Endomorphisms of $\mathcal{O}_n$ which preserve the canonical UHF-subalgebra

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Abstract

Unital endomorphisms of the Cuntz algebra $\mathcal{O}_n$ which preserve the canonical UHF-subalgebra $\mathcal{F}_n \subseteq \mathcal{O}_n$ are investigated. We give examples of such endomorphisms $\lambda = \lambda_u$ for which the associated unitary element u in $\mathcal{O}_n$ (which satisfies $\lambda(S_j) = uS_j$ for all j) does not belong to $\mathcal{F}_n$. One such example, in the case where $n = 2$, arises from a construction of a unital endomorphism on $\mathcal{O}_2$ which preserves the canonical UHF-subalgebra and where the relative commutant of its image in $\mathcal{O}_2$ contains a copy of $\mathcal{O}_2$.

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1 Introduction

The study of endomorphisms of Cuntz algebras continues to attract attention of researchers. On the one hand, such endomorphisms naturally arise in a number of contexts including index theory and subfactors, entropy, and classical dynamical systems on the Cantor set. On the other hand, they exhibit interesting and intriguing features while being concrete enough to allow explicit albeit sometimes complicated computations.

It is a fundamental fact that there is a one-to-one correspondence between unitaries in $\mathcal{O}_n$ and unital endomorphisms on $\mathcal{O}_n$ whereby u in $\mathcal{O}_n$ corresponds to the endomorphism λ_u which maps the jth canonical generator S_j of $\mathcal{O}_n$ onto uS_j for $j = 1, 2, \ldots, n$. In the ground breaking paper by Cuntz on this subject, [6], it is noted that λ_u maps the canonical UHF-subalgebra $\mathcal{F}_n$ of $\mathcal{O}_n$ into itself whenever u belongs to $\mathcal{F}_n$; and the question if the converse also holds is considered. This indeed is true in many cases (as one can deduce from [6]), for example if one knows in advance that the range of the endomorphism λ_u is globally invariant under the gauge action of $\mathbb{T}$. This assumption is already sufficient to cover several interesting cases, e.g. if λ_u is an automorphism of $\mathcal{O}_n$.

We show in this paper that this converse statement is false in general, i.e., there is a unitary element u in $\mathcal{O}_n$ which does not belong to $\mathcal{F}_n$ but where λ_u maps $\mathcal{F}_n$ into itself.

The paper is organized in the following way. In section 2, after some preliminaries, we present a general framework for finding the announced counterexamples and we discuss a specific example in the case of $\mathcal{O}_2$ that arises in a combinatorial way. In section 3, we exhibit a unitary u in the UHF-subalgebra of $\mathcal{O}_2$ such that the image of the corresponding endomorphism λ_u has relative commutant containing a copy of $\mathcal{O}_2$. One can then easily find another unitary v in $\mathcal{O}_2$ such that v does not belong to $\mathcal{F}_2$ but where λ_v agrees with λ_u on $\mathcal{F}_2$, whence in particular λ_v maps $\mathcal{F}_2$ into itself. From this construction one gets as a byproduct an embedding of $\mathcal{O}_2 \otimes \mathcal{O}_2$ into $\mathcal{O}_2$ that maps $\mathcal{F}_2 \otimes \mathcal{F}_2$ into $\mathcal{F}_2$.

It remains an interesting open problem if for every unital endomorphism λ on $\mathcal{O}_n$ which maps $\mathcal{F}_n$ into itself there exists a unitary element u in $\mathcal{F}_n$ such that λ and λ_u agree on $\mathcal{F}_n$.

In section 4, we expand our initial observations on endomorphisms preserving the canonical UHF-subalgebra in a more systematic manner. In section 5, we study a particularly interesting class of such endomorphisms related to certain elements in the normalizer of the canonical MASA.

Finally, we would like to mention that endomorphisms preserving the core AF-subalgebras of certain C^*-algebras corresponding to rank-2 graphs (generalizing the Cuntz algebras) have been very recently considered in [15].

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2 A counterexample

If n is an integer greater than 1, then the Cuntz algebra $\mathcal{O}_n$ is a unital, simple C^*-algebra generated by n isometries $S_1, \ldots, S_n$, satisfying $\sum_{i=1}^n S_i S_i^* = I$, [5]. We denote by W_n^k the set of k-tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_m \in \{1, \ldots, n\}$, and by W_n the union $\cup_{k=0}^\infty W_n^k$, where $W_n^0 = \{0\}$. We call elements of W_n multi-indices. If $\alpha = (\alpha_1, \ldots, \alpha_k) \in W_n$, then $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$ ($S_0 = I$ by convention) and $P_\alpha = S_\alpha S_\alpha^*$. Every word in $\{S_i, S_i^* \mid i = 1, \ldots, n\}$ can be uniquely expressed as $S_\alpha S_\beta^*$, for $\alpha, \beta \in W_n$, [5, Lemma 1.3]. If $\alpha \in W_n^k$ then $|\alpha| = k$ is the length of α.

The C^*-algebra generated by all words of the form $S_\alpha S_\beta^*$, $\alpha, \beta \in W_n^k$, is denoted by $\mathcal{F}_n^k$ and it is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. The norm closure of $\cup_{k=0}^\infty \mathcal{F}_n^k$ is a UHF-algebra of type n^∞, called the core UHF-subalgebra of $\mathcal{O}_n$, [5], and is denoted by $\mathcal{F}_n$. It is the fixed point algebra for the periodic gauge action of the reals: $\alpha\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_n)$ defined on generators as $\alpha_t(S_i) = e^{it} S_i$, $t \in \mathbb{R}$. The commutative C^*-algebra generated by all projections P_α, $\alpha \in W_n^k$, is denoted by $\mathcal{D}_n^k$, and it is isomorphic to $\mathbb{C}^{n^k}$. Finally, $\mathcal{D}_n$, the norm closure of $\cup_{k=0}^\infty \mathcal{D}_n^k$, is referred to as the canonical MASA in $\mathcal{O}_n$.

We denote by $\mathcal{S}_n$ the group of those unitaries in $\mathcal{O}_n$ which can be written as finite sums of words, i.e., in the form $u = \sum_{j=1}^m S_{\alpha_j} S_{\beta_j}^*$ for some $\alpha_j, \beta_j \in W_n$. It turns out that $\mathcal{S}_n$ is isomorphic to the Higman-Thompson group $G_{n,1}$ [10]. We also denote $\mathcal{P}_n = \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n)$. Then $\mathcal{P}_n = \cup_k \mathcal{P}_n^k$, where $\mathcal{P}_n^k$ are permutation unitaries in $\mathcal{U}(\mathcal{F}_n^k)$. That is, for each $u \in \mathcal{P}_n^k$ there is a unique permutation σ of multi-indices W_n^k such that $u = \sum_{\alpha \in W_n^k} S_{\sigma(\alpha)} S_\alpha^*$.

For u a unitary in $\mathcal{O}_n$ we denote by λ_u the unital endomorphism of $\mathcal{O}_n$

determined by $\lambda_u(S_i) = uS_i$, $i = 1, \ldots, n$. We denote by φ the canonical shift: $\varphi(x) = \sum_i S_i x S_i^*$, $x \in \mathcal{O}_n$. Note that φ commutes with the action α. If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we denote

$$u_k = u\varphi(u) \cdots \varphi^{k-1}(u).$$

We agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i a = \varphi(a) S_i$, valid for all $i = 1, \ldots, n$ and $a \in \mathcal{O}_n$.

Proposition 2.1. *Let u be a unitary in $\mathcal{O}_n$ and let v be a unitary in the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Then the three endomorphisms λ_u, λ_{vu}, and $\lambda_{u\varphi(v)}$ coincide on $\mathcal{F}_n$.*

Proof. We only show that the restrictions of λ_u and $\lambda_{u\varphi(v)}$ to $\mathcal{F}_n$ coincide (the proof that the restrictions of λ_u and λ_{vu} to $\mathcal{F}_n$ concide is similar and easier). Put $w = u\varphi(v)$. It suffices to show that λ_w and λ_u agree on all elements of the form $S_{\alpha_1} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_1}^*$ for every integer $k \geq 1$, all α_i and β_j in $\{1, \ldots, n\}$, and for all $1 \leq i, j \leq k$. We prove this by induction on k.

