Purely infinite C^*-algebras: ideal-preserving zero homotopies

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Abstract

We show that if A is a separable, nuclear, $\mathcal{O}_\infty$-absorbing (or strongly purely infinite) C^*-algebra which is *homotopic to zero in an ideal-system preserving way*, then A is the inductive limit of C^*-algebras of the form $C_0(\Gamma, v) \otimes M_k$, where Γ is a finite connected graph (and $C_0(\Gamma, v)$ is the algebra of continuous functions on Γ that vanish at a distinguished point $v \in \Gamma$).

We show further that if B is any separable, nuclear C^*-algebra, then $B \otimes \mathcal{O}_2 \otimes \mathcal{K}$ is isomorphic to a crossed product $D \rtimes_\alpha \mathbb{Z}$, where D is an inductive limit of C^*-algebras of the form $C_0(\Gamma, v) \otimes M_k$ (and D is $\mathcal{O}_2$-absorbing and homotopic to zero in an ideal-system preserving way).

1 Introduction

Cuntz defined in [2] a simple C^*-algebra to be purely infinite if each of its non-zero hereditary sub-C^*-algebras contain an infinite projection. The class of simple purely infinite C^*-algebras has since then received much attention; in parts because many interesting and naturally occurring C^*-algebras, such as the Cuntz algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, are purely infinite, and in parts because strong classification theorems have been obtained for this class of C^*-algebras (see [6], [14], and [16]).

The notion of being purely infinite was extended to non-simple C^*-algebras in [8]. A (possibly non-simple) C^*-algebra A is said to be purely infinite if A has no character and if for every pair of positive elements $a, b \in A$, such that a belongs to the closed two-sided ideal in A generated by b, there is a sequence $\{x_n\}$ in A with $x_n^* b x_n \to a$. Every purely infinite C^*-algebra is traceless (in the sense that no algebraic ideal of the C^*-algebra admits a non-zero trace or a quasitrace), and $A \otimes \mathcal{O}_\infty$ is purely infinite for every C^*-algebra A. It was shown in [9] that if A is a nuclear, separable C^*-algebra that is either stable or unital, then the following three statements are equivalent: $A \cong A \otimes \mathcal{O}_\infty$, A is approximately divisible and traceless, and A is *strongly purely infinite* (an internal algebraic condition, see [9, Definition 5.1]). If A is of real rank zero, then these conditions are satisfied if and only if every non-zero projection in A is properly infinite.

There is a classification (in terms of an ideal related version of Kasparov's KK-theory) for separable, stable, nuclear C^*-algebras that absorb the Cuntz algebra $\mathcal{O}_\infty$; and in the special case of $\mathcal{O}_2$-absorbing C^*-algebras the classifying invariant is nothing but the primitive ideal space (or equivalently, the ideal lattice). These results were obtained by the first named author in [6]. (A C^*-algebra A is said to be $\mathcal{O}_\infty$- and $\mathcal{O}_2$-absorbing, respectively, if $A \cong A \otimes \mathcal{O}_\infty$ and $A \cong A \otimes \mathcal{O}_2$, respectively.) The following question is open:

Question 1.1 Is every point-complete[1], second countable, locally quasi-compact T_0-space the primitive ideal space of a separable (or separable and nuclear) C^*-algebra.

Going a step further, it would be desirable to have an algorithm that to each admissible topological space assigns a separable, nuclear, $\mathcal{O}_2$-absorbing C^*-algebra with that space as its primitive ideal space. We take a step in this direction at the end of Section 6 by showing the following result: For every C^*-algebra B there is a commutative sub-C^*-algebra $C \subseteq B \otimes \mathcal{O}_2$ such that $\Psi\colon J \mapsto C \cap (J \otimes \mathcal{O}_2)$ is an injective lattice-morphism from the Hausdorff lattice of closed ideals of B into the lattice of closed ideals of C (in particular, $\Psi(J_1 + J_2) = \Psi(J_1) + \Psi(J_2)$).

In [15] a C^*-algebra $\mathcal{A}_{[0,1]}$ was constructed that has primitive ideal space $[0,1)$ with the non-Hausdorff T_0 topology $\{[0,\alpha) : \alpha \in (0,1]\} \cup \{\emptyset\}$. The C^*-algebra $\mathcal{A}_{[0,1]}$ is $\mathcal{O}_\infty$-absorbing, and at the same time an inductive limit of C^*-algebras of the form $C_0([0,1), M_{2^k})$. We observe in Section 6 that $\mathcal{A}_{[0,1]}$ is zero homotopic in an ideal-system preserving way, i.e., that there is a continuous path $\{\rho_t\}_{t\in[0,1]}$ of *-endomorphisms on $\mathcal{A}_{[0,1]}$ such that $\rho_0 = 0$, $\rho_1 = \mathrm{id}$, and $\rho_t(I) \subseteq I$ for every closed two-sided ideal I in $\mathcal{A}_{[0,1]}$ and for each t. By the classification of nuclear $\mathcal{O}_\infty$-absorbing C^*-algebras we can then conclude that $\mathcal{A}_{[0,1]}$ is $\mathcal{O}_2$-absorbing (see Theorem 5.2), and hence is the unique such C^*-algebra, up to isomorphism, with that primitive ideal space.

A question we seek to answer in this paper is to what extend this example is a special case of a more general theory. In particular we ask the following:

[1] A T_0-space is *point-complete* (or *spectral*) if every prime closed subset is the closure of a point. A closed subset F of a T_0-space X is called *prime* if $F = G \cup H$ implies $F = G$ or $F = H$, when G and H are closed.

Question 1.2 Are the following conditions equivalent for a nuclear, separable, stable, $\mathcal{O}_\infty$-absorbing C^*-algebra A:

(i) A is zero homotopic in an ideal-system preserving way.

(ii) Each quotient of A is homotopic to zero.

(iii) Each quotient of A is projectionless.

(iv) A is an AH_0-algebra.

An AH_0-algebra is a C^*-algebra that is the inductive limit of direct sums of C^*-algebras of the form $C_0(X, M_n)$, where X is a locally compact space. The spaces X, that occur in the direct limit, can in our case be taken to be (one-dimensional) graphs. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are true and trivial. We show here (Theorem 5.12) that (i) $\Rightarrow$ (iv) if A is $\mathcal{O}_\infty$-absorbing. We do not know if (iii) $\Rightarrow$ (i).

If A satisfies (i) above, then so does $A \otimes B$ for every C^*-algebra B (as long as one of A and B is nuclear). In particular, $\mathcal{A}_{[0,1]} \otimes B$ satisfies (i), and hence (iv), in Question 1.2 for every separable nuclear C^*-algebra B. We show in Section 6 that there is an automorphism α on $\mathcal{A}_{[0,1]}$ such that $\mathcal{A}_{[0,1]} \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$. Hence, if B is any separable, nuclear C^*-algebra, then $B \otimes \mathcal{O}_2 \otimes \mathcal{K}$ is isomorphic to $D \rtimes_\beta \mathbb{Z}$, where $D = \mathcal{A}_{[0,1]} \otimes B$ and $\beta = \alpha \otimes \mathrm{id}_B$, and D is an AH_0-algebra. In other words, every separable, nuclear, stable, $\mathcal{O}_2$-absorbing C^*-algebra is the crossed product of an AH_0-algebra by an action of the integers, thus confirming the following fundamental question in the special case of $\mathcal{O}_2$-absorbing C^*-algebras.

Question 1.3 Is every separable, nuclear, stable C^*-algebra isomorphic to a crossed product $D \rtimes_\alpha \mathbb{Z}$, where D is an inductive limit of type I C^*-algebras, and where α is an automorphism on D?

An affirmative answer to this question will imply that the Universal Coefficient Theorem (UCT) holds for all separable nuclear C^*-algebras. (To this end, it will suffice to affirm Question 1.3 for C^*-algebras that absorb the Cuntz algebra $\mathcal{O}_\infty$.) Unfortunately, our result for $\mathcal{O}_2$-absorbing C^*-algebras does not add new information to the UCT.

In Section 2 we remind the reader of some results about completely positive maps, and prove some sharpened versions of known results (in parts from our earlier paper [9]).

In Section 3 we show that any Abelian sub-C^*-algebra B of an $\mathcal{O}_2$-absorbing C^*-algebra A is approximately contained in an Abelian sub-C^*-algebra C of A, where the spectrum of C is a (one-dimensional) graph. We also show that if X is any locally compact metrizable space, then $C(X) \otimes 1 \subseteq D \subseteq C(X) \otimes \mathcal{O}_2$ for some Abelian C^*-algebra D whose spectrum is one-dimensional.

In Section 4 we prove a Hahn-Banach type separation theorem (implicitly contained in our earlier paper [9]) for completely positive maps from a nuclear C^*-algebra A into an arbitrary C^*-algebra B: If $\mathscr{K}$ is an operator convex set of completely positive maps from A to B, and if $T\colon A \to B$ is a completely positive map, then T belongs to the point-norm closure of $\mathscr{K}$ if and only if $T(a)$ belongs to the closure of $\{V(a) : V \in \mathscr{K}\}$ for all $a \in A$. This result, and the approximation result from Section 3, are the main technical ingredients in the proof in Section 5 of our main result. There are also other applications of our Hahn-Banach theorem. We give a new proof of the fundamental uniqueness theorem for $\mathcal{O}_2$-absorbing C^*-algebras: If φ and ψ are *-homomorphisms from a separable, nuclear C^*-algebra A to an $\mathcal{O}_2$-absorbing, stable C^*-algebra B, then φ and ψ are approximately unitarily equivalent if and only if they induce the same map from the ideal lattice of B to the ideal lattice of A.

Section 5 contains the proof of our main result described in the abstract, and Section 6 contains some applications thereof, already mentioned.

2 Preliminaries

We remind the reader of concepts and notations related to limit algebras. A filter ω on $\mathbb{N}$ is called free if $\bigcap_{A \in \omega} A = \emptyset$. The filter ω_∞ of all co-finite subsets of $\mathbb{N}$ is free; and the free filters are precisely those that contain ω_∞. In this paper ω always means a *free* filter on $\mathbb{N}$.

If A is a C^*-algebra and ω is a filter on $\mathbb{N}$, then A_ω is defined to be the quotient-C^*-algebra $\ell_\infty(A)/I_\omega$, where $\ell_\infty(A)$ is the C^*-algebra of all bounded sequences $\{a_n\}_{n=1}^\infty$, with $a_n \in A$, and I_ω is the closed two-sided ideal in $\ell_\infty(A)$ consisting of those sequences $\{a_n\}_{n=1}^\infty$ for which $\lim_\omega \|a_n\| = 0$. We let π_ω denote the quotient mapping $\ell_\infty(A) \to A_\omega$. A sequence converges along the filter ω_∞ (defined above) if and only if it converges in the usual sense as a sequence. Therefore $I_{\omega_\infty} = c_0(A)$, which is the set of sequences $\{a_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \|a_n\| = 0$. We let A_∞ denote the C^*-algebra A_{ω_∞}, i.e., $A_\infty = \ell_\infty(A)/c_0(A)$.

We shall often have occasion to consider the C^*-algebra $\mathcal{M}(A)_\omega$, where A is a non-unital C^*-algebra. Note that there are natural inclusions $A \subseteq A_\omega \subseteq \mathcal{M}(A)_\omega \subseteq \mathcal{M}(A_\omega)$.

We remind the reader of the following:

Definition 2.1 (Approximately inner completely positive maps) A completely positive map T from a C^*-algebra A to another C^*-algebra B is called *n-step inner* if there are elements $b_1, \dots, b_n$ in B such that $T(a) = \sum_{j=1}^n b_j^* a b_j$ for all $a \in A$. T is called *inner* if it is n-step inner for some n. If T is the point-norm limit of a sequence of n-step inner or inner completely positive maps, then T is called *approximately n-step inner* and *approximately inner*, respectively.

Lemma 2.2 (cf. Lemma 7.4 of [9]) *Let A be a stable C^*-algebra, let ω be a filter on $\mathbb{N}$, and let B be a separable sub-C^*-algebra of A_ω. Then each approximately 1-step inner completely positive contraction $T\colon B \to A_\omega$ is of the form $T(b) = t^*bt$, $b \in B$, for some isometry $t \in \mathcal{M}(A)_\omega$.*

Proof: By assumption and [9, Lemma 7.2] there is a sequence of *contractions* $\{f_n\}$ in A_ω such that $f_n^* b f_n \to T(b)$ for all b in B. By [9, Lemma 2.5] there is a single contraction d in A_ω for which we have the exact relation $T(b) = d^*bd$ for all b in B. Write $d = \pi_\omega(d_1, d_2, \dots)$, where each d_n is a contraction in A. Let $\{b_1, b_2, \dots\}$ be a countable dense subset of B and write $b_j = \pi_\omega(b_{j,1}, b_{j,2}, \dots)$, where $b_{j,n} \in A$. As in the proof of [9, Lemma 7.4], using stability of A, we can find isometries $s_{n,1}$ and $s_{n,2}$ in $\mathcal{M}(A)$ for each $n \in \mathbb{N}$ satisfying $s_{n,1}s_{n,1}^* + s_{n,2}s_{n,2}^* = 1$ and $\|b_{j,n}s_{n,2}s_{n,2}^*\| \le 1/n$ for $j = 1, 2, \dots, n$. As in the proof of [9, Lemma 7.4], set

$$t_n = s_{n,1}s_{n,1}^*d_n + s_{n,2}\big(1 - d_n^*s_{n,1}s_{n,1}^*d_n\big)^{1/2} \in \mathcal{M}(A), \qquad t = \pi_\omega(t_1, t_2, \dots) \in \mathcal{M}(A)_\omega.$$

Each t_n is an isometry and $\|t_n^*b_{j,n}t_n - d_n^*b_{j,n}d_n\| \to 0$ as $n \to \infty$ for all j. It follows that t is an isometry and that

$$t^*b_jt = \pi_\omega(t_1^*b_{j,1}t_1, t_2^*b_{j,2}t_2, \dots) = \pi_\omega(d_1^*b_{j,1}d_1, d_2^*b_{j,2}d_2, \dots) = d^*b_jd = T(b_j)$$

for all j, and so $T(b) = t^*bt$ for all b in B. □

Lemma 2.3 *Let A be a C^*-algebra, let ω be a free filter on $\mathbb{N}$, and let B be a separable sub-C^*-algebra of A_ω.*

(i) *If $A \cong A \otimes \mathcal{O}_2$, then there is a unital embedding of $\mathcal{O}_2$ into $B' \cap \mathcal{M}(A)_\omega$.*

(ii) *If $A \cong A \otimes \mathcal{O}_\infty$, then there is a unital embedding of $\mathcal{O}_\infty$ into $B' \cap \mathcal{M}(A)_\omega$.*

Proof: We prove (i) and (ii) simultaneously. Let m represent either 2 or ∞ and assume that $A \cong A \otimes \mathcal{O}_m$. Then there is a sequence $\varphi_n\colon \mathcal{O}_m \to \mathcal{M}(A)$ of unital *-homomorphisms that asymptotically commutes with A, ie., $\varphi_n(x)a - a\varphi_n(x) \to 0$ for all $x \in \mathcal{O}_m$ and all $a \in A$ (see eg. [16, Theorem 7.2.2 and Remark 7.2.3]). Let $\pi_\omega\colon \ell_\infty(A) \to A_\omega$ be the quotient mapping, let $\{b_1, b_2, b_3, \dots\}$ be a dense subset of B, let $\{x_1, x_2, x_3, \dots\}$ be a dense subset of $\mathcal{O}_m$, and write $b_j = \pi_\omega(b_{j1}, b_{j2}, b_{j3}, \dots)$, where $b_{ji} \in A$. For each natural number k find a natural number n_k such that $\|\varphi_{n_k}(x_i)b_{jk} - b_{jk}\varphi_{n_k}(x_i)\| < 1/k$ for $i, j = 1, 2, \dots, k$. Define $\varphi\colon \mathcal{O}_m \to \mathcal{M}(A)_\omega$ by

$$\varphi(x) = \pi_\omega(\varphi_{n_1}(x), \varphi_{n_2}(x), \varphi_{n_3}(x), \dots), \qquad x \in \mathcal{O}_m.$$

Because ω is free and

$$\lim_{k\to\infty} \|\varphi_{n_k}(x_i)b_{jk} - b_{jk}\varphi_{n_k}(x_i)\| = 0, \qquad i, j \in \mathbb{N},$$

it follows that $\varphi(x_i)b_j = b_j\varphi(x_i)$ for all i and j; and this shows that the image of φ commutes with B. $\square$

A C^*-algebra A is called *$\mathcal{O}_2$-absorbing* or *$\mathcal{O}_\infty$-absorbing*, if $A \cong A \otimes \mathcal{O}_2$ or $A \cong A \otimes \mathcal{O}_\infty$, respectively. We have $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathcal{O}_\infty$ and $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ (see [7, Theorem 3.8 and Theorem 3.15]), so $A \otimes \mathcal{O}_2$ is both $\mathcal{O}_2$- and $\mathcal{O}_\infty$-absorbing, and $A \otimes \mathcal{O}_\infty$ is $\mathcal{O}_\infty$-absorbing for every C^*-algebra A

Let D be a unital C^*-algebra and suppose that v_1, v_2 are isometries in D satisfying the $\mathcal{O}_2$-relation: $v_1v_1^* + v_2v_2^* = 1$. We shall then consider the *Cuntz sum* of elements $a, b \in D$ defined by $a \oplus_{v_1,v_2} b = v_1av_1^* + v_2bv_2^*$.

