Projections in free product C*–algebras, II

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Abstract

Let (A, φ) be the reduced free product of infinitely many C*–algebras (A_ι, φ_ι) with respect to faithful states. Assume that the A_ι are not too small, in a specific sense. If φ is a trace then the positive cone of $K_0(A)$ is determined entirely by $K_0(\varphi)$. If, furthermore, the image of $K_0(\varphi)$ is dense in $\mathbb{R}$, then A has real rank zero. On the other hand, if φ is not a trace then A is simple and purely infinite.

Introduction

Let I be a set having at least two elements and, for every $\iota \in I$, let A_ι be a unital C*–algebra with a state, φ_ι, whose GNS representation is faithful. Their reduced free product,

$$(A, \varphi) = \underset{\iota \in I}{*} (A_\iota, \varphi_\iota) \tag{1}$$

was introduced by Voiculescu [20] and independently (in a more restricted way) by Avitzour [1]. Thus A is a unital C*–algebra with canonical, injective, unital *–homomorphisms, $\pi_\iota \colon A_\iota \to A$, and φ is a state on A such that $\varphi \circ \pi_\iota = \varphi_\iota$ for all ι. It is the natural construction in Voiculescu's free probability theory (see [21]), and Voiculescu's theory has been vital to the study of these C*–algebras.

In [12], for reduced free product C*–algebras A as in (1), when all the φ_ι are faithful, we investigated projections in A and the related topic of positive elements in $K_0(A)$. The behaviour we discovered, under mild conditions specifying that the A_ι are not too small, depended broadly on whether φ is a trace, (i.e. on whether all the φ_ι are traces). If φ is a not trace then by [12] A is properly infinite. It remained open whether A must be purely infinite. (Some special classes of reduced free product C*–algebras have in [13] and [9] been shown to be purely infinite.) When φ is a trace, then it follows from [12] that for every element, x, of the subgroup, G, of $K_0(A)$ generated by $\bigcup_{\iota\in I} K_0(\pi_\iota)(K_0(A_\iota))$, if $K_0(\varphi)(x) > 0$ then $x \geq 0$ and if $0 < K_0(\varphi)(x) < 1$ then there is a projection $p \in A$ such that $x = [p]_0$. By work of E. Germain [14], [15], [16], if each A_ι is an amenable C*–algebra then $K_0(A) = G$ and G can be found from the groups $K_0(A_\iota)$ by using exact sequences, (and by taking inductive limits if I is infinite); hence under the hypothesis of amenability, we used Germain's results to give a complete characterization of the positive cone of $K_0(A)$ and of its elements corresponding to projections in A.

In the present paper we investigate similar questions for reduced free product C*–algebras, (1), when I is infinite and when, for infinitely many $\iota \in I$, there is a unitary, $u \in A_\iota$ such that $\varphi_\iota(u) = 0$. We show that in this case, if φ is not a trace then A is purely infinite and simple. If φ is a trace then, although we do not know in general if the subgroup G described above exhausts $K_0(A)$, we nonetheless show that for every $x \in K_0(A)$, if $K_0(\varphi)(x) > 0$ then $x \geq 0$; furthermore, we show that if $x \in K_0(A)$ and if $0 < K_0(\varphi)(x) < 1$ then there is a projection $p \in A$ such that $x = [p]_0$. We also show that if the image of $K_0(\varphi)$ is dense in $\mathbb{R}$ then A has real rank zero.

The *real rank* of a C*–algebra, A, is denoted RR(A) and was invented by L.G. Brown and G.K. Pedersen [4]. Of particular interest is the case RR$(A) = 0$, which is defined, for a unital C*–algebra A, to mean that the invertible self–adjoint elements are dense in the set of all self–adjoint elements of A. If φ is a faithful state on an infinite dimensional, simple C*–algebra A, then a necessary condition for RR$(A) = 0$ is that there be projections, p, in A such that $\varphi(p)$ is arbitrarily small and positive; hence in particular, the image of $K_0(\varphi)$ must be dense in $\mathbb{R}$. We show that this condition is sufficient when A is a reduced free product of infinitely many algebras, as above, and when φ is a trace. (Moreover, when φ is not a trace then A is purely infinite, so by a result of S. Zhang [22], A has real rank zero.)