For $k = 1$ we have

$$\lambda_w(S_{\alpha_1} S_{\beta_1}^*) = wS_{\alpha_1} S_{\beta_1}^* w^* = u\varphi(v) S_{\alpha_1} S_{\beta_1}^* \varphi(v)^* u^* = uS_{\alpha_1} S_{\beta_1}^* u^* = \lambda_u(S_{\alpha_1} S_{\beta_1}^*),$$

since $\varphi(v)$ and $S_{\alpha_1} S_{\beta_1}^*$ commute. (Use the identity $S_i v = \varphi(v) S_i$ mentioned above to see this.) Now, assuming that the identity holds for $k-1$, we have

$$\begin{aligned}
\lambda_w(S_{\alpha_1} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_1}^*) &= \lambda_w(S_{\alpha_1}) \lambda_w(S_{\alpha_2} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_2}^*) \lambda_w(S_{\beta_1})^* \\
&= u\varphi(v) S_{\alpha_1} \lambda_u(S_{\alpha_2} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_2}^*) S_{\beta_1}^* \varphi(v)^* u^* \\
&= uS_{\alpha_1} v \lambda_u(S_{\alpha_2} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_2}^*) v^* S_{\beta_1}^* u^* \\
&= uS_{\alpha_1} \lambda_u(S_{\alpha_2} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_2}^*) S_{\beta_1}^* u^* \\
&= \lambda_u(S_{\alpha_1} \cdots S_{\alpha_k} S_{\beta_k}^* \cdots S_{\beta_1}^*),
\end{aligned}$$

since $\varphi(v) S_{\alpha_1} = S_{\alpha_1} v$, and v is in the commutant of $\lambda_u(\mathcal{F}_n)$. □

Corollary 2.2. *Under the hypothesis of Proposition 2.1, assume further that $u \in \mathcal{F}_n$. Then $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and thus $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$. However, w belongs to $\mathcal{F}_n$ if and only if v does.*

The crucial role in the above construction is played by $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. It turns out that this relative commutant can be calculated as follows (compare [9, Proposition 3.1]).

Proposition 2.3. *Let u be a unitary in $\mathcal{O}_n$, then*

$$\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \bigcap_{k \geq 1} (\mathrm{Ad}u \circ \varphi)^k(\mathcal{O}_n). \tag{1}$$

Proof. Clearly an element $x \in \mathcal{O}_n$ lies in $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ if and only if, for all $k \geq 1$ and all $y \in \mathcal{F}_n^k$, x commutes with $\lambda_u(y) = u_k y u_k^*$, i.e.

$$u_k^* x u_k \in (\mathcal{F}_n^k)' \cap \mathcal{O}_n = \varphi^k(\mathcal{O}_n) \ .$$

This means precisely that, for each $k \geq 1$, x lies in the range of $\mathrm{Ad}(u_k)\varphi^k = (\mathrm{Ad}u \circ \varphi)^k$. □

It is also useful to observe that $\mathrm{Ad}u \circ \varphi$ restricts to an automorphism of $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. This follows from the following simple lemma.

Lemma 2.4. *Let $\mathfrak{A}$ be a unital C^*-algebra and ρ an injective unital $*$-endomorphism of $\mathfrak{A}$, then ρ restricts to a $*$-automorphism of*

$$\mathfrak{A}_\rho := \bigcap_{k \in \mathbb{N}} \rho^k(\mathfrak{A}) \ .$$

Proof. One has a descending tower of unital C^*-subalgebras of $\mathfrak{A}$,

$$\mathfrak{A} \supset \rho(\mathfrak{A}) \supset \rho^2(\mathfrak{A}) \supset \dots \ ,$$

thus $\mathfrak{A}_\rho$ is a unital C^*-subalgebra of $\mathfrak{A}$. An element $x \in \mathfrak{A}_\rho$ satisfies

$$x = \rho(x_1) = \rho^2(x_2) = \cdots = \rho^k(x_k) = \dots$$

for elements $x_1, \dots, x_k, \dots$ in $\mathfrak{A}$. It is then clear that ρ maps $\mathfrak{A}_\rho$ into itself, and moreover $x_1, \dots, x_k, \dots \in \mathfrak{A}_\rho$ so that in particular $\rho(\mathfrak{A}_\rho) = \mathfrak{A}_\rho$. □

Endomorphisms ρ for which $\mathfrak{A}_\rho = \mathbb{C}1$ are often called *shifts*.

Corollary 2.2 shows how to construct examples of unitaries w outside $\mathcal{F}_n$ for which nevertheless $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$. To this end, it suffices to find a unitary $u \in \mathcal{F}_n$ such that the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ is not contained in

$\mathcal{F}_n$. This is possible. In fact, one can even find unitaries in a matrix algebra $\mathcal{F}_n^k$ such that $\lambda_u(\mathcal{O}_n)' \cap \mathcal{O}_n$ is not contained in $\mathcal{F}_n$. The existence of such unitaries was demonstrated in [3]. The relative commutant $\lambda_u(\mathcal{O}_n)' \cap \mathcal{O}_n$ coincides with the space (λ_u, λ_u) of self-intertwiners of the endomorphism λ_u, which can be computed as

$$(\lambda_u, \lambda_u) = \{x \in \mathcal{O}_n : x = (\mathrm{Ad}u \circ \varphi)(x)\} \ .$$

Indeed, an element $x \in \mathcal{O}_n$ lies in the C^*-subalgebra (λ_u, λ_u) if and only if $xuS_i = uS_i x$ for all $i = 1, \dots, n$. But this means that $u^* x u S_i = S_i x = \varphi(x) S_i$ for all $i = 1, \dots, n$, and the latter condition is clearly equivalent to $u^* x u = \varphi(x)$.

Example 2.5. We give an explicit example of a permutation unitary $u \in \mathcal{P}_2^4$ and a unitary v in $\mathcal{S}_2 \setminus \mathcal{P}_2$ such that $v \in (\lambda_u, \lambda_u)$. Indeed, one can check by a lengthy but straightforward computation that the pair:

$$\begin{aligned} u \ = \ & S_{111}S_{111}^* + S_{1211}S_{1121}^* + S_{1221}S_{1122}^* + S_{2111}S_{1211}^* \\ & + S_{1212}S_{1212}^* + S_{221}S_{122}^* + S_{112}S_{211}^* + S_{2121}S_{2121}^* \\ & + S_{1222}S_{2122}^* + S_{2112}S_{2211}^* + S_{2122}S_{2212}^* + S_{222}S_{222}^* \ \in \ \mathcal{P}_2^4, \end{aligned}$$

$$\begin{aligned} v \ = \ & S_{122}S_{11}^* + S_{111}S_{121}^* + S_{211}S_{122}^* + S_{22}S_{211}^* \\ & + S_{112}S_{212}^* + S_{121}S_{221}^* + S_{212}S_{222}^* \ \in \ \mathcal{S}_2 \setminus \mathcal{P}_2, \end{aligned}$$

does the job. (Here, for simplicity, we have used the multi-index notation.)

The way the examples in this and the following section have been constructed leaves open the possibility that for each endomorphism of $\mathcal{O}_n$ globally preserving the core UHF-subalgebra there exists another one, induced by a unitary in $\mathcal{F}_n$, which restricts to the same endomorphism of $\mathcal{F}_n$. So far, this question has not been settled in full generality and we would like to leave it as an open problem.

3 An endomorphism on $\mathcal{O}_2$ with relative commutant containing $\mathcal{O}_2$

As observed in the previous section, if u is a unitary element in a Cuntz algebra $\mathcal{O}_n$ and v is a unitary element in the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, then λ_u and λ_{vu} will agree on the canonical UHF-algebra $\mathcal{F}_n$ contained in $\mathcal{O}_n$. Hence, if u belongs to $\mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ is not contained in $\mathcal{F}_n$, then one can choose v as above such that v, and hence vu, do not belong to $\mathcal{F}_n$; whereas λ_{vu} will map $\mathcal{F}_n$ into itself. We constructed an example of such a unitary element u in Example 2.5 above. In this section we shall construct another example, in the case where $n = 2$, where the relative commutant $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ contains a copy of $\mathcal{O}_2$ and therefore is not contained in $\mathcal{F}_2$.