Lemma 2.4 *Let D be a unital C^*-algebra, let s be an isometry in D, and let $V(a) = s^*as$ be the corresponding unital completely positive map on D. Suppose that v_1, v_2 are isometries in D satisfying the $\mathcal{O}_2$-relation. Put $w_1 = (1 - ss^*) + sv_1s^*$ and $w_2 = sv_2$.*

Then w_1, w_2 are isometries satisfying the $\mathcal{O}_2$-relation, and

$$\begin{aligned}
\|a \oplus_{w_1,w_2} V(a) - a\| &\le \|[v_1, V(a)]\| + \|[v_2, V(a)]\| + 2\|[a, ss^*]\|, \\
\|[a, ss^*]\| &= \max\{\|V(a^*a) - V(a)^*V(a)\|^{1/2}, \|V(aa^*) - V(a)V(a)^*\|^{1/2}\},
\end{aligned}$$

for all $a \in D$.

Proof: It is straightforward to check that w_1, w_2 satisfy the $\mathcal{O}_2$-relation. Next,

$$\begin{aligned}
\|a \oplus_{w_1,w_2} V(a) - a\| &\le \|(1 - ss^*)a(1 - ss^*) + s\big(s^*as \oplus_{v_1,v_2} s^*as\big)s^* - a\| \\
&\quad + \|(1 - ss^*)asv_1^*s^* + sv_1s^*a(1 - ss^*)\| \\
&\le \|(1 - ss^*)a(1 - ss^*) + ss^*ass^* - a\| \\
&\quad + \|V(a) \oplus_{v_1,v_2} V(a) - V(a)\| + \|[a, ss^*]\| \\
&\le \|[v_1, V(a)]\| + \|[v_2, V(a)]\| + 2\|[a, ss^*]\|.
\end{aligned}$$

The last statement follows from the identities

$$\begin{aligned}
\|[a,ss^*]\| &= \max\{\|(1-ss^*)ass^*\|, \|ss^*a(1-ss^*)\|\},\\
\|V(a^*a)-V(a)^*V(a)\| &= \|s^*a^*(1-ss^*)as\| \;=\; \|(1-ss^*)as\|^2 \;=\; \|(1-ss^*)ass^*\|^2,\\
\|V(aa^*)-V(a)V(a)^*\| &= \|(1-ss^*)a^*ss^*\|^2 \;=\; \|ss^*a(1-ss^*)\|^2.
\end{aligned}$$

□

Lemma 2.5 (cf. Lemma 1.12 of [7]) *Suppose that A is a separable, stable, $\mathcal{O}_2$-absorbing C^*-algebra and that s and t are isometries in $\mathcal{M}(A)$. Let $V(a) = s^*as$ and $W(a) = t^*at$ be the corresponding unital completely positive maps on $\mathcal{M}(A)$. Then for every finite subset F of A and for every $\varepsilon > 0$ there is a unitary element u in $\mathcal{M}(A)$ with*

$$\|u^*V(a)u - a\| \leq 5\kappa + \varepsilon$$

for all $a \in F$, where

$$\kappa = \max_{a\in F\cup F^*}\{\|V(a^*a)-V(a^*)V(a)\|^{1/2}, \|WV(a^*a)-WV(a)^*WV(a)\|^{1/2}, \|WV(a)-a\|\}.$$

Proof: Find isometries r_1, r_2 in $\mathcal{M}(A)$ satisfying the $\mathcal{O}_2$-relation such that

$$\|[r_1,V(a)]\| + \|[r_2,V(a)]\| \leq \varepsilon/2, \qquad \|[r_1,WV(a)]\| + \|[r_2,WV(a)]\| \leq \varepsilon/2, \tag{2.1}$$

for all $a \in F$ (see eg. [16, Theorem 7.2.2 and Remark 7.2.3]). Define new sets of isometries (t_1,t_2) and (s_1,s_2) satisfying the $\mathcal{O}_2$-relation by

$$t_1 = (1-tt^*) + tr_1t^*, \quad t_2 = tr_2, \qquad s_1 = (1-ss^*) + sr_1s^*, \quad s_2 = sr_2,$$

(cf. Lemma 2.4), and set $u = t_1s_2^* + t_2s_1^*$. Then u is unitary element in $\mathcal{M}(A)$, and $u^*(a \oplus_{t_1,t_2} b)u = b \oplus_{s_1,s_2} a$ for all $a, b \in \mathcal{M}(A)$. Thus

$$\begin{aligned}
\|u^*V(a)u - a\| &= \|u^*V(a)u - u^*\big(V(a)\oplus_{t_1,t_2} WV(a)\big)u + WV(a)\oplus_{s_1,s_2} V(a) - a\|\\
&\leq \|V(a) - V(a)\oplus_{t_1,t_2} WV(a)\| + \|a\oplus_{s_1,s_2} V(a) - a\| + \|WV(a)-a\|
\end{aligned}$$

for $a \in \mathcal{M}(A)$. By the inequality

$$0 \leq W(V(a)^*V(a)) - WV(a)^*WV(a) \leq WV(a^*a) - WV(a)^*WV(a),$$

and by Lemma 2.4 and (2.1) we see that the two first terms above are each bounded by $2\kappa+\varepsilon/2$, when $a \in F$, and the last term is at most κ. This proves the lemma. $\square$

Remark 2.6 Lemma 2.5 holds with $\mathcal{M}(A)$ replaced with $\mathcal{M}(A)_\omega$ for any free filter ω, even with $\varepsilon = 0$. Indeed, (2.1) is satisfied with $\varepsilon = 0$ by Lemma 2.3 (i) (when B is the separable sub-C^*-algebra of A_ω generated by $V(A) \cup WV(A)$). The rest of the proof of Lemma 2.5 works with $\mathcal{M}(A)$ replaced with $\mathcal{M}(A)_\omega$.

It follows in particular that if $\{s_n\}$ and $\{t_n\}$ are isometries in $\mathcal{M}(A)_\omega$ such that

$$\lim_{n\to\infty} \|s_n^* a b s_n - s_n^* a s_n s_n^* b s_n\| = 0, \qquad \lim_{n\to\infty} \|t_n^* s_n^* a s_n t_n - a\| = 0 \quad \text{for all } a, b \in A,$$

then there is a sequence $\{u_n\}$ of unitaries in $\mathcal{M}(A)_\omega$ such that $\|u_n^* s_n^* a s_n u_n - a\| \to 0$ for all $a \in A$.

3 Local and global AH_0-algebras

The main result of this section is Theorem 3.6 which states that any $\mathcal{O}_2$-absorbing C^*-algebra, that locally is an AH_0-algebra, is actually an inductive limit of C^*-algebras of the form $C_0(\Gamma, v) \otimes M_k$, where (Γ, v) is a finite pointed graph. Along the way we prove a perturbation result, which states that a subalgebra B of an $\mathcal{O}_2$-absorbing C^*-algebra, with $B \cong C_0(X, M_k)$, is approximately contained in a subalgebra that is isomorphic to $C_0(Y, M_k)$ for some one-dimensional space Y. We end the section by showing that for every compact metrizable space X there is a one-dimensional compact metrizable space Y such that $C(X) \otimes 1 \subseteq C(Y) \subseteq C(X) \otimes \mathcal{O}_2$.

Lemma 3.1 *Let X be a compact subset of the complex plane and let $f \in C(X, \mathcal{O}_2)$ be the function $f(z) = z1$. Then for each $\lambda \in \mathbb{C}$ and for each $\varepsilon > 0$ there is a normal element $g \in C(X, \mathcal{O}_2)$ that satisfies $\lambda \notin \mathrm{sp}(g)$, $\|f - g\| \le 2\varepsilon$, and $g(z) = f(z)$ when $|z - \lambda| \ge \varepsilon$.*

Proof: Let $D = \{z \in \mathbb{C} : |z - \lambda| \le \varepsilon\}$ be the closed disk with radius ε and center λ, and let ∂D denote its boundary. The map $u_0 \in C(\partial D, \mathcal{O}_2)$ given by $u_0(z) = (z - \lambda)/\varepsilon$ is unitary. The restriction map $C(D, \mathcal{O}_2) \to C(\partial D, \mathcal{O}_2)$ is a *-epimorphism, and the unitary group of $C(\partial D, \mathcal{O}_2)$ is connected (cf. [2]), so there is an element $u \in C(D, \mathcal{O}_2)$ with values in the unitary group of $\mathcal{O}_2$ such that $u(z) = u_0(z)$ for $z \in \partial D$. The map $g \colon X \to \mathcal{O}_2$ given by

$$g(z) = \begin{cases} \varepsilon u(z) + \lambda 1, & z \in X \cap D, \\ z1, & z \in X \setminus D, \end{cases}$$

defines a normal element in $C(X, \mathcal{O}_2)$ (because $g(z)$ is normal for every $z \in X$). Use that $g(z) - \lambda 1$ is invertible and $\|g(z) - f(z)\| = \|g(z) - z1\| \leq 2\varepsilon$ for every $z \in D$ to see that g has the desired properties. $\square$

Let $a_1, a_2, \dots, a_n$ be commuting normal elements in a C^*-algebra A. Their joint spectrum, $\mathrm{sp}(a_1, a_2, \dots, a_n)$, is the compact subset of $\mathbb{C}^n$ of all n-tuples $(\rho(a_1), \rho(a_2), \dots, \rho(a_n))$, where ρ is a character on $C^*(a_1, a_2, \dots, a_n)$. Each continuous function $f\colon \mathrm{sp}(a_1, a_2, \dots, a_n) \to \mathbb{C}$ defines an element $f(a_1, a_2, \dots, a_n) \in C^*(a_1, a_2, \dots, a_n)$ that satisfies $\rho(f(a_1, a_2, \dots, a_n)) = f(\rho(a_1), \rho(a_2), \dots, \rho(a_n))$ for all characters ρ on $C^*(a_1, a_2, \dots, a_n)$.

Lemma 3.2 *Let A be a C^*-algebra, let a_1, a_2 be two commuting self-adjoint elements in A, and suppose there is a unital *-homomorphism $\varphi\colon \mathcal{O}_2 \to \mathcal{M}(A) \cap \{a_1, a_2\}'$. Then for each point $(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and for each $\varepsilon > 0$ there are self-adjoint elements b_1, b_2 in A such that $(s,t) \notin \mathrm{sp}(b_1, b_2)$ and $\|a_j - b_j\| < \varepsilon$ for $j = 1, 2$.*

Proof: Put $a = a_1 + ia_2$ and put $\lambda = s + it \neq 0$. The C^*-algebra $C^*(a, \varphi(\mathcal{O}_2)) \subseteq \mathcal{M}(A)$ is isomorphic to $C(X, \mathcal{O}_2)$, where $X = \mathrm{sp}(a)$, via an isomorphism that maps a to the function f given by $f(z) = z1$. Take $\eta > 0$ such that $\eta < \varepsilon/2$ and $\eta \leq |\lambda|$. Use Lemma 3.1 to find a normal element g in $C(X, \mathcal{O}_2)$ with $\lambda \notin \mathrm{sp}(g)$, $\|f - g\| \leq 2\eta$, and $g(z) = f(z)$ when $|z - \lambda| \geq \eta$. Let b in $C^*(a, \varphi(\mathcal{O}_2))$ correspond to g, so that $\|a - b\| \leq 2\eta < \varepsilon$ and $\lambda \notin \mathrm{sp}(b)$. Since $g(0) = f(0) = 0$, because $|0 - \lambda| \geq \eta$, we conclude that g belongs to the ideal in $C(X, \mathcal{O}_2)$ generated by f. Thus b belongs to the ideal in $C^*(a, \varphi(\mathcal{O}_2))$ generated by a, and this ideal is contained in A, whence b belongs to A.

Write $b = b_1 + ib_2$ where b_1 and b_2 are self-adjoint. Then b_1 commutes with b_2 (because b is normal), $\|a_j - b_j\| \leq \|a - b\| < \varepsilon$, and $(s,t) \notin \mathrm{sp}(b_1, b_2)$—the latter because $(s,t) \in \mathrm{sp}(b_1, b_2)$ if and only if $s + it \in \mathrm{sp}(b_1 + ib_2)$. $\square$

For each $\varepsilon > 0$ and for each natural number n, let $\Gamma_{\varepsilon,n}$ denote the 1-dimensional grid in $\mathbb{R}^n$ consisting of those n-tuples $(t_1, t_2, \dots, t_n)$ for which t_j belongs to $\varepsilon\mathbb{Z}$ for all but at most one j. Every connected compact subset of $\Gamma_{\varepsilon,n}$ is (homeomorphic to) a finite graph.

A finite graph is (here) a topological space that consists of finitely many vertices, each homeomorphic to a point, and finitely many edges, each homeomorphic to an interval, so that each edge connects two vertices. There are no crossings of edges (except at vertices). Any finite graph Γ is a compact Hausdorff space. In this paper we shall exclusively be concerned with finite graphs, and any graph will tacitly be understood to be finite.

If v is a point in Γ (a vertex or not), then $C_0(\Gamma, v)$ denotes the set of continuous functions on Γ that vanish at v. Equivalently, $C_0(\Gamma, v) = C_0(\Gamma \setminus \{v\})$. The pair (Γ, v) is called a *pointed graph.*

We state a topological lemma which is needed for Lemma 3.4 below. For each $t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n$ let $\nu_n(t)$ be the number of j's for which $t_j \in (\mathbb{Z} + 1/2)\varepsilon$. Let $X_{\varepsilon,n}$ be the set of all $t \in \mathbb{R}^n$ for which $\nu_n(t) \leq 1$. Observe that $\Gamma_{\varepsilon,n} \subseteq X_{\varepsilon,n} \subseteq \mathbb{R}^n$.

Lemma 3.3 *For each natural number $n \geq 2$ and for each $\varepsilon > 0$ there is a continuous retract $F\colon X_{\varepsilon,n} \to \Gamma_{\varepsilon,n}$ that satisfies $\|F(t) - t\|_\infty \leq \varepsilon$ for all $t \in X_{\varepsilon,n}$.*

Proof: For $1 \leq k \leq n$, let $Z_{\varepsilon,n,k}$ be the set of all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ such that the number of j's for which t_j belongs to $\varepsilon\mathbb{Z}$ is at least $n - k$. Put $Y_{\varepsilon,n,k} = Z_{\varepsilon,n,k} \cap X_{\varepsilon,n}$. Then

$$X_{\varepsilon,n} = Y_{\varepsilon,n,n} \supseteq Y_{\varepsilon,n,n-1} \supseteq \cdots \supseteq Y_{\varepsilon,n,2} \supseteq Y_{\varepsilon,n,1} = Z_{\varepsilon,n,1} = \Gamma_{\varepsilon,n}.$$

We define the function F to be the composition of continuous retracts

$$Y_{\varepsilon,n,n} \xrightarrow{F_n} Y_{\varepsilon,n,n-1} \xrightarrow{F_{n-1}} Y_{\varepsilon,n,n-2} \xrightarrow{F_{n-2}} \cdots \longrightarrow Y_{\varepsilon,n,2} \xrightarrow{F_2} Y_{\varepsilon,n,1},$$

that are to be constructed. It will map each cube $\prod_{j=1}^n [m_j\varepsilon, (m_j + 1)\varepsilon] \cap X_{\varepsilon,n}$ (where m_j are integers) into itself, and any function F with this property will satisfy $\|F(t) - t\|_\infty \leq \varepsilon$.

We construct first a continuous retract $G_k\colon X_{\varepsilon,k} \to Y_{\varepsilon,k,k-1}$ for each $k = 2, 3, \dots, n$. Take $t \in X_{\varepsilon,k}$ and find integers m_j such that t belongs to the cube $C = \prod_{j=1}^k [m_j\varepsilon, (m_j+1)\varepsilon]$. Observe that $X_{\varepsilon,k} \cap \partial C = Y_{\varepsilon,k,k-1} \cap C$, where ∂C is the boundary of C. Put

$$q_C = \big((m_1 + 1/2)\varepsilon, (m_2 + 1/2)\varepsilon, \dots, (m_k + 1/2)\varepsilon\big) \in C,$$

and let ℓ_{t,q_C} be the half-line in $\mathbb{R}^k$ that starts at q_C and runs through t (note that $t \neq q_C$ because $\nu_k(t) \leq 1$ and $\nu_k(q_C) = k > 1$). The half-line ℓ_{t,q_C} intersects ∂C in exactly one point, and this point we define to be $G_k(t)$. The point $G_k(t)$ belongs to $X_{\varepsilon,n}$ because $\nu_k(G_k(t)) = \nu_k(t) \leq 1$. Note that $G_k(t) = t$ if and only if $t \in Y_{\varepsilon,k,k-1}$. (For $t \in C \cap X_{\varepsilon,k}$ this happens if and only if $t \in \partial C$.) This shows that G_k is a continuous retract from $X_{\varepsilon,k}$ onto $Y_{\varepsilon,k,k-1}$.

