, denoted $a \sim b$, if $a = xx^*$ and $b = x^*x$ for some element $x \in A$.

Remark 2.3 (The cone over a matrix algebra)**.** Let m be a positive integer, and let $CM_m(\mathbb{C})$ denote the cone over $M_m(\mathbb{C})$, i.e., the C^*-algebra of all continuous functions $f\colon [0,1] \to M_m(\mathbb{C})$ that vanish at 0.

For each $i, j = 1, 2, \dots, m$, let e_{ij} denote the (i,j)th matrix unit in $M_m(\mathbb{C})$, and denote by $e_{ij} \otimes \iota$ the function $t \mapsto te_{ij}$ in $CM_m(\mathbb{C})$. Then $(e_{ii} \otimes \iota)_{i=1}^m$ are positive contractions in $CM_m(\mathbb{C})$ which are pairwise equivalent (in the sense defined in Remark 2.2 above) and orthogonal.

We recall the following well-known universal property of $CM_m(\mathbb{C})$ (see for example [RW10, Propositions 2.3 and 2.4]): Let A be any C^*-algebra and let $a_1, a_2, \dots, a_m$ be positive contractions in A. Then there exists a *-homomorphism $\varphi\colon CM_m(\mathbb{C}) \to A$ satisfying $\varphi(e_{jj} \otimes \iota) = a_j$ if and only if $a_1, a_2, \dots, a_m$ are pairwise orthogonal and pairwise equivalent in A.

The following lemma is well-known:

Lemma 2.4. *Let A be a C^*-algebra and let $a, b_1, b_2, \dots, b_n$ be positive elements in A. Then:*

(i) $\langle a \rangle \le \sum_{i=1}^n \langle b_i \rangle$ *if and only if for each $\varepsilon > 0$ there exist $d_1, d_2, \dots, d_n \in A$ such that $(a-\varepsilon)_+ = \sum_{i=1}^n d_i b_i d_i^*$.*

(ii) $\sum_{i=1}^n \langle b_i \rangle \le \langle a \rangle$ *if and only if for each $\varepsilon > 0$ there exist mutually orthogonal positive elements $a_1, a_2, \dots, a_n$ in $\overline{aAa}$ such that $a_i \sim (b_i - \varepsilon)_+$ for all i.*

Proof. (i). If $\langle a \rangle \le \sum_{i=1}^n \langle b_i \rangle$, then $a \precsim b_1 \oplus b_2 \oplus \cdots \oplus b_n$, whence

$$(a-\varepsilon)_+ = d^*(b_1 \oplus b_2 \oplus \cdots \oplus b_n)d = \sum_{i=1}^n d_i^* b_i d_i$$

for some $d = (d_1, d_2, \dots, d_n)^t \in M_{n,1}(A)$. The converse statement is trivial.

(ii). Suppose that $\sum_{i=1}^n \langle b_i \rangle \le \langle a \rangle$. Then $(b_1-\varepsilon)_+ \oplus (b_2-\varepsilon)_+ \oplus \cdots \oplus (b_n-\varepsilon)_+ = d^*ad$ for some $d = (d_1, d_2, \dots, d_n)$ in $M_{1,n}(A)$. Thus $d_j^* a d_i = 0$ if $j \ne i$ and $d_i^* a d_i = (b_i - \varepsilon)_+$ for all i. Put $a_i = a^{1/2} d_i d_i^* a^{1/2}$. It is now straightforward to verify that the a_i's are as desired. The converse statement is trivial. $\square$

Here is another lemma that we will use frequently:

Lemma 2.5. *Let a and b be positive elements in $A \otimes \mathcal{K}$ with $\|a\| \le 1$, and let $m \in \mathbb{N}$. The following are equivalent:*

(i) $m\langle a \rangle \le \langle b \rangle$,

(ii) *for each $\varepsilon > 0$ there exist mutually orthogonal positive elements $b_1, b_2, \dots, b_m$ in $\overline{b(A \otimes \mathcal{K})b}$ such that $\langle b_i \rangle = \langle (a-\varepsilon)_+ \rangle$ for all i.*

(iii) *for each $\varepsilon > 0$ there exists a *-homomorphism $\varphi\colon CM_m(\mathbb{C}) \to \overline{b(A \otimes \mathcal{K})b}$ such that $\langle \varphi(e_{11} \otimes \iota) \rangle = \langle (a-\varepsilon)_+ \rangle$.*

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear, cf. Remark 2.3 and Lemma 2.4. Let us show that (i) implies (iii). Let $\varepsilon > 0$ be given. By Lemma 2.4 (ii) there are mutually orthogonal positive elements $b_1, b_2, \dots, b_m$ in $\overline{b(A \otimes \mathcal{K})b}$ such that each b_j is equivalent to $(a-\varepsilon)_+$. By the universal property of the cone $CM_m(\mathbb{C})$, see Remark 2.3, there is a *-homomorphism $\varphi\colon CM_m(\mathbb{C}) \to \overline{b(A \otimes \mathcal{K})b}$ satisfying $\varphi(e_{jj} \otimes \iota) = b_j$. Hence (iii) holds. $\square$

3 Three divisibility properties

Definitions and basic properties

Definition 3.1. Let A be a C^*-algebra and fix $u \in \mathrm{Cu}(A)$. Let $m, n \geq 1$ be integers. Then:

(i) u is (m,n)-*divisible* if for every $u' \in \mathrm{Cu}(A)$ with $u' \ll u$ there exists $x \in \mathrm{Cu}(A)$ such that $mx \leq u$ and $u' \leq nx$.

The least n such that u is (m,n)-divisible is denoted by $\mathrm{Div}_m(u, A)$, with $\mathrm{Div}_m(u, A) = \infty$ if no such n exists.

(ii) u is (m,n)-*decomposable* if for every $u' \in \mathrm{Cu}(A)$ with $u' \ll u$ there exist elements $x_1, x_2, \dots, x_m \in \mathrm{Cu}(A)$ such that $x_1 + x_2 + \dots + x_m \leq u$ and $u' \leq nx_j$ for all $j = 1, 2, \dots, m$.

The least n such that u is (m,n)-decomposable is denoted by $\mathrm{Dec}_m(u, A)$, with $\mathrm{Dec}_m(u, A) = \infty$ if no such n exists.

(iii) u is *weakly* (m,n)-*divisible* if for every $u' \in \mathrm{Cu}(A)$ with $u' \ll u$ there exist elements $x_1, x_2, \dots, x_n \in \mathrm{Cu}(A)$ such that $mx_j \leq u$ for all $j = 1, 2, \dots, m$ and $u' \leq x_1 + x_2 + \dots + x_n$.

The least n such that u is weakly (m,n)-divisible is denoted by $\text{w-Div}_m(u, A)$, with $\text{w-Div}_m(u, A) = \infty$ if no such n exists.

Remark 3.2. In the case that u in Definition 3.1 is compact (e.g., when A is unital and $u = \langle 1_A \rangle$), the conditions above read a little easier:

(i) u is (m,n)-*divisible* if there exists $x \in \mathrm{Cu}(A)$ such that $mx \leq u \leq nx$.

(ii) u is (m,n)-*decomposable* if there exist elements $x_1, x_2, \dots, x_m \in \mathrm{Cu}(A)$ such that $x_1 + x_2 + \dots + x_m \leq u \leq nx_j$ for all $j = 1, 2, \dots, m$.

(iii) u is *weakly* (m,n)-*divisible* if there exist elements $x_1, x_2, \dots, x_n \in \mathrm{Cu}(A)$ such that $mx_j \leq u \leq x_1 + x_2 + \dots + x_n$.

The three divisibility properties above are related as follows:

Proposition 3.3. *Let $m, n \in \mathbb{N}$ and $u \in \mathrm{Cu}(A)$. Then*

$$\text{w-Div}_m(u, A) \leq \mathrm{Div}_m(u, A), \quad \mathrm{Dec}_m(u, A) \leq \mathrm{Div}_m(u, A), \quad \text{w-Div}_m(u, A) \leq \mathrm{Dec}_m(u, A)^m.$$

Proof. The two first inequalities are clear (take $x_i = x$ in both cases).

To prove the last inequality, suppose that u is (m,n)-decomposable. We show that u is weakly (m,n^m)-divisible. Let $u' \ll u$ and find u'' such that $u' \ll u'' \ll u$. There exist elements $x_1, \dots, x_m$ in $\mathrm{Cu}(A)$ such that $\sum_{i=1}^m x_i \le u$ and $u'' \le nx_i$ for all i. We proceed to find elements

$$\tilde{y}(i_1, \dots, i_k),\, y(i_1, \dots, i_k) \in \mathrm{Cu}(A), \quad k = 1, \dots, m,\, i_j = 1, \dots, n,$$

satisfying

(a) $\tilde{y}(i_1, \dots, i_k) \ll y(i_1, \dots, i_k)$,

(b) $\tilde{y}(i_1, \dots, i_{k-1}) \ll \sum_{i=1}^n y(i_1, \dots, i_{k-1}, i)$ if $k \ge 2$, and $u' \ll \sum_{i=1}^n y(i)$,

(c) $\tilde{y}(i_1, \dots, i_{k-1}) \le \sum_{i=1}^n \tilde{y}(i_1, \dots, i_{k-1}, i)$ if $k \ge 2$, and $u' \le \sum_{i=1}^n \tilde{y}(i)$,

(d) $y(i_1, \dots, i_k) \le x_k$,

(e) $y(i_1, \dots, i_{k-1}, i_k) \le y(i_1, \dots, i_{k-1})$ if $k \ge 2$, and $y(i) \le u''$.

The elements above are constructed inductively after k using the following fact:

($*$) if $x' \ll x \le nz$ in $\mathrm{Cu}(A)$, then there exist $y_1, \dots, y_n \in \mathrm{Cu}(A)$ such that $x' \ll \sum_{i=1}^n y_i$, $y_i \le x$, and $y_i \le z$,

which follows from Property (P1) of the Cuntz semigroup stated in the previous section.

Take first $k = 1$. The existence of $y(i)$, with $i = 1, \dots, n$, satisfying (b), (d) and (e) follows from ($*$) applied to $u' \ll u'' \le nx_1$. The existence of $\tilde{y}(i) \ll y(i)$ satisfying (a) and (c) then follows from Axiom (A2) of the Cuntz semigroup from the previous section. Assume that $2 \le k \le m$ and that $\tilde{y}(i_1, i_2, \dots, i_{k-1})$ and $y(i_1, i_2, \dots, i_{k-1})$ have been found. The existence of $y(i_1, \dots, i_{k-1}, i)$, with $i = 1, \dots, n$, satisfying (b), (d) and (e) follows from ($*$) applied to

$$\tilde{y}(i_1, \dots, i_{k-1}) \ll y(i_1, \dots, i_{k-1}) \le nx_k.$$

(To see that the latter inequality holds, note that $y(i_1, \dots, i_{k-1}) \ll u''$, which follows by repeated use of (e).) The existence of $\tilde{y}(i_1, i_2, \dots, i_k)$ satisfying (a) and (c) follows from Axiom (A2).

We claim that the n^m elements $\big(y(i_1, \dots, i_m)\big)$ witness the weak (m,n^m)-divisibility of u. Indeed, it follows from (d) and (e) that $y(i_1, \dots, i_m) \le x_j$ for all $j = 1, \dots, m$, whence

$$m \cdot y(i_1, \dots, i_m) \le x_1 + x_2 + \cdots + x_m \le u.$$

It follows from (b) and (c) that the sum of the elements $y(i_1, \dots, i_m)$ is greater than or equal to u'. $\square$

If any of the divisibility numbers $\mathrm{Div}_m(u,A)$, $\mathrm{Dec}_m(u,A)$, and $\text{w-}\mathrm{Div}_m(A)$ is less than m, then u (or a multiple of u) must be properly infinite, as shown below. We shall pursue this and related questions in more detail in Section 9.

Proposition 3.4. *Let A be a C^*-algebra and let $u \in \mathrm{Cu}(A)$.*

(i) *If u is properly infinite, then $\mathrm{Div}_m(u, A) = 1$ for all integers $m \geq 1$.*

(ii) *If $1 \leq n < m$ are integers and if u is either (m, n)-divisible, (m, n)-decomposable or weakly (m, n)-divisible, then nu is properly infinite, i.e., $nu = 2nu$.*

(iii) *If $1 \leq n < m$ are integers and if u is compact and (m, n)-divisible, then u is properly infinite.*

Proof. (i). If u is properly infinite, then $mu \leq u$ for all m, whence $\mathrm{Div}_m(u, A) = 1$.

(ii). Assume that u is weakly (m, n)-divisible and take $u' \ll u$. Then there exist $x_1, \dots, x_n$ such that $mx_i \leq u$ for all i, and $u' \leq \sum_{i=1}^n x_i$. Thus,

$$mu' \leq \sum_{i=1}^{n} mx_i \leq nu.$$

As this holds for all $u' \ll u$, we get $((m - n) + n)u = mu \leq nu$. This entails that $(k(m - n) + n)u \leq nu$ for all positive integers k, whence $\ell u \leq nu$ for all positive integers ℓ. In particular, $2nu \leq nu$, which implies that nu is properly infinite.

Next, suppose that u is (m, n)-decomposable and let $u' \ll u$. Then there exist $x_1, \dots, x_m$ such that $\sum_{i=1}^m x_i \leq u$ and $u' \leq nx_i$ for all i. Thus,

$$mu' \leq n \sum_{i=1}^{m} x_i \leq nu.$$

Arguing as before, we conclude that nu is properly infinite.

Finally note that if u is (m, n)-divisible, then it is both (m, n)-decomposable and weakly (m, n)-divisible, whence nu is properly infinite.

(iii). Since $\mathrm{Div}_m(u, A) = n < m$ and $u \ll u$, there exists x such that $mx \leq u \leq nx$. Arguing as in the proof of part (ii) this implies that $\ell x \leq nx$ for all positive integers ℓ. This shows that $2u \leq 2nx \leq nx \leq mx \leq u$, whence u is properly infinite. ☐

Remark 3.5. By functoriality, each *-homomorphism $\varphi \colon A \to B$ between C^*-algebras A and B induces a morphism $\mathrm{Cu}(\varphi) \colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ which preserves order, addition, and the relation of compact containment. Thus, for each $u \in \mathrm{Cu}(A)$, and with $v = \mathrm{Cu}(\varphi)(u)$, we have:

$$\mathrm{Div}_m(v, B) \leq \mathrm{Div}_m(u, A), \quad \mathrm{Dec}_m(v, B) \leq \mathrm{Dec}_m(u, A), \quad \text{w-}\mathrm{Div}_m(v, B) \leq \text{w-}\mathrm{Div}_m(u, A).$$

In particular, if A and B are unital C^*-algebras, and if $\mathrm{Div}_m(\langle 1_B \rangle, B) > \mathrm{Div}_m(\langle 1_A \rangle, A)$ for some m (or if the corresponding inequality holds for one of the other two divisibility numbers), then one can not find a unital embedding of A into B. Divisibility numbers thus serve as an obstruction for embedding a unital C^*-algebra with nice divisibility properties into a unital C^*-algebra with less nice divisibility properties.

The three divisibility properties behave well with respect to inductive limits thanks to the sequential continuity of the functor $\mathrm{Cu}(\cdot)$:

Proposition 3.6. *Let $A = \varinjlim(A_i, \varphi_{i,j})$ be a sequential inductive limit of C^*-algebra. Let $u \in \mathrm{Cu}(A_1)$ and, for each i, denote by $u_i \in \mathrm{Cu}(A_i)$ and $u_\infty \in \mathrm{Cu}(A)$ the images of u in $\mathrm{Cu}(A_i)$ and $\mathrm{Cu}(A)$, respectively. Then:*

$$\mathrm{Div}_m(u_\infty, A) \le \inf_i \mathrm{Div}_m(u_i, A_i), \qquad \mathrm{Dec}_m(u_\infty, A) \le \inf_i \mathrm{Dec}_m(u_i, A_i),$$

$$\text{w-}\mathrm{Div}_m(u_\infty, A) \le \inf_i \text{w-}\mathrm{Div}_m(u_i, A_i).$$

If u is compact (i.e., if $u \ll u$), then the above inequalities are equalities.

Proof. We will only prove the statements above in the former case; the proofs for the two other cases are similar.

The inequalities $\mathrm{Div}_m(u_\infty, A) \le \mathrm{Div}_m(u_i, A_i)$, with $i = 1, 2, \dots$, follow from Remark 3.5. Suppose now that u is compact. Set $\mathrm{Div}_m(u_\infty, A) = n$. Then there exists $x \in \mathrm{Cu}(A)$ such that $mx \le u_\infty \le nx$. By Proposition 2.1 (i) and compactness of u_∞, it follows that x is the image of some $y \in \mathrm{Cu}(A_i)$ for some i. By Axiom (A2) of the Cuntz semigroup and by compactness of u_∞ there exists $y' \ll y$ in $\mathrm{Cu}(A_i)$ such that $u_\infty \le n\,\mathrm{Cu}(\varphi_{i,\infty})(y')$. Since the u_i's are compact, Proposition 2.1 (ii) implies that there exists $j > i$ such that

$$m\,\mathrm{Cu}(\varphi_{i,j})(y') \le u_j \le n\,\mathrm{Cu}(\varphi_{i,j})(y').$$

Thus u_j is (m, n)-divisible in $\mathrm{Cu}(A_j)$. □

Definition 3.7. Let A be a σ-unital C^*-algebra. Then A contains a strictly positive element. This element represents a class in $\mathrm{Cu}(A)$, which is independent of the choice of the strictly positive element, and which we shall denote by $\langle A \rangle$. If A is unital, then $\langle A \rangle = \langle 1_A \rangle$. We shall write $\text{w-}\mathrm{Div}_m(A)$, $\mathrm{Dec}_m(A)$, and $\mathrm{Div}_m(A)$ for $\text{w-}\mathrm{Div}_m(\langle A \rangle, A)$, $\mathrm{Dec}_m(\langle A \rangle, A)$, and $\mathrm{Div}_m(\langle A \rangle, A)$, respectively.

If A and B are unital C^*-algebras such that there exist unital *-homomorphisms $A \to B$ and $B \to A$, then, by Remark 3.5, we must have

$$\mathrm{Div}_m(A) = \mathrm{Div}_m(B), \quad \mathrm{Dec}_m(A) = \mathrm{Dec}_m(B), \quad \text{w-}\mathrm{Div}_m(A) = \text{w-}\mathrm{Div}_m(B)$$

for all m. This applies in particular to the situation where A is any unital C^*-algebra and $B = A \otimes D$ for some unital C^*-algebra D which has a character. In general, if D is any unital C^*-algebra, possibly without characters, the divisibility numbers associated with $A \otimes D$ are smaller than or equal to those of A.

Examples and remarks

Let us first examine the divisibility numbers for matrix algebras:

Example 3.8. Let $m \geq 2$ and $k \geq 2$ be integers. Using that

$$\big(\mathrm{Cu}(M_k(\mathbb{C})), \langle 1 \rangle\big) \cong \big(\{0, 1, 2, 3, \dots, \infty\}, k\big),$$

an elementary algebraic argument yields that

$$\text{w-Div}_m(M_k(\mathbb{C})) = \mathrm{Dec}_m(M_k(\mathbb{C})) = \mathrm{Div}_m(M_k(\mathbb{C})),$$

and

$$\mathrm{Div}_m(M_k(\mathbb{C})) = \begin{cases} \left\lceil \frac{k}{\lfloor \frac{k}{m} \rfloor} \right\rceil, & \text{if } m \leq k, \\ \infty, & \text{if } m > k. \end{cases} \tag{3.1}$$

Here $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the "ceiling" and "floor" functions. In particular, $\mathrm{Div}_m(M_k(\mathbb{C})) = m$ if and only if $m \,|\, k$, and $\mathrm{Div}_m(M_k(\mathbb{C})) = m + 1$ if $m \nmid k$ and $m(m-1) \leq k$.

Definition 3.9 (The rank of a C^*-algebra)**.** Let A be a C^*-algebra. Let $\mathrm{rank}(A)$ denote the smallest positive integer n for which A has an irreducible representation on a Hilbert space of dimension n, and set $\mathrm{rank}(A) = \infty$ if A has no finite dimensional (irreducible) representation.