It is well-known, [12], that $\mathcal{O}_2$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_2$. In particular there is a unital embedding $\mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$. If one composes that with the embedding $\mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ given by $x \mapsto x \otimes 1$, then one obtains an endomorphism $\lambda \colon \mathcal{O}_2 \to \mathcal{O}_2$ such that $\lambda(\mathcal{O}_2)' \cap \mathcal{O}_2$ contains a unital copy of $\mathcal{O}_2$. We show that one can choose this endomorphism λ such that it is of the form $\lambda = \lambda_u$ for some unitary u in $\mathcal{F}_2$.

Let $\eta \colon \mathcal{O}_2 \to D$ be a unital $*$-homomorphism. Let S_1, S_2 be the two canonical generators of $\mathcal{O}_2$. Define unital endomorphisms φ on $\mathcal{O}_2$ and ψ on D by

$$\varphi(x) = S_1 x S_1^* + S_2 x S_2^*, \qquad \psi(y) = \eta(S_1) y \eta(S_1)^* + \eta(S_2) y \eta(S_2)^*, \tag{2}$$

for $x \in \mathcal{O}_2$ and $y \in D$. One has that $\psi \circ \eta = \eta \circ \varphi$. Close inspection of the proof of Theorem 3.6 from [11] shows that the following holds:

Theorem 3.1 (cf. [11])**.** *Let D be a unital properly infinite C^*-algebra, let $\eta \colon \mathcal{O}_2 \to D$ be a unital $*$-homomorphism, and let D_0 be a unital sub-C^*-algebra of D such that*

(i) *D_0 is K_1-injective and has bounded exponential length,*

(ii) *D_0 is invariant under the endomorphism ψ on D associated with η (as defined in* (2) *above).*

(iii) *D_0 contains $\eta(\mathcal{F}_2)$.*

It follows that $\{v\psi(v)^* \mid v \in \mathcal{U}(D_0)\}$ is dense in $\mathcal{U}(D_0)$.

Combining the theorem above with [12, Lemma 1] we get the following:

Proposition 3.2. *There is a sequence $\{v_n\}$ of unitaries in $\mathcal{F}_2$ such that the corresponding sequence $\{\lambda_{v_n}\}$ of endomorphisms on $\mathcal{O}_2$ is asymptotically central.*

(A sequence ρ_n of endomorphisms on a C^*-algebra A is said to be *asymptotically central* if $[\rho_n(x), y] \to 0$ (in norm) as $n \to \infty$ for all x, y in A.)

Proof. As in the proof of [12, Lemma 1], if $\{v_n\}$ is a sequence of unitaries in $\mathcal{O}_2$, then $\{\lambda_{v_n}\}$ is asymptotically central if and only if

$$\varphi(v_n)^* v_n \to \sum_{i,j=1}^{2} S_i S_j S_i^* S_j^*.$$

The unitary on the right-hand side belongs to $\mathcal{F}_2$. The existence of the desired sequence $\{v_n\}$ of unitaries in $\mathcal{F}_2$ therefore follows from Theorem 3.1 with $D = \mathcal{O}_2$, $\eta = \mathrm{Id}$, and with $D_0 = \mathcal{F}_2$. □

Proposition 3.3. *In $\mathcal{O}_2 \otimes \mathcal{O}_2$ consider the unitary element*

$$u_0 = S_1^* \otimes S_1 + S_2^* \otimes S_2,$$

and let B be the C^*-algebra generated by $\mathcal{F}_2 \otimes \mathcal{F}_2 \cup \{u_0\}$. Then

$$B = C^*(\mathcal{F}_2 \otimes \mathcal{F}_2, u_0) \cong \Big(\bigotimes_{n\in\mathbb{Z}} M_2\Big) \rtimes_{\mathrm{shift}} \mathbb{Z},$$

whence B is a simple $A\mathbb{T}$-algebra of real rank zero. In particular, B is K_1-injective and has finite exponential rank. Also, B is invariant under the endomorphism $\varphi \otimes \mathrm{Id}$ on $\mathcal{O}_2 \otimes \mathcal{O}_2$ (where φ is as defined in (2)*)*

The "half flip" on $\mathcal{O}_2 \otimes \mathcal{O}_2$ is approximately inner with unitaries belonging to B, i.e., there is a sequence $\{z_n\}$ of unitaries in B such that $z_n(x \otimes 1)z_n^* \to 1 \otimes x$ for all $x \in \mathcal{O}_2$.

Proof. With φ as above, put

$$E_{ij}^{(n)} = \begin{cases} \varphi^{(-n)}(S_i S_j^*) \otimes 1, & n \le 0, \\ 1 \otimes \varphi^{n-1}(S_i S_j^*), & n \ge 1, \end{cases}$$

for $i, j = 1, 2$ and for $n \in \mathbb{Z}$. Then $C^*\big(E_{ij}^{(n)} \mid i, j = 1, 2\big)$, $n \in \mathbb{Z}$, is a commuting family of C^*-algebras each isomorphic to M_2, and

$$\mathcal{F}_2 \otimes \mathcal{F}_2 = C^*\big(E_{ij}^{(n)} \mid i, j = 1, 2, n \in \mathbb{Z}\big).$$

Moreover, $u_0 E_{ij}^{(n)} u_0^* = E_{ij}^{(n+1)}$ for all $i, j = 1, 2$ and for all $n \in \mathbb{Z}$. This proves that

$$C^*(\mathcal{F}_2 \otimes \mathcal{F}_2, u_0) \cong \Big(\bigotimes_{n\in\mathbb{Z}} M_2\Big) \rtimes_{\text{shift}} \mathbb{Z}.$$

(Off hand, without going into issues of proper outerness of the shift action, one would only get that the C^*-algebra on the left-hand side is a quotient of the crossed product on the right-hand side, but since the latter is simple, we get the isomorphism for free.)

The crossed product on the right-hand side is known to be a simple $A\mathbb{T}$-algebra of real rank zero, [1]. We must show that $C^*(\mathcal{F}_2 \otimes \mathcal{F}_2, u_0)$ is invariant under $\kappa := \varphi \otimes \mathrm{Id}$. It is clear that $\mathcal{F}_2 \otimes \mathcal{F}_2$ is invariant under κ. A brief calculation shows that $\kappa(u_0) = zu_0$, where

$$z = S_1S_1^* \otimes S_1S_1^* + S_1S_2^* \otimes S_2S_1^* + S_2S_1^* \otimes S_1S_2^* + S_2S_2^* \otimes S_2S_2^*.$$

Notice that z belongs to $\mathcal{F}_2 \otimes \mathcal{F}_2$. Therefore $\kappa(u_0)$ belongs to $C^*(\mathcal{F}_2 \otimes \mathcal{F}_2, u_0)$. As $\kappa^n(u_0) = \kappa^{n-1}(z)\kappa^{n-1}(u_0)$ it follows by induction that $\kappa^n(u_0)$ belongs to $C^*(\mathcal{F}_2 \otimes \mathcal{F}_2, u_0)$ for all n. This proves that $C^*(\mathcal{F}_2 \otimes \mathcal{F}_2, u_0)$ is invariant under κ.

It follows from Theorem 3.1, with $D = \mathcal{O}_2 \otimes \mathcal{O}_2$, with $D_0 = B$, and with $\eta \colon \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ given by $\eta(x) = x \otimes 1$, that there is a sequence $\{z_n\}$ of unitaries in B such that $z_n\kappa(z_n)^* \to u_0$. It is straightforward to check that $1 \otimes S_j = u_0(S_j \otimes 1)$ and a standard calculation, cf. [11], then shows that $z_n(S_j \otimes 1)z_n^* \to 1 \otimes S_j$ for $j = 1, 2$. □

Corollary 3.4. *Let D be a unital C^*-algebra, and suppose that $\eta_1, \eta_2 \colon \mathcal{O}_2 \to D$ are unital $*$-homomorphisms with commuting images. There is a sequence $\{w_n\}$ of unitaries in the sub-C^*-algebra $D_0 = C^*(\eta_1(\mathcal{F}_2), u)$, where*

$$u = \eta_2(S_1)\eta_1(S_1)^* + \eta_2(S_2)\eta_1(S_2)^*,$$

such that $w_n\eta_1(x)w_n^* \to \eta_2(x)$ for all $x \in \mathcal{O}_2$.