Put $F_n = G_n$. Let now $2 \leq k \leq n - 1$ be given. The space $Z_{\varepsilon,n,k} \supset Y_{\varepsilon,n,k}$ is the (non-disjoint) union of infinitely many subsets each of which is homeomorphic to $\mathbb{R}^k$. We define below F_k so that it corresponds to the retract G_k on each of these subsets. In more detail, for each subset α of $\{1, 2, \dots, n\}$ with $|\alpha| = k$ let $\pi_\alpha\colon \mathbb{R}^n \to \mathbb{R}^k$ and $\pi'_\alpha\colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ be the projection-maps onto the coordinates corresponding to α and to the complement of α, respectively. Then $Z_{\varepsilon,n,k}$ is the union of the collection of sets $D_{\alpha,p} = (\pi'_\alpha)^{-1}(p)$, with α as above and $p \in (\varepsilon\mathbb{Z})^{n-k}$; and $\pi_\alpha|_{D_{\alpha,p}}\colon D_{\alpha,p} \to \mathbb{R}^k$ is a homeomorphism. As

$\pi_\alpha(D_{\alpha,p} \cap X_{\varepsilon,n}) = X_{\varepsilon,k}$ and $\pi_\alpha(D_{\alpha,p} \cap Y_{\varepsilon,n,k-1}) = Y_{\varepsilon,k,k-1}$, there is a unique continuous retract $H_{\alpha,p}$ making the diagram

$$\begin{tikzcd} D_{\alpha,p} \cap X_{\varepsilon,n} \arrow[r, "H_{\alpha,p}"] \arrow[d, "\pi_\alpha"'] & D_{\alpha,p} \cap Y_{\varepsilon,n,k-1} \arrow[d, "\pi_\alpha"] \\ X_{\varepsilon,k} \arrow[r, "G_k"'] & Y_{\varepsilon,k,k-1} \end{tikzcd}$$

commutative. Note that $H_{\alpha,p}(D_{\alpha,p} \cap C) \subseteq C$ whenever C is a cube $\prod_{j=1}^n [k_j\varepsilon, (k_j+1)\varepsilon]$ with $k_j \in \mathbb{Z}$. If $(\alpha, p) \neq (\alpha', p')$, then $D_{\alpha,p} \cap D_{\alpha',p'}$ is (empty or) contained in $Y_{\varepsilon,n,k-1}$, whence $H_{\alpha,p}(t) = H_{\alpha',p'}(t) = t$ for t in $D_{\alpha,p} \cap D_{\alpha',p'}$. The maps $H_{\alpha,p}$ therefore extend to a continuous retract F_k from $Y_{\varepsilon,n,k} = \bigcup_{\alpha,p} D_{\alpha,p} \cap X_{\varepsilon,n}$ onto $Y_{\varepsilon,n,k-1} = \bigcup_{\alpha,p} D_{\alpha,p} \cap Y_{\varepsilon,n,k-1}$. □

Lemma 3.4 *Let $a_1, a_2, \ldots, a_n$ be commuting self-adjoint elements in a C^*-algebra A, and let $\varepsilon > 0$. Suppose that*

$$\mathrm{sp}(a_i, a_j) \cap (\varepsilon/2 + \varepsilon\mathbb{Z}) \times (\varepsilon/2 + \varepsilon\mathbb{Z}) = \emptyset$$

for all pairs (i, j) with $i \neq j$. Then there are commuting self-adjoint elements $b_1, b_2, \ldots, b_n$ in $C^*(a_1, a_2, \ldots, a_n) \subseteq A$ such that $\|a_j - b_j\| \leq \varepsilon$ for all j and $\mathrm{sp}(b_1, b_2, \ldots, b_n) \subseteq \Gamma_{\varepsilon,n}$, where $\Gamma_{\varepsilon,n}$ is the one-dimensional grid in $\mathbb{R}^n$ defined above Lemma 3.3.

Proof: If $n = 1$, then we can take $b_1 = a_1$ (because $\Gamma_{\varepsilon,1} = \mathbb{R}$). We assume below that $n \geq 2$. The condition on the pairwise joint spectra of the elements $a_1, a_2, \ldots, a_n$ ensures that the joint spectrum $\mathrm{sp}(a_1, a_2, \ldots, a_n)$ is contained in the set $X_{\varepsilon,n}$ defined above Lemma 3.3. Let $F\colon X_{\varepsilon,n} \to \Gamma_{\varepsilon,n}$, with $\|F(t) - t\|_\infty \leq \varepsilon$ for all $t \in X_{\varepsilon,n}$, be the continuous retract found in Lemma 3.3. Write $F(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, where each $f_j\colon X_{\varepsilon,n} \to \mathbb{R}$ is a continuous function. Let $p_j\colon \mathbb{R}^n \to \mathbb{R}$ be the jth coordinate function so that $a_j = p_j(a_1, a_2, \ldots, a_n)$ for $j = 1, 2, \ldots, n$. Put $b_j = f_j(a_1, a_2, \ldots, a_n)$. Then $b_1, b_2, \ldots, b_n$ are self-adjoint (and necessarily commuting) elements in $C^*(a_1, a_2, \ldots, a_n)$,

$$\|a_j - b_j\| = \sup\{|p_j(t) - f_j(t)| : t \in \mathrm{sp}(a_1, a_2, \ldots, a_n)\} \leq \sup\{\|F(t) - t\|_\infty : t \in X_{\varepsilon,n}\} \leq \varepsilon,$$

and

$$\mathrm{sp}(b_1, b_2, \ldots, b_n) = F(\mathrm{sp}(a_1, a_2, \ldots, a_n)) \subseteq F(X_{\varepsilon,n}) \subseteq \Gamma_{\varepsilon,n}$$

as desired. □

Proposition 3.5 *Let A be a separable C^*-algebra that absorbs $\mathcal{O}_2$, and let ω be a free filter on $\mathbb{N}$. Suppose that B is a sub-C^*-algebra of A_ω and that B is isomorphic to $C_0(X, M_k)$ for some locally compact Hausdorff space X and for some natural number k. Let $b_1, b_2, \dots, b_n$ be elements in B and let $\varepsilon > 0$.*

Then there exist a sub-C^*-algebra B_1 of A_ω and elements $c_1, c_2, \dots, c_n \in B_1$ such that $\|b_j - c_j\| < \varepsilon$ for all j, and such that B_1 is isomorphic to $C_0((\Gamma, v), M_k)$, or to $C(\Gamma, M_k)$, for some compact subset Γ of $\Gamma_{\varepsilon,m}$, for some $v \in \Gamma$, and for some natural numbers m, k.

Proof: Use Lemma 2.3 to find a unital *-homomorphism $\varphi\colon \mathcal{O}_2 \to \mathcal{M}(A)_\omega \cap B'$. Let I be the closed two-sided ideal in $C^*(B, \varphi(\mathcal{O}_2)) \subseteq \mathcal{M}(A)_\omega$ generated by B. Then

$$I \cong M_k \otimes C_0(X) \otimes \mathcal{O}_2 \quad \text{and} \quad I \subseteq A_\omega.$$

The elements $b_1, b_2, \dots, b_n$ in B correspond to elements $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n$ in $M_k \otimes C_0(X) \otimes 1$. Find self-adjoint elements $f_1, f_2, \dots, f_m$ in $C_0(X)$ such that $M_k \otimes C^*(f_1, f_2, \dots, f_m) \otimes 1$ contains the elements $\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n$. We find pairwise commuting self-adjoint elements $g_1, g_2, \dots, g_m$ in $C_0(X) \otimes \mathcal{O}_2$ with $\|f_j \otimes 1 - g_j\| < \delta$, for $j = 1, 2, \dots, m$ and for some small enough $\delta > 0$, such that $(\Gamma =)\ \mathrm{sp}(g_1, g_2, \dots, g_m) \subseteq \Gamma_{\varepsilon,m}$. The C^*-algebra $C^*(g_1, \dots, g_m)$ is isomorphic to $C_0(\Gamma, v)$ if the origin $v = (0, \dots, 0)$ belongs to Γ, and it is isomorphic to $C(\Gamma)$ if v does not belong to Γ. In both cases, the C^*-algebra $B_1 \subseteq A_\omega$ that corresponds to $M_k \otimes C^*(g_1, g_2, \dots, g_m) \subseteq M_k \otimes C_0(X) \otimes \mathcal{O}_2$ will be as desired.

To find $g_1, g_2, \dots, g_m$ it suffices by Lemma 3.4 to find self-adjoint elements $h_1, h_2, \dots, h_m$ in $C_0(X) \otimes \mathcal{O}_2$ such that $\|f_j \otimes 1 - h_j\| < \varepsilon/2$ for all j and such that $\mathrm{sp}(h_i, h_j) \cap \Lambda = \emptyset$ for $i \neq j$, where $\Lambda = (1/2 + \mathbb{Z})\frac{\varepsilon}{2} \times (1/2 + \mathbb{Z})\frac{\varepsilon}{2}$.

Let R be the set of all commuting m-tuples of self-adjoint elements in $C_0(X) \otimes \mathcal{O}_2$, and for $i, j = 1, \dots, m$, with $i \neq j$, and for $\lambda \in \Lambda$ let $R_{i,j,\lambda}$ be the set of all $(h_1, \dots, h_m)$ in R such that $\lambda \notin \mathrm{sp}(h_i, h_j)$. Then R is a closed subset of $\big(C_0(X) \otimes \mathcal{O}_2\big)^m$, whence R is a complete metric space. Each $R_{i,j,\lambda}$ is clearly open in R. We show below that $R_{i,j,\lambda}$ is also dense in R.

There is a sequence $\{\psi_\ell\}_{\ell=1}^\infty$ of *-endomorphisms on $\mathcal{O}_2$ such that $\mathcal{O}_2 \cap \mathrm{Im}(\psi_\ell)'$ contains a unital copy of $\mathcal{O}_2$ for each ℓ, and such that $\psi_\ell(x) \to x$ as $\ell \to \infty$ for all $x \in \mathcal{O}_2$. (Indeed, since $\mathcal{O}_2 \cong \bigotimes_{r=1}^\infty \mathcal{O}_2$ (cf. [16, Corollary 5.2.4]) we can take the ψ_ℓ that corresponds to the endomorphism on $\bigotimes_{r=1}^\infty \mathcal{O}_2$ that fixes the first ℓ copies of $\mathcal{O}_2$ and shifts the remaining copies one place to the right.) Put $\varphi_\ell = \mathrm{id}_{C_0(X)} \otimes \psi_\ell$. Then there is a unital copy of $\mathcal{O}_2$ in $\mathcal{M}(C_0(X) \otimes \mathcal{O}_2) \cap \mathrm{Im}(\varphi_\ell)'$ for every ℓ.

Let $(d_1, \dots, d_m) \in R$ and $\varepsilon > 0$ be given. Choose ℓ such that $(\varphi_\ell(d_1), \dots, \varphi_\ell(d_m)) \in R$ is within distance $\varepsilon/2$ from $(d_1, \dots, d_m)$. By Lemma 3.2 there is a commuting pair (e_i, e_j)

of self-adjoint elements in $C_0(X) \otimes \mathcal{O}_2 \cap \mathrm{Im}(\varphi_\ell)'$ within distance $\varepsilon/2$ from $(\varphi_\ell(d_i), \varphi_\ell(d_j))$ and with $\lambda \notin \mathrm{sp}(e_i, e_j)$. Put $e_k = \varphi_\ell(d_k)$ if $k \neq i, j$. Then $(e_1, \dots, e_m)$ belongs to $R_{i,j,\lambda}$ and has distance at most ε to $(d_1, \dots, d_m)$.

Now, R has the Baire property, being a complete metric space, so the intersection R_0 of all $R_{i,j,\lambda}$, where $i, j = 1, \dots, m$, $i \neq j$, and $\lambda \in \Lambda$, is dense in R. This ensures the existence of $(h_1, \dots, h_m) \in R_0$ such that $\|f_j \otimes 1 - h_j\| < \varepsilon/2$ for all j, thus completing the proof. $\square$

Theorem 3.6 *Let A be a separable C^*-algebra and let A_∞ be the limit algebra $\ell_\infty(A)/c_0(A)$. The following three conditions are equivalent:*

(i) *A is the inductive limit of a sequence*

$$C_0((\Gamma_1, v_1), M_{k_1}) \xrightarrow{\varphi_1} C_0((\Gamma_2, v_2), M_{k_2}) \xrightarrow{\varphi_2} C_0((\Gamma_3, v_3), M_{k_3}) \xrightarrow{\varphi_3} \cdots \longrightarrow A,$$

where each (Γ_j, v_j) is a pointed graph, k_j is a natural number, and each connecting map φ_j is a *-homomorphism.

(ii) *For every finite set $a_1, a_2, \dots, a_n$ in A and for every $\varepsilon > 0$ there exist a C^*-algebra $B \cong C_0((\Gamma, v), M_k)$, for some pointed graph (Γ, v) and for some $k \in \mathbb{N}$, a *-homomorphism $\varphi \colon B \to A$, and elements $b_1, b_2, \dots, b_n$ in B, such that $\|a_j - \varphi(b_j)\| < \varepsilon$ for all j.*

(iii) *For every finite set $a_1, a_2, \dots, a_n$ in A and for every $\varepsilon > 0$ there exist a C^*-algebra $B \cong C_0((\Gamma, v), M_k)$, for some pointed graph (Γ, v) and for some $k \in \mathbb{N}$, a *-homomorphism $\varphi \colon B \to A_\infty$, and elements $b_1, b_2, \dots, b_n$ in B, such that $\|a_j - \varphi(b_j)\| < \varepsilon$ for all j.*

Suppose now that A is a separable $\mathcal{O}_2$-absorbing C^*-algebra. Then conditions (i)–(iii) *are equivalent to condition* (iv) *below. Moreover, the *-homomorphism φ in* (ii) *and* (iii) *can be taken to be injective. Also, the connecting maps φ_j in* (i) *can be taken to be injective and the graphs Γ_j in* (i) *will be connected if the connecting maps φ_j are injective.*

(iv) *For every finite set $a_1, a_2, \dots, a_n$ in A and for every $\varepsilon > 0$ there exist a sub-C^*-algebra B of A_∞ and elements $b_1, b_2, \dots, b_n$ in B such that*

   (a) *$\|a_j - b_j\| < \varepsilon$ for all j, and*

   (b) *B is isomorphic to $C_0(X, M_k)$ for some locally compact Hausdorff space X and for some natural number k.*

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

Assume that (iv) holds and that $A \cong A \otimes \mathcal{O}_2$. Then A cannot contain a non-zero projection, and as every non-zero projection in A_∞ lifts to a non-zero projection in $\ell^\infty(A)$, there are no non-zero projections in A_∞. Indeed, any non-zero projection in A would be infinite (because A is purely infinite), and, by (iv), any infinite projection in A is equivalent to an infinite projection in a sub-C^*-algebra B of A where B is isomorphic to $C_0(X, M_k)$ (or to $C(X, M_k)$ if X is compact). But $C_0(X, M_k)$ (and $C(X, M_k)$) contain no infinite projections.

Let $a_1, \dots, a_n$ in A and $\varepsilon > 0$ be given. Find $B \subseteq A_\infty$ and $c_1, \dots, c_n \in B$ with $B \cong C_0(X, M_k)$ and $\|c_j - a_j\| < \varepsilon/2$. Use Proposition 3.5 to find a C^*-algebra $B_1 \subseteq A_\infty$ and $b_1, \dots, b_n \in B_1$ with $B_1 \cong C_0((\Gamma, v), M_k)$, or with $B_1 \cong C(\Gamma, M_k)$, and $\|c_j - b_j\| < \varepsilon/2$, for some closed subset Γ of $\Gamma_{\varepsilon/2,m}$, some m, and some $v \in \Gamma$. We have observed that A_∞ has no non-zero projections, so B_1 has no non-zero projections. This entails that Γ must be connected and it excludes the unital possibility: $B_1 \cong C(\Gamma, M_k)$. Any connected closed subset of $\Gamma_{\varepsilon/2,m}$ is (homeomorphic to) a graph. We have therefore proved that (iv) implies (iii) and that φ in (iii) can be taken to be injective (actually the inclusion mapping) when A absorbs $\mathcal{O}_2$.

Still assuming that A is $\mathcal{O}_2$-absorbing, one can in (i) and (ii) arrange that φ (in (ii)) and the connecting maps φ_j (in (i)) are injective. Indeed, replacing B and A_j by $B/\mathrm{Ker}(\varphi)$ and $A_j/\mathrm{Ker}(\varphi_{\infty,j})$, respectively, where $\varphi_{\infty,j} \colon A_j \to A$ is the inductive limit map, it follows that B and A_j remain of the form $C_0((\Gamma, v), M_k)$, with Γ a connected graph, because A (and hence B and A_j, respectively) have no non-zero projections.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). These implications follow by the work of Loring on semiprojective C^*-algebras. A finitely generated C^*-algebra is semiprojective if and only if it has a finite set of stable generators (Loring, [12, Theorem 14.1.4]). It follows from [11, Theorem 5.1] (applied with $n = 1$) together with [12, Theorem 14.1.7] that $C_0(\Gamma, v)$ is semiprojective when (Γ, v) is a pointed graph. By [12, Theorem 14.2.2], any matrix algebra over $C_0(\Gamma, v)$ is semiprojective.