Note that $\mathrm{rank}(A) = 1$ if and only if A has a character. We remind the reader about the following classical result due to Glimm. (We include a short proof, as this is an important result for our paper, and because we didn't find an exact reference for this statement.)

Proposition 3.10 (Glimm)**.** *Let A be a (not necessarily unital) C^*-algebra and let $n \geq 1$ be an integer. Then there is a non-zero *-homomorphism $CM_n(\mathbb{C}) \to A$ if and only if A admits at least one irreducible representation on a Hilbert of dimension $\geq n$.*

Proof. "Only if" is clear. To prove the "if"-part of the proposition, let π be an irreducible representation of A on a Hilbert space H of dimension $\geq n$. Let P be an n-dimensional projection on H. By Kadison's transitivity theorem for each unitary operator $U \colon P(H) \to P(H)$ there is a contraction $a \in A$ such that $\pi(a)P = U$. As a is a contraction, this entails $\pi(a)P = P\pi(a)$. Let B be the sub-C^*-algebra of A consisting of all elements a such that $\pi(a)P = P\pi(a)$, and define a *-homomorphism $\varphi \colon B \to M_n(\mathbb{C})$ by $\varphi(a) = P\pi(a)P$, $a \in B$. The argument above shows that φ is surjective. Hence, there is an isomorphism $M_n(\mathbb{C}) \to B/\ker(\varphi)$. By projectivity of $CM_n(\mathbb{C})$ this isomorphism lifts to a (non-zero) *-homomorphism $CM_n(\mathbb{C}) \to B \subseteq A$. □

It follows from Remark 3.5 that $\text{w-Div}_m(A)$, $\mathrm{Dec}_m(A)$, and $\mathrm{Div}_m(A)$ are greater than or equal to $\mathrm{Div}_m(M_n(\mathbb{C}))$ if $\mathrm{rank}(A) = n$. In particular, these three quantities are infinite when $m > \mathrm{rank}(A)$.

Example 3.11 (Simple C^*-algebras)**.** If A is a simple, unital, infinite dimensional C^*-algebra, then $\mathrm{Div}_m(A)$, $\mathrm{Dec}_m(A)$, and $\text{w-Div}_m(A)$ are finite for all positive integers m. Indeed, by the assumption that A is infinite dimensional, it follows that there is a non-zero $x \in \mathrm{Cu}(A)$ such that $mx \leq \langle 1_A \rangle$. As every simple unital C^*-algebra is algebraically simple, it follows that $\langle 1_A \rangle \leq nx$ for some positive integer n, i.e., $\langle 1_A \rangle$ is (m, n)-divisible. Hence $\mathrm{Div}_m(A) \leq n$, which entails that also $\mathrm{Dec}_m(A) \leq n$ and $\text{w-Div}_m(A) \leq n$.

Example 3.12. The dimension drop C^*-algebra $Z_{p,q}$, associated with the positive integers p and q, is defined to be

$$Z_{p,q} = \{f \in C([0,1], M_p \otimes M_q \mid f(0) \in M_p \otimes \mathbb{C}1_q, \;\; f(1) \in \mathbb{C}1_p \otimes M_q\}.$$

Note that $\mathrm{rank}(Z_{p,q}) = \min\{p,q\}$. It was shown in [Rør04, Lemma 4.2] (and its proof) that $\mathrm{Div}_m(Z_{m,m+1}) = m+1$. By Remark 3.5, it follows that if $Z_{m,m+1}$ maps unitally into A, then $\mathrm{Div}_m(A) \le m+1$. Moreover, as shown in [RW10, Proposition 5.1], if A is a unital C^*-algebra of stable rank one, then $\mathrm{Div}_m(A) \le m+1$ if and only if $Z_{m,m+1}$ maps unitally into A.

Remark 3.13 (Almost divisibility)**.** The property "almost divisibility" of a C^*-algebra is expressed by saying that $\mathrm{Div}_m(A) \le m+1$ for all integers $m \ge 1$. If every dimension drop algebra $Z_{m,m+1}$ maps unitally into A, or if the Jiang-Su algebra maps unitally into A, then A is almost divisible.

Remark 3.14 (Non-embeddability of the Jiang-Su algebra)**.** It was shown in [DHTW09] that there is a simple unital infinite dimensional nuclear C^*-algebra A such that the dimension drop C^*-algebra $Z_{3,4}$, and hence the Jiang-Su algebra $\mathcal{Z}$, do not embed unitally into A. The divisibility properties of A were not explicitly mentioned in [DHTW09], but it is easily seen (using Lemma 6.1, that is paraphrased from [Rør04, Lemma 4.3]) that $\mathrm{Div}_3(A) > 4$. We shall in Section 7 give further examples of simple unital infinite dimensional C^*-algebras where the divisibility numbers attain non-trivial values.

Remark 3.15 (Real rank zero C^*-algebras)**.** It was shown in [PR04, Proposition 5.7] that if A is a unital C^*-algebra of real rank zero, then $\mathrm{rank}(A) \ge n$ if and only if there exists a unital embedding of a finite dimensional C^*-algebra of rank $\ge n$ into A. Combining this with Remark 3.8 we see that $\mathrm{Div}_m(A) \le m+1$ whenever A is a unital C^*-algebra of real rank zero and with $\mathrm{rank}(A) \ge m(m-1)$. In particular, every unital C^*-algebra A of real rank zero and with $\mathrm{rank}(A) = \infty$ is almost divisible.

Kirchberg considered in [Kir06] a covering number of a unital C^*-algebra B. Let us recall the definition:

Definition 3.16 (Kirchberg)**.** Let $m \in \mathbb{N}$. The covering number of a unital C^*-algebra B, denoted by $\mathrm{cov}(B,m)$, is the least positive integer n such that there exist finite dimensional C^*-algebras $F_1, F_2, \dots, F_n$ with $\mathrm{rank}(F_i) \ge m$, *-homomorphisms $\varphi_i \colon CF_i \to B$, and $d_1, d_2, \dots, d_n \in B$ such that $1_B = \sum_{i=1}^n d_i^* \varphi_i(1_{F_i} \otimes \iota) d_i$, where $CF_i = F_i \otimes C_0((0,1])$ and $\iota(t) = t$.

Kirchberg's covering number $\mathrm{cov}(B,m)$ relates to our $\text{w-Div}_m(B)$ as follows.

Proposition 3.17. *Let B be a unital C^*-algebra and let m be a positive integer.*

(i) $\mathrm{cov}(B,m)$ *is the least n for which there exist $x_1, x_2, \dots, x_n \in \mathrm{Cu}(B)$ such that*

$$x_i \le \langle 1_B \rangle \le x_1 + x_2 + \cdots + x_n, \qquad x_i = \sum_{j=i}^{k_i} m_{ij} y_{ij} \tag{3.2}$$

for some integers $m_{ij} \ge m$, some positive integers k_i, and some $y_{ij} \in \mathrm{Cu}(A)$.

(ii) $\operatorname{cov}(B,m) \leq \text{w-Div}_m(B) \leq (2m-1)\operatorname{cov}(B,m)$.

Proof. (i). Assume that $n \geq \operatorname{cov}(B,m)$ and let F_i, $\varphi_i \colon CF_i \to B$, and $d_i \in B$ be as in Definition 3.16. Write $F_i = \bigoplus_{j=1}^{k_i} M_{m_{ij}}(\mathbb{C})$ with $m_{ij} \geq m$. Let $e^{(ij)}$ be a one-dimensional projection in $M_{m_{ij}}(\mathbb{C})$. It then follows from Lemma 2.4 that the elements

$$x_i = \langle \varphi_i(1_{F_i} \otimes \iota) \rangle, \qquad y_{ij} = \langle \varphi_i(e^{(ij)} \otimes \iota) \rangle$$

satisfy the relations in (3.2).

Suppose, conversely, that $n \geq 1$ is chosen such that there are elements x_i and y_{ij} in $\mathrm{Cu}(B)$ satisfying (3.2). Put $F_i = \bigoplus_{j=1}^{k_i} M_{m_{ij}}(\mathbb{C})$. By the assumption that $\sum_{j=i}^{k_i} m_{ij} y_{ij} \leq \langle 1_B \rangle$ it follows from Lemma 2.4 (ii) that there are mutually orthogonal positive elements a_{ijr} in B, where $1 \leq i \leq n$, $1 \leq j \leq k_i$, $1 \leq r \leq m_{ij}$, such that $\langle a_{ijr} \rangle = y_{ij}$. We can further assume that the r positive elements $a_{ij1}, \dots, a_{ijr}$ are pairwise equivalent. It then follows from the universal property of the cone over a matrix algebra (see Remark 2.3) that there are *-homomorphisms $\varphi_i \colon CF_i \to B$ such that $\langle \varphi_i(e^{(ij)} \otimes \iota) \rangle = y_{ij}$, where $e^{(ij)}$ is a one-dimensional projection in the summand $M_{m_{ij}}(\mathbb{C})$ of F_i. The existence of $d_i \in B$ with $1_B = \sum_{i=1}^n d_i^* \varphi_i(1_{F_i} \otimes \iota) d_i$ follows from Lemma 2.4 (i). Thus $\operatorname{cov}(B,m) \leq n$.

(ii). To prove the first inequality, assume that $\text{w-Div}_m(B) = n < \infty$ and take $y_1, \dots, y_n$ such that $m y_j \leq \langle 1_B \rangle \leq y_1 + \cdots + y_n$. Then (3.2) holds with $k_i = 1$ and $x_i = m y_i$.

Assume next that $\operatorname{cov}(B,m) = n < \infty$, and find elements x_i and y_{ij} satisfying the relations in (3.2). Upon replacing y_{ij} with an integral multiple of y_{ij} we can assume that $m \leq m_{ij} < 2m$ for all i and j. Let z_{ik}, $1 \leq k \leq 2m-1$, be the sum of a suitable subset of the y_{ij}'s such that $\sum_{k=1}^{2m-1} z_{ik} = \sum_{j=i}^{k_i} m_{ij} y_{ij} = x_i$. The $(2m-1)n$ elements (z_{ik}) will then witness that $\text{w-Div}_m(B) \leq (2m-1)n$. □

4 The asymptotic divisibility numbers

One can collect the sequence of divisibility numbers $\big(\mathrm{Div}_m(A)\big)_{m=2}^{\infty}$ of a unital C^*-algebra A into a single divisibility number as follows:

$$\mathrm{Div}_*(A) = \liminf_{m \to \infty} \frac{\mathrm{Div}_m(A)}{m}.$$

In a similar way one can define $\mathrm{Dec}_*(A)$ and $\text{w-Div}_*(A)$. Propositions 4.1 and 4.2 below hold verbatim for those quantities as well. However, to keep the exposition bounded, we only treat the case of "Div".

It follows from Proposition 3.4 that $\mathrm{Div}_*(A) = 0$ if and only if A is properly infinite and that $\mathrm{Div}_*(A) \geq 1$ if A is not properly infinite.

Proposition 4.1. *Let A be a unital C^*-algebra.*

(i) $\mathrm{Div}_m(A) \leq m\,\mathrm{Div}_*(A) + 1$ *for all integers* $m \geq 2$.

(ii) $\mathrm{Div}_*(A) = \lim_{m \to \infty} \mathrm{Div}_m(A)/m$ *(the limit always exists, but is possibly equal to ∞).*

(iii) *If A is not properly infinite, then* $\mathrm{Div}_*(A) = 1$ *if and only if* $\mathrm{Div}_m(A) \le m+1$ *for all integers* $m \ge 2$.

It follows from Proposition 3.4 and from (iii) above, that A is almost divisible if and only if $\mathrm{Div}_*(A) \le 1$ (i.e., if and only if $\mathrm{Div}_*(A) = 0$ or $\mathrm{Div}_*(A) = 1$).

Proof. (i). If $\mathrm{Div}_*(A) = \infty$ there is nothing to prove. If $\mathrm{Div}_*(A) = 0$, then A is properly infinite and so the inequality holds trivially. Assume that $1 \le \mathrm{Div}_*(A) < \infty$. Let $m \ge 2$ be given. Let L be the smallest integer strictly greater than $m\,\mathrm{Div}_*(A)$. We show that $\mathrm{Div}_m(A) \le L$. Choose $\alpha > 1$ and a positive integer r_0 such that

$$\alpha\,\frac{r_0+1}{r_0}\,m\,\mathrm{Div}_*(A) \le L.$$

By the definition of $\mathrm{Div}_*(A)$ there is $k \ge r_0 m$ such that $\ell := \mathrm{Div}_k(A) \le \alpha k\,\mathrm{Div}_*(A)$. Take $x \in \mathrm{Cu}(A)$ such that $kx \le \langle 1_A \rangle \le \ell x$. Write $k = rm+d$, with $0 \le d < m$ and $r \ge r_0$. Also, write $\ell = tr - d'$, with $0 \le d' < r$ and $t \ge 1$. Put $y = rx \in \mathrm{Cu}(A)$. Then $my \le \langle 1_A \rangle \le ty$. With $\lceil\,\cdot\,\rceil$ denoting the ceiling function, we have

$$\begin{aligned}
\mathrm{Div}_m(A) \;\le\; t \;&=\; \lceil \frac{\ell}{r} \rceil \\
&=\; \left\lceil \frac{\ell}{k-d}\,m \right\rceil \\
&\le\; \left\lceil \alpha\,\frac{k}{k-d}\,m\,\mathrm{Div}_*(A) \right\rceil \\
&\le\; \left\lceil \alpha\,\frac{r+1}{r}\,m\,\mathrm{Div}_*(A) \right\rceil \\
&\le\; \left\lceil \alpha\,\frac{r_0+1}{r_0}\,m\,\mathrm{Div}_*(A) \right\rceil \;\le\; L.
\end{aligned}$$

(ii). It follows from (i) that

$$\limsup_{m\to\infty} \frac{\mathrm{Div}_m(A)}{m} \;\le\; \mathrm{Div}_*(A) \;=\; \liminf_{m\to\infty} \frac{\mathrm{Div}_m(A)}{m},$$

and so the claims follows.

(iii). The "if" part is trivial, and the "only if" part follows from (i). □

We proceed to discuss how $\mathrm{Div}_*(\,\cdot\,)$ behaves under forming matrix algebras:

Proposition 4.2. *Let A be a unital C^*-algebra.*

(i) $\mathrm{Div}_*(M_n(A)) \le \mathrm{Div}_*(A)$ *for all integers* $n \ge 2$.

(ii) *If* $\mathrm{Cu}(A)$ *is almost unperforated, then* $\mathrm{Div}_*(M_n(A)) = \mathrm{Div}_*(A)$ *for all integers* $n \ge 2$.

Proof. (i) follows from Remark 3.5 (as A embeds unitally into $M_n(A)$).

(ii). Assume that $\mathrm{Cu}(A)$ is almost unperforated. We show first that

$$\mathrm{Div}_*(A) \le \frac{n+1}{n-1}\,\mathrm{Div}_*(M_n(A)) \tag{4.1}$$

for all $n \ge 2$. To see this take any integer $m \ge 2$, and use Proposition 4.1 (i) to see that $\ell := \mathrm{Div}_m(M_n(A)) \le m\,\mathrm{Div}_*(M_n(A)) + 1$. Write $m = r(n+1) + d$ and $\ell = t(n-1) - d'$, where r and t are positive integers, $0 \le d < n+1$, and $0 \le d' < n-1$.

Identify $\mathrm{Cu}(M_n(A))$ with $\mathrm{Cu}(A)$ in the canonical way, where $\langle 1_{M_n(A)}\rangle \in \mathrm{Cu}(M_n(A))$ is identified with $n\langle 1_A\rangle$. Under this identification we can find $x \in \mathrm{Cu}(A)$ such that $mx \le n\langle 1_A\rangle \le \ell x$. In particular,

$$(n+1)rx \le n\langle 1_A\rangle \le (n-1)tx,$$

which by the assumption that $\mathrm{Cu}(A)$ is almost unperforated implies that $rx \le \langle 1_A\rangle \le tx$. This shows that

$$\begin{aligned} \frac{\mathrm{Div}_r(A)}{r} \;\le\; \frac{t}{r} \;&=\; r^{-1}\Big\lceil \frac{\ell}{n-1}\Big\rceil \;\le\; r^{-1}\big(\frac{\ell}{n-1}+1\big) \\ &\le\; r^{-1}\Big(\frac{m\,\mathrm{Div}_*(M_n(A))+1}{n-1}+1\Big) \\ &\le\; \frac{n+1}{n-1}\,\mathrm{Div}_*(M_n(A)) + r^{-1}\frac{n}{n-1}\,\mathrm{Div}_*(M_n(A)) + r^{-1}\frac{n}{n-1}. \end{aligned}$$

Now, $r \to \infty$ as $m \to \infty$, and so (4.1) follows by letting m tend to infinity.

To complete the proof of (ii), take $n \ge 2$. By (i) and (4.1) we have:

$$\mathrm{Div}_*(A) \;\le\; \frac{kn+1}{kn-1}\,\mathrm{Div}_*(M_{nk}(A)) \;\le\; \frac{kn+1}{kn-1}\,\mathrm{Div}_*(M_n(A))$$

for all $k \ge 1$, which shows that $\mathrm{Div}_*(A) \le \mathrm{Div}_*(M_n(A))$. □

We have previously remarked that $\mathrm{Div}_m(A) = \infty$ whenever $m > \mathrm{rank}(A)$. It follows that $\mathrm{Div}_*(A) = \infty$ whenever $\mathrm{rank}(A) < \infty$, i.e., whenever A admits a non-zero finite dimensional representation.

Remark 4.3. It can happen that $\mathrm{Div}_*(M_n(A)) < \mathrm{Div}_*(A)$. Take for example A such that $M_n(A)$ is properly infinite, but A itself is not properly infinite, cf. [Rør03]. Then $\mathrm{Div}_*(M_n(A)) = 0$ and $\mathrm{Div}_*(A) \ge 1$.

It is an important open problem if $\mathrm{Div}_*(A) \le 1$ (i.e., if A is almost divisible) for every (simple) unital infinite dimensional C^*-algebra A for which $\mathrm{Cu}(A)$ is almost unperforated.

5 Finite-, infinite-, and ω-divisibility

The property that any of the divisibility numbers $\mathrm{Div}_m(A)$, $\mathrm{Dec}_m(A)$, and $\text{w-Div}_m(A)$ is finite, when A is a unital C^*-algebra, has interpretations in terms of structural properties of the C^*-algebra A. We have already noted that the divisibility numbers always are finite when A is a simple C^*-algebra, and the corresponding structural properties of the C^*-algebra are, as we shall see, trivially satisfied for simple C^*-algebras. The correct definition of "finite divisibility" in the non-unital case is what we call (m,ω)-divisibility as defined below.

Definition 5.1. Let A be a C^*-algebra, let $u \in \mathrm{Cu}(A)$, and let m be a positive integer. Then:

(i) u is (m,ω)-*divisible* if for all $u' \in \mathrm{Cu}(A)$ with $u' \ll u$ there exists $x \in \mathrm{Cu}(A)$ such that $mx \le u$ and $u' \le nx$ for some positive integer n.

(ii) u is (m,ω)-*decomposable* if for all $u' \in \mathrm{Cu}(A)$ with $u' \ll u$ there exist elements $x_1, x_2, \dots, x_m \in \mathrm{Cu}(A)$ such that $x_1 + x_2 + \cdots + x_m \le u$ and $u' \le nx_j$ for some positive integer n and for all j.

(iii) u is *weakly* (m,ω)-*divisible* if for all $u' \in \mathrm{Cu}(A)$ with $u' \ll u$ there exist elements $x_1, x_2, \dots, x_n$ in $\mathrm{Cu}(A)$ such that $mx_j \le u$ for all j and $u' \le x_1 + x_2 + \cdots + x_n$.

Remark 5.2. If u in Definition 5.1 is compact, then u is (m,ω)-divisible, (m,ω)-decomposable, respectively, weakly (m,ω)-divisible if and only if $\mathrm{Div}_m(u,A) < \infty$, $\mathrm{Dec}_m(u,A) < \infty$, respectively, $\text{w-Div}_m(u,A) < \infty$, cf. Remark 3.2.