Proof. The $*$-homomorphisms η_1 and η_2 induce a $*$-homomorphism $\eta \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to D$ given by

$$\eta(x \otimes y) = \eta_1(x)\eta_2(y), \qquad x, y \in \mathcal{O}_2.$$

In the notation of Proposition 3.3 we have

$$\eta(u_0) = u, \qquad \eta(\mathcal{F}_2 \otimes 1) = \eta_1(\mathcal{F}_2), \qquad \eta(1 \otimes \mathcal{F}_2) = \eta_2(\mathcal{F}_2).$$

It follows from Proposition 3.3 and its proof that $1 \otimes \mathcal{F}_2$ is contained in the C^*-algebra generated by $\{E_{ij}^{(0)}\}$ and u_0 and hence is contained in $C^*(\mathcal{F}_2 \otimes 1, u_0)$. The C^*-algebra B from that proposition is therefore generated by $\mathcal{F}_2 \otimes 1$ and u_0, which shows that $\eta(B) = D_0$.

Let $\{z_n\}$ be as in Proposition 3.3 and put $w_n = \eta(z_n) \in D_0$. Then

$$w_n \eta_1(x) w_n^* = \eta(z_n(x \otimes 1)z_n^*) \to \eta(1 \otimes x) = \eta_2(x)$$

for all $x \in \mathcal{O}_2$. □

Proposition 3.5. *There are sequences $\{v_n\}$ and $\{w_n\}$ of unitaries in $\mathcal{F}_2$ such that*

(i) *$\{\lambda_{v_n}\}$ is asymptotically central in $\mathcal{O}_2$,*

(ii) *$\|w_n \lambda_{v_{n+1}}(S_j) w_n^* - \lambda_{v_n}(S_j)\| < 2^{-n}$ for all $n \in \mathbb{N}$ and for $j = 1, 2$,*

(iii) *$\|w_n S_j w_n^* - S_j\| < 2^{-n}$ for all $n \in \mathbb{N}$ and for $j = 1, 2$.*

Proof. Let $\{v_n\}$ be as in Proposition 3.2. Then $\{v_n\}$ and any subsequence thereof will satisfy (i). Upon passing to a subsequence we can assume that

$$\|\lambda_{v_m}(S_i)\lambda_{v_n}(S_j) - \lambda_{v_n}(S_j)\lambda_{v_m}(S_i)\| < 1/n, \tag{3}$$

for all $m > n \geq 1$ and for all $i, j = 1, 2$. We claim that one can find a sequence $\{w_n\}$ of unitaries in $\mathcal{F}_2$ satisfying (ii) and (iii) above—provided that we again pass to a subsequence of $\{v_n\}$. It suffices to show that for each $\delta > 0$ there exists a natural number n such that for each natural number $m > n$ there is a unitary $w \in \mathcal{F}_2$ for which

$$\|w\lambda_{v_m}(S_j)w^* - \lambda_{v_n}(S_j)\| < \delta, \qquad \|wS_jw^* - S_j\| < \delta$$

for $j = 1, 2$. We give an indirect proof of the latter statement. If it were false, then there would exist $\delta > 0$ and a sequence $1 \leq n_1 < n_2 < n_3 < \cdots$ such that one of

$$\|w\lambda_{v_{n_{k+1}}}(S_i)w^* - \lambda_{v_{n_k}}(S_i)\|, \qquad \|wS_iw^* - S_i\|,$$

$i = 1, 2$, is greater than δ for every k and for all unitaries w in $\mathcal{F}_2$. We proceed to show that this will lead to a contradiction.

Choose a free ultrafilter ω on $\mathbb{N}$ and consider the relative commutant $\mathcal{O}_2' \cap (\mathcal{O}_2)_\omega$ inside the ultrapower $(\mathcal{O}_2)_\omega$. This C^*-algebra is purely infinite and simple (see [7, Proposition 3.4]). Consider the unital $*$-homomorphisms $\eta_1, \eta_2 \colon \mathcal{O}_2 \to \mathcal{O}_2' \cap (\mathcal{O}_2)_\omega$ given by

$$\eta_1(x) = \pi_\omega\big(\lambda_{v_{n_2}}(x), \lambda_{v_{n_3}}(x), \lambda_{v_{n_4}}(x), \dots\big),$$
$$\eta_2(x) = \pi_\omega\big(\lambda_{v_{n_1}}(x), \lambda_{v_{n_2}}(x), \lambda_{v_{n_3}}(x), \dots\big),$$

$x \in \mathcal{O}_2$, where $\pi_\omega \colon \ell^\infty(\mathcal{O}_2) \to (\mathcal{O}_2)_\omega$ is the quotient mapping. The images of η_1 and η_2 commute by (3). Put

$$u = \eta_2(S_1)\eta_1(S_1)^* + \eta_2(S_2)\eta_1(S_2)^* = \pi_\omega(v_{n_1}v_{n_2}^*, v_{n_2}v_{n_3}^*, v_{n_3}v_{n_4}^*, \dots),$$

and notice that u is a unitary element in $\mathcal{O}_2' \cap (\mathcal{F}_2)_\omega \subseteq \mathcal{O}_2' \cap (\mathcal{O}_2)_\omega$. Use Corollary 3.4 to obtain a sequence $\{w_n\}$ of unitaries in $C^*(\eta_1(\mathcal{F}_2), u) \subseteq \mathcal{O}_2' \cap (\mathcal{F}_2)_\omega$ such that $w_n \eta_1(S_j) w_n^* \to \eta_2(S_j)$ for $j = 1, 2$. By [8, Lemma 2.5] there is a single unitary w in $\mathcal{O}_2' \cap (\mathcal{F}_2)_\omega$ such that $w\eta_1(S_j)w^* = \eta_2(S_j)$ for $j = 1, 2$ (and hence such that $w\eta_1(x)w^* = \eta_2(x)$ for all $x \in \mathcal{O}_2$).

Each unitary element in the ultrapower $(\mathcal{F}_2)_\omega$ lifts to a unitary element in $\ell^\infty(\mathcal{F}_2)$, so we can write

$$w = \pi_\omega(w_1, w_2, w_3, \dots),$$

where each w_n is a unitary element in $\mathcal{F}_2$. This establishes the desired contradiction, as

$$\lim_{n\to\omega} \|S_j w_n - w_n S_j\| = 0, \qquad \lim_{n\to\omega} \|w_n \lambda_{v_{n_{k+1}}}(S_j) w_n^* - \lambda_{v_{n_k}}(S_j)\| = 0,$$

for $j = 1, 2$ and for all k. □

Theorem 3.6. *There is a unitary element $u \in \mathcal{F}_2$ such that the relative commutant $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$ contains a unital copy of $\mathcal{O}_2$.*