To prove (iii) $\Rightarrow$ (ii), let $a_1, \dots, a_n \in A$ and $\varepsilon > 0$ be given. Take $\varphi \colon B \to A_\infty$ and $b_1, \dots, b_n \in B$ as in (iii). Let I_m be the ideal in $\ell_\infty(A)$ consisting of those sequences $\{a_k\}_{k=1}^\infty$ for which $a_k = 0$ for all $k > m$. Then $\{I_m\}$ is an increasing sequence and $c_0(A) = \overline{\bigcup_{m=1}^\infty I_m}$. By the definition of semiprojectivity, for some large enough m there is a *-homomorphism $\psi \colon B \to \ell_\infty(A)/I_m = \prod_{k=m+1}^\infty A$ with $\nu_m(\psi(b)) = \varphi(b)$, where $\nu_m \colon \ell_\infty(A)/I_m \to A_\omega$ is the quotient mapping. Write $\psi = (\psi_{m+1}, \psi_{m+2}, \dots)$, where each $\psi_k \colon B \to A$ is a *-homomorphism. Now, for $j = 1, 2, \dots, n$,

$$\varepsilon > \|a_j - \varphi(b_j)\| = \|\nu_m\big(a_j - \psi_{m+1}(b_j), a_j - \psi_{m+2}(b_j), \dots\big)\| = \limsup_{\substack{k\to\infty \\ k>m}} \|a_j - \psi_k(b_j)\|.$$

It follows that $\psi_k \colon B \to A$ satisfies the conditions of (ii) for some large enough k.

The implication (ii) $\Rightarrow$ (i) is contained in [12, Lemma 15.2.2]. $\square$

We conclude this section with a sharpened version of Proposition 3.5, that will not be used for the proof of our main result, but is needed for an application in Section 6. It may also have some independent interest.

Proposition 3.7 *Let X be any metrizable compact space. Then there is a metrizable one-dimensional compact space Y and there is an embedding $\psi \colon C(Y) \to C(X) \otimes \mathcal{O}_2$ such that $C(X) \otimes 1 \subseteq \psi(C(Y))$.*

Proof: To make the strategy of the somewhat lengthy and technical proof more transparent, the proof is divided into three steps and is carried out backwards.

Step 1. It is shown in Steps 2 and 3 that there is a sequence of graphs, Γ_n, and *-homomorphisms $\varphi_n \colon C(\Gamma_n) \to C(X) \otimes \mathcal{O}_2$ such that

(a) $\varphi_1(C(\Gamma_1)) \subseteq \varphi_2(C(\Gamma_2)) \subseteq \varphi_3(C(\Gamma_3)) \subseteq \cdots$,

(b) $C(X) \otimes 1 \subseteq \overline{\bigcup_{n=1}^{\infty} \varphi_n(C(\Gamma_n))}$.

The *-homomorphisms φ_n need not be injective, but if we pass to the quotient of $C(\Gamma_n)$ by the kernel of φ_n we obtain injective *-homomorphisms $\psi_n \colon C(Y_n) \to C(X) \otimes \mathcal{O}_2$ for suitable closed subsets Y_n of Γ_n, and $\mathrm{Im}(\psi_n) = \mathrm{Im}(\varphi_n)$. This leads to the commutative diagram

$$\begin{array}{ccccccccc}
C(Y_1) & \xrightarrow{\lambda_1} & C(Y_2) & \xrightarrow{\lambda_2} & C(Y_3) & \xrightarrow{\lambda_3} & \cdots & \longrightarrow & C(Y) \\
\downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \downarrow{\scriptstyle\psi_3} & & & & \downarrow{\scriptstyle\psi} \\
C(X)\otimes\mathcal{O}_2 & = & C(X)\otimes\mathcal{O}_2 & = & C(X)\otimes\mathcal{O}_2 & = & \cdots & = & C(X)\otimes\mathcal{O}_2
\end{array}$$

where $\lambda_n = \psi_{n+1}^{-1} \circ \psi_n$ (we are here using that $\mathrm{Im}(\psi_n) \subseteq \mathrm{Im}(\psi_{n+1})$). The inductive limit of the top row of the diagram above is (isomorphic to) the Abelian C^*-algebra $C(Y)$, where Y is the inverse limit of the sequence of spaces $\{Y_n\}$ with connecting maps $\widehat{\lambda}_n \colon Y_{n+1} \to Y_n$. In particular, Y is a compact metrizable one-dimensional space (recall that each of the spaces Y_n is one-dimensional being a closed subset of the one-dimensional space Γ_n). The image of ψ contains $C(X) \otimes 1$ by (b).

Step 2. Choose a dense sequence $\{f_1, f_2, f_3, \dots\}$ in $C(X)$. The *-homomorphisms φ_n from Step 1 are constructed from sequences $\{\varphi_n^{(k)}\}_{k=n}^{\infty}$ of *-homomorphisms $\varphi_n^{(k)} \colon C(\Gamma_n) \to C(X) \otimes \mathcal{O}_2$, together with a collection of finite sets G_n and $F_n^{(k)}$, for $k \geq n$, all to be constructed in Step 3, with the following properties,

(i) G_n is a set of stable generators, with respect to some relations, for the semiprojective C^*-algebra $C(\Gamma_n)$,

(ii) $G_n \subseteq F_n^{(n)} \subseteq F_n^{(n+1)} \subseteq \dots \subseteq C(\Gamma_n)$ for all n,

(iii) $\varphi_n^{(k)}(G_n) \subseteq \varphi_{n+1}^{(k)}(F_{n+1}^{(k)})$ when $n+1 \leq k$,

(iv) $\|\varphi_n^{(k)}(f) - \varphi_n^{(k+1)}(f)\| \leq 2^{-k}$ for all $n \leq k$ and $f \in F_n^{(k)}$,

(v) $\mathrm{dist}(f_j \otimes 1, \varphi_n^{(n)}(F_n^{(n)})) \leq 1/n$ for all n and for $j = 1, 2, \dots, n$.

It follows from (ii) and (iv) that $\{\varphi_n^{(k)}(f)\}_{k=n}^{\infty}$ is a Cauchy sequence for every n and for every $f \in G_n$. The set of elements $f \in C(\Gamma_n)$ for which $\{\varphi_n^{(k)}(f)\}_{k=n}^{\infty}$ is a Cauchy sequence is a C^*-algebra, and as this C^*-algebra contains a set of generators for $C(\Gamma_n)$, we conclude that there is a *-homomorphism $\varphi_n \colon C(\Gamma_n) \to C(X) \otimes \mathcal{O}_2$ such that $\varphi_n(f) = \lim_{k\to\infty} \varphi_n^{(k)}(f)$ for every $f \in C(\Gamma_n)$. We show that (a) and (b) in Step 1 hold. Note first that

(vi) $\|\varphi_n(f) - \varphi_n^{(k)}(f)\| \leq 2^{-k+1}$ for $f \in F_n^{(k)}$,

by (iv). To see that (a) holds, let n and $g \in G_n$ be given. For any $k > n$ there is by (iii) an element $h_k \in F_{n+1}^{(k)}$ such that $\varphi_n^{(k)}(g) = \varphi_{n+1}^{(k)}(h_k)$, and

$$\|\varphi_n(g) - \varphi_{n+1}(h_k)\| \leq \|\varphi_n(g) - \varphi_n^{(k)}(g)\| + \|\varphi_{n+1}^{(k)}(h_k) - \varphi_{n+1}(h_k)\| \leq 2^{-k+2}.$$

It follows that $\varphi_n(G_n) \subseteq \varphi_n(C(\Gamma_n))$, and in turns that $\varphi_{n+1}(C(\Gamma_{n+1})) \subseteq \varphi_n(C(\Gamma_n))$. To prove (b), take natural numbers $n \geq j$ and find according to (v) an element $h_n \in F_n^{(n)}$ such that

$$\|f_j \otimes 1 - \varphi_n(h_n)\| \leq \|f_j \otimes 1 - \varphi_n^{(n)}(h_n)\| + \|\varphi_n^{(n)}(h_n) - \varphi_n(h_n)\| \leq 1/n + 2^{-n+1},$$

where (vi) has been used. (b) follows from the above estimate.

Step 3. We construct the graphs Γ_n, the *-homomorphisms $\varphi_n^{(k)} \colon C(\Gamma_n) \to C(X) \otimes \mathcal{O}_2$ and the sets $F_n^{(k)} \subseteq C(\Gamma_n)$ so that (i)–(v) in Step 2 are satisfied; and this is done by induction on k. Since $\mathcal{O}_2 \cong \bigotimes_{r=1}^{\infty} \mathcal{O}_2$ we can find a sequence of C^*-algebras $\mathbb{C} = D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots \subseteq \mathcal{O}_2$ with $D_n \cap D_{n-1}' \cong \mathcal{O}_2$. The *-homomorphism $\varphi_n^{(k)}$ will be constructed so that its image is contained in $C(X) \otimes D_k$ whenever $n \leq k$.

For $k = n = 1$ use Proposition 3.5 to find an injective *-homomorphism $\varphi \colon B \to C(X) \otimes D_1$ such that $B = C(\Gamma')$ or $B = C_0(\Gamma', v)$ for some closed subset Γ' of $\Gamma_{1/2,m}$ (for some m), and $\|1 - \varphi(b_1)\| < 1/2$ and $\|f_1 \otimes 1 - \varphi(b_2)\| \leq 1$ for some $b_1, b_2 \in B$. We infer from $\|1 - \varphi(b_1)\| < 1/2$ that B is unital and that φ is unit-preserving. Hence $B = C(\Gamma')$. Now, Γ' is contained in a compact connected subset Γ_1 of $\Gamma_{1/2,m}$, and Γ_1 is necessarily (homeomorphic to) a graph. Take $\varphi_1^{(1)} \colon C(\Gamma_1) \to C(X) \otimes D_1$ to be the composition of the restriction mapping $C(\Gamma_1) \to C(\Gamma')$ and $\varphi \colon C(\Gamma') \to C(X) \otimes D_1$. Then $\|f_1 \otimes 1 - \varphi_1^{(1)}(h)\| \leq 1$ for some $h \in C(\Gamma_1)$. Let G_1 be a set of stable generators for the semiprojective C^*-algebra $C(\Gamma_1)$ with respect to some relations (cf. Loring [11, Theorem 5.1]), and set $F_1^{(1)} = G_1 \cup \{h\}$.

Suppose now that $k \geq 1$ and that Γ_n, $\varphi_n^{(k)}$, G_n, and $F_n^{(k)}$ have been found meeting conditions (i)–(v) for all $n = 1, 2, \ldots, k$. Then, by (i) and (iii),

$$\varphi_1^{(k)}(C(\Gamma_1)) \subseteq \varphi_2^{(k)}(C(\Gamma_2)) \subseteq \varphi_3^{(k)}(C(\Gamma_3)) \subseteq \cdots \subseteq \varphi_k^{(k)}(C(\Gamma_k)) \subseteq C(X) \otimes D_k.$$

We proceed to find Γ_{k+1} and *-homomorphisms

$$\varphi_{k+1}^{(k+1)} \colon C(\Gamma_{k+1}) \to C(X) \otimes D_{k+1}, \quad \varphi_n^{(k+1)} \colon C(\Gamma_n) \to \varphi_{n+1}^{(k+1)}(C(\Gamma_{n+1})), \ 1 \leq n \leq k,$$

such that

$$\mathrm{dist}(f_j \otimes 1, \mathrm{Im}(\varphi_{k+1}^{(k+1)})) \leq 1/(k+1), \qquad \|\varphi_n^{(k+1)}(f) - \varphi_n^{(k)}(f)\| \leq 2^{-k},$$

for $1 \leq j \leq k+1$, $1 \leq n \leq k$, and for all $f \in F_n^{(k)}$. Once this is done, it is straightforward to find the sets $F_n^{(k+1)}$, $n = 1, \ldots, k+1$, such that (ii)–(v) are satisfied.

As the generating set G_n for $C(\Gamma_n)$ is stable, $n = 1, \ldots, k$, one can for each $\varepsilon > 0$ and for each finite subset F of $C(\Gamma_n)$ find $\delta = \delta(\varepsilon, F, \Gamma_n, G_n) > 0$ such that whenever $B \subseteq A$ are unital C^*-algebras and $\rho \colon C(\Gamma_n) \to A$ is a *-homomorphism with $\mathrm{dist}(\rho(g), B) \leq \delta$ for all $g \in G_n$, then there is a *-homomorphism $\sigma \colon C(\Gamma_n) \to B$ with $\|\rho(f) - \sigma(f)\| \leq \varepsilon$ for all $f \in F$.

Set $\delta_1 = 2^{-k}$ and set

$$\delta_{n+1} \quad = \quad \delta(\delta_n, F_n^{(k)}, \Gamma_n, G_n) \wedge 2^{-k}, \quad n = 1, 2, \ldots, k.$$

Arguing as in the case $k = n = 1$ we use Proposition 3.5 to find a graph Γ_{k+1} and a *-homomorphism $\varphi_{k+1}^{(k+1)}\colon C(\Gamma_{k+1}) \to C(X) \otimes D_{k+1}$ such that

$$\operatorname{dist}(f_j \otimes 1, \operatorname{Im}(\varphi_{k+1}^{(k+1)})) \leq 1/(k+1), \qquad \operatorname{dist}(\varphi_k^{(k)}(g), \operatorname{Im}(\varphi_{k+1}^{(k+1)})) \leq \delta_{k+1},$$

for $1 \leq j \leq k+1$ and for all $g \in G_k$. Take a finite set G_{k+1} of stable generators for the semiprojective C^*-algebra $C(\Gamma_{k+1})$. By the choice of δ_{k+1} we can then find a *-homomorphism $\varphi_k^{(k+1)}\colon C(\Gamma_k) \to \operatorname{Im}(\varphi_{k+1}^{(k+1)}) \subseteq C(X) \otimes \mathcal{O}_2$ such that $\|\varphi_k^{(k+1)}(f) - \varphi_k^{(k)}(f)\| \leq \delta_k$ ($\leq 2^{-k}$) for all f in $F_k^{(k)}$. In particular, $\operatorname{dist}(\varphi_{k-1}^{(k)}(g), \operatorname{Im}(\varphi_k^{(k+1)})) \leq \delta_k$ for all $g \in G_{k-1}$. We can in this way continue the construction of $\varphi_{k-1}^{(k+1)}, \varphi_{k-2}^{(k+1)}, \dots, \varphi_1^{(k+1)}$ with the desired properties. $\square$

There is a curious topological corollary to Proposition 3.7 that each compact metrizable space is the continuous-open image of a one-dimensional space. This relates to the classical Peano Curve: a continuous surjection from the interval onto the square (or onto more arbitrary compact spaces). The Peano Curve is not open, and there exists no open continuous surjection from the interval onto the square. Indeed, suppose that f were a continuous and open mapping from the interval onto the square, let I be a closed line segment in the square, and let $C = f^{-1}(I)$. The interior of C is empty because the interior of I is empty, so C is compact and totally disconnected. The restriction mapping $f\colon C \to I$ is continuous, open, and maps compact sets to compact sets; hence it maps clopen sets into clopen sets. Thus $f(C_0) = I$ for every non-empty clopen subset of C_0 of C. This contradicts the continuity of f because C has clopen subsets of arbitrary small diameter.

We thank Etienne Blanchard for suggesting the continuous bundle proof below.

Corollary 3.8 *For each compact metrizable space X there is a compact metrizable one-dimensional space Y and a continuous, open, surjection $Y \to X$.*

Proof: Use Proposition 3.7 to find a compact metrizable one-dimensional space Y and an embedding $\varphi\colon C(Y) \to C(X) \otimes \mathcal{O}_2$ such that $C(X) \otimes 1 \subseteq \varphi(C(Y))$. Then $f \mapsto \varphi^{-1}(f \otimes 1)$ defines an injection from $C(X)$ into $C(Y)$ which can be realized by a continuous surjective function $\lambda\colon Y \to X$; i.e., $f \circ \lambda = \varphi^{-1}(f \otimes 1)$, or $\varphi(f \circ \lambda) = f \otimes 1$, for all f in $C(X)$.

We proceed to show that λ is open. The argument uses implicitly that $C(Y)$ is a continuous field over X. Let $\pi_x\colon C(X) \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be the point evaluation at x, and put $\nu_x(g) = \pi_x(\varphi(g))$ for $g \in C(Y)$. Then $x \mapsto \|\nu_x(g)\|$ is continuous for every g in $C(Y)$.

Fix x in X and choose a decreasing sequence $\{f_n\}_{n=1}^{\infty}$ of positive contractions in $C(X)$

such that $f_n(x) = 1$ for all n and $f_n(x') = 0$ eventually when $x' \neq x$. Then

$$\|\nu_x(g)\| = \lim_{n\to\infty} \|(f_n \otimes 1)\varphi(g)\| = \lim_{n\to\infty} \|(f_n \circ \lambda)g\| = \sup\{|g(y)| : y \in \lambda^{-1}(x)\}.$$

Let U be an open subset of Y. Because Y is a metrizable space there is a function $g \in C(Y)$ such that $g(y) \neq 0$ if and only if $y \in U$. As

$$x \in \lambda(U) \iff \lambda^{-1}(x) \cap U \neq \emptyset \iff g|_{\lambda^{-1}(x)} \neq 0 \iff \|\nu_x(g)\| > 0,$$

it follows that $\lambda(U) = \{x \in X : \|\nu_x(g)\| > 0\}$, and this proves that $\lambda(U)$ is open. $\square$

4 A Hahn–Banach separation theorem for completely positive maps

This section contains a Hahn–Banach type separation theorem for completely positive maps—a result which is implicitly contained in [9]—and some applications thereof.

Definition 4.1 Let A and B be C^*-algebras. Denote by $CP(A, B)$ the cone of all completely positive maps from A to B. A subset $\mathscr{K}$ of $CP(A, B)$ is called *operator convex* if it satisfies:

(i) $\mathscr{K}$ is a cone.

(ii) If $V \in \mathscr{K}$ and $b \in B$, then the map $a \mapsto b^*V(a)b$ belongs to $\mathscr{K}$.