In the next result we express (m,ω)-divisibility in terms of structural properties of the C^*-algebra. Part (iii) is almost contained in [Kir06] (see [Kir06, Definition 3.1] and [Kir06, Remark 3.3 (7)] and compare with Definition 3.16 and Proposition 3.17). Recall the definition of the rank of a C^*-algebra from Definition 3.9, and that $CM_m(\mathbb{C}) = M_m(\mathbb{C}) \otimes C_0((0,1])$ is the cone over $M_m(\mathbb{C})$.

Theorem 5.3. *Let A be a σ-unital C^*-algebra and let e be a strictly positive element of A. (If A is unital, we can take e to be the unit of A.) Put $u = \langle e \rangle = \langle A \rangle$.*

(i) *u is (m,ω)-divisible if and only if for every $\varepsilon > 0$ there exists a *-homomorphism $\varphi\colon CM_m(\mathbb{C}) \to A$ such that $(e-\varepsilon)_+$ belongs to the closed two-sided ideal generated by the image of φ.*

(ii) *u is (m,ω)-decomposable if and only if for every $\varepsilon > 0$ there exist mutually orthogonal positive elements $b_1, b_2, \dots, b_m$ in A such that $(e-\varepsilon)_+$ belongs to the closed two-sided ideal generated by b_i for each i.*

(iii) *The following are equivalent:*

(a) *u is weakly (m,ω)-divisible,*

(b) $\operatorname{rank}(A) \geq m$,

(c) *there exist* **-homomorphisms* $\varphi_i \colon CM_m(\mathbb{C}) \to A$, $i = 1, 2, \ldots, n$, *for some* n, *such that* $(e - \varepsilon)_+$ *belongs to the closed two-sided ideal generated by the union of the images of the* φ_i*'s.*

Proof. (i). Let us assume that u is (m,ω)-divisible. Let $\varepsilon > 0$. Find $x \in \mathrm{Cu}(A)$ and a positive integer n such that $mx \leq u$ and $\langle (e - \varepsilon/2)_+ \rangle \leq nx$. Choose a positive element a in $A \otimes \mathcal{K}$ such that $x = \langle a \rangle$, and choose $\eta > 0$ such that $\langle (e-\varepsilon)_+ \rangle \leq n\langle (a-\eta)_+ \rangle$. By Lemma 2.5 there exists $\varphi \colon CM_m(\mathbb{C}) \to A$ such that $\langle \varphi(e_{11} \otimes \iota) \rangle = \langle (a - \eta)_+ \rangle$. Then $\langle (e-\varepsilon)_+ \rangle \leq n \langle \varphi(e_{11} \otimes \iota) \rangle$ which implies that $(e-\varepsilon)_+$ belongs to the closed two-sided ideal generated by the image of φ.

Suppose conversely that for every $\varepsilon > 0$ there exists $\varphi \colon CM_m \to A$ such that $(e - \varepsilon)_+$ is in the closed two-sided ideal generated by $\varphi(e_{11} \otimes \iota)$. Set $\langle \varphi(e_{11} \otimes \iota) \rangle = x$. Then $mx \leq u$ by Lemma 2.5, while $\langle (e - 2\varepsilon)_+ \rangle \leq nx$ for some positive integer n. This shows that u is (m,ω)-divisible.

(ii). "Only if". Let $\varepsilon > 0$ and suppose that $b_1, b_2, \ldots, b_m$ in A exist with the stipulated properties. Set $\langle b_j \rangle = x_j \in \mathrm{Cu}(A)$. Then

$$x_1 + x_2 + \cdots + x_m = \langle b_1 + b_2 + \cdots + b_m \rangle \leq u.$$

Since $(e - \varepsilon)_+$ belongs to the closed two-sided ideal generated by b_j, $\langle (e - 2\varepsilon)_+ \rangle \leq nx_j$ for some integer $n \geq 1$. It follows that u is (m,ω)-decomposable.

"If". If $u = \langle e \rangle$ is (m,ω)-decomposable and if $\varepsilon > 0$, then there are positive elements $a_1, a_2, \ldots, a_m$ in $A \otimes \mathcal{K}$ such that $\langle a_1 \rangle + \langle a_2 \rangle + \cdots + \langle a_m \rangle \leq u$ and $\langle (e-\varepsilon/2)_+ \rangle \leq n \langle a_j \rangle$ for some positive integer n. Choose $\eta > 0$ such that $\langle (e-\varepsilon)_+ \rangle \leq n\langle (a_j - \eta)_+ \rangle$ for all j. By Lemma 2.4 (ii) there are pairwise orthogonal positive elements $b_1, b_2, \ldots, b_m$ in A such that $b_j \sim (a_j - \eta)_+$. Then the closed two-sided ideal generated by b_j contains $(e - \varepsilon)_+$ for each j.

(iii). (a) $\Rightarrow$ (b). Assume that u is weakly (m,ω)-divisible. Suppose that A has an irreducible representation $\pi \colon A \to B(\mathbb{C}^k) = M_k(\mathbb{C})$ of finite positive dimension k. Then π is necessarily surjective. Since (m,ω)-divisibility is preserved by *-homomorphisms (cf. Remark 3.5), we conclude that $M_k(\mathbb{C})$ is weakly (m,ω)-divisible. But then $k \geq m$, cf. Example 3.8. Hence (b) holds.

(b) $\Rightarrow$ (c). Assume that (b) holds. Let $(\varphi_i)_{i \in \mathbb{I}}$ be the family of all non-zero *-homomorphisms $\varphi_i \colon CM_m(\mathbb{C}) \to A$ and let I be the closed two-sided ideal in A generated by the images of all φ_i's. Thus each φ_i maps $CM_m(\mathbb{C})$ into I. We claim that $I = A$. Assume, to reach a contradiction, that $I \neq A$. By the assumption that $\operatorname{rank}(A) \geq m$, all irreducible representations of A/I have dimension at least m. It follows from Glimm's lemma (Proposition 3.10) that there is a non-zero *-homomorphism $CM_m(\mathbb{C}) \to A/I$, which by projectivity lifts to a *-homomorphism $\varphi \colon CM_m(\mathbb{C}) \to A$. But the image of φ is not contained in I, which is a contradiction.

For each finite subset F of $\mathbb{I}$ consider the closed two-sided ideal I_F of A generated by $\bigcup_{i \in F} \varphi_i(CM_m(\mathbb{C}))$. Then A is the closure of the union of the upwards directed family of ideals (I_F). Hence, for each $\varepsilon > 0$, there is a finite subset F of $\mathbb{I}$ such that $(e-\varepsilon)_+$ belongs to I_F. Thus (c) holds.

(c) $\Rightarrow$ (a). Assume that (c) holds. Set $z_i = \langle \varphi_i(e_{11} \otimes \iota) \rangle$ for $i = 1, 2, \dots, n$. Then $mz_i \leq u$ for all i. Moreover, $(a-\varepsilon)_+$ belongs to the algebraic ideal generated by the n elements $\varphi_i(e_{11} \otimes \iota)$, whence $\langle (a-\varepsilon)_+ \rangle \leq \sum_{j=1}^n n_j z_j$ for suitable positive integers n_j. Put $N = \sum n_j$ and let $x_1, x_2, \dots, x_N$ be a listing of the elements $z_1, \dots, z_n$, with z_j repeated n_j times. Then $mx_j \leq u$ and $(a-\varepsilon)_+ \leq x_1 + x_2 + \cdots + x_N$. This shows that u is weakly (m, ω)-divisible. $\square$

The theorem above can be simplified in the case where u is compact, and in particular in the case where A is unital:

Corollary 5.4. *Let A be a unital C^*-algebra, and let m be a positive integer. Then:*

(i) $\mathrm{Div}_m(A) < \infty$ *if and only if there exists a *-homomorphism $\varphi \colon CM_m(\mathbb{C}) \to A$ whose image is full in A.*

(ii) $\mathrm{Dec}_m(A) < \infty$ *if and only if there exist full, pairwise orthogonal positive elements $b_1, b_2, \dots, b_m$ in A.*

(iii) *The following are equivalent:*

  (a) $\text{w-Div}_m(A) < \infty$,

  (b) $\mathrm{rank}(A) \geq m$,

  (c) *there exist *-homomorphisms $\varphi_i \colon CM_m(\mathbb{C}) \to A$, $i = 1, 2, \dots, n$, for some n, such that the union of their images is full in A.*

Propositions 3.10 (Glimm), Proposition 3.3, and Corollary 5.4 (i) immediately imply:

Corollary 5.5. *If A is a unital, infinite dimensional, simple C^*-algebra, then the three divisibility numbers* $\mathrm{Div}_m(A)$, $\mathrm{Dec}_m(A)$, *and* $\text{w-Div}_m(A)$ *are finite for every integer $m \geq 1$.*

Let us also note what it means to have infinite $\text{w-Div}_m(\,\cdot\,)$ numbers:

Corollary 5.6. *Let A be a unital C^*-algebra.*

(i) *A admits a character if and only if* $\text{w-Div}_2(A) = \infty$.

(ii) *A admits a finite dimensional representations if and only if* $\text{w-Div}_m(A) = \infty$ *for some integer $m \geq 2$.*

Remark 5.7 (The Global Glimm Halving Problem)**.** Glimm's lemma (Proposition 3.10) says that there exists a non-zero *-homomorphism from $CM_n(\mathbb{C})$ into a C^*-algebra A if and only if A admits an irreducible representation of dimension at least n. It is not known how "large" one can make the image of such a *-homomorphism. In particular, it is not known for which C^*-algebras A one can find a *-homomorphism $CM_n(\mathbb{C}) \to A$ whose image is full in A (i.e., the image is not contained in any proper closed two-sided ideal in A). For $n = 2$

this problem is known as the "Global Glimm Halving Problem" (see [BK04a], [BK04b] and [KR02]). A unital C^*-algebra A is said to have the *Global Glimm Halving Property* if there is a *-homomorphism $CM_2(\mathbb{C}) \to A$ with full image.

More specifically, one can ask if any (unital) C^*-algebra, which admits no finite dimensional representation, satisfies the Global Glimm Halving Property. In view of Corollary 5.4, this problem for unital C^*-algebras A may be restated as follows: Does $\mathrm{Div}_2(A) = \infty$ imply that $\text{w-Div}_m(A) = \infty$ for some positive integer m? For a non-unital C^*-algebra A, the one can restate the problem in the following way: Does $\mathrm{Div}_2(A) = \infty$ imply that $\langle A \rangle$ fails to be (m, ω)-divisible for some positive integer m.

It is shown in [KR02] that if A is a weakly purely infinite C^*-algebra, then A is purely infinite if and only if all hereditary sub-C^*-algebras of A have the Global Glimm Halving Property. (It is easy to see that the rank of any weakly purely infinite C^*-algebra is infinite.) It is an open problem if all weakly purely infinite C^*-algebras are purely infinite.

Remark 5.8. Let A be a unital C^*-algebra. It follows from Proposition 3.3 (and also from Corollary 5.4) that

$$\mathrm{Div}_m(A) < \infty \implies \mathrm{Dec}_m(A) < \infty \implies \text{w-Div}_m(A) < \infty$$

for all positive integers m. None of the two reverse implications hold in general.

For each integer $m \geq 2$ let p in $C(S^{2m}) \otimes \mathcal{K}$ be a projection of (complex) dimension m and with non-zero Euler class (the existence of such p is guaranteed by [Hus94, p. 278, Corollary 9.8]). As all projections in $C(S^{2m}) \otimes \mathcal{K}$ of dimension $< m$ have trivial Euler class, it follows that p has no non-trivial subprojection. The unital C^*-algebra $A = p(C(S^{2m}) \otimes \mathcal{K})p$ is a homogeneous C^*-algebra of rank m. Hence $\text{w-Div}_m(A) < \infty$. Suppose that $\mathrm{Dec}_m(A) < \infty$. Then, by Corollary 5.4 (ii), there would exist full, pairwise orthogonal, positive elements $b_1, \dots, b_m$ in A. This would entail that each b_j is one-dimensional in each fiber of A, and hence that $f_j.b_j$ is a one-dimensional projection for some $f_j \in C(S^{2m})$. But this contradicts the fact that p has no proper subprojections.

To see that $\mathrm{Dec}_m(A) < \infty$ does not imply $\mathrm{Div}_m(A) < \infty$, consider the C^*-algebra $B = C(S^2) \otimes \mathcal{K}$, let $p \in B$ be the "Bott projection" (i.e., the rank one projection associated to the Hopf line bundle over S^2) and let $q \in B$ be a trivial $(m-1)$-dimensional projection orthogonal to p. Put $A = (p+q)B(p+q)$. It follows from a K-theoretical argument that $p+q$ cannot be written as the sum of m pairwise orthogonal and equivalent projections (because $[p+q]$ is not divisible by m in $K_0(A)$). The unit of A can be written as the sum of m (necessarily full) projections, so $\mathrm{Dec}_m(A) < \infty$. Assume that $\mathrm{Div}_m(A) < \infty$. Then, by Corollary 5.4 (i), there is a *-homomorphism $\varphi \colon CM_m(\mathbb{C}) \to A$ whose image is full in A. As explained in Remark 2.3, this entails that there exist full, pairwise orthogonal, pairwise equivalent, positive elements $b_1, b_2, \dots, b_m$ in A. Arguing as in the paragraph above, we can assume that each b_j is in fact a projection. But that contradicts the fact that $1_A = p + q$ is not the sum of m pairwise equivalent projections.

6 Divisibility and comparability

Let A and B be C^*-algebras. Then there is a natural bi-additive map

$$\mathrm{Cu}(A) \times \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B), \qquad (x, y) \mapsto x \otimes y,$$

defined as follows: If $x = \langle a \rangle$ and $y = \langle b \rangle$ with a a positive element in $A \otimes \mathcal{K}$ and b a positive element in $B \otimes \mathcal{K}$, then $x \otimes y = \langle a \otimes b \rangle$, where we identify $(A \otimes \mathcal{K}) \otimes (B \otimes \mathcal{K})$ with $A \otimes B \otimes \mathcal{K}$. Note that $x_1 \otimes y_1 \leq x_2 \otimes y_2$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.

Part (i) of the following result was (implicitly) proved in [Rør04, Lemma 4.3], and was used to prove that $\mathrm{Cu}(A \otimes \mathcal{Z})$ is almost unperforated for all unital C^*-algebras A.

Lemma 6.1. *Let A and B be unital C^*-algebras and let $1 \leq m < n$ be integers.*

(i) *Let $x, y \in \mathrm{Cu}(A)$ be such that $nx \leq my$. If B is (m,n)-divisible, then $x \otimes \langle 1_B \rangle \leq y \otimes \langle 1_B \rangle$.*

(ii) *Let $x_1, x_2, \dots, x_m, y \in \mathrm{Cu}(A)$ be such that $nx_j \leq y$ for all j. If B is (m,n)-decomposable, then*
$$(x_1 + x_2 + \cdots + x_m) \otimes \langle 1_B \rangle \leq y \otimes \langle 1_B \rangle.$$

(iii) *Let $x, y_1, y_2, \dots, y_n \in \mathrm{Cu}(A)$ be such that $x \leq my_j$ for all j. If B is weakly (m,n)-divisible, then*
$$x \otimes \langle 1_B \rangle \leq (y_1 + y_2 + \cdots + y_n) \otimes \langle 1_B \rangle.$$

Proof. (i). Take $z \in \mathrm{Cu}(B)$ such that $mz \leq \langle 1_B \rangle \leq nz$. Then

$$x \otimes \langle 1_B \rangle \leq x \otimes nz = nx \otimes z \leq my \otimes z = y \otimes mz \leq y \otimes \langle 1_B \rangle.$$

(ii). Take $z_1, z_2, \dots, z_m \in \mathrm{Cu}(B)$ such that $z_1 + z_2 + \cdots + z_m \leq \langle 1_B \rangle \leq nz_j$. Then

$$\begin{aligned}(x_1 + x_2 + \cdots + x_m) \otimes \langle 1_B \rangle &\leq x_1 \otimes nz_1 + x_2 \otimes nz_2 + \cdots + x_m \otimes nz_m \\ &= nx_1 \otimes z_1 + nx_2 \otimes z_2 + \cdots + nx_m \otimes z_m \\ &\leq y \otimes (z_1 + z_2 + \cdots + z_m) \\ &\leq y \otimes \langle 1_B \rangle.\end{aligned}$$

(iii). Take $z_1, z_2, \dots, z_n \in \mathrm{Cu}(B)$ such that $mz_j \leq \langle 1_B \rangle \leq z_1 + z_2 + \cdots + z_n$. Then

$$\begin{aligned}(y_1 + y_2 + \cdots + y_n) \otimes \langle 1_B \rangle &\geq y_1 \otimes mz_1 + y_2 \otimes mz_2 + \cdots + y_n \otimes mz_n \\ &= my_1 \otimes z_1 + my_2 \otimes z_2 + \cdots + my_n \otimes z_n \\ &\geq x \otimes (z_1 + z_2 + \cdots + z_n) \\ &\geq x \otimes \langle 1_B \rangle. \quad \square\end{aligned}$$

The lemma above can loosely be paraphrased as follows: Good divisibility properties of B ensure good comparability properties of $A \otimes B$, and bad comparability properties of $A \otimes B$ entail bad divisibility properties of B.

We proceed to show that infinite tensor products of (suitable) unital C^*-algebras cannot have very bad comparability properties.

Lemma 6.2. *Let $n \geq 1$ be an integer and let $A_1, A_2, \ldots, A_n$ be unital C^*-algebras such that $N := \max_{1 \leq k \leq n} \text{w-Div}_2(A_k) < \infty$. Then*

$$\text{w-Div}_m\Big(\bigotimes_{k=1}^n A_k\Big) \leq N^{\lceil \log_2 m \rceil},$$

whenever $2 \leq m \leq 2^n$.

Proof. Put $r = \lceil \log_2 m \rceil$. We show that $\text{w-Div}(\bigotimes_{k=1}^r A_k) \leq N^r$. This will show that $\text{w-Div}(\bigotimes_{k=1}^n A_k) \leq N^r$ by Remark 3.5 because $r \leq n$.

For each $k = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, N$, find $z_i^{(k)} \in \mathrm{Cu}(A_k)$ such that $2z_i^{(k)} \leq \langle 1_{A_k} \rangle \leq \sum_{i=1}^N z_i^{(k)}$. Given a multi-index $(i_1, i_2, \ldots, i_r) \in \{1, \ldots, N\}^r$, put

$$z_{i_1,i_2,\ldots,i_r} = z_{i_1}^{(1)} \otimes z_{i_2}^{(2)} \otimes \cdots \otimes z_{i_r}^{(r)} \in \mathrm{Cu}\Big(\bigotimes_{k=1}^r A_k\Big).$$

Then

$$\begin{aligned} m \cdot z_{i_1,i_2,\ldots,i_r} \leq 2^n \cdot z_{i_1,i_2,\ldots,i_r} &= (2z_{i_i}^{(1)}) \otimes (2z_{i_2}^{(2)}) \otimes \cdots \otimes (2z_{i_r}^{(r)}) \\ &\leq \big\langle 1_{A_1} \otimes 1_{A_2} \otimes \cdots \otimes 1_{A_r} \big\rangle \\ &\leq \Big(\sum_{i=1}^N z_i^{(1)}\Big) \otimes \Big(\sum_{i=1}^N z_i^{(2)}\Big) \otimes \cdots \otimes \Big(\sum_{i=1}^N z_i^{(r)}\Big) \\ &= \sum_{i_1,i_2,\ldots,i_r=1}^N z_{i_1,i_2,\ldots,i_r}. \end{aligned}$$ □

The N^r elements $(z_{i_1,i_2,\ldots,i_r})$ thus witness that $\bigotimes_{k=1}^r A_k$ is weakly (m, N^r)-divisible.

Recall that a C^*-algebra A has the Corona Factorization Property if and only if all full projections in $\mathcal{M}(A \otimes \mathcal{K})$ are properly infinite. It was shown in [OPR] that if A is a separable C^*-algebra, then A and all its closed two-sided ideals have the Corona Factorization Property if and only if for every integer $m \geq 2$, and for all $x', x, y_1, y_2, \ldots$ in $\mathrm{Cu}(A)$ such that $x' \ll x$ and $x \leq my_j$ for all j, one has $x' \leq y_1 + y_2 + \cdots + y_N$ for some integer $N \geq 1$.