Proof. Let $\{v_n\}$ and $\{w_n\}$ be as in Proposition 3.5 and define endomorphisms on $\mathcal{O}_2$ by

$$\lambda_n(x) = w_1 w_2 \cdots w_n \lambda_{v_{n+1}}(x) w_n^* \cdots w_2^* w_1^*, \quad \rho_n(x) = w_1 w_2 \cdots w_n x w_n^* \cdots w_2^* w_1^*,$$

for $x \in \mathcal{O}_2$. Then

$$\|\lambda_n(S_j) - \lambda_{n-1}(S_j)\| < 2^{-n}, \qquad \|\rho_n(S_j) - \rho_{n-1}(S_j)\| < 2^{-n}$$

for $j = 1, 2$, and $\lambda_n(x)\rho_n(y) - \rho_n(y)\lambda_n(x) \to 0$ for all $x, y \in \mathcal{O}_2$. Using that

$$w\lambda_u(x)w^* = \lambda_{wu\varphi(w)^*}(x)$$

whenever w is a unitary in $\mathcal{O}_2$ and $x \in \mathcal{O}_2$, we see that $\lambda_n = \lambda_{u_n}$ for some unitary u_n in $\mathcal{F}_2$. It follows from the estimates above that the sequences $\{\lambda_n(S_j)\}$ and $\{\rho_n(S_j)\}$, $j = 1, 2$, and hence also the sequence $\{u_n\}$, are Cauchy and therefore convergent. Let $\lambda \colon \mathcal{O}_2 \to \mathcal{O}_2$ and $\rho \colon \mathcal{O}_2 \to \mathcal{O}_2$ be the (pointwise-norm) limits of the sequences $\{\lambda_n\}$ and $\{\rho_n\}$, respectively, and let $u \in \mathcal{F}_2$ be the limit of the sequence $\{u_n\}$. Then $\lambda = \lambda_u$ and the images of λ and ρ commute. □

Corollary 3.7. *There is a unitary $v \in \mathcal{O}_2$ such that $\lambda_v(\mathcal{F}_2) \subseteq \mathcal{F}_2$ but $v \notin \mathcal{F}_2$.*

Proof. Let $u \in \mathcal{O}_2$ be as in Theorem 3.6 and take a unitary element z in $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$ that does not belong to $\mathcal{F}_2$. Put $v = zu$. Then v does not belong to $\mathcal{F}_2$, and λ_u and λ_v coincide on $\mathcal{F}_2$ by Proposition 2.1, whence λ_v maps $\mathcal{F}_2$ into itself. □

Corollary 3.8. *There is a unital $*$-homomorphism $\sigma \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ such that $\sigma(\mathcal{F}_2 \otimes \mathcal{F}_2) \subseteq \mathcal{F}_2$.*

Proof. Take $\lambda \colon \mathcal{O}_2 \to \mathcal{O}_2$ and $\rho \colon \mathcal{O}_2 \to \mathcal{O}_2$ as in the proof of Theorem 3.6. Recall that λ and ρ have commuting images and that $\lambda(\mathcal{F}_2) \subseteq \mathcal{F}_2$ and $\rho(\mathcal{F}_2) \subseteq \mathcal{F}_2$. We can therefore define a $*$-homomorphism $\sigma \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ by

$$\sigma(x \otimes y) = \lambda(x)\rho(y), \qquad x, y \in \mathcal{O}_2.$$

Then

$$\sigma(\mathcal{F}_2 \otimes \mathcal{F}_2) = \lambda(\mathcal{F}_2)\rho(\mathcal{F}_2) \subseteq \mathcal{F}_2.$$

□

We know that $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $\mathcal{O}_2$ are isomorphic, but we do not know if one can find an *isomorphism* $\sigma \colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ such that $\sigma(\mathcal{F}_2 \otimes \mathcal{F}_2)$ is contained in (or better, equal to) $\mathcal{F}_2$.

4 Endomorphisms preserving the canonical UHF-subalgebra

Below, ϕ denotes the standard left inverse of φ, i.e., the unital, completely positive map given by $\phi(x) := \frac{1}{n} \sum S_i^* x S_i$, $x \in \mathcal{O}_n$.

Lemma 4.1. *Let $u \in \mathcal{U}(\mathcal{O}_n)$, then the following conditions are equivalent:*

(i) $\phi(u) \in \mathcal{U}(\mathcal{O}_n)$;

(ii) $u \in \varphi(\mathcal{O}_n)$;

(iii) $S_i^* u S_i = S_j^* u S_j \in \mathcal{U}(\mathcal{O}_n)$, *for all* $i, j \in \{1, \dots, n\}$.

Proof. (i) $\Rightarrow$ (ii): it follows from (i) that u lies in the multiplicative domain of ϕ and therefore, by Choi's theorem (see e.g. [14, 1.5.6]), $\phi(S_i u) = \phi(S_i)\phi(u)$, that is $uS_i = S_i\phi(u)$ for all $i = 1, \dots, n$. Thus, $u = \varphi(\phi(u))$.
The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are obvious. □

Lemma 4.2. *For $v, w \in \mathcal{U}(\mathcal{O}_n)$ the following three conditions are equivalent.*

(i) *Endomorphisms λ_v and λ_w coincide on $\mathcal{F}_n$.*

(ii) *For each $k \geq 1$ we have $w_k^* v_k \in \varphi^k(\mathcal{O}_n)$.*

(iii) *There exists a sequence of unitaries $z_k \in \mathcal{U}(\mathcal{O}_n)$ such that $z_1 = \phi(w^* v)$ and $z_{k+1} = \phi(w^* z_k v)$ for all $k \geq 1$.*

Proof. The endomorphisms λ_v and λ_w coincide on $\mathcal{F}_n$ if and only if they coincide on each $\mathcal{F}_n^k$. Now if α and β are two multi-indices of length k then $\lambda_v(S_\alpha S_\beta^*) = v_k S_\alpha S_\beta^* v_k^*$ and $\lambda_w(S_\alpha S_\beta^*) = w_k S_\alpha S_\beta^* w_k^*$. Thus $\lambda_v(S_\alpha S_\beta^*) = \lambda_w(S_\alpha S_\beta^*)$ for all such α, β if and only if $w_k^* v_k$ is in the commutant of $\mathcal{F}_n^k$, that is when $w_k^* v_k \in \varphi^k(\mathcal{O}_n)$. Now it easily follows from Lemma 4.1 that this holds for all k if and only if condition (iii) above is satisfied. □

Proposition 4.3. *If $w \in \mathcal{U}(\mathcal{O}_n)$ then $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ if and only if λ_w and $\lambda_{\alpha_t(w)}$ coincide on $\mathcal{F}_n$ for all $t \in \mathbb{R}$. This in turn takes place if and only if one can inductively define unitaries $z_t^{(k)}$, $k \geq 1$, $t \in \mathbb{R}$ by*

$$\varphi(z_t^{(1)}) = w^* \alpha_t(w), \quad (\mathrm{Ad}w \circ \varphi)(z_t^{(k+1)}) = z_t^{(k)}.$$

Moreover, in that case $t \mapsto z_t^{(1)}$ ($t \in \mathbb{R}$) is a unitary α-cocycle in $\lambda_w(\mathcal{F}_n)' \cap \mathcal{O}_n$. Finally, if $\lambda_w(\mathcal{F}_n) \subset \mathcal{F}_n$ then $w \in \mathcal{F}_n$ if and only if λ_w and $\lambda_{\alpha_t(w)}$ have the same range for all $t \in \mathbb{R}$.

Proof. Given a unitary w in $\mathcal{O}_n$ one has, by a direct computation,

$$\lambda_{\alpha_t(w)} = \alpha_t \circ \lambda_w \circ \alpha_t^{-1} \ ,$$

for all $t \in \mathbb{R}$. Since $\mathcal{F}_n$ is precisely the fixed point algebra under the gauge action, the first claim is now clear. The second equivalence in terms of the existence of the unitaries $z_t^{(k)}$ is then deduced from Lemma 4.2 (see also Remark 4.4, below). Now notice that if such unitaries exist one has, for any $s, t \in \mathbb{R}$,

$$\begin{aligned}\varphi(z_{t+s}^{(1)}) &= w^*\alpha_{t+s}(w) = w^*\alpha_t(\alpha_s(w)) = w^*\alpha_t(ww^*\alpha_s(w)) \\ &= w^*\alpha_t(w)\alpha_t(w^*\alpha_s(w)) = \varphi(z_t^{(1)})\alpha_t(\varphi(z_s^{(1)})) \ ,\end{aligned}$$

from which the cocycle equation for $z^{(1)}$ follows immediately, since α and φ commute. Moreover, $z_t^{(1)} \in \lambda_w(\mathcal{F}_n)' \cap \mathcal{O}_n$ for all $t \in \mathbb{R}$ by identity (1).