(iii) If $V \in \mathscr{K}$, $c_1, \dots, c_n \in A$, and $b_1, \dots, b_n \in B$, then the map given by

$$a \mapsto \sum_{i=1}^{n} \sum_{j=1}^{n} b_i^* V(c_i^* a c_j) b_j,$$

belongs to $\mathscr{K}$.

The completely positive map displayed in (iii) is equal to the map $a \mapsto b^*(V \otimes \mathrm{id}_{M_n})(c^*ac)b$, where b is the row matrix $(b_1, \dots, b_n)$ and c is the column matrix $(c_1, \dots, c_n)^t$.

Proposition 4.2 (cf. Lemma 7.18 of [9]) *Let A be a separable, nuclear C^*-algebra and let B be any C^*-algebra. Suppose that $\mathscr{K}$ is an operator convex subset of $CP(A, B)$, and let $T \in CP(A, B)$. Then T belongs to the point-norm closure of $\mathscr{K}$ if and only if $T(a)$ belongs to the closed two-sided ideal in B generated by $\{V(a) \mid V \in \mathscr{K}\}$ for every a in A.*

Proof: The "only if" part is trivial. The proof of the "if" part is almost verbatim identical with the proof of [9, Lemma 7.18], but since there are some changes (in particular in the last paragraph of the proof and in the notation), we repeat below most of the argument for the convenience of the reader.

For each finite set $a_1, \dots, a_n$ in A and for each $\varepsilon > 0$ we must find V in $\mathscr{K}$ such that $\|T(a_j) - V(a_j)\| \leq \varepsilon$ for all j. Equivalently, we must show that

$$(T(a_1), \dots, T(a_n)) \in \overline{\{\big(V(a_1), \dots, V(a_n)\big) \mid V \in \mathscr{K}\}}. \tag{4.1}$$

By a Hahn–Banach separation argument, and because the set on the right-hand side of (4.1) is a cone (by assumption (i)), and hence convex, (4.1) is implied by the following: For each set $f_1, \dots, f_n$ in B^* and for each $\varepsilon > 0$ there is V in $\mathscr{K}$ such $|f_j(T(a_j)) - f_j(V(a_j))| \leq \varepsilon$ for $j = 1, \dots, n$.

Choose a cyclic representation $\pi \colon B \to B(H)$, a cyclic vector ξ in H, and elements $c_1, \dots, c_n$ in $\pi(B)' \cap B(H)$ such that $f_j(a) = \langle \pi(a)\xi, c_j^*\xi \rangle$ for all a in A and for all j; cf. [9, Lemma 7.17 (i)]. Let C be the sub-C^*-algebra of $\pi(B)' \cap B(H)$ generated by $c_1, \dots, c_n$. Keeping π and ξ fixed, associate to each completely positive map $V \colon A \to B$ the positive functional φ_V on $A \otimes C$ given by

$$\varphi_V(a \otimes c) = \big\langle \pi(V(a))\xi, c^*\xi \big\rangle, \qquad a \in A, \;\; c \in C. \tag{4.2}$$

(A priori, φ_V defines a functional on the maximal tensor product $A \otimes_{\max} C$, but the maximal and the minimal tensor products on $A \odot C$ coincide because A is nuclear.) Let $\mathscr{C}$ be the weak-* closure of the cone $\{\varphi_V : V \in \mathscr{K}\}$. Observe that $f_j(V(a_j)) = \langle \pi(V(a_j))\xi, c_j^*\xi \rangle = \varphi_V(a_j \otimes c_j)$ for every completely positive map $V \colon A \to B$. Hence it will suffice to show that φ_T belongs to $\mathscr{C}$.

It follows as in the proof of [9, Lemma 7.18] (using assumption (iii)!) that $d^*\rho d$ belongs to $\mathscr{C}$ for all ρ in $\mathscr{C}$ and for all d in $A \otimes C$.

As in [9, Lemma 7.17 (ii)], let J be the closed two-sided ideal in $A \otimes C$ consisting of those elements z for which $\varphi_V(z^*z) = 0$ for all V in $\mathscr{K}$. By [9, Lemma 7.17 (ii)] one can conclude that φ_T belongs to $\mathscr{C}$ if we know that $\varphi_T(z^*z) = 0$ for all z in J. Using again the assumption that A is nuclear, we can use a theorem of Blackadar, [1, Theorem 3.3] (see also [6, Proposition 2.13]) that J is the closed linear span of the set of elementary tensors $x \otimes y$ in J. The left kernel L of φ_T, consisting of all z in $A \otimes C$ such that $\varphi_T(z^*z) = 0$, is a closed linear subspace of $A \otimes C$, and so it suffices to show that $\varphi_T(x^*x \otimes y^*y) = 0$ whenever $x \in A$ and $y \in C$ are such that $x \otimes y$ belongs to J

Fix $x \otimes y$ in J as above. By assumption, $T(x^*x)$ belongs to the closed two-sided ideal in B generated by $\{V(x^*x) \mid V \in \mathcal{K}\}$. For any $\varepsilon > 0$ we can therefore find $V_1, \dots, V_m$ in $\mathscr{K}$ and $b_1, \dots, b_m$ in B such that

$$\|T(x^*x) - b_1^*V_1(x^*x)b_1 - \cdots - b_m^*V_m(x^*x)b_m\| \leq \varepsilon.$$

In other words, $\|T(x^*x) - W(x^*x)\| \leq \varepsilon$, when $W(a) = b_1^*V_1(a)b_1 + \cdots + b_m^*V_m(a)b_m$. By assumptions (i) and (ii) we see that W belongs to $\mathscr{K}$. Hence,

$$\begin{aligned} |\varphi_T(x^*x \otimes y^*y)| &= |\varphi_T(x^*x \otimes y^*y) - \varphi_W(x^*x \otimes y^*y)| \\ &= \left|\left\langle \pi(T(x^*x) - W(x^*x))\xi, y^*y\xi \right\rangle\right| \;\leq\; \varepsilon\|\xi\|^2\|y\|^2. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we get $\varphi_T(x^*x \otimes y^*y) = 0$ as desired. □

The nuclearity assumption in Proposition 4.2 is necessary. Indeed, for any pair of C^*-algebras A and B, the set $CP_{\text{nuc}}(A, B)$ of all nuclear completely positive maps from A to B is operator convex; $a = \text{id}_A(a)$ belongs to the closed two-sided ideal generated by $\{V(a) \mid V \in CP_{\text{nuc}}(A, A)\}$ for every $a \in A$; but id_A belongs to the point-norm closure of $CP_{\text{nuc}}(A, A)$ if and only if A is nuclear.

To prove the second claim above, take a non-zero $a \in A$, take a positive linear functional ρ on A such that $\rho(a) \neq 0$, and set $V(x) = \rho(x)a^*a$, $x \in A$. Then V belongs to $CP_{\text{nuc}}(A, A)$ (because V factors through the complex numbers via the positive functional ρ), and a belongs to the closed two-sided ideal in A generated by $V(a) = \rho(a)a^*a$.

Corollary 4.3 *Let B be any C^*-algebra, let A be a separable nuclear sub-C^*-algebra of B, and let $T \colon A \to B$ be a completely positive map. Then T is approximately inner (cf. Definition 2.1) if and only if $T(a) \in \overline{BaB}$ for all $a \in A$.*

Proof: The set, $\mathscr{K}$, of all inner completely positive maps from A into B is operator convex. Indeed, it is trivial that (i) and (ii) hold of Definition 4.1 hold. To see that (iii) holds, take V in $\mathscr{K}$, take elements $c_1, \dots, c_m \in A$, take elements $b_1, \dots, b_m \in B$, and set $W(a) = \sum_{i,j=1}^n b_i^*V(c_i^*ac_j)b_j$. If $V(a) = \sum_{k=1}^n d_k^*ad_k$, then $W(a) = \sum_{k=1}^n f_k^*af_k$ when $f_k = \sum_{j=1}^n c_jd_kb_j$.

Observe that $T \colon A \to B$ is approximately inner if and only if T belongs to the point-norm closure of $\mathscr{K}$. The set $\{V(a) \mid V \in \mathscr{K}\}$ is contained in the two-sided ideal in B generated by a (it actually coincides with the positive part of the algebraic two-sided ideal generated by a, when a is positive), and so the lemma follows from Proposition 4.2. □

We do not know if the conclusion of Corollary 4.3 actually characterizes nuclear C^*-algebras, as phrased more formally below:

Question 4.4 Suppose that A is a C^*-algebra for which every completely positive map $V\colon A \to A$, that satisfies $V(a) \in \overline{AaA}$ for all $a \in A$, is approximately inner. Does it follow that A is nuclear?

We conclude this section with a (new) proof of the fundamental uniqueness theorem for $\mathcal{O}_2$-absorbing C^*-algebras. Two *-homomorphisms $\varphi, \psi\colon A \to B$ are said to induce the same map at the level of ideals, if $\varphi^{-1}(I) = \psi^{-1}(I)$ for all closed two-sided ideals I in B. This is the case if and only if the closed two-sided ideals in B generated by $\varphi(a)$ and $\psi(a)$, respectively, coincide for all $a \in A$.

Theorem 4.5 *Let A be a separable, nuclear C^*-algebra, let B be an $\mathcal{O}_2$-absorbing stable C^*-algebra, and let $\varphi, \psi\colon A \to B$ be *-homomorphisms. Then φ and ψ induce the same map at the level of ideals if and only if they are approximately unitarily equivalent, i.e., there is a sequence $\{u_n\}_{n=1}^\infty$ of unitary elements of $\mathcal{M}(B)$ such that $u_n^*\psi(a)u_n \to \varphi(a)$ for all $a \in A$.*

Proof: The "if" part is trivial. To prove the "only if" part, let $\mathscr{K}$ denote the set of all completely positive maps $A \to B$ of the form $V \circ \varphi$, where $V\colon \varphi(A) \to B$ is approximately inner. Then $\mathscr{K}$ is operator convex (cf. the proof of Corollary 4.3). By assumption, $\psi(a)$ belongs to $\overline{B\varphi(a)B}$ for all $a \in A$. The closed two-sided ideal in B generated by $\{W(a) \mid W \in \mathscr{K}\}$ contains $\overline{B\varphi(a)B}$, and hence $\psi(a)$, for all $a \in A$. Thus ψ belongs to the point-norm closure of $\mathscr{K}$ by Proposition 4.2. It follows from [9, Proposition 8.4] that all approximately inner completely positive maps $\varphi(A) \to B$ are approximately 1-step inner. Accordingly, there is a sequence $\{d_n\}$ in B such that $d_n^*\varphi(a)d_n \to \psi(a)$ for all $a \in A$.

Let $\{e_n\}_{n=1}^\infty$ be an approximate unit for A consisting of positive contractions, and put $c_n = \varphi(e_n)d_n$. Then $c_n^*c_n = d_n^*\varphi(e_n^2)d_n \to \psi(e_n^2)$, whence $\|c_n\| \to 1$, and $c_n^*\varphi(a)c_n \to \psi(a)$ for all $a \in A$. The contractions $b_n = \|c_n\|^{-1}c_n$ satisfy $b_n^*\varphi(a)b_n \to \psi(a)$ for all $a \in A$. By [9, Lemma 2.4] there is a sequence $\{s_n\}$ of isometries in $\mathcal{M}(B)$ such that $s_n^*\varphi(a)s_n \to \psi(a)$ for all $a \in A$.

A similar argument shows that there is a sequence $\{t_n\}$ of isometries in $\mathcal{M}(B)$ satisfying $t_n^*\psi(a)t_n \to \varphi(a)$ for all $a \in A$.

For all $x, y \in \varphi(A)$,

$$\lim_{n\to\infty} \|s_n^*xys_n - s_n^*xs_ns_n^*ys_n\| = 0, \qquad \lim_{n\to\infty} \|t_n^*s_n^*xs_nt_n - x\| = 0,$$

and so it follows from Lemma 2.5 that there is a sequence $\{u_n\}$ of unitary elements in $\mathcal{M}(B)$ such that $u_n^*\psi(a)u_n \to \varphi(a)$ for all $a \in A$. $\square$

5 The main result: ideal-system preserving zero homotopic purely infinite C^*-algebras

In this section we prove our main result. As we shall spend a great deal of effort to keep track of ideals, it is convenient to set up some relevant notation. If F is a subset of a C^*-algebra A, then $\overline{AFA}$ and $I_A(F)$ will both denote the closed two-sided ideal in A generated by F. In the case where $F = \{a\}$ is a singleton, we write $\overline{AaA}$ and $I_A(a)$, respectively, instead of $\overline{AFA}$ and $I_A(F)$.

Definition 5.1 A C^*-algebra A is said to be *zero-homotopic in an ideal-system preserving way* if there is a pointwise-norm continuous path $\{\rho_t\}_{t\in[0,1]}$ of *-endomorphisms on A such that $\rho_0 = 0$, $\rho_1 = \mathrm{id}_A$, and $\rho_t(J) \subseteq J$ for all closed two-sided ideals J in A.

A word of warning: The cone $CA = C_0((0,1], A)$ over a C^*-algebra A is usually not zero-homotopic in an ideal-system preserving way (although it is zero-homotopic). The standard homotopy path $\varphi_t\colon CA \to CA$ from the identity to zero is not ideal preserving. In Section 6 we shall give examples of C^*-algebras that are zero-homotopic in an ideal-system preserving way.

We emphasize below an important consequence of the classification of separable, nuclear, $\mathcal{O}_\infty$-absorbing, stable C^*-algebras by a an ideal respecting version of Kasparov's KK-theory.

Theorem 5.2 (Kirchberg [6]) *Let A be a separable, stable, $\mathcal{O}_\infty$-absorbing, nuclear C^*-algebra that is zero-homotopic in an ideal-system preserving way. Then $A \cong A \otimes \mathcal{O}_2$.*

Lemma 5.3 *Let A be a separable stable C^*-algebra. Then A contains an Abelian sub-C^*-algebra C that separates ideals of A.*

That C separates ideals of A means that $I \cap C = J \cap C \Rightarrow I = J$ for any two closed two-sided ideals I, J of A.

Proof: Because A is stable and separable we can write $A = A_0 \otimes \mathcal{K}$ for some separable C^*-algebra A_0. Let $\{a_1, a_2, \dots\}$ be a dense subset of the positive cone in A_0. Let $\{p_{n,m}\}_{n,m=1}^{\infty}$

be pairwise orthogonal 1-dimensional projections in $\mathcal{K}$, and put

$$C = C^*\big((a_n - 1/m)_+ \otimes p_{n,m} \mid n, m \in \mathbb{N}\big).$$

Clearly, C is an Abelian sub-C^*-algebra of A. We must check that C separates ideals, so let I and J be closed two-sided ideals in $A = A_0 \otimes \mathcal{K}$ with $I \cap C = J \cap C$. Note that $I = I_0 \otimes \mathcal{K}$ and $J = J_0 \otimes \mathcal{K}$ for some ideals I_0 and J_0 in A_0. Let a be a positive element in I_0 and let $\varepsilon > 0$. Find $n, m \in \mathbb{N}$ such that $\|a - a_n\| < 1/m < \varepsilon$. Then $(a_n - 1/m)_+ = x^*ax$ for some x in A_0 (see [9, Lemma 2.2]), whence

$$(a_n - 1/m)_+ \otimes p_{n,m} \in I_A(a \otimes p_{n,m}) \cap C \subseteq I \cap C = J \cap C,$$

and so $(a_n - 1/m)_+$ belongs to J_0. As

$$\|a - (a_n - 1/m)_+\| \leq \|a - a_n\| + \|a_n - (a_n - 1/m)_+\| < 1/m + 1/m \leq 2\varepsilon,$$

and because $\varepsilon > 0$ was arbitrary, it follows that a belongs to J_0. This proves $I_0 \subseteq J_0$, and a symmetric argument yields the reverse inclusion. □

The reader is referred to [9, Definition 5.1] for a formal definition being "*strongly purely infinite*". Every $\mathcal{O}_\infty$-absorbing C^*-algebra is strongly purely infinite, and it is proved in [9] that every separable, nuclear, strongly purely infinite C^*-algebra that is either unital or stable is $\mathcal{O}_\infty$-absorbing.

Proposition 5.4 (Proposition 7.13 of [9]) *Suppose that A is a separable strongly purely infinite C^*-algebra. Let C be an Abelian sub-C^*-algebra of A and let ω be a free filter on $\mathbb{N}$. Then there is an Abelian C^*-algebra C_1 with $C \subseteq C_1 \subseteq A_\omega$ and a 1-step inner completely positive contraction $V\colon A \to C_1$ with $V(c) = c$ for all c in C.*

From now on, let $V\colon A \to C_1 \subseteq A_\omega$ be the completely positive contraction as in Proposition 5.4 wrt. the Abelian sub-C^*-algebra C found in Lemma 5.3.

The Kasparov–Stinespring dilation. Kasparov's version of the Stinespring dilation, [5, Theorem 3], shows that every completely positive contraction $T\colon A \to B$ has a dilation to a *-homomorphism $\psi_T\colon A \to \mathcal{M}(B \otimes \mathcal{K})$ which satisfies

$$(1 \otimes e_{11})\psi_T(a)(1 \otimes e_{11}) = T(a) \otimes e_{11}, \qquad a \in A, \tag{5.1}$$

where e_{11} is a 1-dimensional projection in $\mathcal{K}$. We shall apply this to the completely positive contraction $V\colon A \to C_1$ considered above.