Proposition 6.3. *Let $\big(A_k\big)_{k=1}^\infty$ be a sequence of unital C^*-algebras such that*

$$\sup_k \text{w-Div}_2(A_k) < \infty.$$

It follows that the C^*-algebra $\bigotimes_{k=1}^\infty A_k$ and all its closed two-sided ideals have the Corona Factorization Property. In particular, if A is a unital C^*-algebra without characters then $\bigotimes_{k=1}^\infty A$ and all its closed two-sided ideals have the Corona Factorization Property.

Proof. Put $B = \bigotimes_{k=1}^{\infty} A_k$, and for each $n \geq 1$ put $B_n = \bigotimes_{k=1}^{n} A_k$ and $D_n = \bigotimes_{k=n+1}^{\infty} A_k$. We shall view $(B_n)_{n=1}^{\infty}$ as an increasing sequence of sub-C^*-algebras of B such that $\bigcup_{n=1}^{\infty} B_n$ is dense in B, and we shall identify B with $B_n \otimes D_n$ for all n.

Let $m \geq 1$ be an integer and let $x', x, y_1, y_2, y_3, \dots$ in $\mathrm{Cu}(B)$ be such that $x' \ll x$ and $x \leq my_j$ for all j. By Lemma 6.2 there is a positive integer N such that $\text{w-Div}_m(D_n) \leq N$ for all n. We show that $x' \leq y_1 + y_2 + \cdots + y_N$. This will prove that B has the Corona Factorization Property.

Repeated use of Proposition 2.1 (i) and (ii) shows that there exists a positive integer n, and elements $x'', y_1', y_2', \dots, y_N'$ in $\mathrm{Cu}(B_n)$ such that

$$x' \leq x'' \otimes \langle 1_{D_n} \rangle \leq x, \quad y_j' \otimes \langle 1_{D_n} \rangle \ll y_j \text{ in } \mathrm{Cu}(B); \qquad x'' \ll my_j' \text{ in } \mathrm{Cu}(B_n),$$

where $x \mapsto x \otimes \langle 1_{D_n} \rangle$ denotes the canonical embedding $\mathrm{Cu}(B_n) \to \mathrm{Cu}(B)$. We can now apply Lemma 6.1 (iii) to deduce that

$$x' \leq x'' \otimes \langle 1_{D_n} \rangle \leq (y_1' + y_2' + \cdots + y_N') \otimes \langle 1_{D_n} \rangle \leq y_1 + y_2 + \cdots + y_N$$

as desired. □

7 Obstructions to Divisibility

A trivial obstruction to (weak) divisibility of a C^*-algebra is its rank: $\text{w-Div}_m(A) < \infty$ if and only if $m \leq \mathrm{rank}(A)$ (by Corollary 5.4 (iii)). In this section we shall discuss ways of obtaining homogeneous C^*-algebras with large rank and large weak divisibility constant. We use these techniques to construct unital simple C^*-algebras with large weak divisibility constants.

We remark first that Lemma 6.1 provides non-trivial obstructions to divisibility in B. Indeed, it follows by that lemma that if there exists a unital C^*-algebra A and $x, y \in \mathrm{Cu}(A)$ such that $nx \leq my$ but $x \otimes \langle 1_B \rangle \not\leq y \otimes \langle 1_B \rangle$, then $\mathrm{Div}_m(B) > n$. Similarly, if there exist $x_1, \dots, x_m, y$ in $\mathrm{Cu}(A)$ such that $nx_j \leq y$ for all j while

$$(x_1 + x_2 + \cdots + x_m) \otimes \langle 1_B \rangle \not\leq y \otimes \langle 1_B \rangle,$$

then $\mathrm{Dec}_m(B) > n$. Finally, if there exist $x, y_1, \dots, y_n \in \mathrm{Cu}(A)$ such that $x \leq my_j$ for all j while

$$x \otimes \langle 1_B \rangle \not\leq (y_1 + y_2 + \cdots + y_n) \otimes \langle 1_B \rangle,$$

then $\text{w-Div}_m(B) > n$.

We introduce below another way to obtain bad divisibility behaviour:

Lemma 7.1. *Let $u, v \in \mathrm{Cu}(A)$ be compact elements. If $\mathrm{Div}_2(u + v, A) \leq N$ then there exist $x_1, x_2, \dots, x_N$ in $\mathrm{Cu}(A)$ such that $2x_i \leq v$ for all i and*

$$2v \leq v + (2N + 1)u + \sum_{i=1}^{N} 2x_i.$$

Proof. By assumption there exists x such that $2x \leq u+v \leq Nx$. By compactness of $u+v$ we can find $x'' \ll x' \ll x$ such that $u+v \leq Nx''$. Since $x' \ll x \leq u+v$, it follows from Property (P1) of the Cuntz semigroup (see Section 2) (leaving u unchanged) that there exists v_1 such that

$$x' \leq u + v_1, \qquad v_1 \leq x, \qquad v_1 \leq v.$$

As $x'' \ll u + v_1$ there is $v_1' \ll v_1$ such that $x'' \leq u + v_1'$. Apply (P2) to the relation $v_1' \ll v_1 \leq v$ to obtain v_2 satisfying

$$v_1' + v_2 \leq v \leq v_1 + v_2.$$

By compactness of v we can find $v_2' \ll v_2$ such that $v \leq v_1 + v_2'$. Now,

$$v_2' \ll v_2 \leq u + v \leq Nx'' \leq Nu + Nv_1',$$

and so we can use (P1) (leaving Nu unchanged) to find $x_1, \dots, x_N$ such that

$$v_2' \leq Nu + \sum_{j=1}^{N} x_j, \qquad x_j \leq v_1', \qquad x_j \leq v_2.$$

It follows that $2x_j \leq v_1' + v_2 \leq v$ and that

$$2v \leq 2v_1 + 2v_2' \leq 2x + 2Nu + 2\sum_{j=1}^{N} x_j \leq v + (2N+1)u + 2\sum_{j=1}^{N} x_j,$$

as desired. $\square$

The corollary below illustrates how the preceding lemma can be used to find elements with bad divisibility properties:

Corollary 7.2. *Let X be a compact Hausdorff space and suppose that $p \in C(X) \otimes \mathcal{K}$ is a projection such that $[1] \not\leq (2N+1)[p]$ in $K_0(C(X))$, where 1 denotes the unit of $C(X)$. Then $\mathrm{Div}_2(\langle 1 \rangle + \langle p \rangle, C(X)) > N$.*

Proof. Consider the compact elements $u = \langle p \rangle$ and $v = \langle 1 \rangle$ of $\mathrm{Cu}(C(X))$. Note that $2x \leq \langle 1 \rangle$ implies $x = 0$ and that $2\langle 1 \rangle \not\leq \langle 1 \rangle + (2N+1)\langle p \rangle$. The desired conclusion now follows from Lemma 7.1. $\square$

The relation $[1] \not\leq (2N+1)[p]$ in $K_0(C(X))$ is satisfied whenever the $(2N+1)$-fold direct sum of p with itself is a projection with non-trivial Euler class (as explained in more detail below). It is known that for each integer $d \geq 1$ and for each positive integer N there exist X and $p \in C(X) \otimes \mathcal{K}$ such that $(2N+1)[p]$ has non-trivial Euler class and p has rank d. The unital C^*-algebra $A = (p \oplus 1)\big(C(X) \otimes \mathcal{K}\big)(p \oplus 1)$ with this choice of X and p will then satisfy $\mathrm{rank}(A) = d + 1$ and $\mathrm{Div}_2(A) > N$.

We will now give a different method for constructing elements with large divisibility constants and large rank, where we get upper bounds and where we also can give sharper lower bounds for the weak divisibility constant. Let S^2 denote the 2-dimensional sphere. Let p denote the "Bott-projection" in $C(S^2) \otimes M_2 \subseteq C(S^2) \otimes \mathcal{K}$, i.e., the projection associated to the Hopf line bundle over S^2. For each $1 \le j \le N$, let $p_j \in C((S^2)^N) \otimes \mathcal{K}$ be given by

$$p_j(x_1, x_2, \dots, x_N) = p(x_j), \qquad (x_1, \dots, x_N) \in (S^2)^N.$$

Since $\langle 1 \rangle \le 2\langle p \rangle$ in $\mathrm{Cu}(C(S^2))$, we have

$$N\langle 1 \rangle \le 2\Big\langle \bigoplus_{i=1}^{N} p_i \Big\rangle.$$

As another obstruction to weak divisibility, we shall use the following corollary of Lemma 6.1 (iii), cf. the remarks at the beginning of this section, applied to the relations $\langle 1 \rangle \le 2\langle p_i \rangle$, $i = 1, 2 \dots, N$.

Corollary 7.3. *Let X be a locally compact Hausdorff space, and let $q \in C_0(X) \otimes \mathcal{K}$ be a projection. Let $(p_i)_{i=1}^N$ be the projections in $C((S^2)^N) \otimes \mathcal{K}$ defined in the preceding paragraph. Suppose that*

$$q \otimes 1 \;\not\precsim\; q \otimes \bigoplus_{i=1}^{N} p_i.$$

Then $\text{w-Div}_2(\langle q \rangle, C_0(X)) > N$.

Let us now give examples of projections to which the corollary above can be applied. We will make use of characteristic classes of vector bundles. Recall that projections in $C(X) \otimes \mathcal{K}$, with X compact and Hausdorff, give rise to vector bundles over X: if p is a projection, then $\eta_p = (E_p, X, \pi)$, with $E_p = \{(x, v) \in X \times l_2(\mathbb{N}) \mid p(x)v = v\}$ is the vector bundle associated to p. Up to Murray-von Neumann equivalence of projections and isomorphism of vector bundles, this correspondence is a bijection. We denote by $e(\eta_p) \in H^*(X)$, or simply $e(p)$, the Euler class of η_p. For the cartesian product of spheres $(S^2)^N$ we have (e.g., by the Künneth formula) that

$$H^*((S^2)^N) \cong \mathbb{C}[z_1, z_2, \dots, z_N]/(z_1^2, z_2^2, \dots, z_N^2).$$

With this identification, the Euler classes of the projections $p_i \in C((S^2)^N) \otimes \mathcal{K}$ defined earlier can be shown to be $e(p_i) = z_i$.

Proposition 7.4. *Let X be a compact Hausdorff space and let $q \in C(X) \otimes \mathcal{K}$ be a projection such that $e(q)^N \neq 0$. Then*

$$\text{w-Div}_2(\langle 1 \oplus q \rangle, C(X)) > N.$$

Proof. By Corollary 7.3 it suffices to show that

$$(1_X \oplus q) \otimes 1_{(S^2)^N} \not\precsim (1_X \oplus q) \otimes \bigoplus_{i=1}^{N} p_i \tag{7.1}$$

in $\mathrm{Cu}(C(X) \otimes C((S^2)^N))$, where 1_X denotes the unit in $C(X)$. (In the formulation of the proposition above, we denoted 1_X simply by 1.) Observe that the trivial rank one projection is a subprojection of the projection on the left-hand side of (7.1). Thus, it suffices to show that the right side of (7.1) has non-zero Euler class.

Set $\mathrm{rank}(q) = k$. For each positive integer i, let $c_i(q) \in H^{2i}(X)$ denote the ith characteristic class of q (so that $c_k(q) = e(q)$). By the Künneth Theorem ([MS74, Theorem A.6]), we can identify $H^*(X \times (S^2)^N)$ with $H^*(X) \otimes H^*((S^2)^N)$. Then

$$\begin{aligned} e\big((1_X \oplus q) \otimes \bigoplus_{i=1}^N p_i\big) &= e\big(1_X \otimes \bigoplus_{i=1}^N p_i\big)\, e\big(q \otimes \bigoplus_{i=1}^N p_i\big) \\ &= \prod_{i=1}^N e(1_X \otimes p_i) \prod_{i=1}^N e(q \otimes p_i) \\ &= \prod_{i=1}^N e(p_i) \prod_{i=1}^N \sum_{j=0}^k c_{k-j}(q) e(p_i)^j \\ &= \prod_{i=1}^N e(p_i)\, e(q)^N \neq 0. \end{aligned}$$

In the above computation we have used that $e(1_X \otimes p_i) = e(1_X) + e(p_i) = e(p_i)$ and that $e(q \otimes p) = \sum_{j=0}^k c_{k-j}(q) e(p)^j$, for q a projection of rank k and p a projection of rank 1. To obtain the last equality we have used that $e(p_i)^2 = 0$ for all i. □

Let us now give examples of families of projections to which the above proposition can be applied. We shall here and in the following, whenever p is a projection (in a C^*-algebra) and n is a positive integer, let $n{\cdot}p$ denote the n-fold direct sum, $p \oplus p \oplus \cdots \oplus p$, (in a matrix algebra over the given C^*-algebra) of the projection p.

Example 7.5. Let N be a positive integer, and let $\mathbb{C}\mathrm{P}^N$ denote the $2N$-dimensional complex projective space. Let η denote the tautological line bundle over $\mathbb{C}\mathrm{P}^N$ and p_η the rank 1 projection associated to it. It is known that $e(p_\eta) = z^2 \in C[z^2]/(z^{2N})$, where we have identified $H^*(\mathbb{C}\mathrm{P}^N)$ with $C[z^2]/(z^{2N})$. Let d, d' be positive integers such that $dd' < N$. Then $e(d{\cdot}p_\eta)^{d'} = z^{2dd'} \neq 0$. It follows that

$$\left\lfloor \frac{N-1}{d} \right\rfloor < \mathrm{w}\text{-}\mathrm{Div}_2(\langle 1 \oplus d{\cdot}p_\eta \rangle, C(\mathbb{C}\mathrm{P}^N)) \leq \mathrm{Div}_2(\langle 1 \oplus d{\cdot}p_\eta \rangle, C(\mathbb{C}\mathrm{P}^N)) \leq \left\lceil \frac{N+d+1/2}{\lfloor d/2 \rfloor} \right\rceil.$$

Indeed, the first inequality follows from Proposition 7.4 and the calculations made above. The second inequality follows from Proposition 3.3. The last inequality can be proved as follows: Put $x = \lfloor d/2 \rfloor \langle p_\eta \rangle$. Then $2x \leq \langle 1 \oplus d{\cdot}p_\eta \rangle$. By a classical result about vector bundles (see [Hus94, Chapter 9, Proposition 1.1]) we have that $1 \precsim k{\cdot}p_\eta$ if $2N \leq 2k-1$. It follows that

$$1 \oplus d{\cdot}p_\eta \precsim n \lfloor d/2 \rfloor p_\eta,$$

or, equivalently, that $\langle 1 \oplus d{\cdot}p_\eta \rangle \leq nx$, if $n \lfloor d/2 \rfloor \geq N + d + 1/2$.

Simple C^*-algebras with bad divisibility

In this and the following two subsections we give examples of unital simple C^*-algebras with bad divisibility behaviour. We use the Euler class obstruction described in the following example.

Example 7.6. Let d be a positive integer. Following the notation in [Rør03], for each set $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, d\}$, let p_I be the one-dimensional projection in $C((S^2)^d) \otimes \mathcal{K}$ given by

$$p_I(x) = p_{i_1}(x) \otimes p_{i_2}(x) \otimes \cdots \otimes p_{i_k}(x), \qquad x \in (S^2)^d,$$

where p_i is as defined above Corollary 7.3. It is shown in [Rør03, Proposition 4.5] that if $I_1, I_2 \dots, I_r$ are subsets of $\{1, 2, \dots, d\}$ that admit a matching (i.e., $|\bigcup_{i \in F} I_i| \geq |F|$ for all subset F of $\{1, 2, \dots, r\}$) then the Euler class of $p_{I_1} \oplus p_{I_2} \oplus \cdots \oplus p_{I_r}$ is non-zero.

The examples constructed in this and the following two subsections are built on the same template described in the following lemma (which is a variation of one of Villadsen's constructions). We retain the terminology from the example above throughout the rest of this section.

Lemma 7.7. *Let $\big(J_j\big)_{j=1}^{\infty}$ be a sequence of pairwise disjoint finite subsets of $\mathbb{N}$. Choose $d_n \in \mathbb{N}$ large enough so that all J_j, $j = 1, 2, \dots, n$, are contained in the set $\{1, 2, \dots, d_n\}$. Consider the projection q_n of rank 2^n in $C((S^2)^{d_n}) \otimes \mathcal{K}$ given by*

$$q_n = 1 \oplus p_{J_1} \oplus 2 \cdot p_{J_2} \oplus \cdots \oplus 2^{n-1} \cdot p_{J_n}.$$

It follows that there is a simple unital AH-algebra A which is the inductive limit of the sequence

$$q_1\big(C((S^2)^{d_1}) \otimes \mathcal{K}\big)q_1 \xrightarrow{\varphi_1} q_2\big(C((S^2)^{d_2}) \otimes \mathcal{K}\big)q_2 \xrightarrow{\varphi_2} \cdots \longrightarrow A,$$

where the connecting mappings φ_n are unital.

Proof. Set $X_n = (S^2)^{d_n}$ and $A_n = q_n\big(C(X_n) \otimes \mathcal{K}\big)q_n$. Write

$$X_{n+1} = X_n \times (S^2)^{d_{n+1} - d_n},$$

let $\pi_n \colon X_{n+1} \to X_n$ be the projection mapping, and let $\pi_{m,n} \colon X_n \to X_m$ denote the composition map $\pi_m \circ \pi_{m+1} \circ \cdots \circ \pi_{n-1}$. Choose $x_n \in X_n$ for each n such that the set $\{\pi_{m,n}(x_n) \mid n \geq m\}$ is dense in X_m for all $m \geq 1$.

Define a *-homomorphism $\varphi_n^0 \colon C(X_n, \mathcal{K}) \to C(X_{n+1}, \mathcal{K})$ by

$$\varphi_n^0(f)(x) = f(\pi_n(x)) \,\oplus\, \big(f(x_n) \otimes p_{J_{n+1}}(x)\big), \qquad f \in C(X_n, \mathcal{K}), \quad x \in X_{n+1},$$

where we in a suitable way have identified $\mathcal{K} \oplus (\mathcal{K} \otimes \mathcal{K})$ with a subalgebra of $\mathcal{K}$. We also identify $C(X) \otimes \mathcal{K}$ with $C(X, \mathcal{K})$. We make another identification: if

$$J \subseteq \{1, 2, \dots, d_n\} \subseteq \{1, 2, \dots, d_{n+1}\},$$

then the projection p_J is defined both in $C(X_n)\otimes\mathcal{K}$ and in $C(X_{n+1})\otimes\mathcal{K}$ and $p_{J_n} = p_{J_n}\circ\pi_n$ (where the former occurrence of p_{J_n} is viewed as an element in the former algebra, and the latter in the latter). We shall use the same notation for the two projections. Taking these identification a step further, we have $q_n = q_n \circ \pi_n$ and that $q_{n+1} = q_n \oplus 2^n\cdot p_{J_{n+1}}$. (These identification hold, strictly speaking, only up to conjugation with an inner automorphism on $\mathcal{K}$.) In this notation we get

$$\begin{aligned}\varphi_n^0(q_n)(x) &= q_n(\pi_n(x)) \,\oplus\, \big(q_n(x_n)\otimes p_{J_{n+1}}(x)\big) \\ &= q_n(\pi_n(x)) \,\oplus\, \operatorname{rank}(q_n)\cdot p_{J_{n+1}}(x) \;=\; q_{n+1}(x),\end{aligned}$$

for all $x \in X_{n+1}$, i.e., $\varphi_n^0(q_n) = q_{n+1}$ (possibly after composing φ_n^0 with an inner automorphism on $\mathcal{K}$). This shows that φ_n^0 maps A_n unitally into A_{n+1}. Let φ_n denote the unital *-homomorphism that arises in this way, i.e., φ_n is the restriction of φ_n^0 to A_n (and the co-restriction to A_{n+1}), and let A be the inductive limit of the sequence

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots \longrightarrow A.$$

Let $\varphi_{m,n}\colon A_m \to A_n$ denote the composition map $\varphi_{n-1}\circ\varphi_{n-2}\circ\cdots\circ\varphi_m$ when $m \le n$. One can check that $\varphi_{m,n}(f)(x)$ is non-zero for all $x \in X_n$ if f is a function in $A_m = q_mC(X_m,\mathcal{K})q_m$ which is non-zero on at least one point in the set $\{\pi_{m,k}(x_k) \mid m \le k \le n\}$. By the choice of the points x_n, it follows that for each m and for each non-zero f in A_m there is $n \ge m$ such that $\varphi_{m,n}(f)$ is full in A_n (i.e., that $\varphi_{m,n}(f)(x) \neq 0$ for all $x \in X_n$). This entails that A is simple. □

Lemma 7.8. *Let N be a positive integer. In the notation of Lemma 7.7 choose the sequence $(J_j)_{j=1}^\infty$ such that $|J_j| = N\cdot 2^{n-1}$. It then follows that*

$$\text{w-Div}_2\big(\langle q_n\rangle, C((S^2)^{d_n})\big) > N, \qquad \text{Div}_2\big(\langle q_2\rangle, C((S^2)^{d_2}\big) \le 3N+4.$$

for all n.