Finally, suppose that $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and $\lambda_w(\mathcal{O}_n) = \lambda_{\alpha_t(w)}(\mathcal{O}_n)$. Then, for each t, define a map β_t from $\mathcal{O}_n$ into itself via $\lambda_{\alpha_t(w)}(x) = \lambda_w(\beta_t(x)), x \in \mathcal{O}_n$. It must necessarily be that $\beta_t \in \mathrm{Aut}_{\mathcal{F}_n}(\mathcal{O}_n)$ and the argument in the proof of [6, Proposition 2.1(b)] goes through. □

In particular, if $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ or, more generally, $\lambda_w(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$ then $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ if and only if $w \in \mathcal{F}_n$.

Remark 4.4. Existence of unitaries $z_t^{(k)}$, as defined in Proposition 4.3, is easily seen to be equivalent to existence of unitaries $\tilde{z}_t^{(k)}$, $k \geq 1$, $t \in \mathbb{R}$, defined inductively by

$$\varphi(\tilde{z}_t^{(1)}) = w^*\alpha_t(w), \quad \varphi(\tilde{z}_t^{(k+1)}) = w^*\tilde{z}_t^{(k)}\alpha_t(w).$$

Proposition 4.5. *Let $w \in \mathcal{U}(\mathcal{O}_n)$ be such that $\lambda_w(\mathcal{F}_n^1) \subseteq \mathcal{F}_n$. Then the unitary α-cocycle $z_t^{(1)} := \phi(w^*\alpha_t(w))$ is a coboundary, i.e. there exists a unitary z such that $z_t^{(1)} = z\alpha_t(z^*)$ for all $t \in \mathbb{R}$.*

Proof. Indeed, since $\lambda_w(\mathcal{F}_n^1) \subseteq \mathcal{F}_n$ there exists a unitary $u \in \mathcal{F}_n$ such that λ_w and λ_u coincide on $\mathcal{F}_n^1$. In fact, we could take as λ_u an inner automorphism implemented by a unitary in $\mathcal{F}_n$. Then w^*u commutes with $\mathcal{F}_n^1$, and thus there exists a unitary z such that $w^*u = \varphi(z)$. Now we have $\varphi(z\alpha_t(z^*)) = w^*u\alpha_t(u^*)\alpha_t(w) = w^*\alpha_t(w)$, since $\alpha_t(u^*) = u^*$. □

Proposition 4.6. *If w is a unitary in $\mathcal{O}_n$ such that $w\mathcal{D}_n w^* \subseteq \mathcal{F}_n$ then $w^*\alpha_t(w) \in \mathcal{D}_n$ for all $t \in \mathbb{R}$. If, in addition, $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ then $t \mapsto z_t^{(1)}$ is a one-parameter unitary group in $\lambda_w(\mathcal{F}_n)' \cap \mathcal{D}_n$.*

Proof. By assumption, for any $x \in \mathcal{D}_n$ one has $wxw^* = \alpha_t(wxw^*)$ for all $t \in \mathbb{R}$. Therefore, $\mathcal{D}_n$ being a MASA in $\mathcal{O}_n$ (see the introduction to Section 2), $w^*\alpha_t(w) \in \mathcal{D}_n' \cap \mathcal{O}_n = \mathcal{D}_n$. Now, notice that $\mathcal{D}_n \cap \varphi(\mathcal{O}_n) = \varphi(\mathcal{D}_n)$, so that indeed if $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ the cocycle given by Proposition 4.3 lies in $\mathcal{D}_n \subseteq \mathcal{F}_n$, and the conclusion follows at once from the cocycle equation and Proposition 4.3. □

Of course, the first part of the preceding proposition applies to all elements of the group $\mathcal{S}_n$, as they normalize $\mathcal{D}_n$.

The following result is a slight reformulation of Proposition 2.1, enhanced for our needs, put in a more symmetric form and taking also into account Proposition 2.3 and Lemma 2.4.

Proposition 4.7. *Let u and w be two unitaries in $\mathcal{O}_n$. If λ_u and λ_w coincide on $\mathcal{F}_n$ then, for every nonnegative integer h,*

$$(\mathrm{Ad}u \circ \varphi)^h(wu^*) \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n \ .$$

Conversely, if $wu^* \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ then $\lambda_u(x) = \lambda_w(x)$ for any $x \in \mathcal{F}_n$.

Proof. Concerning the first implication, by the above it clearly suffices to show only the case $h = 0$. Indeed, for every $k \geq 1$ one has

$$\begin{aligned} wu^*\lambda_u(S_{\alpha_1}\ldots S_{\alpha_k}S_{\beta_k}^*\cdots S_{\beta_1}^*)uw^* &= wu^*uS_{\alpha_1}\ldots uS_{\alpha_k}S_{\beta_k}^*u^*\cdots S_{\beta_1}^*u^*uw^* \\ &= wS_{\alpha_1}uS_{\alpha_2}\ldots uS_{\alpha_k}S_{\beta_k}^*u^*\cdots S_{\beta_2}^*u^*S_{\beta_1}^*w^* \\ &= wS_{\alpha_1}\lambda_u(S_{\alpha_2}\ldots S_{\alpha_k}S_{\beta_k}^*\ldots S_{\beta_2}^*)S_{\beta_1}^*w^* \\ &= wS_{\alpha_1}\lambda_w(S_{\alpha_2}\ldots S_{\alpha_k}S_{\beta_k}^*\ldots S_{\beta_2}^*)S_{\beta_1}^*w^* \\ &= \lambda_w(S_{\alpha_1}\ldots S_{\alpha_k}S_{\beta_k}^*\cdots S_{\beta_1}^*) \\ &= \lambda_u(S_{\alpha_1}\ldots S_{\alpha_k}S_{\beta_k}^*\cdots S_{\beta_1}^*) \ . \end{aligned}$$

The opposite implication can be easily checked by induction on k, just repeating the argument in Proposition 2.1 after noticing that if $wu^* \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ then, by Lemma 2.4, $wu^* = u\varphi(z)u^*$ for some unitary $z \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, that is $w = u\varphi(z)$. □

In particular, it follows that if $w \in \mathcal{U}(\mathcal{O}_n) \setminus \mathcal{F}_n$ and there exists some $u \in \mathcal{U}(\mathcal{F}_n)$ such that λ_w and λ_u coincide on $\mathcal{F}_n$ then w must necessarily be of the form $w = u\varphi(z)$ for some $z \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$, which is exactly the situation discussed in section 2.

Corollary 4.8. *Let w be a unitary in $\mathcal{O}_n$ and suppose that $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$. Then $w \in \mathcal{F}_n$ and $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$.*

Proof. By Proposition 4.3, λ_w and $\lambda_{\alpha_t(w)}$ coincide on $\mathcal{F}_n$. Therefore, by Proposition 4.7 one has $w\alpha_t(w^*) \in \mathcal{U}(\mathcal{F}_n' \cap \mathcal{O}_n) = \mathbb{T}$ and thus w is an eigenvector for α. Hence w belongs to $\mathcal{F}_n$ and the conclusion follows from [2, Proposition 1.1 (a)]. □

Combining [2, Proposition 1.1 (a)], Proposition 4.3 and Corollary 4.8, we obtain the following.

Corollary 4.9. *For a unitary $w \in \mathcal{O}_n$, the following three conditions are equivalent:*

(i) $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$;

(ii) $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ *and* $w \in \mathcal{F}_n$;

(iii) $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ *and* $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$.