1 Comparison between positive elements and projections

Most of this section is a reformulation of results from [19]. The proof of Theorem 1.5 below is almost identical to the proof of [19, Theorem 7.2], but the statements of these two theorems are quite different.

We recall the notion of comparison of positive elements as introduced by J. Cuntz in [5] and [6] (see also [19]). Let A be a C*-algebra, and let a, b be positive elements in A.

Then $a \precsim b$ will mean that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in A with

$$\lim_{n\to\infty} \|a - x_n b x_n^*\| = 0.$$

If $p, q \in A$ are projections, then the definition above of $p \precsim q$ agrees with the usual definition: $p = vv^*$ and $v^*v \leq q$ for some partial isometry $v \in A$.

If A is unital, and if φ is a state on A, then define $D_\varphi \colon A^+ \to [0,1]$ by

$$D_\varphi(a) = \lim_{\varepsilon\to 0^+} \varphi(f_\varepsilon(a)),$$

where $f_\varepsilon \colon \mathbb{R}^+ \to [0,1]$ is the continuous function, which is zero on $[0, \varepsilon/2]$, linear on $[\varepsilon/2, \varepsilon]$, and equal to 1 on $[\varepsilon, \infty)$. If φ is a trace, then D_φ is a dimension function (in the sense of Cuntz, [6]). Notice that $D_\varphi(p) = \varphi(p)$ for all projections $p \in A$.

We shall use the following facts:

$$f_{2\varepsilon}(a) \leq f_\delta(f_\varepsilon(a)) \leq f_{\varepsilon/2}(a), \qquad f_{\varepsilon/2}(a) f_\varepsilon(a) = f_\varepsilon(a), \tag{2}$$

when $\varepsilon > 0$ and $0 < \delta \leq 1/2$, and, consequently, $D_\varphi(f_\varepsilon(a)) \leq \varphi(f_{\varepsilon/2}(a))$. Moreover, if $0 \leq a \leq 1$, then $\varphi(a) \leq D_\varphi(a)$. Recall from [18] that the *stable rank* of a unital C*–algebra A is equal to 1 if and only if the set invertible elements of A is dense in A.

Lemma 1.1 *Let A be a C*-algebra of stable rank one, let B be a hereditary subalgebra of A, let a be a positive element in B, and let q be a projection in B such that $a \precsim q$. Then for each $\varepsilon > 0$ there is a projection $p \in B$ such that $f_\varepsilon(a) \leq p \sim q$.*

Proof: Observe first that the comparisons $a \precsim p$ and $p \sim q$ are independent of whether they are relative to A, B, or $\tilde{B}$, where $\tilde{B}$ denotes the C*-algebra obtained by adjoining a unit to B. If follows from [18] and [3] that if A has stable rank one, then so do B and $\tilde{B}$. By [19, Proposition 2.4], there is for each $\varepsilon > 0$ a unitary u in $\tilde{B}$ with $u f_\varepsilon(a) u^* \in q\tilde{B}q$ $(= qBq)$. Put $p = u^*qu$. Then p is as desired. □

Lemma 1.2 *Let A be a C*-algebra, let a be a positive element in A, and let p be a projection in A. Then the following are equivalent:*

(i) $p \precsim a$,

(ii) $p = xax^*$ *for some* $x \in A$,

(iii) *p is equivalent to some projection in the hereditary subalgebra of A generated by a.*

Proof: (i) $\Rightarrow$ (ii). If $p \precsim a$, then $\|p - yay^*\| < 1/2$ for some $y \in A$. Hence $pyay^*p$ is invertible (and positive) in pAp, and therefore $p = zpyay^*pz^*$ for some $z \in pAp$.

(ii) $\Rightarrow$ (iii). Put $u = xa^{1/2}$. Then $uu^* = p$, and hence u is a partial isometry. Put $q = u^*u = a^{1/2}x^*xa^{1/2}$. Then q is a projection in the hereditary subalgebra generated by a, and $q \sim p$.