Proof. We use Proposition 7.4 to prove the first claim. It suffices to show that the Euler class of the projection $N\cdot p_{J_1}\oplus 2N\cdot p_{J_2}\oplus\cdots\oplus 2^{n-1}N\cdot p_{J_n}$ is non-zero. But this follows from [Rør03, Proposition 4.5], cf. Example 7.6 above, and from the choice of the sets J_n.

To prove the second claim, put $x = \langle p_{J_2}\rangle$ and note that $2x \le \langle q_2\rangle$. It follows from [Dup76, Proposition 1] that $q_2 \precsim M\cdot p_{J_2}$ if $M - 4 \ge (2d_2-1)/2 = 3N - 1/2$. This shows that $\langle q_2\rangle \le (3N+4)x$. □

Theorem 7.9. *For each positive integer N there exists a simple unital infinite dimensional AH-algebra A such that $N < \text{w-Div}_2(A) \le \text{Div}_2(A) \le 3N+4$.*

Proof. Let A be the simple C^*-algebra constructed in Lemma 7.7 based on the choice of $(J_j)_{j=1}^\infty$ made in Lemma 7.8. Then A is the inductive limit of the sequence of C^*-algebras $A_n = q_n\big(C((S^2)^{d_n})\otimes\mathcal{K}\big)q_n$ with unital connecting mappings. It follows from Lemma 7.8 that $\text{w-Div}_2(A_n) > N$ for all n, and that $\text{Div}_2(A_2) \le 3N+4$.

By Proposition 3.6 and Remark 3.5,

$$\text{w-Div}_2(A) = \inf_{n\in\mathbb{N}} \text{w-Div}_2(A_n) > N,$$

and $\mathrm{Div}_2(A) \leq \mathrm{Div}_2(A_2) \leq 3N + 4$. $\square$

Remark 7.10 (Initial objects)**.** Suppose that $\mathcal{C}$ is a class of unital C^*-algebra. An element A in $\mathcal{C}$ is an *inital object in* $\mathcal{C}$ if there exists a unital *-homomorphism $A \to B$ for every B in $\mathcal{C}$.

It is well-known that the Cuntz algebra $\mathcal{O}_\infty$ is an initial object in the class of unital properly infinite C^*-algebras. In fact, a unital C^*-algebra is properly infinite if and only if it contains $\mathcal{O}_\infty$ as a unital sub-C^*-algebra. Every properly infinite unital sub-C^*-algebra of $\mathcal{O}_\infty$ is then also an initial object in the class of unital properly infinite C^*-algebras. Hence the Cuntz-Toeplitz algebras, $\mathcal{T}_n$, $n \geq 2$, are initial objects and so are all unital Kirchberg algebras A for which the assignment $[1_A] \mapsto 1$ extends to a homomorphism $K_0(A) \to \mathbb{Z}$.

It was shown in [ER06] that also the class of unital C^*-algebras of real rank zero and of infinite rank has initial objects. One can even find initial objects to this class which are simple AF-algebras (necessarily with infinite dimensional trace simplex). It follows in particular that the class of unital simple infinite dimensional C^*-algebras of real rank zero has initial objects.

Clearly, $\mathbb{C}$ is an initial object in the category of all unital C^*-algebras, and so is any unital C^*-algebra that admits a character. (Note that we do not require the unital *-homomorphism $A \to B$ to be injective.)

The corollary below shows that initial objects do not exist in the general non-real rank zero case.

Corollary 7.11. *The class of unital simple infinite dimensional C^*-algebras and the class of unital C^*-algebras of infinite rank do not have initial objects. In fact, there is no unital C^*-algebra without characters that maps unitally into every unital simple infinite dimensional C^*-algebra.*

Proof. If A is a unital C^*-algebra that maps unitally into every unital simple infinite dimensional C^*-algebra, then $\text{w-Div}_2(A) \geq \text{w-Div}_2(B)$ for all unital simple infinite dimensional C^*-algebras B, cf. Remark 3.5, whence $\text{w-Div}_2(A) = \infty$ by Theorem 7.9. On the other hand, if A has no character, then $\text{w-Div}_2(A) < \infty$ by Corollary 5.6. $\square$

The asymptotic divisibility numbers

We can give a lower and an upper bound on the asymptotic divisibility constant (discussed in Section 4) for the C^*-algebra considered above:

Corollary 7.12. *Let N be a positive integer, and let A be the simple AH-algebra constructed in Theorem* 7.9 *associated with N. It follows that*

$$(N-1)/2 < \mathrm{Div}_*(A) \leq 2N + 2.$$

Proof. By Proposition 4.1 we get that $\mathrm{Div}_*(A) \geq (\mathrm{Div}_2(A) - 1)/2 > (N-1)/2$. To prove the reserve inequality, take any positive integer n and put $m = 2^{n-1}$. We show that $\mathrm{Div}_m(A_n) \leq (2N+2)m$, where A_n is as in the proof of Theorem 7.9. In the notation of Lemma 7.8, let $x = \langle p_{J_n} \rangle$, put $u = \langle q_n \rangle$, and recall that q_n is the unit of the C^*-algebra A_n. By the definition of q_n (in Lemma 7.8) it follows that $mx \leq u$. As

$$\dim((2+2N)m \cdot p_{J_n}) - \dim(q_n) = (2N+2)m - 2^n = 2^n N \geq \frac{\dim(X_n) - 1}{2},$$

if follows from [Dup76, Proposition 1] that $(2+2N)m \cdot p_{J_n} \precsim q_n$, whence $u \leq (2N+2)mx$. This proves that $\mathrm{Div}_m(A_n) \leq (2N+2)m$.

It follows from Remark 3.5 that $\mathrm{Div}_m(A) \leq (2N+2)m$ whenever m is a power of 2, and this entails that

$$\mathrm{Div}_*(A) = \liminf_{m \to \infty} \mathrm{Div}_m(A)/m \leq \liminf_{n \to \infty} \mathrm{Div}_{2^{n-1}}(A)/2^{n-1} \leq 2N+2,$$

as desired. $\square$

We can use Lemma 7.7 to construct a unital, simple AH-algebra A such that $\mathrm{Div}_*(A) = \infty$. The proof requires the following sharpening of Corollary 7.3 that may have independent interest.

Corollary 7.13. *Let X be a locally compact Hausdorff space, and let $q \in C_0(X) \otimes \mathcal{K}$ be a projection. Let m and N be positive integers, let $I_1, I_2, \dots, I_N$ be pairwise disjoint subsets of $\mathbb{N}$ with $|I_i| = m-1$ for all i, and let $(p_{I_i})_{i=1}^N$ be the associated projections in $C((S^2)^{(m-1)N}) \otimes \mathcal{K}$ defined in Example 7.6. Suppose that*

$$q \otimes 1 \not\precsim q \otimes \bigoplus_{i=1}^N p_{I_i}.$$

Then $\text{w-Div}_m(\langle q \rangle, C_0(X)) > N$.

Proof. Apply Lemma 6.1 (iii) to $x = \langle 1 \rangle$ and $y_i = \langle p_{I_i} \rangle$, and note that $x \leq m y_i$, cf. Example 7.6. $\square$

Lemma 7.14. *Let $(J_j)_{j=1}^\infty$ be a sequence of pairwise disjoint subsets of $\mathbb{N}$ with $|J_j| = 2^{2j-1} j$. Then, in the notation of Lemma 7.7, we have*

$$2^k k < \text{w-Div}_{2^k}\big(\langle q_n \rangle, C((S^2)^{d_n})\big) < \infty$$

if $n \geq k$, and that $\text{w-Div}_{2^k}\big(\langle q_n \rangle, C((S^2)^{d_n})\big) = \infty$ *if $n < k$.*

Proof. We use Corollary 7.13 with $N = 2^k k$ and $m = 2^k$ to prove the first claim. As $1 \precsim q_n$ it suffices to show that $q_n \otimes \bigoplus_{i=1}^N p_{I_i}$ has non-trivial Euler class, when $I_1, \dots, I_N$ are as in Corollary 7.13. Write

$$q_n \otimes \bigoplus_{i=1}^N p_{I_i} = \bigoplus_{j=0}^n \bigoplus_{i=1}^N 2^{\max\{0,j-1\}} \cdot p_{J_j} \otimes p_{I_i} = \bigoplus_{j=0}^n \bigoplus_{i=1}^N 2^{\max\{0,j-1\}} \cdot p_{J_j \cup I_i}.$$

As explained in Example 7.6, to prove non-triviality of the Euler class of the projection $q_n \otimes \bigoplus_{i=1}^N p_{I_i}$ one needs to verify the combinatorial fact that the family of sets $(J_j \cup I_i)$, $j = 0, \dots, n$, $i = 1, \dots, N$, and with the set $J_j \cup I_i$ repeated $2^{\max\{0,j-1\}}$ times, satisfies the Marriage Lemma condition.

By first exhausting the elements in the sets I_i, and using that $\sum_{j=0}^{k-1} 2^{\max\{0,j-1\}} = 2^{k-1} < |I_i|$, it suffices to show that the family of sets (J_j), $j = k, \dots, n$, with each set repeated $2^{j-1}N = 2^{j+k-1}k$ times, satisfies the Marriage Lemma condition. However, this holds because $|J_j| = 2^{2j-1}j \geq 2^{j+k-1}k$ when $j \geq k$.

The second claim follows from the fact that the dimension of the projection q_n is 2^n and that $\text{w-Div}_m(\langle q_n \rangle, C((S^2)^{d_n})) = \infty$ whenever $m > \dim(q_n)$. $\square$

Theorem 7.15. *There is a simple unital infinite dimensional AH-algebra A which satisfies $\mathrm{Div}_*(A) = \infty$.*

Proof. Let A be the simple AH-algebra constructed in Lemma 7.7 with respect to the choice of (J_j) from Lemma 7.14. Recall that A is an inductive limit of a sequence of unital C^*-algebras $A_1 \to A_2 \to \cdots$, where

$$A_n = q_n\big(C((S^2)^{d_n}) \otimes \mathcal{K}\big)q_n.$$

It follows from Lemma 7.14 that $\text{w-Div}_{2^k}(A_n) > 2^k\, k$ when $n \geq k$, and $\text{w-Div}_{2^k}(A_n) = \infty$ when $n < k$. This entails that $\mathrm{Div}_{2^k}(A) > 2^k\, k$ by Proposition 3.6. Finally, by Proposition 4.1 (ii),

$$\mathrm{Div}_*(A) = \limsup_{k\to\infty} \mathrm{Div}_{2^k}(A)/2^k = \infty. \qquad \square$$

As remarked in Section 4, if A is any unital C^*-algebra, then $\mathrm{Div}_*(A) = 0$ if and only if A is properly infinite, and $\mathrm{Div}_*(A) \in [1, \infty]$ otherwise. Moreover, $\mathrm{Div}_*(A) = 1$ if and only if A is almost divisible (and not properly infinite). In other words, the range of the invariant $\mathrm{Div}_*(\,\cdot\,)$ is contained in the set $\{0\} \cup [1, \infty]$, and $\mathrm{Div}_*(A) \leq 1$ if and only if A is almost divisible.

We can easily produce examples of simple, unital, infinite dimensional C^*-algebras A such that $\mathrm{Div}_*(A) = 0$ (eg., A could be a Cuntz algebra), or such that $\mathrm{Div}_*(A) = 1$ (eg., A is any simple, unital, infinite-dimensional C^*-algebra of real rank zero, cf. Example 3.15). The theorem above provides an example of a simple, unital, infinite dimensional C^*-algebra A where $\mathrm{Div}_*(A) = \infty$.

It follows from Corollary 7.12 that $\mathrm{Div}_*(\,\cdot\,)$ attains infinitely many values in the interval $(1, \infty)$, when restricted to the class of unital, simple, infinite dimensional C^*-algebras, and that the possible values of $\mathrm{Div}_*(\,\cdot\,)$ in this interval is upwards unbounded. We do not know if all values in the interval $(1, \infty)$ are thus attained. For that matter we cannot exhibit any number in the interval $(1, \infty)$ which for sure is the value of $\mathrm{Div}_*(A)$ for some simple unital infinite dimensional C^*-algebra A.

Divisibility of infinite tensor products

We end this section by giving yet another class of examples of simple unital C^*-algebras with bad divisibility properties. The ones we construct below are of the form $\bigotimes_{j=1}^\infty A_j$, where the A_j's are unital simple infinite dimensional C^*-algebras. In particular, such C^*-algebras need not absorb the Jiang-Su algebra tensorially. It remains an open problem if $\bigotimes_{j=1}^\infty A$ absorbs the Jiang-Su algebra whenever A is a simple unital infinite dimensional C^*-algebra (or a unital C^*-algebra without characters), cf. [DT09].

It was shown in [HRW07, Example 4.8] that there exists a sequence (A_n) of homogeneous C^*-algebras of rank two such that $\bigotimes_{n=1}^\infty A_n$ does not absorb the Jiang-Su algebra tensorially. (It is an easy consequence of this that the Jiang-Su algebra cannot embed unitally into $\bigotimes_{n=k}^\infty A_n$ for some k.) Of course, one can regroup the tensor factors A_n to get a new sequence (B_n) of unital C^*-algebras each of which has infinite rank and where the Jiang-Su algebra does not embed into $\bigotimes_{n=1}^\infty B_n$. It is not known if every unital C^*-algebra of infinite rank admits an embedding of a unital simple infinite dimensional C^*-algebra. If it were true, then Theorem 7.17 below would follow from [HRW07, Example 4.8] .

We introduce some notation to keep track of the combinatorics. Define a total order on the set $\mathbb{N} \times \mathbb{N}_0$ by

$$(k,j) \le (\ell,i) \iff k+j < \ell+i \quad \text{or} \quad (k+j = \ell+i \text{ and } k \le \ell).$$

For each $(k,j) \in \mathbb{N} \times \mathbb{N}_0$ and for each integer $m \ge k$ let $S(m;k,j)$ denote the set of all m-tuples $(i_1, i_2, \dots, i_m) \in \mathbb{N}_0^m$ such that $i_k = j$ and $(\ell, i_\ell) < (k,j)$ for all $\ell \neq k$.

Lemma 7.16. *Let $N \ge 1$ be an integer. For each integer $k \ge 1$, let $(J_j^{(k)})_{j=1}^\infty$ be a sequence of subsets of $\mathbb{N}$ such that $J_j^{(k)} \cap J_i^{(\ell)} = \varnothing$ whenever $(k,j) \neq (\ell,i)$, and such that*

$$|J_j^{(k)}| \;=\; \max_{m \ge k} \sum_{(i_1,\dots,i_m) \in S(m;k,j)} N \prod_{t=1}^m 2^{\max\{i_t - 1, 0\}}.$$

(The quantity on the right-hand side is finite because $S(m;k,j)$ is finite for all (k,j) and all $m \ge k$, and $S(m;k,j) = \varnothing$ when $m > k+j$.) Let $d_n^{(k)} \in \mathbb{N}$ and $q_n^{(k)} \in C((S^2)^{d_n^{(k)}}) \otimes \mathcal{K}$ be as defined in Lemma 7.7 associated with the sequence $(J_j^{(k)})_{j=1}^\infty$. It then follows that

$$\text{w-Div}_2\Big(\big\langle q_n^{(1)} \otimes q_n^{(2)} \otimes \cdots \otimes q_n^{(m)} \big\rangle, C((S^2)^{d_n^{(1)}}) \otimes C((S^2)^{d_n^{(2)}}) \otimes \cdots \otimes C((S^2)^{d_n^{(m)}})\Big) > N,$$

for all positive integers n and m.

Proof. Let $T(n,m)$ be the set of all non-zero m-tuples $(i_1, i_2, \dots, i_m) \in \mathbb{N}_0^m$ such that $i_k \le n$ for all $k = 1, 2, \dots, m$. Adopt the convention $J_0^{(k)} = \varnothing$ for all k and let $p_\varnothing$ denote the trivial (= constant) one-dimensional projection. We can then express the projection $q_n^{(1)} \otimes q_n^{(2)} \otimes \cdots \otimes q_n^{(m)}$ as follows:

$$1 \oplus \sum_{(i_1,\dots,i_m) \in T(n,m)} \Big(\prod_{t=1}^m 2^{\max\{i_t-1,0\}}\Big) \cdot p_{J_{i_1}^{(1)}} \otimes p_{J_{i_2}^{(2)}} \otimes \cdots \otimes p_{J_{i_m}^{(m)}}.$$

By Proposition 7.4 it suffices to show that the Euler class of the projection

$$\sum_{(i_1,\dots,i_m)\in T(n,m)} \big(N \prod_{t=1}^{m} 2^{\max\{i_t-1,0\}}\big) \cdot p_{J^{(1)}_{i_1}} \otimes p_{J^{(2)}_{i_2}} \otimes \cdots \otimes p_{J^{(m)}_{i_m}}$$

is non-zero, or, equivalently, that the Euler class of the projection

$$\sum_{(i_1,\dots,i_m)\in T(n,m)} \big(N \prod_{t=1}^{m} 2^{\max\{i_t-1,0\}}\big) \cdot p_{J^{(1)}_{i_1} \cup J^{(2)}_{i_2} \cup \cdots \cup J^{(m)}_{i_m}}$$

is non-zero. By [Rør03, Proposition 4.5], cf. Example 7.6, it suffices to show that the family of sets $J^{(1)}_{i_1} \cup J^{(2)}_{i_2} \cup \cdots \cup J^{(m)}_{i_m}$, where $(i_1,\dots,i_m) \in T(n,m)$ and where the set $J^{(1)}_{i_1} \cup J^{(2)}_{i_2} \cup \cdots \cup J^{(m)}_{i_m}$ is repeated $N \cdot \prod_{t=1}^{m} 2^{\max\{i_t-1,0\}}$ times, admits a matching.

Construct a matching by selecting the matching elements for the $N \cdot \prod_{t=1}^{m} 2^{\max\{i_t-1,0\}}$ copies of the set $J^{(1)}_{i_1} \cup J^{(2)}_{i_2} \cup \cdots \cup J^{(m)}_{i_m}$ inside the subset $J^{(k)}_j$, where (k,j) is the largest of the elements $(1,i_1),(2,i_2),\dots,(m,i_m)$. To check that this works, i.e., to see that $J^{(k)}_j$ is large enough, let $T(n,m;k,j)$ be the set of those m-tuples $(i_1,i_2,\dots,i_m)$ in $T(n,m)$ for which

$$(k,j) = \max\{(1,i_1),(2,i_2),\dots,(m,i_m)\}.$$

Then $T(n,m;k,j) \subseteq S(m;k,j)$, and so it follows by the assumption on $|J^{(k)}_j|$ that

$$|J^{(k)}_j| \geq \sum_{(i_1,\dots,i_m)\in T(n,m;k,j)} N \prod_{t=1}^{m} 2^{\max\{i_t-1,0\}}.$$

The suggested matching is therefore possible. □

The theorem below shows that an infinite tensor product of simple unital infinite dimensional C^*-algebras does not necessarily have good divisibility properties. Any such C^*-algebra $A = \bigotimes_{n=1}^{\infty} A_n$ will have (many) non-trivial central sequences, i.e., the central sequence algebra $A_\omega \cap A'$ with respect to an ultrafilter ω on $\mathbb{N}$ is non-trivial. For example, $CM_m(\mathbb{C})$ embeds into $A_\omega \cap A'$ for all m, albeit, not necessarily with full image. However, in the example constructed below, one cannot embed the Jiang-Su algebra into $A_\omega \cap A'$.