Corollary 4.10. *Assume that λ_w is an endomorphism of $\mathcal{O}_n$ that restricts to the identity on $\mathcal{F}_n$. Then λ_w is a gauge automorphism.*

Proof. By Corollary 4.8, $\lambda_w \in \mathrm{Aut}_{\mathcal{F}_n}(\mathcal{O}_n) = \{\alpha_t : t \in \mathbb{R}\}$. □

Corollary 4.11. *Let w be a unitary in $\mathcal{O}_n$ such that $w^*\mathcal{D}_n w \subseteq \mathcal{F}_n$. If $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ then $w\alpha_t(w^*) \in \lambda_w(\mathcal{F}_n)' \cap \mathcal{D}_n$ so that, in particular, $w \in \mathcal{F}_n$ whenever λ_w is irreducible in restriction to $\mathcal{F}_n$.*

Proof. This readily follows from Propositions 4.3, 4.6 and 4.7. □

Corollary 4.12. *Let $w \in \mathcal{S}_n$ be such that $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ or, more generally, such that $\mathcal{D}_n \subseteq \lambda_w(\mathcal{F}_n)$. Then $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ if and only if $w \in \mathcal{P}_n$.*

Proof. An element of $\mathcal{S}_n$ normalizes $\mathcal{D}_n$ and thus satisfies the first assumption in the previous corollary. Then the only nontrivial assertion follows from the fact that an endomorphism λ_w of $\mathcal{O}_n$ such that $\lambda_w(\mathcal{F}_n) \supseteq \mathcal{D}_n$ is necessarily irreducible in restriction to $\mathcal{F}_n$ by an argument similar to the one in [2, Proposition 1.1], using the facts that $\mathcal{D}_n$ is a MASA in $\mathcal{F}_n$ and $\mathcal{F}_n$ is simple. □

Example 4.13. In order to provide a simple example, we consider the following situation. Let $w' \in \mathcal{P}_n$ be such that $\lambda_{w'}(\mathcal{D}_n) = \mathcal{D}_n$ but $\lambda_{w'} \notin \mathrm{Aut}(\mathcal{O}_n)$ (many examples of such permutation unitaries were provided in [4]). Let $w'' \in \mathcal{S}_n \setminus \mathcal{P}_n$ be such that $\lambda_{w''} \in \mathrm{Aut}(\mathcal{O}_n)$ (e.g., an inner one), so that $\lambda_{w''}(\mathcal{D}_n) = \mathcal{D}_n$. Set $\lambda_w := \lambda_{w'}\lambda_{w''}$, then $w = \lambda_{w'}(w'')w' \in \mathcal{S}_n \setminus \mathcal{P}_n$ and $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$. Such λ_w is irreducible on $\mathcal{O}_n$ and $\lambda_w(\mathcal{F}_n) \nsubseteq \mathcal{F}_n$.

Remark 4.14. In view of the above, it would also be very useful to have a general criterion for $w \in \mathcal{S}_n$ to satisfy

(i) $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$, or

(ii) $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$.

Such criteria for $w \in \mathcal{P}_n$ were given in [4, 13].

5 Analysis of cocycles for unitaries in $\mathcal{S}_n$

Let $w \in \mathcal{S}_n$. Then w is of the form

$$w = \sum_{(\alpha,\beta)} S_\alpha S_\beta^* \ ,$$

where the sum runs over a certain family $\mathcal{J}$ of pairs of multi-indices (α, β). For convenience, we also introduce a set $\mathcal{J}_2 := \{\beta \mid (\alpha, \beta) \in \mathcal{J}\}$, which is in bijective correspondence with $\mathcal{J}$ via $(\alpha, \beta) \leftrightarrow \beta$. The fact that w as above is unitary is equivalent to that both collections of the P_α's and of the P_β's form partitions of unity, i.e.

$$\sum_{(\alpha,\beta)} P_\alpha = \sum_{(\alpha,\beta)} P_\beta = 1 \ .$$

Then, for each $i = 1, \ldots, n$, one has $S_i^* \sum P_\alpha S_i = 1$ and similarly for the P_β's and therefore, after summing over all i's,

$$\sum_{i=1}^n S_i^* \Big(\sum_{(\alpha,\beta)} P_\alpha \Big) S_i = \sum_{i=1}^n S_i^* \Big(\sum_{(\alpha,\beta)} P_\beta \Big) S_i = n1 \ .$$

Consequently, denoting by $\tilde{\alpha}$ (resp. $\tilde{\beta}$) the multi-index obtained from α (resp. β) after deleting the first entry, we have

$$\sum_{(\alpha,\beta)} P_{\tilde{\alpha}} = \sum_{(\alpha,\beta)} P_{\tilde{\beta}} = n1 \ .$$

In other words, both collections of projections $\{P_{\tilde{\alpha}}\}$ and $\{P_{\tilde{\beta}}\}$ form an n-covering of unity.

In the sequel, we repeatedly make use of Proposition 4.3 without further mention. We compute for $t \in \mathbb{R}$

$$\begin{aligned} w^*\alpha_t(w) &= \Big(\sum_{(\alpha,\beta)} S_\beta S_\alpha^*\Big)\Big(\sum_{(\alpha',\beta')} e^{it(|\alpha'|-|\beta'|)} S_{\alpha'} S_{\beta'}^*\Big) \\ &= \sum_{(\alpha,\beta)} e^{it(|\alpha|-|\beta|)} P_\beta \in \mathcal{U}(\mathcal{D}_n) \end{aligned}$$

by orthogonality of the ranges of S_α's. Throughout the reminder of this section, we assume that $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$. Therefore, $w^*\alpha_t(w)$ must be a unitary in $\mathcal{D}_n \cap \varphi(\mathcal{O}_n) = \varphi(\mathcal{D}_n)$ and hence $z_t^{(1)} := \phi(w^*\alpha_t(w))$ must be a unitary in $\mathcal{D}_n$. We have

$$z_t^{(1)} = \frac{1}{n}\sum_{i=1}^n S_i^*\Big(\sum_{(\alpha,\beta)} e^{it(|\alpha|-|\beta|)} P_\beta\Big) S_i = \frac{1}{n}\sum_{(\alpha,\beta)} e^{it(|\alpha|-|\beta|)} P_{\tilde{\beta}} \ .$$

The last expression turns out to be unitary precisely when $|\alpha|-|\beta|$ is constant over the classes of $\mathcal{J}_2$ with respect to the equivalence relation "generated by nontrivial overlaps of the $P_{\tilde{\beta}}$'s". Namely, for $\beta, \beta' \in \mathcal{J}_2$, define

$$\beta \sim \beta' \Leftrightarrow \exists \beta_1 = \beta, \ldots, \beta_r = \beta' \in \mathcal{J}_2, \ \ P_{\tilde{\beta}_s} P_{\tilde{\beta}_{s+1}} \neq 0, \ \forall s = 1, \ldots, r-1 \ .$$

Thus, $z_t^{(1)}$ is unitary if and only if the function $\psi_1 \colon \mathcal{J}_2 \to \mathbb{Z}$ such that $\psi_1(\beta) = |\alpha| - |\beta|$ is constant on the equivalence classes of relation $\sim$. Unfortunately, such combinatorial analysis of "higher cocycles" $z_t^{(k)}$ quickly becomes rather cumbersome. Thus from now on we make a simplifying assumption that for all $(\alpha, \beta) \in \mathcal{J}$ we have $|\alpha| - |\beta| \in \{-1, 0, +1\}$.

Proposition 5.1. *Let* $w = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^* \in \mathcal{S}_n$ *be such that* $|\alpha| - |\beta| \in \{-1, 0, +1\}$ *for all* $(\alpha, \beta) \in \mathcal{J}$*. Then* $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ *if and only if there exists a sequence of functions* $\psi_k \colon \mathcal{J}_2 \to \mathbb{Z}$, $k = 1, 2, \ldots$, *such that*

(1) ψ_k is constant on the equivalence classes of relation $\sim$,

(2) $\psi_1(\beta) = |\alpha| - |\beta|$ and $\psi_{k+1}(\beta) = \psi_k(\beta')$, where $(\alpha, \beta) \in \mathcal{J}$ and β' is any element of $\mathcal{J}_2$ such that $\widetilde{\beta'}$ is an initial segment of α.