Because $V(c) = c$ for all $c \in C \subseteq C_1$ the two mappings $c \mapsto \psi_V(c)$ and $c \mapsto c \otimes 1$ from C into $\mathcal{M}(C_1 \otimes \mathcal{K})$ are approximately unitarily equivalent. In particular, $I_{\mathcal{M}(C_1\otimes\mathcal{K})}(\psi_V(c)) = I_{\mathcal{M}(C_1\otimes\mathcal{K})}(c \otimes 1)$ for all $c \in C$. We note for later use that ψ_C is ideal preserving as follows:

Lemma 5.5 *For each closed two-sided ideal J in A, set $J_1 = I_{C_1}(J \cap C)$, and denote by $\mathcal{M}(C_1 \otimes \mathcal{K}, J_1 \otimes \mathcal{K})$ the closed two-sided ideal in $\mathcal{M}(C_1 \otimes \mathcal{K})$ consisting of those multipliers $x \in \mathcal{M}(C_1 \otimes \mathcal{K})$ for which $xy, yx \in J_1 \otimes \mathcal{K}$ for all $y \in C_1 \otimes \mathcal{K}$. Then*

$$\psi_V(J) \subseteq \mathcal{M}(C_1 \otimes \mathcal{K}, J_1 \otimes \mathcal{K}). \tag{5.2}$$

Proof: Take first $c \in J \cap C$. Then

$$I_{\mathcal{M}(C_1\otimes\mathcal{K})}(\psi_V(c)) \cap C_1 \otimes \mathcal{K} \;=\; I_{\mathcal{M}(C_1\otimes\mathcal{K})}(c \otimes 1) \cap C_1 \otimes \mathcal{K} \;\subseteq\; J_1 \otimes \mathcal{K},$$

whence $\psi_V(c)$ belongs to $\mathcal{M}(C_1 \otimes \mathcal{K}, J_1 \otimes \mathcal{K})$. It follows that $\psi_V(a) \in \mathcal{M}(C_1 \otimes \mathcal{K}, J_1 \otimes \mathcal{K})$ for all $a \in I_A(J \cap C)$. But $I_A(J \cap C) = J$ for every closed two-sided ideal J in A because C separates ideals in A. $\square$

We shall from now on consider a C^*-algebra A that has the following properties:

separable, strongly purely infinite, and zero-homotopic in an ideal preserving way. (†)

Property (†) passes to quotients. In particular, A has no non-zero projections, and no quotient of A contains a non-zero projection. As A in particular is purely infinite and has no unital quotient, it follows from [8, Theorem 4.24] that A is stable. If A besides having property (†) is nuclear, then $A \cong A \otimes \mathcal{O}_\infty$ (by [9, Theorem 8.6]). Furthermore, when A is nuclear, Theorem 5.2 implies that $A \cong A \otimes \mathcal{O}_2$.

Fix a path $\{\rho_t\}_{t\in[0,1]}$ of *-endomorphism that implements the zero-homotopy and define the *-homomorphism

$$\rho\colon A \to C_0(]0,1], A) \quad \text{by} \;\; \rho(a)(t) = \rho_t(a), \;\; a \in A.$$

Note that $\rho(J) \subseteq C_0(]0,1], J)$ for all closed two-sided ideals J of A by assumption on the path ρ_t.

In the following lemmas we show that if A is nuclear (besides having property (†)), then it satisfies condition (iv) of Theorem 3.6 wrt. any free filter ω on $\mathbb{N}$. (We can take

$\omega = \omega_\infty$.) The idea of the proof is outlined in the diagram:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\psi_V"'] & C_0(]0,1], A) \arrow[d, dashed, "W_n"] & & \\ \mathcal{M}(C_1 \otimes \mathcal{K}) & C_1 \otimes \mathcal{K} \arrow[l, hook'] \arrow[r, dashed, "\iota"'] & A_\omega \arrow[r, dashed, "T_n"'] & A_\omega \end{tikzcd} \tag{5.3}$$

The maps W_n, ι, and T_n (that are to be constructed) are respectively an almost multiplicative completely positive contraction, a *-homomorphism, and a completely positive contraction, chosen such that the sequence $\{T_n \circ \iota \circ W_n \circ \rho\}$ converges pointwise to the inclusion mapping $A \to A_\omega$. Using Remark 2.6 (among other results) one finds a sequence $\{u_n\}$ of unitaries in $\mathcal{M}(A)_\omega$ such that $\|u_n(\iota \circ W_n \circ \rho)(a)u_n^* - a\| \to 0$ for all $a \in A$. As $C_1 \otimes \mathcal{K}$ is the completion of $\bigcup_{k=1}^\infty C_1 \otimes M_k$, we can choose the sub-$C^*$-algebra B of A_ω as in Theorem 3.6 (iv) to be $u_n(\iota(C_1 \otimes M_k))u_n^*$ for some large enough n and k.

We begin by constructing the mapping W_n. For each natural number n choose a continuous function $g_n \colon [0,1]^2 \to \mathbb{R}^+$ such that

$$\int_0^1 g_n(s,t)^2 dt = 1 \;\text{ for all }\; s \in [0,1], \quad \text{and} \quad g_n(s,t) = 0 \;\text{ when }\; |s-t| \geq 1/n. \tag{5.4}$$

Choose a quasi-central approximate unit $\{e_n\}_{n=1}^\infty$ for $C_1 \otimes \mathcal{K}$ consisting of positive contractions that satisfy $e_n e_{n+1} = e_n = e_{n+1} e_n$ and $e_n(1 \otimes e_{11}) = (1 \otimes e_{11})e_n$ for all n, and

$$\lim_{n\to\infty} \sup_{t \in [0,1]} \|g_n(e_n,t)\psi_V(a) - \psi_V(a)g_n(e_n,t)\| = 0, \qquad a \in A.$$

For each $f \in C_0(]0,1], A)$ and for each natural number n put

$$\widetilde{W}_n(f) = \int_0^1 g_n(e_n,t)\psi_V(f(t))g_n(e_n,t)dt, \quad W_n(f) = \int_{1/n}^1 g_n(e_n,t)\psi_V(f(t))g_n(e_n,t)dt.$$

We record below some properties of the maps W_n.

Lemma 5.6 *Let f, g be in $C_0(]0,1], A)$ and let $m < n$ be natural numbers.*

(i) $\lim_{n\to\infty} \|\widetilde{W}_n(f) - W_n(f)\| = 0$.

(ii) *W_n is a completely positive contraction from $C_0(]0,1], A)$ into $C_1 \otimes \mathcal{K}$.*

(iii) *$W_n(f)$ belongs to the ideal in $C_1 \otimes \mathcal{K}$ generated by $\{e_n\psi_V(f(t))e_n \mid t \in [0,1]\}$.*

(iv) $\lim_{n\to\infty} \|W_n(fg) - W_n(f)W_n(g)\| = 0$.

(v) $\|e_m W_n(f) e_m - e_m \psi_V(f(1)) e_m\| \leq \sup\{\|f(t) - f(1)\| : t \in [1-1/n, 1]\}$.

Proof: (i). Since $f(t) \to 0$ as $t \to 0$ we get

$$\begin{aligned} \|\widetilde{W}_n(f) - W(f)\| &= \Big\| \int_0^{1/n} g_n(e_n,t)\psi_V(f(t))g_n(e_n,t)dt \Big\| \\ &\leq \sup_{t\in[0,1/n]} \|f(t)\| \cdot \Big\| \int_0^1 g_n(e_n,t)^2 dt \Big\| = \sup_{t\in[0,1/n]} \|f(t)\| \to 0. \end{aligned}$$

(ii) and (iii). Approximating the integral that defines W_n by a Riemann sum we see that W_n is a completely positive map. Note that $g(0,t) = 0$ for $t \geq 1/n$. It follows that $g(e_n,t)$ belongs to $C^*(e_n)$ for $t \geq 1/n$, whence $g(e_n,t)\psi_V(f(t))g(e_n,t)$ belongs to the ideal in $C_1 \otimes \mathcal{K}$ generated by $e_n \psi_V(f(t)) e_n$.

(iv). By (i) it will suffice to show that $\|\widetilde{W}_n(fg) - \widetilde{W}_n(f)\widetilde{W}_n(g)\| \to 0$. Note that $g_n(x,t)g_n(x,s) = 0$ if $x,t,s \in [0,1]$ and $|t-s| \geq 2/n$. It follows that $g_n(e_n,t)g_n(e_n,s) = 0$ when $|t-s| \geq 2/n$. Hence

$$\begin{aligned} \widetilde{W}_n(f)\widetilde{W}_n(g) &= \int_0^1 \int_0^1 g_n(e_n,t)\psi_V(f(t))g_n(e_n,t)g_n(e_n,s)\psi_V(g(s))g_n(e_n,s)\, ds\, dt \\ &= \int_0^1 \int_{(t-2/n)\vee 0}^{(t+2/n)\wedge 1} g_n(e_n,t)\psi_V(f(t))g_n(e_n,t)g_n(e_n,s)\psi_V(g(s))g_n(e_n,s)\, ds\, dt. \end{aligned}$$

By the choice of the quasi-central approximate unit $\{e_n\}$,

$$\lim_{n\to\infty} \sup_{s,t\in[0,1]} \|g_n(e_n,t)g_n(e_n,s)\psi_V(g(s)) - \psi_V(g(s))g_n(e_n,t)g_n(e_n,s)\| = 0;$$

and g is uniformly continuous, whence

$$\lim_{n\to\infty} \sup\{\|g(s) - g(t)\| : s,t \in [0,1], |s-t| \leq 2/n\} = 0.$$

It follows that

$$\begin{aligned} \widetilde{W}_n(f)\widetilde{W}_n(g) &= \int_0^1 \int_{(t-2/n)\vee 0}^{(t+2/n)\wedge 1} g_n(e_n,t)\psi_V(f(t)g(t))g_n(e_n,t)g_n(e_n,s)^2\, ds\, dt + \mathrm{o}(1/n) \\ &= \int_0^1 \int_0^1 g_n(e_n,t)\psi_V(f(t)g(t))g_n(e_n,t)g_n(e_n,s)^2\, ds\, dt + \mathrm{o}(1/n) \\ &= \widetilde{W}_n(fg) + \mathrm{o}(1/n), \end{aligned}$$

where $\{o(1/n)\}$ is a sequence of elements whose norm tend to zero as n tends to infinity.

(v). Because $e_m e_n = e_m$ we have $e_m g_n(e_n, t) = e_m g_n(1, t)$, and hence,

$$\begin{aligned} e_m W_n(f) e_m - e_m \psi_V(f(1)) e_m &= e_m\Big(\int_{1/n}^{1} g_n(1,t)\psi_V\big(f(t) - f(1)\big) g_n(1,t)dt\Big) e_m \\ &= e_m\Big(\int_{1-1/n}^{1} g_n(1,t)\psi_V\big(f(t) - f(1)\big) g_n(1,t)dt\Big) e_m \end{aligned}$$

when $n \geq 2$. The norm of the last expression is less than $\sup_{t\in[1-1/n,1]} \|f(t) - f(1)\|$. $\square$

Lemma 5.7 *Suppose A has property (†). Then there is a *-homomorphism $\iota\colon C_1 \otimes \mathcal{K} \to A_\omega$ such that*

$$I_{A_\omega}\big(\iota(J \otimes \mathcal{K})\big) = I_{A_\omega}(J)$$

for all closed two-sided ideals J in $C_1 \subseteq A_\omega$.

It follows in particular that $I_{A_\omega}(\iota(x \otimes e)) = I_{A_\omega}(x)$ for all $x \in C_1$ and for all non-zero $e \in \mathcal{K}$.

Proof: The multiplier algebra $\mathcal{M}(A)$ is properly infinite by the assumption that A is stable. Thus $\mathcal{M}(A)_\omega$ is properly infinite because it contains $\mathcal{M}(A)$, and so we can find a sequence $s_1, s_2, \dots$ of isometries in $\mathcal{M}(A)_\omega \subseteq \mathcal{M}(A_\omega)$ with orthogonal ranges. Define $\iota\colon C_1 \otimes \mathcal{K} \to A_\omega$ by $\iota(c \otimes e_{kl}) = s_k c s_l^*$, for $c \in C_1$ and where $\{e_{kl}\}_{k,l=1}^{\infty}$ is a system of matrix units for $\mathcal{K}$.

Let J be a closed two-sided ideal in C_1. Take x in J. Then

$$x = s_1^*(s_1 x s_1^*)s_1 = s_1^*\iota(x \otimes e_{11})s_1 \in I_{A_\omega}\big(\iota(J \otimes \mathcal{K})\big).$$

This proves $I_{A_\omega}(J) \subseteq I_{A_\omega}\big(\iota(J \otimes \mathcal{K})\big)$. To show the reverse inclusion, we use that $J \otimes \mathcal{K}$ is the closure of the linear span of elements of the form $x \otimes e_{kl}$, where $x \in J$ and $k, l \in \mathbb{N}$. Now, $\iota(x \otimes e_{kl}) = s_k x s_l^* \in I_{A_\omega}(J)$, when x belongs to J, and this proves $I_{A_\omega}\big(\iota(J \otimes \mathcal{K})\big) \subseteq I_{A_\omega}(J)$. $\square$

Put $S_n = \iota \circ W_n \circ \rho\colon A \to A_\omega$. Each S_n is a completely positive contraction, and $\{S_n\}_{n=1}^{\infty}$ is asymptotically multiplicative, ie., $\lim_{n\to\infty} \|S_n(ab) - S_n(a)S_n(b)\| = 0$ for all $a, b \in A$ by Lemma 5.6 (iv).

Lemma 5.8 *Suppose that A has property (†) and that A is nuclear. Then there are isometries s_n in $\mathcal{M}(A)_\omega$ such that $S_n(a) = s_n^* a s_n$ for all $a \in A$.*

Proof: Doing the proof backwards, use Lemma 2.2 to reduce the problem to showing that S_n is approximately 1-step inner. As A is strongly purely infinite by assumption, so is A_ω, cf. [9, Proposition 5.12]. By [9, Theorem 7.21], every approximately inner completely positive contraction $A \to A_\omega$ is actually approximately 1-step inner, so it suffices to show that S_n is approximately inner.

To show that S_n is approximately inner it suffices to check that $S_n(a)$ belongs to $I_{A_\omega}(a)$ for every $a \in A$, cf. Corollary 4.3. Fix $a \in A$, put $J = I_A(a)$, $J_1 = I_{C_1}(J \cap C)$, $J_2 = I_{A_\omega}(a)$, and observe that $J_1 \subseteq J_2$. As $\rho_t(a) \in J$ for all $t \in [0,1]$ it follows from (5.2) that $\psi_V(\rho_t(a))$ belongs to $\mathcal{M}(C_1 \otimes \mathcal{K}, J_1 \otimes \mathcal{K})$. Lemma 5.6 (iii) now implies that $W_n(\rho(a))$ belongs to $J_1 \otimes \mathcal{K}$, and so $S_n(a) = \iota\big(W_n(\rho(a))\big)$ belongs to $\iota(J_1 \otimes \mathcal{K})$. Finally, $I_{A_\omega}(\iota(J_1 \otimes \mathcal{K})) = I_{A_\omega}(J_1) \subseteq J_2$ by Lemma 5.7. □

Lemma 5.9 *Suppose that A has property (†) and that A is nuclear. There is a sequence of 1-step inner completely positive contractions $T_n \colon A_\omega \to A_\omega$ such that $(T_n \circ S_n)(a) \to a$ for all $a \in A$.*

Proof: To ease the notation we let B denote A_ω (so that $S_n \colon A \to B$). It suffices to show that for each finite subset F of contractions in A and for each $\varepsilon > 0$ there is a natural number n_0 and a sequence $\{g_n\}_{n=n_0}^{\infty}$ of contractions in B such that $\|g_n^* S_n(a) g_n - a\| < \varepsilon$ for all $a \in F$ and all $n \geq n_0$. Upon making a small perturbation of the elements in F we may assume that A contains a positive contraction e such that $ea = ae = a$ for all $a \in F$. Put $F_1 = F \cup \{e\}$.

Recall that B_∞ denotes the quotient algebra $\ell_\infty(B)/c_0(B)$, that $\pi_\infty \colon \ell_\infty(B) \to B_\infty$ is the quotient mapping, and that $A \subseteq A_\omega = B \subseteq B_\infty$ (where the canonical inclusion mappings are suppressed).