(iii) $\Rightarrow$ (i). Assume q is a projection in the hereditary subalgebra generated by a, and that $q \sim p$. Then there exists an $n \in \mathbb{N}$ such that $\|q - a^{1/n}qa^{1/n}\| < 1/2$. It follows that $qa^{1/n}qa^{1/n}q$ and, consequently, $qa^{2/n}q$, are invertible in qAq. Therefore $q = rqa^{2/n}qr^*$ for some $r \in qAq$. This shows that $p \sim q \precsim a^{2/n} \precsim a$. □

Lemma 1.3 ([19, Proposition 2.2]) *Let A be a unital C^*-algebra, let a, b be positive elements in A, and let $\varepsilon > 0$. If $\|a - b\| < \varepsilon$, then $f_\varepsilon(a) \precsim b$.*

Lemma 1.4 *Let A be a unital C^*-algebra, and let $\mathfrak{A}$ be a dense unital *-subalgebra of A. Suppose that for each positive element $a \in \mathfrak{A}$ and each $\varepsilon > 0$ there is a projection $p \in A$ and $0 < \delta < \varepsilon$ such that $f_\varepsilon(a) \le p \le f_\delta(a)$. Then $\mathrm{RR}(A) = 0$.*

Proof: To show that $\mathrm{RR}(A) = 0$ it will suffice (by [4]) to show that all self-adjoint elements in the dense *–subalgebra $\mathfrak{A}$ can be approximated by invertible self-adjoint elements.

Let a be a self-adjoint element in $\mathfrak{A}$, and write $a = a_+ - a_-$. For each $n \in \mathbb{N}$ find $\delta_n > 0$ and projections p_n, q_n in A such that

$$f_{1/n}(a_+) \le p_n \le f_{\delta_n}(a_+), \qquad f_{1/n}(a_-) \le q_n \le f_{\delta_n}(a_-).$$

Then $p_n \perp q_n$, $p_n a_+ p_n \to a_+$, and $q_n a_- q_n \to a_-$. Set

$$b_n = (p_n a_+ p_n + \tfrac{1}{n} p_n) - (q_n a_- q_n + \tfrac{1}{n} q_n) + \tfrac{1}{n}(1 - p_n - q_n).$$

Then each b_n is invertible and self-adjoint, and $b_n \to a$. □

Let $\mathcal{Q}$ be a compact convex subset of the state space of a unital C*-algebra A, and let $\mathrm{Aff}(\mathcal{Q})$ denote the real vector space of all affine continuous functions $\mathcal{Q} \to \mathbb{R}$. Equip this space with the strict ordering, i.e., $f \ge 0$ if $f = 0$ or if $f(\varphi) > 0$ for all $\varphi \in \mathcal{Q}$, and, in turn, with the topology induced by this ordering. All self-adjoint elements $a \in A$ induce an element $\hat{a} \in \mathrm{Aff}(\mathcal{Q})$ through the formula $\hat{a}(\varphi) = \varphi(a)$. We will consider the interval $[0,1]$ of $\mathrm{Aff}(\mathcal{Q})$, defined by

$$[0,1] = \{f \in \mathrm{Aff}(\mathcal{Q}) \mid 0 \leq f \leq 1\}.$$

Theorem 1.5 *Let A be a unital C^*-algebra, let $\mathcal{Q}$ be a compact convex subset of the state space of A, let Π be a subset of of the set of projections in A, and let $\mathfrak{A}$ be a dense *-subalgebra of A, which is closed under continuous function calculus (on its normal elements).*

Assume that each state in $\mathcal{Q}$ is faithful on $\mathfrak{A}$, and that the following comparison properties hold for all positive elements $a \in \mathfrak{A}$ and for all projections $p \in \Pi$:

- (α) *if $D_\varphi(a) < \varphi(p)$ for all $\varphi \in \mathcal{Q}$, then $a \precsim p$,*
- (β) *if $\varphi(p) < D_\varphi(a)$ for all $\varphi \in \mathcal{Q}$, then $p \precsim a$, and*
- (γ) *the subset of $\mathrm{Aff}(\mathcal{Q})$ induced by Π is dense in the interval $[0,1]$ of $\mathrm{Aff}(\mathcal{Q})$.*