Theorem 7.17. *For each integer $N > 2$, there exists a sequence (A_n) of unital simple infinite dimensional C^*-algebras (in fact, AH-algebras) such that*

$$\text{w-Div}_2\Big(\bigotimes_{n=1}^{\infty} A_n\Big) > N.$$

In particular, $\bigotimes_{n=1}^{\infty} A_n$ does not absorb the Jiang-Su algebra $\mathcal{Z}$, and it does not even admit a unital embedding of $\mathcal{Z}$.

Proof. Let A_k be the simple unital AH-algebra constructed in Lemma 7.7 associated with the sequence $\big(J_j^{(k)}\big)_{j=1}^{\infty}$ from Lemma 7.16. Then A_k is an inductive limit of a sequence, $A_k(1) \to A_k(2) \to \cdots$, of unital homogeneous C^*-algebras with unital connecting maps, where

$$A_k(n) = q_n^{(k)}\Big(C((S^2)^{d_n^{(k)}}) \otimes \mathcal{K}\Big)q_n^{(k)}.$$

The infinite tensor product $\bigotimes_{n=1}^{\infty} A_n$ is the inductive limit of the sequence

$$A_1 \to A_1 \otimes A_2 \to A_1 \otimes A_2 \otimes A_3 \to \cdots$$

with unital connecting maps. It therefore suffices to show that $\text{w-Div}_2\big(\bigotimes_{k=1}^{m} A_k\big) > N$ for every m, cf. Proposition 3.6. Now, $\bigotimes_{k=1}^{m} A_k$ is the inductive limit of the sequence

$$\bigotimes_{k=1}^{m} A_k(1) \to \bigotimes_{k=1}^{m} A_k(2) \to \bigotimes_{k=1}^{m} A_k(3) \to \cdots,$$

with unital connecting mappings, and so, again by Proposition 3.6, it suffices to show that

$$\text{w-Div}_2\Big(\bigotimes_{k=1}^{m} A_k(n)\Big) > N$$

for every m and n. The latter is precisely the content of Lemma 7.16. $\square$

8 Ultrapowers

In this section we show that our divisibility properties behave well with respect to taking direct products and ultrapowers of sequences of unital C^*-algebras. This has the surprising consequence that such products and ultrapowers may admit characters even if all the C^*-algebras in the ingoing sequence are unital, simple and infinite dimensional.

We define the notion of "almost characters" and show that the existence of such is related to the invariant $\text{w-Div}_2(\,\cdot\,)$. It follows in particular that simple unital infinite dimensional C^*-algebras can have almost characters.

First we need some technical lemmas:

Lemma 8.1. *Let A be a unital C^*-algebra and let (I_λ) be an upward directed family of ideals of A. Set $\overline{\bigcup I_\lambda} = I$. It follows that*

$$\text{Div}_m(A/I) = \inf_\lambda \text{Div}_m(A/I_\lambda), \quad \text{Dec}_m(A/I) = \inf_\lambda \text{Dec}_m(A/I_\lambda)$$

$$\text{w-Div}_m(A/I) = \inf_\lambda \text{w-Div}_m(A/I_\lambda)$$

for all positive integers m.

Proof. The inequality "$\leq$" in all three cases follows from Remark 3.5 since we have a unital *-homomorphism $A/I_\lambda \to A/I$ for each λ. We prove the reverse inequality "$\geq$" only in the case of $\mathrm{Dec}_m(\,\cdot\,)$; the proofs of the other two instances are similar.

Set $\mathrm{Dec}_m(A/I) = n$, and let us show that $\mathrm{Dec}_m(A/I_\lambda) \leq n$ for some λ. Find $x_1 \dots, x_m$ in $\mathrm{Cu}(A/I)$ be such that

$$x_1 + x_2 + \cdots + x_m \leq \langle 1 \rangle \leq n x_j$$

for all j. Find positive contractions $a_1, \dots, a_m$ in $A \otimes \mathcal{K}$ such that $x_j = \langle b_j \rangle$, where $b_j \in A/I \otimes \mathcal{K}$ is the image of a_j under the quotient mapping $A \to A/I$. Find $\varepsilon > 0$ such that $\langle 1 \rangle \leq n\langle (b_j - \varepsilon)_+ \rangle$ for all j. It follows from [KR00, Lemma 4.12] that there are positive elements $c, c_1', \dots, c_m'$ in $I \otimes \mathcal{K}$ such that

$$\langle (a_1 - \varepsilon/2)_+ \rangle + \cdots + \langle (a_m - \varepsilon/2)_+ \rangle \leq \langle 1 \rangle + \langle c \rangle, \qquad \langle 1 \rangle \leq n\langle (a_j - \varepsilon)_+ \rangle + \langle c_j' \rangle$$

for all j. There is $\delta > 0$ such that

$$\langle (a_1 - \varepsilon)_+ \rangle + \cdots + \langle (a_m - \varepsilon)_+ \rangle \leq \langle 1 \rangle + \langle (c - \delta)_+ \rangle, \quad \langle 1 \rangle \leq n\langle (a_j - \varepsilon)_+ \rangle + \langle (c_j' - \delta)_+ \rangle.$$

Since $\bigcup I_\lambda$ is dense in I, it follows that $(c - \delta)_+$ and $(c_j' - \delta)_+$ all belong to $I_\lambda \otimes \mathcal{K}$ for some λ. Let $z_j \in \mathrm{Cu}(A/I_\lambda)$ be the Cuntz class of the image of the element $(a_j - \varepsilon)_+$ under the quotient mapping $A \to A/I_\lambda$. Then $z_1 + \cdots + z_m \leq \langle 1 \rangle \leq n z_j$, whence $\mathrm{Dec}_m(A/I_\lambda) \leq n$. □

For each $\varepsilon > 0$, let $h_\varepsilon \colon \mathbb{R}^+ \to [0, 1]$ be a continuous functions such that $h_\varepsilon(0) = 0$ and $h_\varepsilon(t) = 1$ when $t \geq \varepsilon$.

Lemma 8.2. *Let A be a unital C^*-algebra. Let $b_1, b_2, \dots, b_n$ be positive elements in A such that $\sum_{j=1}^n \langle b_j \rangle \geq \langle 1_A \rangle$. Then, for some $\varepsilon > 0$, there are contractions y_j in A such that*

$$\sum_{j=1}^n y_j^* h_\varepsilon(b_j) y_j = 1_A.$$

Proof. By assumption, and by compactness of $\langle 1_A \rangle$, there are elements $v_j \in A$ such that $\sum_{j=1}^n v_j^* b_j v_j = 1_A$. Thus $\sum_{j=1}^n v_j^* (b_j - \varepsilon)_+ v_j$ is invertible for some $\varepsilon > 0$, and so there are elements $w_j \in A$ such that $\sum_{j=1}^n w_j^* (b_j - \varepsilon)_+ w_j = 1_A$. Put $y_j = (b_j - \varepsilon)_+^{1/2} w_j$ and notice that $h_\varepsilon(b_j)(b_j - \varepsilon)_+ = (b_j - \varepsilon)_+$ for all j. Thus

$$\sum_{j=1}^n y_j^* h_\varepsilon(b_j) y_j = \sum_{j=1}^n y_j^* y_j = 1_A,$$

which shows that the y_j's are contractions with the desired properties. □

Lemma 8.3. *Let A be a unital C^*-algebra, and let m, n be positive integers.*

(i) *A is weakly (m, n)-divisible if and only if there exist positive contractions a_{ij} and contractions y_j in A, $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, m$, such that $a_{1j}, a_{2j}, \dots, a_{mj}$ are pairwise equivalent and pairwise orthogonal for all j, and such that $1_A = \sum_{j=1}^n y_j^* a_{1j} y_j$.*

(ii) *A is (m,n)-decomposable if and only if there exist pairwise orthogonal positive contractions a_i and contractions y_{ij} in A, $j = 1,2,\dots,n$ and $i = 1,2,\dots,m$, such that $\sum_{j=1}^n y_{ij}^* a_i y_{ij} = 1_A$ for all i.*

(iii) *A is (m,n)-divisible if and only if there exist pairwise equivalent and pairwise orthogonal positive contractions a_i and contractions y_j in A, $j = 1,2,\dots,n$ and $i = 1,2,\dots,m$, such that $\sum_{j=1}^n y_j^* a_1 y_j = 1_A$.*

Proof. We identify A with the sub-C^*-algebra $1_A(A\otimes\mathcal{K})1_A$ of $A\otimes\mathcal{K}$.

(i). "If". Put $x_j = \langle a_{1j}\rangle = \langle a_{ij}\rangle \in \mathrm{Cu}(A)$. Then

$$mx_j = \sum_{i=1}^m \langle a_{ij}\rangle = \langle \sum_{i=1}^m a_{ij}\rangle \le \langle 1_A\rangle = \Big\langle \sum_{j=1}^n y_j^* a_{1j} y_j \Big\rangle \le \sum_{j=1}^n \langle y_j^* a_{1j} y_j\rangle \le \sum_{j=1}^n \langle a_{1j}\rangle = \sum_{j=1}^n x_j.$$

"Only if". Let $x_1, x_2, \dots, x_n \in \mathrm{Cu}(A)$ be such that $mx_j \le \langle 1_A\rangle \le x_1 + x_2 + \cdots + x_n$. Choose $x_j' \ll x_j$ such that $\langle 1_A\rangle \le x_1' + x_2' + \cdots + x_n'$. For each fixed $j = 1,2,\dots,n$, apply Lemma 2.4 (ii) to the relation $mx_j \le \langle 1_A\rangle$ to obtain pairwise orthogonal and pairwise equivalent positive elements $b_{1j}, b_{2j}, \dots, b_{mj}$ in A such that $x_j' \le \langle b_{ij}\rangle \le x_j$ for all $i = 1,\dots,m$. Then $\sum_{j=1}^n \langle b_{1j}\rangle \ge \langle 1_A\rangle$, and so it follows from Lemma 8.2 that there are $\varepsilon > 0$ and contractions $y_1, y_2, \dots, y_n$ in A such that $\sum_{j=1}^n y_j^* h_\varepsilon(b_{1j}) y_j = 1_A$. The contractions y_j together with the positive contractions $a_{ij} = h_\varepsilon(b_{ij})$ are then as desired.

(ii). "If". Put $x_i = \langle a_i\rangle \in \mathrm{Cu}(A)$. Then

$$\sum_{i=1}^m x_i = \sum_{i=1}^m \langle a_i\rangle = \Big\langle \sum_{i=1}^m a_i \Big\rangle \le \langle 1_A\rangle = \Big\langle \sum_{j=1}^n y_{ij}^* a_i y_{ij} \Big\rangle \le \sum_{j=1}^n \langle y_{ij}^* a_i y_{ij}\rangle \le \sum_{j=1}^n \langle a_i\rangle = nx_i$$

for all i.

"Only if". Let $x_1, x_2, \dots, x_m \in \mathrm{Cu}(A)$ be such that $x_1 + x_2 + \cdots + x_m \le \langle 1_A\rangle \le nx_j$. Choose $x_i' \ll x_i$ such that $\langle 1_A\rangle \le nx_i'$. Apply Lemma 2.4 (ii) to the relation $x_1 + x_2 + \cdots + x_m \le \langle 1_A\rangle$ to find pairwise orthogonal positive elements $b_1, b_2, \dots, b_m$ in A such that $x_i' \le \langle b_i\rangle \le x_i$ for all i. For each $i = 1,2,\dots,m$, we then have $n\langle b_i\rangle \ge \langle 1_A\rangle$, so by Lemma 8.2 there are $\varepsilon > 0$ and contractions $y_{i1}, y_{i2}, \dots, y_{in}$ in A such that $\sum_{j=1}^n y_{ij}^* h_\varepsilon(b_i) y_{ij} = 1_A$. The contractions y_{ij} together with the positive contractions $a_i = h_\varepsilon(b_i)$ are then as desired.

(iii). "If". Put $x = \langle a_1\rangle = \langle a_i\rangle \in \mathrm{Cu}(A)$. Then

$$mx = \sum_{i=1}^m \langle a_i\rangle = \Big\langle \sum_{i=1}^m a_i \Big\rangle \le \langle 1_A\rangle \le \Big\langle \sum_{j=1}^n y_j^* a_1 y_j \Big\rangle \le \sum_{j=1}^n \langle y_j^* a_1 y_j\rangle \le \sum_{j=1}^n \langle a_1\rangle = nx.$$

"Only if". Let $x \in \mathrm{Cu}(A)$ be such that $mx \le \langle 1_A\rangle \le nx$. Choose $x' \ll x$ such that $\langle 1_A\rangle \le nx'$. Apply Lemma 2.4 (ii) to the relation $mx \le \langle 1_A\rangle$ to obtain pairwise orthogonal and pairwise equivalent positive elements $b_1, b_2, \dots, b_m$ in A such that $x' \le \langle b_j\rangle \le x$. Then $n\langle b_1\rangle \ge \langle 1_A\rangle$, and so by Lemma 8.2 there are $\varepsilon > 0$ and contractions $y_1, y_2, \dots, y_n$ in A such that $\sum_{j=1}^n y_j^* h_\varepsilon(b_1) y_j = 1_A$. The contractions y_j together with the positive contractions $a_i = h_\varepsilon(b_i)$ are then as desired. $\square$

If (A_k) is a sequence of C^*-algebras, then we denote by $\prod_{k=1}^\infty A_k$ the C^*-algebra of all bounded sequences (a_k), with $a_k \in A_k$. If ω is a (free) filter on $\mathbb{N}$, then denote by $c_\omega(\{A_k\})$ the closed two-sided ideal in $\prod_{k=1}^\infty A_k$ consisting of those sequences (a_k) for which $\lim_\omega \|a_k\| = 0$. Finally, denote the quotient $\prod_{k=1}^\infty A_k / c_\omega(\{A_k\})$ by $\prod_\omega A_k$.

Proposition 8.4. *Let (A_k) be a sequence of unital C^*-algebras. Then, for all integers $m \geq 2$ and for any free filer ω on $\mathbb{N}$ we have:*

(i) $\mathrm{Div}_m\Big(\prod_{k=1}^\infty A_k\Big) = \sup_k \mathrm{Div}_m(A_k), \quad \mathrm{Div}_m\Big(\prod_\omega A_k\Big) = \limsup_\omega \mathrm{Div}_m(A_k).$

(ii) $\mathrm{Dec}_m\Big(\prod_{k=1}^\infty A_k\Big) = \sup_k \mathrm{Dec}_m(A_k), \quad \mathrm{Dec}_m\Big(\prod_\omega A_k\Big) = \limsup_\omega \mathrm{Dec}_m(A_k).$

(iii) $\text{w-Div}_m\Big(\prod_{k=1}^\infty A_k\Big) = \sup_k \text{w-Div}_m(A_k), \quad \text{w-Div}_m\Big(\prod_\omega A_k\Big) = \limsup_\omega \text{w-Div}_m(A_k).$

Proof. We only prove (i). The proofs of (ii) and (iii) are very similar.

We have unital *-homomorphisms $\prod_{k=1}^\infty A_k \to A_n$ for all n. Therefore the inequality "$\geq$" holds in the first identity in (i) (and also in (ii) and (iii)), cf. Remark 3.5.

We show next that $\mathrm{Div}_m\Big(\prod_{k=1}^\infty A_k\Big) \leq \sup_k \mathrm{Div}_m(A_k)$. Let n be a positive integer such that $\mathrm{Div}_m(A_k) \leq n$ for all k. Then, by Lemma 8.3 (i), for each k we can find positive contractions $a_{ij}^{(k)}$ and contractions $y_j^{(k)}$ in A_k, for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, m$, such that $a_{1j}^{(k)}, a_{2j}^{(k)}, \dots, a_{mj}^{(k)}$ are pairwise orthogonal and equivalent for all j, and such that

$$1_{A_k} = \sum_{j=1}^n (y_j^{(k)})^* a_{1j}^{(k)} y_j^{(k)}.$$

Put

$$a_{ij} = (a_{ij}^{(k)}) \in \prod_{k=1}^\infty A_k, \qquad y_j = (y_j^{(k)}) \in \prod_{k=1}^\infty A_k.$$

Then $a_{1j}, a_{2j}, \dots, a_{mj}$ are pairwise orthogonal and equivalent, and $\sum_{j=1}^n y_j^* a_{1j} y_j$ is equal to the unit of $\prod_{k=1}^\infty A_k$. By Lemma 8.3 (i), this shows that $\prod_{k=1}^\infty A_k$ is (m,n)-divisible, whence $\mathrm{Div}_m\Big(\prod_{k=1}^\infty A_k\Big) \leq n$.

To prove the second part of (i) note first that we have a natural unital (surjective) *-homomorphism $\prod_{k \in I} A_k \to \prod_\omega A_k$ for each $I \in \omega$. We can therefore use Remark 3.5 and the first identity in (i) to conclude that

$$\mathrm{Div}_m\big(\prod_\omega A_k\big) \leq \mathrm{Div}_m\big(\prod_{k \in I} A_k\big) = \sup_{k \in I} \mathrm{Div}_m(A_k),$$

which shows that $\mathrm{Div}_m\big(\prod_\omega A_k\big) \leq \limsup_\omega \mathrm{Div}_m(A_k)$.

We proceed to prove the reverse inequality. For each $I \in \omega$ consider the ideal $J(I)$ in $\prod_{k=1}^{\infty} A_k$ consisting of those sequences (a_k) for which $a_k = 0$ for all $k \in I$. Then

$$c_\omega(\{A_k\}) = \overline{\bigcup_{I \in \omega} J(I)},$$

(where ω is ordered by reverse inclusion). We can now use Lemma 8.1 and the first identity in (i) to conclude that

$$\begin{aligned} \mathrm{Div}_m\big(\prod_\omega A_k\big) &= \inf_{I \in \omega} \mathrm{Div}_m\Big(\big(\prod_{k=1}^{\infty} A_k\big)/J(I)\Big) = \inf_{I \in \omega} \mathrm{Div}_m\big(\prod_{k \in I} A_k\big) \\ &= \inf_{I \in \omega} \sup_{k \in I} \mathrm{Div}_m(A_k) = \limsup_\omega \mathrm{Div}_m(A_k). \end{aligned}$$

$\square$

If we combine the proposition above with Corollary 5.6 (i) we obtain:

Corollary 8.5. *Let (A_k) be a sequence of unital C^*-algebras such that* $\lim_{k\to\infty} \text{w-}\mathrm{Div}_2(A_k) = \infty$*. Then $\prod_{k=1}^{\infty} A_k$ has a character, and so does $\prod_\omega A_k$ for each free filter ω on $\mathbb{N}$.*

If we combine the corollary above with Theorem 7.9, then we obtain the following surprising fact:

Corollary 8.6. *There is a sequence (A_k) of unital simple infinite dimensional C^*-algebras such that $\prod_{k=1}^{\infty} A_k$ and $\prod_\omega A_k$ have characters for each free filter ω on $\mathbb{N}$.*

Clearly, none of the C^*-algebras A_k in the corollary above can have a character. However, they have "almost characters" in the sense defined below. This is one way of understanding how the product C^*-algebra can have a character when none of the individual C^*-algebras has one.

Definition 8.7. Let $N \geq 2$ be an integer and let $\varepsilon > 0$. A unital C^*-algebra A is said to have (N, ε)-*characters* if for every N-tuple $u_1, u_2, \dots, u_N$ of unitaries in A there exists a state ρ on A such that $|\rho(u_j)| \geq 1 - \varepsilon$ for $j = 1, 2, \dots, N$.