Let $S \colon A \to B_\infty$ be given by

$$S(a) = \pi_\infty(S_1(a), S_2(a), S_3(a), \dots), \qquad a \in A.$$

Then S is a *-homomorphism, cf. Lemma 5.6 (iv). We show first that S is ideal preserving in the sense that

$$a \in I_{B_\infty}(S(a)), \qquad a \in A. \tag{5.5}$$

It suffices to verify (5.5) for elements c in C (because C separates ideals in A), so take such an element. Retain the notation $\{e_m\}$ for the approximate unit for $C_1 \otimes \mathcal{K}$ that is used to define W_m. Then

$$\|e_{n-1}W_n(\rho(c))e_{n-1} - e_{n-1}\psi_V(c)e_{n-1}\| \le \sup\{\|\rho_t(c) - \rho_1(c)\| : t \in [1-1/n, 1]\} \tag{5.6}$$

for all $n \ge 2$ by Lemma 5.6 (v). By (5.1), and since e_{n-1} commutes with $1 \otimes e_{11}$, we get

$$(1 \otimes e_{11})e_{n-1}\psi_V(c)e_{n-1}(1 \otimes e_{11}) = e_{n-1}(V(c) \otimes e_{11})e_{n-1} = e_{n-1}(c \otimes e_{11})e_{n-1} \underset{n\to\infty}{\longrightarrow} c \otimes e_{11}.$$

Multiply (5.6) by $1 \otimes e_{11}$ from the left and from the right:

$$\lim_{n\to\infty} \|(1 \otimes e_{11})e_{n-1}W_n(\rho(c))e_{n-1}(1 \otimes e_{11}) - c \otimes e_{11}\| = 0. \tag{5.7}$$

Apply the map $\iota\colon C_1 \otimes \mathcal{K} \to A_\omega$ defined in Lemma 5.7 to (5.7), and recall from the proof of that lemma that $\iota(c \otimes e_{11}) = s_1 c s_1^*$ for some isometry s_1 in $\mathcal{M}(B)$:

$$\lim_{n\to\infty} \|x_n^* S_n(c) x_n - s_1 c s_1^*\| = 0, \quad \text{when } x_n = \iota(e_{n-1}(1 \otimes e_{11})) \in B.$$

Hence $s_1^* x^* S(c) x s_1 = c$ when $x = \pi_\infty(x_1, x_2, \dots)$. This proves (5.5).

Let $\mathscr{K}$ be the set of completely positive maps $A \to B_\infty$ of the form $a \mapsto d^*S(a)d$ for some d in B_∞. We show next that $\mathscr{K}$ is operator convex, cf. Definition 4.1. It is trivial that (ii) in Definition 4.1 holds and that αW belongs to $\mathscr{K}$ whenever $\alpha \in \mathbb{R}^+$ and $W \in \mathscr{K}$.

By Lemma 2.3 there is a unital embedding of $\mathcal{O}_\infty$ into $\mathcal{M}(B_\infty) \cap S(A)'$, and so there are isometries t_1, t_2 in $\mathcal{M}(B_\infty) \cap S(A)'$ with $t_i^* t_j = 0$ when $i \neq j$. Take $W_1, W_2 \in \mathscr{K}$. Then $W_j(a) = d_j^* S(a) d_j$ for some d_1, d_2 in B_∞. Put $d = t_1 d_1 + t_2 d_2 \in B_\infty$. Then

$$d^*S(a)d = \sum_{i,j=1}^{2} d_i^* t_i^* S(a) t_j d_j = \sum_{i,j=1}^{2} d_i^* t_i^* t_j S(a) d_j = W_1(a) + W_2(a),$$

which shows that $W_1 + W_2$ belongs to $\mathscr{K}$ (so (i) in Definition 4.1 holds).

To verify Definition 4.1 (iii), let $W \in \mathscr{K}$, $c_1, \dots, c_n \in A$, and $b_1, \dots, b_n \in B_\infty$ be given. Take d in B_∞ such that $W(a) = d^*S(a)d$, and put $d_1 = \sum_{j=1}^n S(c_j) d b_j$. Then

$$d_1^* S(a) d_1 = \sum_{i,j=1}^{n} b_i^* d^* S(c_i^*) S(a) S(c_j) d b_j = \sum_{i,j=1}^{n} b_i^* d^* S(c_i^* a c_j) d b_j = \sum_{i,j=1}^{n} b_i^* W(c_i^* a c_j) b_j,$$

which shows that the mapping $a \mapsto \sum_{i,j=1}^n b_i^* W(c_i^* a c_j) b_j$ belongs to $\mathscr{K}$.

It now follows from Proposition 4.2 that the inclusion mapping $j\colon A \to B_\infty$ belongs to the point-norm closure of $\mathscr{K}$, because

$$j(a) = a \in I_{B_\infty}(S(a)) = I_{B_\infty}\big(\{W(a) \mid W \in \mathscr{K}\}\big), \qquad a \in A,$$

by (5.5). Hence there exists an element d in B_∞ such that $\|d^*S(a)d - a\| < \varepsilon/3$ for all a in the finite subset F_1 of A.

We show next that there is a *contraction* g in B_∞ with $\|g^*S(a)g - a\| < \varepsilon$ for all a in the given finite subset F of A. Note first that e belongs to F_1 and that $\|d^*S(e)d - e\| < \varepsilon/3$. Put $f = S(e)^{1/2}d$ and put $g = \|f\|^{-1}f$. Then $f^*S(a)f = d^*S(a)d$ for all $a \in F$ (because $ae = ea = a$ for all $a \in F$); and

$$\|f - g\| \;\le\; |\|f\| - 1| \;\le\; |\|f\|^2 - 1| \;=\; |\|d^*S(e)d\| - 1| \;\le\; \|d^*S(e)d - e\| \;<\; \varepsilon/3.$$

It follows that

$$\begin{aligned}\|g^*S(a)g - a\| \;&\le\; \|g^*S(a)g - f^*S(a)f\| + \|f^*S(a)f - a\| \\ &\le\; 2\|f - g\| + \|f^*S(a)f - a\| \;<\; \varepsilon\end{aligned}$$

for all $a \in F$.

Write $g = \pi_\infty(g_1, g_2, \dots)$, where $g_1, g_2, \dots$ are contractions in B. Then

$$\limsup_{n\to\infty} \|g_n^*S_n(a)g_n - a\| = \|g^*S(a)g - a\| < \varepsilon$$

for all $a \in F$. This completes the proof, cf. the remarks in the first paragraph of the proof. $\square$

Lemma 5.10 *Suppose that A has property (†) and that A is nuclear. Then for each n there is an isometry t_n in $\mathcal{M}(A)_\omega$ such that $t_n^*S_n(a)t_n = (T_n \circ S_n)(a)$ for all a in A.*

Proof: Fix n and let B be the separable sub-C^*-algebra of A_ω generated by $S_n(A)$. Let $T_n'\colon B \to A_\omega$ be the restriction of the 1-step inner completely positive contraction $T_n\colon A_\omega \to A_\omega$ from Lemma 5.9. Then T_n' is still a 1-step inner completely positive contraction, so by Lemma 2.2 there is an isometry t_n in $\mathcal{M}(A)_\omega$ such that $T_n'(b) = t_n^*bt_n$ for all $b \in B$. $\square$

Lemma 5.11 *Suppose that A has property (†) and that A is nuclear. Then there is a sequence $\{u_n\}_{n=1}^\infty$ of unitary elements in $\mathcal{M}(A)_\omega$ that satisfies $\|u_n^*S_n(a)u_n - a\| \to 0$ for all $a \in A$.*

Proof: The conditions on A imply that A is $\mathcal{O}_2$-absorbing as explained below Lemma 5.5.

Combining Lemmas 5.8 and 5.10 we obtain sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ of isometries in $\mathcal{M}(A)_\omega$ such that $s_n^* a s_n = S_n(a)$ and $\|t_n^* s_n^* a s_n t_n - a\| \to 0$ for all $a \in A$. The existence of the unitaries $u_n \in \mathcal{M}(A)_\omega$ follows from Remark 2.6 and the fact established in Lemma 5.6 (iv) that the sequence $\{S_n\}$ is asymptotically multiplicative. $\square$

Consult [9, Definition 5.1] for a formal definition being "strongly purely infinite" (or see also the comments above Proposition 5.4).

Theorem 5.12 *Let A be a separable, nuclear, strongly purely infinite C^*-algebra. Suppose that A is zero-homotopic in an ideal-system preserving way (cf. Definition 5.1). Then A is stable and $\mathcal{O}_2$-absorbing, i.e, $A \cong A \otimes \mathcal{O}_2 \otimes \mathcal{K}$, and A is the inductive limit of a sequence*

$$C_0((\Gamma_1, v_1), M_{k_1}) \xrightarrow{\varphi_1} C_0((\Gamma_2, v_2), M_{k_2}) \xrightarrow{\varphi_2} C_0((\Gamma_3, v_3), M_{k_3}) \xrightarrow{\varphi_3} \cdots \longrightarrow A, \qquad (5.8)$$

where each (Γ_j, v_j) is a connected pointed graph, each k_j is a natural number, and each φ_j is an injective *-homomorphism.

In particular, A is an AH_0-algebra.

Proof: It is shown below Lemma 5.5 that the given conditions on A imply that A is stable and $\mathcal{O}_2$-absorbing.

To prove that A is an inductive limit as stipulated, we use Theorem 3.6 whereby it suffices to show that condition (iv) of that theorem holds. Let $F = \{a_1, a_2, \dots, a_m\}$ be a finite subset of A, let $\varepsilon > 0$, and let ω be a free filter on $\mathbb{N}$. Let $S_n \colon A \to A_\omega$ and $u_n \in \mathcal{M}(A)_\omega$ be as constructed earlier in this section. Choose n large enough so that $\|u_n^* S_n(a) u_n - a\| < \varepsilon/2$ for all $a \in F$. Realize $C_1 \otimes \mathcal{K}$ as the closure of $\bigcup_{k=1}^{\infty} C_1 \otimes M_k$, and set $B_{n,k} = u_n^*(\iota(C_1 \otimes M_k))u_n$ (cf. (5.3)). Then $B_{n,k}$ is a sub-C^*-algebra of A_ω isomorphic to $C_1 \otimes M_k$ and hence to $M_k(C_0(X))$, where X is the spectrum of the Abelian C^*-algebra C_1. As the closure of $\bigcup_{k=1}^{\infty} B_{n,k}$ equals the image of $\operatorname{Ad} u_n \circ \iota$ and hence contains $u_n^* S_n(a) u_n$ for all $a \in A$, we can find a natural number k and elements $b_1, b_2, \dots, b_m$ in $B_{n,k}$ such that $\|u_n^* S_n(a_j) u_n - b_j\| < \varepsilon/2$ for all j. Hence $\|a_j - b_j\| < \varepsilon$, so property (iv) of Theorem 3.6 holds with $B = B_{n,k}$. $\square$

6 Applications

We remind the reader of the construction of the C^*-algebra $\mathcal{A}_{[0,1]}$ considered in [15]. Take a dense sequence $\{t_n\}_{n=1}^{\infty}$ in $[0,1)$, and let $\mathcal{A}_{[0,1]}$ be the inductive limit of the sequence

$$C_0([0,1), M_2) \xrightarrow{\varphi_1} C_0([0,1), M_4) \xrightarrow{\varphi_2} C_0([0,1), M_8) \xrightarrow{\varphi_3} \cdots \longrightarrow \mathcal{A}_{[0,1]},$$

where $\varphi_n(f)(t) = \mathrm{diag}(f(t), f(t \vee t_n))$. Put $A_n = C_0([0,1), M_{2^n})$, and let $\varphi_{\infty,n} \colon A_n \to \mathcal{A}_{[0,1]}$ denote the inductive limit map.

It was shown in [15] that $\mathcal{A}_{[0,1]}$ is stable and that $\mathcal{A}_{[0,1]} \cong \mathcal{A}_{[0,1]} \otimes \mathcal{O}_\infty$. It is clear that $\mathcal{A}_{[0,1]}$ is separable and nuclear. Note also that $\mathcal{A}_{[0,1]}$ by its definition is an inductive limit as in the (5.8) in Theorem 5.12.

For each $n \in \mathbb{N}$ and for each $t \in [0,1]$ put $I_t^{(n)} = C_0([0,t), M_{2^n}) \subseteq A_n$, i.e., a function $f \in A_n$ belongs to $I_t^{(n)}$ if and only if $f|_{[t,1]} \equiv 0$. As $\varphi_n^{-1}(I_t^{(n+1)}) = I_t^{(n)}$ for all n and t, there is for each $t \in [0,1]$ a (unique) closed two-sided ideal I_t in $\mathcal{A}_{[0,1]}$ with $\varphi_{\infty,n}^{-1}(I_t) = I_t^{(n)}$ for all n. It is shown in [15] that any closed two-sided ideal in $\mathcal{A}_{[0,1]}$ is equal to I_t for some $t \in [0,1]$.

Proposition 6.1 *The C^*-algebra $\mathcal{A}_{[0,1]}$ is zero-homotopic in an ideal-system preserving way. In particular, $\mathcal{A}_{[0,1]} \otimes \mathcal{O}_2 \cong \mathcal{A}_{[0,1]}$.*

Proof: For each $s \in [0,1]$ let $\psi_n^{(s)} \colon A_n \to A_n$ be the *-homomorphism given by $\psi_n^{(s)}(f)(t) = f(t \vee s)$. Note that $s \mapsto \psi_n^{(s)}(f)$ is continuous for all $f \in A_n$, that $\psi_n^{(s)}(I_t \cap A_n) \subseteq I_t \cap A_n$, and that $\psi_n^{(0)} = \mathrm{id}$ and $\psi_n^{(1)} = 0$. Moreover, as $t \vee t_n \vee s = t \vee s \vee t_n$ for all n, we get a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1^{(s)}"'] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2^{(s)}"'] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "\psi_3^{(s)}"'] & \cdots \arrow[r] & \mathcal{A}_{[0,1]} \arrow[d, dashed, "\psi^{(s)}"] \\
A_1 \arrow[r, "\varphi_1"'] & A_2 \arrow[r, "\varphi_2"'] & A_3 \arrow[r, "\varphi_3"'] & \cdots \arrow[r] & \mathcal{A}_{[0,1]}
\end{tikzcd}$$

that induces a point-norm continuous path $\psi^{(s)}$, $s \in [0,1]$, of *-endomorphisms on $\mathcal{A}_{[0,1]}$ that satisfies $\psi^{(s)}(I_t) \subseteq I_t$ for all s and t, and $\psi^{(0)} = \mathrm{id}$ and $\psi^{(1)} = 0$. This verifies that $\mathcal{A}_{[0,1]}$ is zero-homotopic in an ideal-system preserving way. The isomorphism $A \cong A \otimes \mathcal{O}_2$ now follows from Theorem 5.2. $\square$

Proposition 6.2 *The tensor product $\mathcal{A}_{[0,1]} \otimes B$ is zero-homotopic in an ideal-system preserving way for any separable nuclear C^*-algebra B. Hence the conclusion of Theorem 5.12 holds for $\mathcal{A}_{[0,1]} \otimes B$, and in particular, $\mathcal{A}_{[0,1]} \otimes B$ is an AH_0-algebra as in* (5.8).

Proof: Let $\rho_t \colon \mathcal{A}_{[0,1]} \to \mathcal{A}_{[0,1]}$ be a point-norm continuous path of ideal preserving *-homomorphisms such that $\rho_1 = \mathrm{id}$ and $\rho_0 = 0$. We assert that $\sigma_t = \rho_t \otimes \mathrm{id}_B$ then realizes an ideal-system preserving zero-homotopy of $\mathcal{A}_{[0,1]} \otimes B$. Clearly, $\sigma_1 = \mathrm{id}$ and $\sigma_0 = 0$. We must show that $\sigma_t(I) \subseteq I$ for every closed two-sided ideal I in $\mathcal{A}_{[0,1]} \otimes B$ and for every t. If $x \otimes y \in I$ (where $x \in \mathcal{A}_{[0,1]}$ and $y \in B$), then $\rho_t(x)$ belongs to the ideal in $\mathcal{A}_{[0,1]}$ generated by x, so $\sigma_t(x \otimes y) = \rho_t(x) \otimes y$ belongs to I. This proves the claim, because I is the closure of the linear span of elementary tensors by Blackadar's [1, Theorem 3.3]. $\square$

Proposition 6.3 *For each homeomorphism h on $[0,1]$ with $h(1) = 1$ there is an automorphism α_h on $\mathcal{A}_{[0,1]}$ with $\alpha_h(I_t) = I_{h(t)}$.*

Proof: Retain the notation set up in the construction of the C^*-algebra $\mathcal{A}_{[0,1]}$ at the beginning of this section. For each n consider the *-homomorphism $\beta_{h,n} \colon A_n = C_0([0,1), M_{2^n}) \to \mathcal{A}_{[0,1]}$ given by $\beta_{h,n}(f) = \varphi_{\infty,n}(f \circ h^{-1})$. Fix $f \in A_n$ and set $t = \inf\{s \in [0,1] \mid f|_{[s,1]} \equiv 0\}$. Then $I_{\mathcal{A}_{[0,1]}}(\varphi_{\infty,n}(f)) = I_t$ and $I_{\mathcal{A}_{[0,1]}}(\beta_{h,n}(f)) = I_{h(t)}$. It follows that the closed two-sided ideal in $\mathcal{A}_{[0,1]}$ generated by $\beta_{h,n}(I_t^{(n)})$ is $I_{h(t)}$.

The two *-homomorphisms $\beta_{h,n+1} \circ \varphi_n$ and $\beta_{h,n}$ are approximately unitarily equivalent by Theorem 4.5, because $\beta_{h,n}(f)$ and $(\beta_{h,n+1} \circ \varphi_n)(f)$ generate the same closed two-sided ideal in $\mathcal{A}_{[0,1]}$ for all $f \in A_n$; namely $I_{h(t)}$, where t as above is $\inf\{s \in [0,1] \mid f|_{[s,1]} \equiv 0\}$.

By a one-sided approximate intertwining (after Elliott), see for example [10, Theorem 1.10.14], there is an endomorphism β_h on $\mathcal{A}_{[0,1]}$ such that $\beta_h \circ \varphi_{\infty,n}$ is approximately unitarily equivalent to $\beta_{h,n}$ for all $n \in \mathbb{N}$. The three sets

$$\beta_h(I_t), \qquad (\beta_h \circ \varphi_{\infty,n})(I_t^{(n)}), \qquad \beta_{h,n}(I_t^{(n)})$$

generate the same closed two-sided ideal in $\mathcal{A}_{[0,1]}$, namely $I_{h(t)}$.