It follows that

- (i) *if $\mathrm{sr}(A) = 1$, then $\mathrm{RR}(A) = 0$, and*
- (ii) *if all nonzero projections in Π are infinite and full, then A is simple and purely infinite.*

Proof: (i). We show that the conditions in Lemma 1.4 are satisfied. So let $a \in \mathfrak{A}$ be a positive element, and let $\varepsilon > 0$. We must find $0 < \delta < \varepsilon$ and a projection $p \in A$ with $f_\varepsilon(a) \leq p \leq f_\delta(a)$.

If $\mathrm{sp}(a) \cap (\varepsilon/8, \varepsilon/4) = \emptyset$, then $p = f_{\varepsilon/8}(a)$ and $\delta = \varepsilon/8$ will be as desired.

Assume now that $\mathrm{sp}(a) \cap (\varepsilon/8, \varepsilon/4) \neq \emptyset$. Then $0 \leq \varphi(f_{\varepsilon/4}(a)) < \varphi(f_{\varepsilon/8}(a)) \leq 1$ for all $\varphi \in \mathcal{Q}$ because each such φ is assumed to be faithful on $\mathfrak{A}$. Since Π is dense in the interval $[0,1]$ of $\mathrm{Aff}(\mathcal{Q})$ there is $q \in \Pi$ with $\varphi(f_{\varepsilon/4}(a)) < \varphi(q) < \varphi(f_{\varepsilon/8}(a))$ for all $\varphi \in \mathcal{Q}$. By (2),

$$D_\varphi(f_{\varepsilon/2}(a)) \leq \varphi(f_{\varepsilon/4}(a)) < \varphi(q) < \varphi(f_{\varepsilon/8}(a)) \leq D_\varphi(f_{\varepsilon/8}(a)).$$

By assumptions (α) and (β) this implies that $f_{\varepsilon/2}(a) \precsim q \precsim f_{\varepsilon/8}(a)$.

From Lemma 1.2 there is a projection r in the hereditary subalgebra, B, generated by $f_{\varepsilon/8}(a)$ such that $q \sim r$. By Lemma 1.1 there is a projection $p \in B$ such that $f_{1/2}(f_{\varepsilon/2}(a)) \leq p$ (and $p \sim r$). By (2), this entails that $f_\varepsilon(a) \leq p \leq f_{\varepsilon/16}(a)$. The claim is therefore proved with $\delta = \varepsilon/16$.

(ii). If each non-zero hereditary subalgebra of A contains a full element, then A must be simple. If, moreover, each such hereditary subalgebra contains an infinite projection, then A is purely infinite and simple (c.f. Cuntz' definition of purely infinite simple C*-algebras in [7]). It therefore suffices to show that each non-zero hereditary subalgebra of A contains an infinite full projection.

Let B be a non-zero hereditary subalgebra of A, and let b be a positive element in B with $\|b\| = 1$. Find a positive element $a \in \mathfrak{A}$ with $\|a - b\| < 1/2$. Since each $\varphi \in \mathcal{Q}$ is faithful, since $f_{1/2}(a) \neq 0$, and since Π is dense in the interval $[0,1]$ of $\mathrm{Aff}(\mathcal{Q})$, there is $q \in \Pi$ with $\varphi(q) < \varphi(f_{1/2}(a))$ for all $\varphi \in \mathcal{Q}$. This implies that $\varphi(q) < D_\varphi(f_{1/2}(a))$ for all $\varphi \in \mathcal{Q}$, and by assumption (β) we get $q \precsim f_{1/2}(a)$. Using Lemma 1.3 we obtain that $q \precsim b$, and Lemma 1.2 finally implies that there is a projection p in the hereditary subalgebra of A generated by b (which is contained in B, so that $p \in B$) such that $p \sim q$. Since q is infinite and full, so is p, and the proof is complete. □

2 Application to reduced free products of C*–algebras

Throughout this section, we consider a reduced free product of C*–algebras,

$$(A, \varphi) = \mathop{*}_{\iota \in I} (A_\iota, \varphi_\iota), \tag{3}$$

where I is an infinite set, where each φ_ι is a faithful state and where for infinitely many $\iota \in I$ there is a unitary $u \in A_\iota$ with $\varphi_\iota(u) = 0$. It follows from [8] that φ is faithful on A.