If such functions exist then

$$z_t^{(k)} = \frac{1}{n} \sum_{(\alpha,\beta)\in\mathcal{J}} e^{it\psi_k(\beta)} P_{\tilde{\beta}} \tag{4}$$

are unitary for all $k = 1, 2, \dots$

Proof. By Proposition 4.3, $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ if and only if all "higher cocycles" $z_t^{(k)}$, $k = 1, 2, \dots$, are unitary. We show by induction on k then under our hypothesis on w there exist functions ψ_k, $k = 1, \dots, m$ satisfying conditions (1) and (2) above if and only if cocycles $z_t^{(k)}$, $k = 1, \dots, m$ are unitary and given by formula (4). Case $k = 1$ is established just above this lemma. So assume the inductive hypothesis holds for k. Then a direct calculation yields

$$z_t^{(k+1)} = \phi(w^* z_t^{(k)} w) = \frac{1}{n} \sum_{(\alpha,\beta)\in\mathcal{J}} e^{it\psi_k(\beta')} P_{\tilde{\beta}}, \tag{5}$$

where β' is an element of $\mathcal{J}_2$ such that $\widetilde{\beta'}$ is an initial segment of α. Note that for another such element β'' we have $\psi_k(\beta') = \psi_k(\beta'')$, since function ψ_k is constant on equivalence classes of relation $\sim$. Thus we can define $\psi_{k+1}(\beta) = \psi_k(\beta')$. Formula (5) yields a unitary if and only if function ψ_{k+1} is constant on equivalence classes of $\sim$. This ends the proof of the inductive step and the lemma. □

The conditions of Proposition 5.1 can be given the following graphical interpretation. Let $w = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^* \in \mathcal{S}_n$ be such that $|\alpha| - |\beta| \in \{-1, 0, +1\}$ for all $(\alpha, \beta) \in \mathcal{J}$. We associate with w a finite directed graph E_w as follows. Vertices of E_w are the equivalence classes of relation $\sim$. Given two vertices a_1, a_2, there is a single edge from a_1 to a_2 if and only if there exist $(\alpha, \beta) \in \mathcal{J}$ with β in the equivalence class a_1 and $\beta' \in \mathcal{J}_2$ in the equivalence class a_2 such that $\widetilde{\beta'}$ is an initial segment of α. We denote by E_w^k the collection of all directed paths in E_w of length k, and by $E_w^k(a)$ the collection of those such paths which begin at vertex a.

If the function ψ_1 (corresponding to w) is constant on the equivalence classes of $\sim$ then we can assign labels from $\{-1, 0, +1\}$ to vertices of E_w in such a way that the label of a is $\psi_1(\beta)$ for β in the equivalence class a. Now the remaining conditions of Lemma 5.1 are equivalent to the following *path condition*:

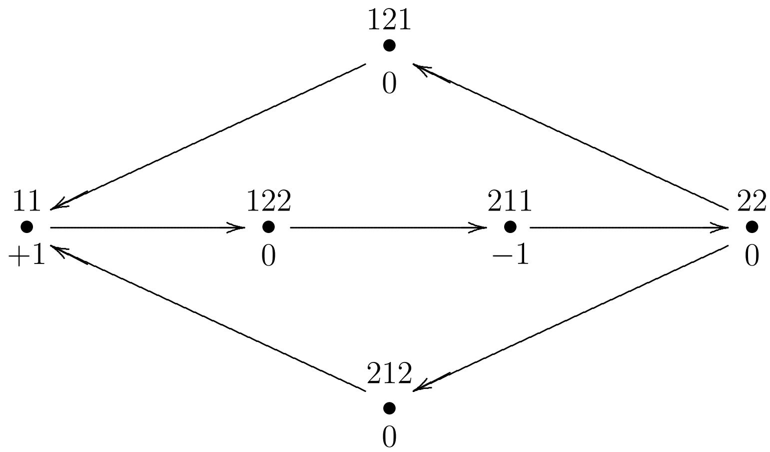
For each vertex a and for each $k \in \mathbb{N}$ the ranges of all directed paths in $E_w^k(a)$ have the same labels.

Since the graph E_w is finite, we obtain the following:

Corollary 5.2. *Let* $w = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^* \in \mathcal{S}_n$ *be such that* $|\alpha| - |\beta| \in \{-1, 0, +1\}$ *for all* $(\alpha, \beta) \in \mathcal{J}$*. Then there exists* $r \in \mathbb{N}$ *such that*

$$\lambda_w(\mathcal{F}_n^r) \subseteq \mathcal{F}_n \;\Rightarrow\; \lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n.$$

Example 5.3. Let $u \in \mathcal{P}_2^4$ and $v \in \mathcal{S}_2$ be as given in Example 2.5. Set $w = vu$. Then the corresponding graph E_w looks as follows.



We denote by A_w the smallest $(\mathrm{Ad}w \circ \varphi)$-invariant C^*-subalgebra of $\mathcal{D}_n$ that contains $\{z_t^{(1)} : t \in \mathbb{R}\}$.

Proposition 5.4. *Let* $w = \sum_{(\alpha,\beta)\in\mathcal{J}} S_\alpha S_\beta^* \in \mathcal{S}_n$ *be such that* $|\alpha| - |\beta| \in \{-1, 0, +1\}$ *for all* $(\alpha, \beta) \in \mathcal{J}$*. Then* $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ *if and only if* $z_t^{(1)}$ *is a unitary cocycle and the algebra* A_w *is finite dimensional. In that case,* A_w *is the* C^**-algebra generated by all cocycles* $\{z_t^{(k)} : k \in \mathbb{N}\}$*.*

Proof. If A_w is finite dimensional then $\mathrm{Ad}w \circ \varphi$ is its automorphism. If, in addition, $z_t^{(1)}$ is unitary then this immediately implies existence of unitary cocycles $z_t^{(k)}$ for all $k \in \mathbb{N}$.

Conversely, if $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ then the C^*-algebra generated by all cocycles $\{z_t^{(k)} : k \in \mathbb{N}\}$ is finite dimensional, since it is contained in $C^*(\{P_{\tilde{\beta}} : \beta \in J_2\})$. It follows that $\mathrm{Ad}w \circ \varphi$ is an automorphism of this algebra. □

References

[1] O. Bratteli, A. Kishimoto, M. Rørdam, and E. Størmer, *The crossed product of a UHF-algebra by a shift*, Ergod. Th. & Dynam. Sys. **13** (1993), 615–626.

[2] R. Conti, J. Kimberley and W. Szymański, *More localized automorphisms of the Cuntz algebras*, arXiv:0808.2843, to appear in Proc. Edinburgh Math. Soc.

[3] R. Conti and C. Pinzari, *Remarks on the index of endomorphisms of Cuntz algebras*, J. Funct. Anal. **142** (1996), 369–405.

[4] R. Conti and W. Szymański, *Labeled trees and localized automorphisms of the Cuntz algebras*, arXiv:0805.4654, to appear in Trans. Amer. Math. Soc.

[5] J. Cuntz, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys. **57** (1977), 173–185.

[6] J. Cuntz, *Automorphisms of certain simple C^*-algebras*, in *Quantum fields-algebras-processes*, ed. L. Streit, Springer 1980.

[7] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras into $\mathcal{O}_2$*, J. reine angew. Math. **525** (2000), 17–53.

[8] E. Kirchberg and M. Rørdam, *Infinite non-simple C^*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Adv. Math. **167** (2002), 195–264.

[9] M. Laca, *Endomorphisms of $\mathcal{B}(\mathcal{H})$ and Cuntz algebras*, J. Operator Theory **30** (1993), 85–108.

[10] V. Nekrashevych, *Cuntz-Pimsner algebras of group actions*, J. Operator Theory **52** (2004), 223–249.

[11] M. Rørdam, *Classification of inductive limits of Cuntz algebras*, J. reine angew. Math. **440** (1993), 175–200.

[12] M. Rørdam, *A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$*, C. R. Math. Rep. Acad. Sci. Canada **16** (1994), 31–36.

[13] W. Szymański, *On localized automorphisms of the Cuntz algebras which preserve the diagonal subalgebra*, in 'New Development of Operator Algebras', R.I.M.S. Kôkyûroku **1587** (2008), 109–115.

[14] S. Wassermann, *Exact C^*-algebras and related topics*, Lecture Notes Series N.19. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1994.

[15] D. Yang, *Endomorphisms and modular theory of 2-graph C^*-algebras*, arXiv:0907.1129, to appear in Indiana Univ. Math. J.

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