A state ρ on a unital C^*-algebra is a character if and only if $|\rho(u)| = 1$ for all unitary elements $u \in A$. Most simple C^*-algebras that we know of do not have $(2, \varepsilon)$-characters for small $\varepsilon > 0$. For example, if A is a C^*-algebra which contains unitaries u, v such that $\|uvu^*v^* - \lambda 1_A\| < \eta$ for some $\lambda \in \mathbb{T}$ and for some $\eta < |1 - \lambda|$, then A does not admit any $(2, \varepsilon)$-character for some small enough $\varepsilon > 0$. Indeed, if ρ is a state on A such that $|\rho(u)|$ and $|\rho(v)|$ are close to 1, then $\rho(uvu^*v^*)$ is close to 1.

Proposition 8.8. *A unital C^*-algebra has a character if and only if it has (N, ε)-characters for all pairs (N, ε), where $N \geq 2$ is an integer and $\varepsilon > 0$.*

Proof. The "only if" part is trivial. Assume that A is a unital C^*-algebra that has (N,ε)-characters for all pairs (N,ε). For each finite subset F of the unitary group of A and for each $\varepsilon > 0$, let $S(F,\varepsilon)$ denote the set of states ρ on A such that $|\rho(u)| \geq 1-\varepsilon$ for all $u \in F$. Then, by assumption, $S(F,\varepsilon)$ is non-empty. It follows that $\bigcap_{(F,\varepsilon)} S(F,\varepsilon)$ is non-empty, and any state in this intersection is a character. $\square$

Proposition 8.9. *Let (A_k) be a sequence of unital C^*-algebras, and let ω be a free ultrafilter on $\mathbb{N}$. Then $\prod_\omega A_k$ has a character if and only if for each integer $N \geq 2$ and for each $\varepsilon > 0$ there exists $I \in \omega$ such that A_k has (N,ε)-characters for each $k \in I$.*

Proof. We prove first the "if" part. By Proposition 8.8 it suffices to show that $\prod_\omega A_k$ has (N,ε)-characters for all (N,ε). Fix (N,ε) and find $I \in \omega$ such that A_k has (N,ε)-characters for each $k \in I$. Let $u_1,\dots,u_N$ be unitaries in $\prod_\omega A_k$, and let $(u_j^{(k)}) \in \prod_{k=1}^\infty A_k$ be a unitary lift of u_j. Then for each $k \in I$ there is a state ρ_k on A_k such that $|\rho_k(u_j^{(k)})| \geq 1-\varepsilon$ for $j = 1,2,\dots,N$. Choose arbitrary states ρ_k on A_k for $k \notin I$ and define a state ρ on $\prod_\omega A_k$ by $\rho(x) = \lim_\omega \rho_k(x_k)$, where $(x_k) \in \prod_{k=1}^\infty A_k$ is a lift of x. (A priori, ρ defines a state on $\prod_{k=1}^\infty A_k$, and one checks that it vanishes on the ideal $c_\omega(\{A_k\})$.) Then

$$|\rho(u_j)| = \lim_{k\to\omega} |\rho_k(u_j^{(k)})| \geq \inf_{k\in I} |\rho_k(u_j^{(k)})| \geq 1-\varepsilon,$$

for $j = 1,2,\dots,N$, which shows that $\prod_\omega A_k$ has (N,ε)-characters.

Suppose next that $\prod_\omega A_k$ has a character ρ. Fix (N,ε), and let J be the set of those $k \in \mathbb{N}$ for which A_k does not have (N,ε)-characters. For each $k \in J$ choose unitaries $u_j^{(k)}$ in A_k, $j = 1,2,\dots,N$, such that there is no state ρ' on A_k for which $|\rho'(u_j^{(k)})| \geq 1-\varepsilon$ for all $j = 1,2,\dots,N$. Choose arbitrary unitaries $u_j^{(k)} \in A_k$ for $k \notin J$, and let u_j be the unitary element $(u_j^{(k)})$ in $\prod_{k=1}^\infty A_k$. Let B be the (separable) sub-C^*-algebra of $\prod_\omega A_k$ generated by the unitaries $\pi_\omega(u_j)$, where π_ω is the quotient mapping $\prod_{k=1}^\infty A_k \to \prod_\omega A_k$. By [Kir06, Lemma 2.5] there is a sequence ρ_k of pure states on A_k such that $\rho(\pi_\omega(x)) = \lim_\omega \rho_k(x_k)$ for all $x = (x_k) \in \prod_{k=1}^\infty A_k$ with $\pi_\omega(x) \in B$. Now,

$$1 = |\rho(\pi_\omega(u_j))| = \lim_\omega |\rho_k(u_j^{(k)})| = \liminf_{k\to\omega} |\rho_k(u_j^{(k)})| = \sup_{I\in\omega} \inf_{k\in I} |\rho_k(u_j^{(k)})|.$$

It follows that there exists $I \in \omega$ such that $|\rho_k(u_j^{(k)})| \geq 1-\varepsilon$ for all $k \in I$ and for all $j = 1,2,\dots,N$. This entails that $I \cap J = \varnothing$. Hence A_k has (N,ε)-characters for all $k \in I$. $\square$

We can relate the existence of (N,ε)-characters on a C^*-algebra A to the divisibility quantity $\text{w-Div}_2(A)$.

Theorem 8.10. *For each pair (N,ε), where $N \geq 2$ is an integer and $\varepsilon > 0$, there exists an integer $n \geq 2$ such that every unital C^*-algebra A which satisfies* $\text{w-Div}_2(A) \geq n$ *has (N,ε)-characters. Conversely, for every integer $n \geq 2$ there exists a pair (N,ε), where $N \geq 2$ is an integer and where $\varepsilon > 0$, such that every unital C^*-algebra A which has (N,ε)-characters satisfies* $\text{w-Div}_2(A) \geq n$.

Proof. Suppose that the first claim were false. Then there would exist a pair (N, ε) and a sequence (A_n) of unital C^*-algebras such that $\text{w-Div}_2(A_n) \geq n$ and none of the A_n's have (N, ε)-characters. However, if ω is any free ultrafilter $\mathbb{N}$, then $\prod_\omega A_n$ has a character by Corollary 8.5, whence A_n has (N, ε)-characters for each n in some subset $I \in \omega$, a contradiction.

Suppose next that the second statement were false. Then there would exist an integer $n \geq 2$ and a sequence (A_k) of unital C^*-algebras such that A_k has $(k, 1/k)$-characters but $\text{w-Div}_2(A_k) < n$. Let ω be a free ultrafilter on $\mathbb{N}$. It then follows from Proposition 8.9 that $\prod_\omega A_k$ has a character. Hence $\text{w-Div}_2(\prod_\omega A_k) = \infty$, whence $\lim_\omega \text{w-Div}_2(A_k) = \infty$ by Proposition 8.4, a contradiction. □

Corollary 8.11. *For each pair (N, ε), where $N \geq 2$ is an integer and $\varepsilon > 0$, there exists a unital simple infinite dimensional C^*-algebra which has (N, ε)-characters.*

We end this section by giving several equivalent formulation of some well-known open problems for C^*-algebras. Recall that a C^*-algebra A has the Global Glimm Halving property if there is a *-homomorphism $CM_2(\mathbb{C}) \to A$ whose image is full in A.

Proposition 8.12. *The following statements are equivalent:*

(i) *Every unital C^*-algebra that has no finite dimensional representation has the Global Glimm Halving property.*

(ii) *For all unital C^*-algebras A, if $\text{w-Div}_m(A) < \infty$ for all $m \geq 2$, then $\text{Div}_2(A) < \infty$.*

(iii) *For every sequence (A_k) of unital C^*-algebras, if $\sup_k \text{w-Div}_m(A_k) < \infty$ for all m, then $\sup_k \text{Div}_2(A_k) < \infty$.*

Proof. (i) $\Leftrightarrow$ (ii). A has no finite dimensional representations if and only if $\text{rank}(A) \geq m$ for all m, which by Corollary 5.4 (iii) is equivalent to $\text{w-Div}_m(A) < \infty$ for all m. It was shown in Corollary 5.4 (i) that the Global Glimm Halving property holds for A if and only if $\text{Div}_2(A) < \infty$.

(ii) $\Rightarrow$ (iii). Given a sequence (A_k) of unital C^*-algebras such that $\sup_k \text{w-Div}_m(A_k) < \infty$ for all m. Consider the C^*-algebra $A = \prod_{k=1}^\infty A_k$. Then $\text{w-Div}_m(A) = \sup_k \text{w-Div}_m(A_k) < \infty$ by Proposition 8.4. Thus $\text{Div}_2(A) < \infty$, which implies that $\sup_k \text{Div}_2(A_k) = \text{Div}_2(A) < \infty$, again by Proposition 8.4.

(iii) $\Rightarrow$ (ii) is trivial: Take $A_k = A$ for all k. □

Proposition 8.13. *The following statements are equivalent:*

(i) *All unital C^*-algebras A that have no finite dimensional representation contain two positive full elements that are orthogonal to each other.*

(ii) *For all unital C^*-algebras A, if $\text{w-Div}_m(A) < \infty$ for all $m \geq 2$, then $\text{Dec}_2(A) < \infty$.*

Proof. (i) $\Leftrightarrow$ (ii). As in the proof of Proposition 8.12, A has no finite dimensional representations if and only if $\text{w-Div}_m(A) < \infty$ for all m. It was shown in Corollary 5.4 (ii) that A contains two positive full elements that are orthogonal to each other if and only if $\text{Dec}_2(A) < \infty$.

(ii) $\Rightarrow$ (iii) is similar to the proof of (ii) $\Rightarrow$ (iii) in Proposition 8.12. (iii) $\Rightarrow$ (ii) is trivial. $\square$

9 Infinite elements

Following [KR00], a Cuntz class u in the Cuntz semigroup of a C^*-algebra A is said to be properly infinite if it satisfies $u = 2u$ (whence $u = \infty \cdot u$). Similarly, a countably generated Hilbert module over A is properly infinite if its Cuntz class is properly infinite. We saw in Proposition 3.4 how infiniteness of a Cuntz class can arise from a certain type of divisibility property. In this section we shall investigate this and related phenomena further with emphasis on the following property:

Definition 9.1. Let A be a C^*-algebra, let $n \geq 1$ be an integer, and let u be an element in $\mathrm{Cu}(A)$. We say that u is (ω, n)-decomposable if there exist $x_1, x_2, \dots$ such that $\sum_{i=1}^{\infty} x_i \leq u$ and $u \leq n x_i$ for all i.

If u is (ω, n)-decomposable, then u is (m, n)-decomposable for all m. In particular, by Proposition 3.4 (ii), it follows that nu is properly infinite.

The condition in the definition above can be reformulated in several different ways:

Lemma 9.2. *Let A be a C^*-algebra, let $n \geq 1$ be an integer, and let u be an element of $\mathrm{Cu}(A)$. Then the following conditions are equivalent:*

(i) *u is (ω, n)-decomposable,*

(ii) *there exist $x_1, x_2, \dots$ such that $\sum_{i=1}^{\infty} x_i \leq u$ and $n x_i = \infty \cdot u$ for all i,*

(iii) *there exist $x_1, x_2, \dots$ and $y_1, y_2, \dots$ such that $\sum_{i=1}^{\infty} x_i \leq u$, $y_{i-1} \leq y_i \leq n x_i$ for all i, and $\infty \cdot u \leq \infty \cdot \sup y_i$,*

(iv) *there exist $x_1, x_2, \dots$ such that $\sum_{i=1}^{\infty} x_i \leq u$ and $n \sum_{j=k}^{\infty} x_j = \infty \cdot u$ for all k.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $x_1, x_2, \dots$ satisfy $\sum_{i=1}^{\infty} x_i \leq u \leq n x_j$. Let $\{I_i\}_{i=1}^{\infty}$ be a partition of the natural numbers into infinite sets. Then the elements $x_i' = \sum_{j \in I_i} x_j$ witness that condition (ii) holds.

To get (ii) $\Rightarrow$ (iii), set $y_i = \infty \cdot u$ for all i and choose $(x_i)_{i=1}^{\infty}$ that satisfies (ii).

(iii) $\Rightarrow$ (iv). Let $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ be as in (iii). Then $n \sum_{i=k}^{\infty} x_i \geq \infty \cdot y_k$. Observe that the left side of this inequality decreases as k increases. Thus, $n \sum_{i=k}^{\infty} x_i \geq \infty \cdot y_{k'}$ for all $k' \geq k$. Taking the supremum over all $k' \geq k$ we get $n \sum_{i=k}^{\infty} x_i \geq \infty \cdot \sup y_i \geq \infty \cdot u$.

(iv) $\Rightarrow$ (i). Suppose that $x_1, x_2, \dots$ satisfy the condition in (iv). Let $(u_i)_{i=1}^{\infty}$ be such that $u_i \ll u_{i+1}$ for all i and $\sup_i u_i = u$. Then there exists a sequence $1 = k_0 < k_1 < \cdots$

such that the elements $x'_i = \sum_{j=k_{i-1}}^{k_i-1} x_j$ satisfy $x'_i \geq u_i$ for all i. Let $\{I_i\}_{i=1}^{\infty}$ be a partition of the natural numbers into infinite sets. Then the elements $x''_i = \sum_{j \in I_i} x'_j$ satisfy the condition in Definition 9.1. $\square$

It was shown in [OPR] that the Corona Factorization Property for a C^*-algebra is equivalent to a condition for its Cuntz semigroup, that we here shall refer to as (CFP4S). A complete ordered abelian semigroup is said to have (CFP4S) if whenever $(x_i)_{i=1}^{\infty}$ is a full sequence, $x' \ll x_1$, and $(y_i)_{i=1}^{\infty}$ is such that $my_i \geq x_i$ for all i and some m, then there exists n such that $\sum_{i=1}^{n} y_i \geq x'$. Recall that a full sequence is one that is increasing and such that $\sup x_i$ is a full element, i.e., $\infty \cdot \sup x_i$ is the largest element of the semigroup (which we shall denote by ∞).

In Section 6 we discussed a related notion, called the strong Corona Factorization Property, and its analog for the Cuntz semigroup.

The proposition below relates the (CFP4S) with the notion of (ω, m)-divisibility. In fact, it is a consequence of this proposition that a semigroup in the category **Cu** has a full elements which is (ω, m)-divisible and not properly infinite if and only if the semigroup does not satisfy (CFP4S).

Proposition 9.3. *The following four conditions are equivalent for any object S in the category* **Cu**.

(i) *S has property (CFP4S).*

(ii) *For every sequence $(y_i)_{i=1}^{\infty}$ in S, if there is a full sequence $(x_i)_{i=1}^{\infty}$ in S such that $my_i \geq x_i$ for some m and for all i, then $\sum_{i=1}^{\infty} y_i = \infty$.*

(iii) *For every sequence $(y_i)_{i=1}^{\infty}$ in S, if $my_i = \infty$ for some m and for all i, then $\sum_{i=1}^{\infty} y_i = \infty$.*

(iv) *For every sequence $(y_i)_{i=1}^{\infty}$ in S, if $\sum_{i=n}^{\infty} my_i = \infty$ for some m and for all n, then $\sum_{i=1}^{\infty} y_i = \infty$.*

(v) *For every full element y in S, if y is (ω, m)-decomposable for some m, then y is properly infinite (whence $y = \infty$).*

Proof. (i) $\Rightarrow$ (ii). Apply the (CFP4S) to the tail sequences $(x_i)_{i=n}^{\infty}$ and $(y_i)_{i=n}^{\infty}$. Then we get $x' \leq \sum_{i=n}^{N} y_i \leq \sum_{i=n}^{\infty} y_i$ for all $x' \ll x_n$, whence $\infty = \sup_n x_n \leq \sum_{i=1}^{\infty} y_i$.

(ii) $\Rightarrow$ (i). If (x_i) is a full sequence and if (y_i) is another sequence such that $x_i \leq my_i$, then $\sum_{i=1}^{\infty} y_i = \infty$ by (ii). In particular, if $x' \ll x_1$, then $x' \leq \sum_{i=1}^{n} y_i$ for some n by the definition of compact containment.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds and that (y_i) is a sequence in S such that $my_i = \infty$ for all i. Then $\infty = \sum_{i=1}^{\infty} y_i$ by (ii) with $x_i = \infty$ for all i.

(iii) $\Rightarrow$ (v). Suppose that (iii) holds, and that $y \in S$ is full and (ω, m)-decomposable. Then y satisfies condition (ii) of Lemma 9.2, so there exists a sequence (y_i) such that $\sum_{i=1}^{\infty} y_i \leq y$ and $my_i = \infty \cdot y = \infty$ for all i. But then $\sum_{i=1}^{\infty} y_i = \infty$ because (iii) holds, whence $y = \infty$.

(v) $\Rightarrow$ (iv). Suppose that (v) holds and let $(y_i)_{i=1}^\infty$ be a sequence in S such that $\sum_{i=n}^\infty my_i = \infty$ for some m and for all n. Put $y = \sum_{i=1}^\infty y_i$. We must show that $y = \infty$. We know that $my = \infty$, so y is full. It is easy to see that y satisfies condition (iv) of Lemma 9.2, so y is (ω, m)-decomposable. Hence $y = \infty$ by the assumption that (v) holds.

(iv) $\Rightarrow$ (ii). Let $(x_i)_{i=1}^\infty$ be a full sequence in S, let $m \geq 1$ be a positive integer, and let (y_i) be such that $my_i \geq x_i$ for all i. Then

$$\sum_{i=n}^{\infty} my_i \geq \sum_{i=n}^{\infty} x_i \geq \sum_{i=k}^{\infty} x_i \geq \infty{\cdot}x_k$$

for all $k \geq n$. As $\infty = \sup_k \infty{\cdot}x_k$, we conclude that $\sum_{i=n}^\infty my_i = \infty$ for all n. By (iv) this entails that $\sum_{i=1}^\infty y_i = \infty$, and in particular that $x_1 \leq \sum_{i=1}^\infty y_i$. □

In the following example we describe a Cuntz semigroup with an element u that is $(\omega, 2)$-decomposable but not properly infinite. In particular, $2u$ is properly infinite while u is not. This example is well known in other contexts, and it was discussed in the paragraph preceding Corollary 7.3.

Example 9.4. Let $X = (S^2)^\infty$ be a countable cartesian product of 2-dimensional spheres, and let $p_i \in C(X, \mathcal{K})$ be the one-dimensional projection arising as the pull back of a non-trivial rank one projection p in $C(S^2) \otimes \mathcal{K}$ along the ith coordinate projection $X \to S^2$, cf. the comments above Corollary 7.3. Let e be a trivial one-dimensional projection. Then $e \not\precsim \bigoplus_{i=1}^N p_j$ for all N, because the Euler class of the projection on the right-hand side is non-trivial.

Put $x_i = \langle p_i \rangle$, $v = \langle e \rangle$, and put $u = \sum_{i=1}^\infty x_i$. Then $v \not\leq u$, $v \leq 2x_i$, and $u + u = \infty{\cdot}v = \infty{\cdot}u$. Hence $2\sum_{j=i}^\infty x_i = \infty{\cdot}v = \infty{\cdot}u$, and so u is $(\omega, 2)$-decomposable; but u is not properly infinite.

We now look more closely at the properties of (ω, n)-decomposable elements.

Proposition 9.5. *Let $(a_i)_{i=0}^\infty$ be a sequence of mutually orthogonal positive elements in a C^*-algebra A such that $\sum_{i=0}^\infty a_i$ converges to a strictly positive element in A. Assume that $n \geq 1$ is an integer such that $\sum_{j \geq i} n\langle a_i \rangle = \infty$ for all i. Then $A \otimes B$ is stable for every σ-unital C^*-algebra B with* $\mathrm{rank}(B) \geq n$.

Proof. Set $\sum_{i=0}^\infty a_i = a \in A$ and let b be a strictly positive element in B. Notice that $a \otimes b$ is a strictly positive element of $A \otimes B$. In order to prove stability of $A \otimes B$ we will use the stability criterion obtained in [HR98]: $A \otimes B$ is stable if for every $\varepsilon > 0$ there exists a positive element c in A which is orthogonal to $(a \otimes b - \varepsilon)_+$ and satisfies $\langle (a \otimes b - \varepsilon)_+ \rangle \leq \langle c \rangle$.