Avitzour's result [1] gives that A is simple if, for example, φ is a trace. Indeed, by partitioning the set I into two suitable subsets, A can be viewed as a reduced free product,

$$(A, \varphi) = (B_1, \psi_1) * (B_2, \psi_2),$$

such that there are unitaries, $u \in B_1$ and $v, w \in B_2$ satisfying that $\psi_1(u) = 0 = \psi_2(v) = \psi_2(w)$ and that v and w are *–free; hence also $\psi_2(v^* w) = 0$. (Avitzour's result also applies in somewhat more general instances.) In addition, if φ is a trace then by [10] the stable rank of A is equal to 1.

The K_0-group, $K_0(D)$, of a C*-algebra D is equipped with a *positive cone* and a *scale* defined respectively by

$$\begin{aligned} K_0(D)^+ &= \{[p]_0 \mid p \in \mathrm{Proj}(D \otimes \mathcal{K})\}, \\ \Sigma(D) &= \{[p]_0 \mid p \in \mathrm{Proj}(D)\}, \end{aligned}$$

where $\mathrm{Proj}(D)$ is the set of projection in D, and where $[\,\cdot\,]_0\colon \mathrm{Proj}(D \otimes \mathcal{K}) \to K_0(D)$ is the canonical map from which K_0 is defined. The positive cone gives rise to an ordering on $K_0(D)$ by $x \le y$ if $y - x \in K_0(D)^+$, and $x < y$ if $y - x \in K_0(D)^+ \setminus \{0\}$. Each (positive) trace φ on D induces a positive group-homomorphism $K_0(\varphi)\colon K_0(D) \to \mathbb{R}$ which satisfies $K_0(\varphi)([p]_0) = \varphi(p)$ for $p \in \mathrm{Proj}(D)$, and $K_0(\varphi)([p]_0) = (\varphi \otimes \mathrm{Tr}_n)(p)$ for $p \in \mathrm{Proj}(D \otimes M_n(\mathbb{C}))$, where Tr_n is the (unnormalized) trace on $M_n(\mathbb{C})$. The ordered abelian group $(K_0(D), K_0(D)^+)$ is called *weakly unperforated* if $nx > 0$ for some $n \in \mathbb{N}$ and some $x \in K_0(D)$ implies that $x \ge 0$.

Theorem 2.1 *Let*

$$(A, \varphi) = \mathop{*}_{\iota \in I} (A_\iota, \varphi_\iota)$$

be the reduced free product C^*-algebra, where each A_ι is a unital C^*-algebra, φ_ι is a faithful state on A_ι, the index set I is infinite, and infinitely many A_ι contain a unitary in the kernel of φ_ι.

If φ is a trace (which is the case if all φ_ι are traces), then

(i) *whenever $p, q \in A \otimes M_n(\mathbb{C})$ are projections such that $(\varphi \otimes \mathrm{Tr}_n)(p) < (\varphi \otimes \mathrm{Tr}_n)(q)$, it follows that $p \precsim q$;*

(ii) *the positive cone and the scale of $K_0(A)$ are given by*

$$\begin{aligned} K_0(A)^+ &= \{0\} \cup \{x \in K_0(D) \mid 0 < K_0(\varphi)(x)\}, \\ \Sigma(A) &= \{0, 1\} \cup \{x \in K_0(D) \mid 0 < K_0(\varphi)(x) < 1\}, \end{aligned}$$

and, as a consequence, $(K_0(A), K_0(A)^+)$ is weakly unperforated;

(iii) $\mathrm{RR}(A) = 0$ *if and only if $K_0(\varphi)(K_0(A))$ is dense in $\mathbb{R}$.*

If φ is not a trace (i.e., if at least one φ_ι is not a trace), then A is simple and purely infinite.

Proof: We consider, for every finite subset $F \subseteq I$, the C*–subalgebra, $\mathfrak{A}_F$, of A generated by $\bigcup_{\iota \in F} \pi_\iota(A_\iota)$, and we let $\mathfrak{A} = \bigcup_{F \ll I} \mathfrak{A}_F$, where the union is over all finite subsets of

I. Note that $\mathfrak{A}$ is a dense, unital *–subalgebra of A that is closed under the continuous functional calculus.

Suppose that φ is a trace, let $n \in \mathbb{N}$ and let $p, q \in A \otimes M_n(\mathbb{C})$ be projections with $(\varphi \otimes \mathrm{Tr}_n)(p) < (\varphi \otimes \mathrm{Tr}_n)(q)$. Using the density of $\mathfrak{A}$ in A and continuous functional calculus, we find a finite subset F of I and projections $\tilde{p}, \tilde{q} \in \mathfrak{A}_F \otimes M_n(\mathbb{C})$ such that $\|\tilde{p} - p\| < 1$ and $\|\tilde{q} - q\| < 1$. This implies $\tilde{p} \sim p$ and $\tilde{q} \sim q$. There are n^2 distinct elements $\iota(1), \iota(2), \ldots, \iota(n^2) \in I \backslash F$ with unitaries $u_k \in A_{\iota(k)}$ such that $\varphi_{\iota(k)}(u_k) = 0$. Let B be the C*–algebra generated by $\{u_1, u_2, \ldots, u_{n^2}\}$. Note that B and $\mathfrak{A}_F$ are free. Then as in the proof of Proposition 3.3 of [12], from the unitaries $u_1, u_2, \ldots, u_{n^2}$ we can construct a Haar unitary, $v \in B \otimes M_n(\mathbb{C})$ such that $\{v\}$ and $\mathfrak{A}_F \otimes M_n(\mathbb{C})$ are *–free (with respect to the tracial state $\varphi \otimes (\frac{1}{n}\mathrm{Tr}_n)$). Now $\tilde{q} \sim v^* \tilde{q} v$ and the pair $\tilde{p}$ and $v^* \tilde{q} v$ is free; moreover, $(\varphi \otimes \mathrm{Tr}_n)(v^* \tilde{q} v) = (\varphi \otimes \mathrm{Tr}_n)(\tilde{q})$. So by Proposition 1.1 of [12], $v^* \tilde{q} v$ is equivalent to a subprojection, r, of $\tilde{p}$; hence $q \precsim p$. We have thus proved (i).

The inclusions $\subseteq$ in (ii) are easy consequences of the fact that φ is faithful. Assume $x \in K_0(A)$ and that $K_0(\varphi)(x) > 0$. Since A is unital, there are $n \in \mathbb{N}$ and projections $p, q \in A \otimes M_n(\mathbb{C})$ such that $x = [p]_0 - [q]_0$. Now,

$$(\varphi \otimes \mathrm{Tr}_n)(p) - (\varphi \otimes \mathrm{Tr}_n)(q) = K_0(\varphi)(x) > 0.$$

Hence, by (i), q is equivalent to a subprojection $\tilde{q}$ of p. Thus $x = [p - \tilde{q}]_0 \in K_0(A)^+$.

Assume next that $x \in K_0(A)$ and that $0 < K_0(\varphi)(x) < 1$. Then, by the argument above, $x = [p]_0$ for some projection $p \in A \otimes M_n(\mathbb{C})$. Let 1_A denote the unit of A, and let $e \in A \otimes M_n(\mathbb{C})$ be the diagonal projection whose upper left corner is 1_A and with all other entries equal to 0. Then $(\varphi \otimes \mathrm{Tr}_n)(p) = K_0(\varphi)(x) < 1 = (\varphi \otimes \mathrm{Tr}_n)(e)$. By (i), this implies that p is equivalent to a subprojection $\tilde{p}$ of e. Hence $x = [\tilde{p}]_0$, and it is easily seen that $[\tilde{p}]_0 \in \Sigma(A)$.