Arguing as in the proof of (iv) $\Rightarrow$ (i) in Lemma 9.2, we may replace $\sum_{j \geq i} n\langle a_i \rangle = \infty$ by the stronger assumption that $n\langle a_i \rangle = \infty$ for all i. By Theorem 5.3 (iii), $\mathrm{rank}(B) \geq n$ is equivalent to weak (n, ω)-divisibility for $\langle b \rangle$. Thus there exist a sequence $(x_i)_{i=1}^\infty$ in $\mathrm{Cu}(B)$ such that $nx_i \leq \langle b \rangle$ for all i and $\sum_{i=1}^\infty x_i = \infty$. We can form a new sequence $(x_i')_{i=1}^\infty$ in which each x_i appears repeated infinitely often. In this way we may assume without loss of generality that $\sum_{i\geq j}^{\infty} x_i = \infty$ for all j. Find positive elements b_i in B such that $x_i = \langle b_i \rangle$ and $\|b_i\| \leq 2^{-i}$. Then

$$(a \otimes b - \varepsilon)_+ = \sum_{i=1}^{\infty} (a_i \otimes b - \varepsilon)_+ = \sum_{i=1}^{N} (a_i \otimes b - \varepsilon)_+,$$

for some integer $N \geq 1$. Set $c = \sum_{i>N} a_i \otimes b$. Then c is orthogonal to $(a \otimes b - \varepsilon)_+$. Also,

$$\langle a_i \otimes b \rangle = \langle a_i \rangle \otimes \langle b \rangle \geq n \, \langle a_i \rangle \otimes \langle b_i \rangle = \infty \cdot \langle a \rangle \otimes \langle b_i \rangle$$

for each i. (Here we have used that $n\langle a_i \rangle = \infty$ for all i). Hence

$$\langle c \rangle = \sum_{i>N} \langle a_i \otimes b \rangle = \sum_{i>N} \infty \cdot \langle a \rangle \otimes \langle b_i \rangle = \infty.$$

Thus, $\langle (a \otimes b - \varepsilon)_+ \rangle \leq \infty = \langle c \rangle$. This shows that $A \otimes B$ is stable. □

The proposition above can be applied to the C^*-algebra $A = P(C(X) \otimes \mathcal{K})P$ arising from Example 9.4 with $P = \bigoplus_{i=1}^{\infty} p_i \in \mathcal{M}(C(X) \otimes \mathcal{K})$. The C^*-algebra A is not stable (because $e \notin A$ while $e \in M_2(A)$), but $A \otimes B$ is stable for every C^*-algebra B that does not have a character by Proposition 9.5 and Example 9.4.

The example obtained in [Rør97] of a simple C^*-algebra A of stable rank 1 such that $M_n(A)$ is stable, while $M_{n-1}(A)$ is not stable, likewise satisfies the hypotheses of Proposition 9.5. In fact, to the authors knowledge, every example of a C^*-algebra that tensored with $M_n(\mathbb{C})$ becomes stable also has the stronger property of becoming stable after being tensored with any C^*-algebra that has no representations of dimension less than n. This raises the following question:

Question 9.6. Is there a C^*-algebra A such that $M_2(A)$ is stable but $A \otimes B$ is not stable for some C^*-algebra B without characters?

Proposition 9.7. *The following statements are equivalent for every C^*-algebra A with a strictly positive element a.*

(i) *$\langle a \rangle$ is (ω, n)-decomposable.*

(ii) *A contains a full hereditary subalgebra B such that $B \otimes C$ is stable for any C^*-algebra C such that* $\mathrm{rank}(C) \geq n$.

(iii) *A contains a full hereditary subalgebra B such that $M_n(B)$ is stable.*

Proof. (i) $\Rightarrow$ (ii). Let $(x_i)_{i=1}^{\infty}$ be such that $\sum_{i=1}^{\infty} x_i \leq \langle a \rangle$ and $nx_i = \infty$. Let $b \in A \otimes \mathcal{K}$ be strictly positive and let $b_i \in A \otimes \mathcal{K}$ be mutually orthogonal elements such that $\langle b_i \rangle = x_i$. We can find mutually orthogonal positive elements a_i in A such that $\langle a_i \rangle \leq \langle b_i \rangle$, $n\langle a_i \rangle \geq \langle (b - 1/i)_+ \rangle$, and such that $\sum_{i=1}^{\infty} a_i$ is convergent. It then follows from Proposition 9.5 that (ii) holds when B is the hereditary sub-C^*-algebra generated by $\sum_{i=1}^{\infty} a_i$.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). Since A is σ-unital, and B is stably isomorphic to A, B is σ-unital too. Let b be a strictly positive element in B.

Use [OPR, Lemma 5.3] to find a sequence (b_k) of pairwise orthogonal positive elements in B such that $\langle (b-1/k)_+\rangle \leq n\langle (b-1/k)_+\rangle \leq n\langle b_k\rangle$ for all k. Then condition (iii) of Lemma 9.2 is satisfied with $u = \langle b\rangle$, $x_k = \langle b_k\rangle$, and $y_k = \langle (b-1/k)_+\rangle$, whence $\langle b\rangle$ is (ω, n)-decomposable.

Finally, by the fact that $\langle b\rangle \leq \langle a\rangle \leq \infty\cdot\langle b\rangle$, it follows by the equivalence of (i) and (ii) in Lemma 9.2 that $\langle a\rangle$ is (ω, n)-decomposable. □

Definition 9.8. Let $n \in \mathbb{N}$ and $u \in \mathrm{Cu}(A)$. We call u weakly (ω, n)-divisible if for every $u' \ll u$ there exist $x_i \in \mathrm{Cu}(A)$, $i = 1, 2, \ldots, n$, such that $\infty\cdot x_i \leq u$ and $u' \leq \sum_{i=1}^n x_i$.

Observe that if u is weakly (ω, n)-divisible, then u is weakly (m, n)-divisible for all $m \in \mathbb{N}$, whence nu is properly infinite by Proposition 3.4.

We will next give an example of a Cuntz semigroup element that is weakly $(\omega, 2)$-divisible but not properly infinite. This example needs some preparatory results. Let us first recall an example given by Dixmier and Doaudy in [DD63].

Example 9.9 (Dixmier–Douady, [DD63, §17])**.** Let B_∞ denote the closed unit ball of $l_2(\mathbb{N})$ endowed with the weak topology. Let $l_2(B_\infty)$ denote the $C(B_\infty)$-Hilbert module of continuous maps from B_∞ to $l_2(\mathbb{N})$. We will construct a countably generated $C(B_\infty)$-Hilbert module D such that $l_2(B_\infty) \hookrightarrow D \hookrightarrow l_2(B_\infty)$ but $D \not\cong l_2(B_\infty)$.

Let $x\colon B_\infty \to l_2(\mathbb{N}) \oplus \mathbb{C}$ be the function given by

$$x(z) = z + \sqrt{1 - \|z\|^2}\cdot e, \text{ for } z \in B_\infty.$$

Here e is a generator of the summand $\mathbb{C}$ in the direct sum $l_2(\mathbb{N}) \oplus \mathbb{C}$. Consider the $C(B_\infty)$-module D_0 of functions from B_∞ to $l_2(\mathbb{N}) \oplus \mathbb{C}$ that have the form $y + x\lambda$, with $y \in l_2(B_\infty)$ and $\lambda \in C(B_\infty)$ (the action of $C(B_\infty)$ is defined by pointwise scalar multiplication). The module D_0 is a pre-Hilbert C*-module over $C(B_\infty)$ when endowed with the pointwise inner product. Indeed, if $y_1 + x\lambda_1$ and $y_2 + x\lambda_2$ are vectors in D_0 then

$$\langle y_1 + x\lambda_1, y_2 + x\lambda_2\rangle = \langle y_1, y_2\rangle + \langle y_1, z\rangle\lambda_2 + \overline{\lambda_1}\langle z, y\rangle + \overline{\lambda_1}\lambda_2 \in C(B_\infty).$$

Let D denote the completion of D_0 with respect to the norm induced by its $C(B_\infty)$-valued inner product. Observe that $l_2(B_\infty) \hookrightarrow D_0 \subseteq D$. Since D is countably generated, we also have that $D \hookrightarrow l_2(B_\infty)$ by Kasparov's stabilization theorem. Let us see that $D \not\cong l_2(B_\infty)$. Consider $E \subseteq D$, the orthogonal complement of $\{x\}$. Then $E = \overline{E_0}$, where

$$E_0 = \{y + x\lambda \in D \mid \langle y(z), z\rangle + \lambda(z) = 0 \text{ for all } z \in B_\infty\}. \tag{9.1}$$

It was implicitly shown by Dixmier and Douady, and explicitly pointed out by Blanchard and Kirchberg ([BK04a, Proposition 3.6]), that for any $v \in E$ there exists $z \in B_\infty$ such that $\langle v, v\rangle(z) = 0$. That is, every section of E vanishes at some point (we will reprove this fact in Proposition 9.10 below). Notice that

$$D = E + x \cdot C(B_\infty) \cong E \oplus C(B_\infty).$$

It can be deduced from this that $D \ncong l_2(B_\infty)$ (see [DD63, Proposition 19]).

Let B_3 denote the unit ball in $\mathbb{R}^3$. Let $f \in M_2(C(B_3))^+$ be defined as

$$f(x, y, z) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}.$$

(The function f is a homeomorphism from B_3 to the set of positive elements of $M_2(\mathbb{C})$ with trace 1. On the boundary 2-sphere of B_3 it agrees with the tautological rank 1 projection.) Consider the $C(B_3)$-module associated to f:

$$F := \overline{f\begin{pmatrix} C(B_3) \\ C(B_3) \end{pmatrix}}. \tag{9.2}$$

Proposition 9.10. *Let B_∞ and B_3 be as before. Let $X = \prod_{i\in I} X_i$, where each X_i is either B_∞ or B_3 and the index set I is non-empty. For each i, let H_i be the pull-back along the projection map $\pi_i\colon X \to X_i$ of either the module E defined in Example* 9.9 *or the module F defined in* (9.2)*. Finally, let H be the $C(X)$-module defined by $H = \bigoplus_{i\in I} H_i$. Then $C(X)$ does not embed in H as a $C(X)$-module (i.e., for every $v \in H$ there exists $z \in X$ such that $\langle v, v\rangle(z) = 0$).*

Notice that if every X_i agrees with B_3, the above proposition can be proven using standard methods in algebraic topology (e.g., characteristic classes). Indeed, it suffices to restrict to the boundary 2-sphere of each X_i and use that on that set F is the tautological rank 1 projective module. It is the inclusion of the spaces B_∞ in the definition of X that forces us to use a different route in the proof.

Proof. Let $v \in H$, and write $v = \sum_{i\in I} v_i$, with $v_i \in H_i$. In order to show $\langle v, v\rangle(z) = 0$ for some $z \in X$, it suffices to prove this for v belonging to a dense submodule of H. For suppose that $(v^{(n)})$ is a sequence in H such that $v^{(n)} \to v$ and $\langle v^{(n)}, v^{(n)}\rangle(z_n) = 0$ for some $z_n \in X$. Then by the compactness of X there exists a subsequence (z_{n_k}) such that $z_{n_k} \to z \in X$, and so $\langle v, v\rangle(z) = 0$. Thus, we may assume that the index set I is finite. Furthermore, for the indices i such that $H_i = \pi_i^*(E)$, we may assume that $v_i \in H_i'$, where $H_i' \subseteq H_i$ is the pull back along π_i of the dense submodule E_0 defined in (9.1).

In the sequel, we assume that $I = \{1, 2, \ldots, n\}$, $X_i = B_\infty$ for $i = 1, 2, \ldots, n_1$, and $X_i = B_3$ for $i = n_1 + 1, \ldots, n$, where $n_1 \le n$.

We will argue by contradiction that $\langle v, v\rangle(z) = 0$ for some $z \in X$. Suppose that $\langle v, v\rangle$ is invertible, and assume without loss of generality that $\langle v, v\rangle = 1$. Observe that, for each $i \le n_1$, v_i is a function from X into the unit ball of $l_2(\mathbb{N}) \oplus \mathbb{C}$, while for $n_1 < i \le n$ the entry v_i is a function from X into the unit ball of $\mathbb{C} \oplus \mathbb{C}$ (let us denote it by B_4).

Let $h_0\colon l_2(\mathbb{N})\oplus\mathbb{C}\to l_2(\mathbb{N})$ denote the projection onto the first direct summand and let $h_1\colon B_4\to B_3$ denote the Hopf fibration (extended to the unit ball):

$$h_1(z_0,z_1):=(2z_0\overline{z_1},|z_0|^2-|z_1|^2).$$

Let $\lambda\colon[0,1]\to[0,1]$ be such that $\lambda(0)=0$, $\lambda(t)=1$ for $t\in[\frac{1}{n},1]$, and λ is linear in $[0,\frac{1}{n}]$. Define $\tilde h_0,\tilde h_1\colon B_4\to B_3$ by

$$\tilde h_0(w)=h_0\Big(\frac{\lambda(|w|)}{|w|}w\Big),\quad \tilde h_1(w)=-h_1\Big(\frac{\lambda(|w|)}{|w|}w\Big).$$

Consider the continuous map $\Phi\colon X\to X$ given by the vector of functions

$$\Phi:=(\tilde h_0\circ v_1,\tilde h_0\circ v_2,\dots,\tilde h_0\circ v_{n_1},\tilde h_1\circ v_{n_1+1},\dots,\tilde h_1\circ v_n).$$

Since X is a compact convex subset of the vector space $(l_2(\mathbb{N}))^{n_1}\times(\mathbb{R}^3)^{n-n_1}$, the map Φ has a fixed point by the Schauder fixed point theorem. Let $\tilde z:=(\tilde z_i)_{i=1}^n\in X$ be a fixed point of Φ. Since $\|v(\tilde z)\|=1$, we must have $\|v_i(\tilde z)\|\geq\frac{1}{n}$ for at least one index i. Notice that both $\tilde h_0$ and $\tilde h_1$ map all vectors of norm at least $1/n$ into the unit sphere of either B_∞ or B_3. It follows that the fixed point $\tilde z$ satisfies $\|\tilde z_i\|=1$ and $\tilde z_j=0$ for all $j\neq i$.

There are two cases to consider: $i\leq n_1$ and $i>n_1$. Suppose that $i\leq n_1$. The general form of $v_i\in H_i'$ is $f+(z_i+\sqrt{1-\|z_i\|^2}e)\alpha$, for some $f\colon X\to l_2(\mathbb{N})$ and $\alpha\in C(X)$. Since $\|\tilde z_i\|=1$, we have

$$v_i(\tilde z)=f(\tilde z)+\alpha(\tilde z)\tilde z_i=\tilde z_i.$$

But $\langle f(\tilde z),\tilde z_i\rangle+\alpha(\tilde z)=0$. This contradicts that $\|\tilde z_i\|=1$.

Suppose that $i>n_1$. Since $z_i\mapsto v_i(\cdots,z_i,\cdots)$, with $z_i\in S^2$, is a section of the tautological bundle on S^2, we have $h_1\circ v_i(z)=z_i$ whenever $z_i\in S^2$. It follows that $\tilde h_1\circ v_i(\tilde z)=-\tilde z_i$. But $\tilde h_1\circ v_i(\tilde z)=\tilde z_i$, by the fixed point property of $\tilde z$. This again contradicts that $\|\tilde z_i\|=1$. $\square$

We are now prepared to give examples of weakly $(\omega,2)$-divisible elements which are not properly infinite.

Example 9.11. Let $X=B_\infty\times B_3$ and consider the Hilbert module $H=\pi_1^*(E)\oplus\pi_2^*(F)$, described in the statement of the previous proposition. We have shown that $[C(X)]\not\lesssim[H]$. In particular, $[H]$ is not properly infinite (since it is full). Let us show that $[H]$ is weakly $(\omega,2)$-divisible. Consider the open sets $U:=B_\infty\times B_3^+$ and $V:=B_\infty\times B_3^-$, where B_3^+ and $B_3^-=$ are (open) upper and lower hemispheres of B_3 that together cover B_3 (e.g., $B_3^+=B_3\setminus\{(0,0,1)\}$ and $B_3^-=B_3\setminus\{(0,0,-1)\}$). We claim that $l_2(U)\hookrightarrow HC_0(U)$ and $l_2(V)\hookrightarrow HC_0(V)$. Indeed, we have

$$\begin{aligned}HC_0(U)&=\pi_1^*(E)C_0(U)\oplus\pi_2^*(F)C_0(U)\\&=\pi_1^*(E)C_0(U)\oplus\pi_2^*(FC_0(B_3^+)).\end{aligned}$$

On the other hand, $FC_0(B_3^+) \cong C_0(B_3^+) \oplus C_0(B_3^+ \backslash S^2)$. This follows from [RT11, Theorem 4.3], where it is shown that the isomorphism class of a Hilbert module over a space of dimension at most 3 is determined by the restrictions of the Hilbert module to the subsets where it has constant fibrewise rank. Observe that $FC_0(B_3^+)$ has rank 1 on $B_3 \cap S^2$, rank 2 on $B_3^+ \backslash S^2$, and furthermore it induces trivial vector bundles when restricted to those sets. Thus, $FC_0(B_3^+) \cong C_0(B_3^+) \oplus C_0(B_3^+ \backslash S^2)$. Since $\pi_2^*(C_0(B_3^+)) = C_0(U)$, we get that

$$\begin{aligned} HC_0(U) &\cong \pi_1^*(E)C_0(U) \oplus C_0(U) \oplus \pi_2^*(FC_0(B_3^+ \setminus S^2)) \\ &\cong \pi_1^*(E \oplus C(B_\infty))C_0(U) \oplus \pi_2^*(FC_0(B_3^+ \setminus S^2)). \end{aligned}$$

But $l_2(B_\infty) \hookrightarrow D = E \oplus C(B_\infty)$. Thus, $l_2(U) \hookrightarrow HC_0(U)$. Symmetrically, we have that $l_2(V) \hookrightarrow HC_0(V)$. It follows that $[HC_0(U)]$ and $[HC_0(V)]$ are properly infinite, and $[H] \leq [HC_0(U)] + [HC_0(V)]$. Thus, $[H]$ is weakly $(\omega, 2)$-divisible.

Remark 9.12. The previous example answers a question posed in [KR00, Question 3.10]: If a and b are properly infinite positive elements, is $a+b$ properly infinite? In the language of Hilbert modules, this question asks whether H is properly infinite if $H = \overline{H_1 + H_2}$, and $H_1, H_2 \subseteq H$ are properly infinite submodules of H. We obtain a counterexample taking H as in the previous example, $H_1 = HC_0(U)$ and $H_2 = HC_0(V)$.

Example 9.13. In this example we answer (in the negative) the following question, posed in [KR00, Question 3.4]: if $[H]$ is properly infinite, is the unit of $B(H)$ a properly infinite projection? Let $X = B_\infty \times (B_3)^\infty$ and consider the Hilbert $C(X)$-module

$$H = C(X) \oplus \pi_1^*(E) \oplus \bigoplus_{i=2}^{\infty} \pi_i^*(F).$$

The module $C(X) \oplus \pi_1^*(E)$ is the pull back along π_1 of the Dixmier-Douady module D. Since $l_2(C(B_\infty))$ embeds in D, $l_2(C(X))$ embeds in $C(X) \oplus \pi_1^*(E)$. Thus, $[H]$ is properly infinite. Also, the direct sum of the module $\bigoplus_{i=2}^{\infty} \pi_i^*(F)$ with itself gives $l_2(C(X))$ (because $F \oplus F$ contains $C(B_3)$ as a direct summand). Therefore, $H \oplus H \cong l_2(C(X))$. However, H is not isomorphic to $l_2(C(X))$, because every section of $\pi_1^*(E) \oplus \bigoplus_{i=2}^{\infty} \pi_i^*(F)$ vanishes, and so adding the trivial rank 1 module to it cannot yield the trivial Hilbert module $l_2(C(X))$ (see the proof of $D \ncong l_2(C(B_\infty))$ in [DD63, Proposition 19]). It follows that $H \oplus H$ is not a direct summand of H, i.e., the unit of $B(H)$ is not properly infinite.

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