Finally, to see that $(K_0(A), K_0(A)^+)$ is weakly unperforated, assume that $x \in K_0(A)$ and that $nx > 0$ for some $n \in \mathbb{N}$. Then $K_0(\varphi)(x) = \frac{1}{n} K_0(\varphi)(nx) > 0$. Hence $x > 0$. We have thus shown (ii).

Let

$$\Pi = \bigcup_{F \ll I} \mathrm{Proj}(\mathfrak{A}_F).$$

For the set $\mathcal{Q}$ used in Theorem 1.5, we take the singleton $\{\varphi\}$. We now show that, regardless of whether φ is a trace or not, conditions (α) and (β) of Theorem 1.5 hold for every $p \in \Pi$ and every positive element, $a \in \mathfrak{A}$. Given a positive element $a \in \mathfrak{A}$ and given $p \in \Pi$, there is a finite subset F of I such that $a, p \in \mathfrak{A}_F$. Let $u \in A_\iota$, for some $\iota \in I \backslash F$, be a unitary such that $\varphi_\iota(u) = 0$. Then $\{a, p\}$ and $\{u\}$ are *–free with respect to φ. Now it follows that u^*pu is a projection with $\varphi(u^*pu) = \varphi(p)$, and that u^*pu and a are free. Hence by Lemma 5.3 of [12] it follows that $a \lesssim u^*pu$ if $D_\varphi(a) < \varphi(p)$ and $u^*pu \lesssim a$ if $\varphi(p) < D_\varphi(a)$. But $u^*pu \sim p$, so (α) and (β) hold.

Suppose now that φ is a trace and that the image of $K_0(\varphi)$ is dense in $\mathbb{R}$, and let us show that $\mathrm{RR}(A) = 0$. We will show that $\{\varphi(p) \mid p \in \Pi\}$ is dense in $[0, 1]$, which will imply that condition (γ) of Theorem 1.5 holds. Since the image of $K_0(\varphi)$ is dense in $\mathbb{R}$, the intersection of this image with $[0, 1]$ is dense in $[0, 1]$. By (ii), it follows that $\{\varphi(p) \mid p \in \mathrm{Proj}(A)\}$ is dense in $[0, 1]$. Since $\mathfrak{A}$ is dense in A, and using continuous functional calculus, we find for every $p \in \mathrm{Proj}(A)$, a projection, $\tilde{p} \in \mathfrak{A}$ such that $\varphi(\tilde{p}) = \varphi(p)$. But $\tilde{p} \in \Pi$. We have shown that condition (γ) of Theorem 1.5 holds, and we have already shown that conditions (α) and (β) hold. Now using the fact that $\mathrm{sr}(A) = 1$, we get from Theorem 1.5(i) that $\mathrm{RR}(A) = 0$. This implies one direction of (iii), but the other direction follows from more general results. Indeed, the image of $K_0(\varphi)$ will be dense in $\mathbb{R}$ if A contains at least one projection and if A has no minimal projections. Both of these conditions hold if $\mathrm{RR}(A) = 0$, and if A is simple and infinite dimensional, as in our case.

Now suppose that φ is not a trace, and let us show that A is purely infinite and simple. Let F be a finite subset of I such that for at least three distinct $\iota \in F$ there is a unitary $u \in A_\iota$ satisfying $\varphi_\iota(u) = 0$, and such that for some $\iota \in F$, φ_ι is not a trace. Then by Theorem 4 of [12], the unit is a properly infinite projection in $\mathfrak{A}_F$. Let $\Pi' \subseteq \Pi$ be the set of all full, properly infinite projections in $\mathfrak{A}$. We have already shown that conditions (α) and (β) are satisfied for every $a \in \mathfrak{A}$ and every $p \in \Pi'$. Since 1 is a properly infinite projection in some $\mathfrak{A}_F$, using Lemma 2.2 below we get that $\{\varphi(p) \mid p \in \Pi'\}$ is dense in $[0, 1]$, so condition (γ) is satisfied. Tautologically, each $p \in \Pi'$ is infinite and full. Hence by Theorem 1.5(ii), A is purely infinite and simple. □

Lemma 2.2 *Let A be a unital C*–algebra in which 1 is properly infinite and let φ be a state on A. Then for every $t \in \mathbb{R}$, $0 < t \leq 1$, there is a projection $p \in A$ such that $p \sim 1$ and $\varphi(p) = t$.*

Proof: Using that 1 is properly infinite, we find isometries, $v_1, v_2, \ldots$ in A whose range projections are mutually orthogonal. These generate a unital C*–subalgebra of A isomorphic to the Cuntz algebra $\mathcal{O}_\infty$. By Cuntz's paper [7], it follows that if $p, q \in \mathrm{Proj}(\mathcal{O}_\infty) \backslash \{0, 1\}$ and if $[p]_0 = [q]_0$ in $K_0(\mathcal{O}_\infty)$ then p is homotopic to q. (Indeed, it follows that $p \sim q$ and $1 - p \sim 1 - q$, hence p is unitarily similar to q. But the unitary group of $\mathcal{O}_\infty$ is connected.)

Now let $\varepsilon > 0$. For some n we must have $\varphi(v_n v_n^*) < \varepsilon$; let $p = v_n v_n^*$. Then $q \stackrel{\text{def}}{=} 1 - v_n(1-p)v_n^*$ is a projection in $\mathcal{O}_\infty$ with $[q]_0 = [1]_0$, and $\varphi(q) > 1 - \varepsilon$. Thus there is a continuous path r_t in $\text{Proj}(\mathcal{O}_\infty)$ such that $r_0 = p$ and $r_1 = q$. We have that $r_t \sim 1$ for all t and $\{\varphi(r_t) \mid t \in [0,1]\} \supseteq (\varepsilon, 1-\varepsilon)$. □

Let us now state a straightforward application of Theorem 2.1 to reduced group C*–algebras. For a group, G, taken with the discrete topology, the reduced group C*–algebra of G, denoted $C^*_{\text{red}}(G)$, is the C*–algebra generated by the left regular representation of G. The canonical tracial state, τ_G, is the vector state for the characteristic function of the identity element of G. The following corollary was cited in [11], where also a partial converse was included.

Corollary 2.3 *Let I be an infinite set and let*

$$G = \mathop{*}_{\iota \in I} G_\iota$$

be the free product of nontrivial groups, G_ι. Suppose that G has finite subgroups of arbitrarily large order. Then

$$\text{RR}(C^*_{\text{red}}(G)) = 0.$$

Proof: We have

$$(C^*_{\text{red}}(G), \tau_G) \cong \mathop{*}_{\iota \in I} (C^*_{\text{red}}(G_\iota), \tau_{G_\iota}).$$

If x is a nontrivial element of G_ι then the left translation operator, $\lambda_x \in C^*_{\text{red}}(G_\iota)$ is a unitary and $\tau_{G_\iota}(\lambda_x) = 0$. In order to apply Theorem 2.1, it is thus sufficient to show that $C^*_{\text{red}}(G)$ contains projections whose traces (under τ_G) are arbitrarily small and positive. But this is clear, since for a finite group H, $C^*_{\text{red}}(H)$ contains projections of trace $1/|H|$. □

It follows easily from the Kurosh Subgroup Theorem for free products of groups, (see page 178 of [17]), that G has finite subgroups of arbitrarily high order only if for every positive integer n there is $\iota \in I$ such that G_ι has a finite subgroup of order greater than n.

Example 2.4 *If*

$$G = \mathop{*}_{n=2}^{\infty} (\mathbb{Z}/n\mathbb{Z}),$$

then $C^*_{\text{red}}(G)$ has real rank zero. Moreover, this C–algebra is simple, has unique tracial state, has stable rank one and is not approximately divisible.*

Proof: It has real rank zero by the above corollary. It is simple and has unique tracial state by Avitzour [1]. It has stable rank one by [10]. That it is not approximately divisible follows from the argument in Example 4.8 of [2], because $C^*_{\mathrm{red}}(G)$ is weakly dense in the group von Neumann algebra $L(G)$, which does not have central sequences. □

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