Consider now the two *-homomorphisms β_h and $\beta_{h^{-1}}$ of $\mathcal{A}_{[0,1]}$. By the argument above, the closed two-sided ideals in $\mathcal{A}_{[0,1]}$ generated by $(\beta_h \circ \beta_{h^{-1}})(I_t)$ and $(\beta_{h^{-1}} \circ \beta_h)(I_t)$, respectively, are both equal to I_t. We can therefore conclude from Theorem 4.5 that the three *-homomorphisms $\beta_h \circ \beta_{h^{-1}}, \beta_{h^{-1}} \circ \beta_h, \mathrm{id}_{\mathcal{A}_{[0,1]}}$ are approximately unitarily equivalent. By a new approximate intertwining argument, cf. [16, Corollary 2.3.4], we obtain an automorphism α_h on $\mathcal{A}_{[0,1]}$ which is approximately unitarily equivalent to β_h. The closed two-sided ideals generated by $\beta_h(I_t)$ and $\alpha_h(I_t)$ are equal, and so $\alpha_h(I_t) = I_{h(t)}$ as desired. $\square$

Lemma 6.4 *Let h be a homeomorphism on $[0,1]$ with $h(1) = 1$, and let α_h be an automorphism on $\mathcal{A}_{[0,1]}$ that satisfies $\alpha_h(I_t) = I_{h(t)}$ for all $t \in [0,1]$, cf. Proposition 6.3. Then the crossed product $\mathcal{A}_{[0,1]} \rtimes_{\alpha_h} \mathbb{Z}$ is simple if (and only if) h has no fixed points in the open interval $(0,1)$.*

Proof: The assumptions imply that $\mathcal{A}_{[0,1]}$ is $(\mathbb{Z}, \alpha_h)$-simple (i.e., that there are no non-trivial ideals invariant under all powers of α_h). It therefore follows from [13, Theorem 7.2] that $\mathcal{A}_{[0,1]} \rtimes_{\alpha_h} \mathbb{Z}$ is simple if α_h^n is properly outer for all $n \geq 1$. Note that $\alpha_h^n = \alpha_{h^n}$ and that $h^n = h \circ h \circ \cdots \circ h$ has no fixed points in $(0,1)$ when $n \neq 0$. Indeed, h must be a strictly increasing continuous function on $[0,1]$ fixing 0 and 1 which either satisfies $h(t) < t$ for all $t \in (0,1)$ or $h(t) > t$ for all $t \in (0,1)$. Hence $\mathcal{A}_{[0,1]}$ is $(\mathbb{Z}, \alpha_h^n)$-simple when $n \neq 0$. It is a consequence of [13, Theorem 6.6] that an automorphism, with no non-trivial invariant ideals, is properly outer if it is not approximately inner. As an approximately inner automorphism leaves all ideals invariant, we can conclude that α_h^n is properly outer for all $n \neq 0$. This completes the proof. $\square$

Proposition 6.5 *There is an automorphism α on $\mathcal{A}_{[0,1]}$ such that $\mathcal{A}_{[0,1]} \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$.*

Proof: Put $B = \mathcal{A}_{[0,1]} \otimes \mathcal{O}_2$, and recall from Proposition 6.1 that $B \cong \mathcal{A}_{[0,1]}$. Set $h(t) = t^2$, and use Proposition 6.3 to find an automorphism α_h on $\mathcal{A}_{[0,1]}$ with $\alpha_h(I_t) = I_{h(t)}$. Consider the automorphism $\alpha = \alpha_h \otimes \mathrm{id}_{\mathcal{O}_2}$ on B. The crossed product $B \rtimes_\alpha \mathbb{Z}$ is nuclear and separable, and it absorbs $\mathcal{O}_2$:

$$B \rtimes_\alpha \mathbb{Z} = (\mathcal{A}_{[0,1]} \otimes \mathcal{O}_2) \rtimes_{\alpha_h \otimes \mathrm{id}_{\mathcal{O}_2}} \mathbb{Z} \cong (\mathcal{A}_{[0,1]} \rtimes_{\alpha_h} \mathbb{Z}) \otimes \mathcal{O}_2.$$

Moreover, $B \rtimes_\alpha \mathbb{Z}$ is stable because B is stable (see for example [4, Corollary 4.5]).

Identify $\mathcal{A}_{[0,1]}$ and B, so that α is an automorphism on $\mathcal{A}_{[0,1]}$. Then $\alpha(I_t) = I_{t^2}$ for all $t \in [0,1]$, and Lemma 6.4 yields that $\mathcal{A}_{[0,1]} \rtimes_\alpha \mathbb{Z}$ is simple.

It finally follows from [7, Theorem 3.8] that $\mathcal{A}_{[0,1]} \rtimes_\alpha \mathbb{Z}$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{K}$. $\square$

Theorem 6.6 *For each separable nuclear C^*-algebra B there is an $\mathcal{O}_2$-absorbing C^*-algebra algebra A, which is zero-homotopic in an ideal-system preserving way, and which is an inductive limit*

$$C_0((\Gamma_1, v_1), M_{k_1}) \xrightarrow{\varphi_1} C_0((\Gamma_2, v_2), M_{k_2}) \xrightarrow{\varphi_2} C_0((\Gamma_3, v_3), M_{k_3}) \xrightarrow{\varphi_3} \cdots \longrightarrow A,$$

where every (Γ_j, v_j) is a pointed connected graph, k_j is a natural number, and each connecting map φ_j is a *-monomorphism, and there is an automorphism β on A such that $B \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong A \rtimes_\beta \mathbb{Z}$.

Proof: Take an automorphism α on $\mathcal{A}_{[0,1]}$ as in Proposition 6.5. Put $A = \mathcal{A}_{[0,1]} \otimes B$, and put $\beta = \alpha \otimes \mathrm{id}_B$. Then A is as claimed by Proposition 6.2, and

$$A \rtimes_\beta \mathbb{Z} \;=\; (\mathcal{A}_{[0,1]} \otimes B) \rtimes_{\alpha\otimes \mathrm{id}_B} \mathbb{Z} \;\cong\; (\mathcal{A}_{[0,1]} \rtimes_\alpha \mathbb{Z}) \otimes B \;\cong\; \mathcal{O}_2 \otimes \mathcal{K} \otimes B.$$

$\square$

We shall now use the fact that $B \otimes \mathcal{A}_{[0,1]}$ is an AH_0-algebra for every separable, nuclear, strongly purely infinite C^*-algebra B to prove a step in the direction of a topological characterization of the primitive ideal space of separable nuclear C^*-algebras. The following notion of a regular subalgebra is crucial for this.

Definition 6.7 A sub-C^*-algebra C of a C^*-algebra B is called *regular* in B, if the following conditions hold for all closed ideals I, J of B:

(i) $I \cap C = J \cap C$ implies $I = J$,

(ii) $(I + J) \cap C = (I \cap C) + (J \cap C)$.

If $C \subseteq B$ is regular, then the map from the ideal lattice of B into the ideal lattice of C given by $I \mapsto I \cap C$ is an injective lattice morphism (i.e., this map preserves the two lattice operations $\vee$ and $\wedge$). Moreover, this map also preserves least upper bounds and greatest lower bounds of arbitrary *infinite* families. Indeed, if $\{I_\alpha\}_{\alpha\in\mathbb{A}}$ is a family of ideals in B, then its greatest lower bound is $\bigcap_\alpha I_\alpha$, and $\big(\bigcap_\alpha I_\alpha\big) \cap C = \bigcap_\alpha (I_\alpha \cap C)$. If $\{I_\alpha\}_{\alpha\in\mathbb{A}}$ is upwards directed, then its least upper bound is $\overline{\bigcup_\alpha I_\alpha}$, and

$$\left(\overline{\bigcup_\alpha I_\alpha}\right) \cap C = \overline{\bigcup_\alpha I_\alpha \cap C}.$$

To form the least upper bound of a general family $\{I_\alpha\}_{\alpha\in\mathbb{A}}$ one proceeds as above with the upper directed family consisting of all finite sums of the ideals from the family $\{I_\alpha\}_{\alpha\in\mathbb{A}}$; and this operation is also preserved by our lattice morphism by the results above.

In the case where C is an Abelian regular sub-C^*-algebra of B, so that C is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, then, upon identifying the ideal lattice of C with the lattice, $\mathcal{O}(X)$, of open subsets of X, we get an injective lattice morphism from the ideal lattice of B into $\mathcal{O}(X)$. This morphism maps B to X and 0 to $\emptyset$. This morphism also preserves least upper bounds and greatest lower bounds of infinite families (the greatest lower bound of a family $\{U_\alpha\}_{\alpha\in\mathbb{A}}$ of open sets is the interior of $\bigcap_\alpha U_\alpha$).

Note that $(I \cap C) + (J \cap C) \subseteq (I + J) \cap C$ holds for any sub-C^*-algebra $C \subseteq B$ and for all closed two-sided ideals I, J in B. This inclusion can be strict, even when (i) above is satisfied.

We need some elementary lemmas to get our final result, Theorem 6.11.

Lemma 6.8 *Suppose that A, B, D are C^*-algebras, such that $A \subseteq D$ and D is simple and nuclear.*

(i) *If $C_1 \subseteq C_2 \subseteq B$ and C_1 is regular in C_2, then C_2 is regular in B if and only if C_1 is regular in B. In particular, if C_1 is a full hereditary sub-C^*-algebra of C_2, then C_1 is regular in B if and only if C_2 is regular in B.*

(ii) *If $C_k \subseteq B_k$ are regular in B_k for $k = 1, 2$, then $C_1 \oplus C_2$ is regular in $B_1 \oplus B_2$.*

(iii) *If $C \subseteq B$ is regular in B, then $C \otimes D$ is regular in $B \otimes D$.*

(iv) *If F is a C^*-algebra with $B \otimes A \subseteq F \subseteq B \otimes D$ and $B \otimes A$ is full in F, then F is regular in $B \otimes D$.*

(v) *If C is a regular sub-C^*-algebra of $B \otimes A$, then C is regular in $B \otimes D$.*

(vi) *If E is the closure of the union of a sequence of C^*-algebras $E_1 \subseteq E_2 \subseteq \ldots$, and if C is a sub-C^*-algebra of E for which $E_n \cap C$ is regular in E_n for every $n \in \mathbb{N}$, and $\bigcup_n (E_n \cap C)$ is dense in C, then C is regular in E.*

Proof: (i). To see that C_2 is regular in B if C_1 is regular in B, take two ideals I, J of B and use regularity of $C_1 \subseteq C_2$ and of $C_1 \subseteq B$ to see that

$$C_1 \cap (C_2 \cap I + C_2 \cap J) = C_1 \cap I + C_1 \cap J = C_1 \cap (I + J) = C_1 \cap (C_2 \cap (I + J)),$$

whence $C_2 \cap I + C_2 \cap J = C_2 \cap (I + J)$ by regularity of $C_1 \subseteq C_2$. The remaining claims of (i) and (ii) are straightforward.

In (iii), (iv), and (v) we need the following observation. Since D is exact and simple, every closed two-sided ideal of $B \otimes D$ is of the form $I \otimes D$ for some closed two-sided ideal I of B. This was first proved by Blackadar, [1, Theorem 3.3], in the case where D is nuclear (all we need here).

(iii). This follows easily from the characterization of ideals of $B \otimes D$ derived above and from the identities: $I \otimes D + J \otimes D = (I + J) \otimes D$ and $(I \otimes D) \cap (C \otimes D) = (I \cap C) \otimes D$, that hold for all closed two-sided ideals I and J in B.

(iv). Property (i) of Definition 6.7 holds for $B \otimes A \subseteq B \otimes D$, and hence also when $B \otimes A$ is replaced by the larger algebra F.

Let F_1 be the (full) hereditary sub-C^*-algebra of F generated by $B \otimes A$. We show below that F_1 is regular in $B \otimes D$, and by (i) this will entail that F is regular in $B \otimes D$.

We note first that

$$(I \otimes D) \cap F_1 = \overline{(I \otimes 1)F_1}$$

for every closed two-sided ideal I of B (with $I \otimes 1$ viewed as a subalgebra of $B \otimes \mathcal{M}(D)$). If x belongs to $(I \otimes D) \cap F_1$, then x belongs to the closure of $(I \otimes 1)x$ which is contained in $\overline{(I \otimes 1)F_1}$. In the other direction, $(I \otimes 1)F_1 \subseteq I \otimes D$. By construction of F_1 we have $(B \otimes 1)F_1 \subseteq F_1$, whence $(I \otimes 1)F_1 \subseteq F_1$.

Let now I, J be two closed two-sided ideals of B. Then

$$\begin{aligned} (I \otimes D + J \otimes D) \cap F_1 &= \big((I + J) \otimes D\big) \cap F_1 \;=\; \overline{\big((I + J) \otimes 1\big)F_1} \\ &= \overline{(I \otimes 1)F_1 + (J \otimes 1)F_1} \;=\; \overline{(I \otimes 1)F_1} + \overline{(J \otimes 1)F_1} \\ &= (I \otimes D) \cap F_1 + (J \otimes D) \cap F_1, \end{aligned}$$

as desired.

(v). Combine (i) and (iv) (with $F = B \otimes A$).

(vi). By a standard property of inductive limits, K is the closure of $\bigcup_{n=1}^{\infty} K \cap E_n$ whenever K is a closed two-sided ideal of E or of C. Let now I, J be two closed two-sided ideals in E. If $I \cap C = J \cap C$, then $I \cap C \cap E_n = J \cap C \cap E_n$ and hence $I \cap E_n = J \cap E_n$ for all n (because $C \cap E_n \subseteq E_n$ is regular), and this implies $I = J$. Using again that $C \cap E_n \subseteq E_n$ is regular, we get

$$\big(I \cap E_n + J \cap E_n\big) \cap C \;=\; I \cap E_n \cap C + J \cap E_n \cap C \;\subseteq\; I \cap C + J \cap C,$$

for all n, whence

$$(I + J) \cap C \;=\; \overline{\Big(\bigcup_{n=1}^{\infty} I \cap E_n + J \cap E_n\Big)} \cap C \;\subseteq\; I \cap C + J \cap C.$$

$\square$

Lemma 6.9 *Suppose that A is a building block for an AH_0-algebra, i.e., A is isomorphic to $\bigoplus_{k=1}^{r} C_0(X_k, M_{n_k})$ for some locally compact spaces X_k and natural numbers n_k, $k = 1, \dots, r$. Then every maximal Abelian sub-C^*-algebra (masa) of A is regular in A.*

Proof: By Lemma 6.8 (ii) it suffices to consider the case where A has a single summand, i.e., $A = C_0(X, M_n)$. If C is a masa in A, then C necessarily contains the center $C_0(X, \mathbb{C}1_n)$ of $C_0(X, M_n)$. Thus C_k is regular in $C_0(X_k, M_{n_k})$ by Lemma 6.8 (iv), with $D = M_{n_k}$ and $A = \mathbb{C}$. $\square$

In general masa's need not be regular, there are even masa's in sub-homogeneous C^*-algebras that are not regular.

For the proof of the next lemma note that any quotient of a standard building block $\bigoplus_{k=1}^{r} C_0(X_k, M_{n_k})$ is isomorphic to $\bigoplus_{k=1}^{r} C_0(Y_k, M_{n_k})$ for some closed (possibly empty) subsets Y_k of X_k (with the convention $C_0(\emptyset, M_n) = 0$).

Lemma 6.10 *Every (separable) AH_0-algebra A contains an Abelian sub-C^*-algebra $C \subseteq A$ which is regular in A.*

Proof: An AH_0-algebra is by definition an inductive limit of a sequence of algebras $B_1 \to B_2 \to B_3 \to \cdots$ where each B_k is a building block (as described in Lemma 6.9). Let $\mu_{\infty,n} \colon B_n \to A$ be the inductive limit map and put $A_n = \mu_{\infty,n}(B_n)$. Then $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A$, $\bigcup_{n=1}^{\infty} A_n$ is dense in A, and each A_n, being isomorphic to a quotient of B_n, is a building block as in Lemma 6.9.

Find inductively maximal Abelian sub-C^*-algebras C_n of A_n such that $C_n \subseteq C_{n+1}$ for all n. The C_n is regular in A_n by Lemma 6.9 (i). The closure C of the union $\bigcup_{n=1}^{\infty} C_n$ is an Abelian sub-C^*-algebra of A, and $C \cap A_n = C_n$ because C_n is maximal Abelian. Hence C is regular in A by Lemma 6.9 (vi). $\square$

Theorem 6.11 *For every separable, nuclear C^*-algebra B there exist an Abelian regular sub-C^*-algebra C of $B \otimes \mathcal{O}_2$ such that the maximal ideal space $\widehat{C} = \mathrm{Prim}(C)$ has dimension at most one.*

Proof: The C^*-algebra $B \otimes \mathcal{A}_{[0,1]}$ is an AH_0-algebra by Proposition 6.2. Therefore, by Lemma 6.10, $B \otimes \mathcal{A}_{[0,1]}$ contains a regular Abelian sub-C^*-algebra C_1. Use any embedding of $\mathcal{A}_{[0,1]}$ into $\mathcal{O}_2$ to obtain inclusions $C_1 \subseteq B \otimes \mathcal{A}_{[0,1]} \subseteq B \otimes \mathcal{O}_2$. By Lemma 6.8 (v), C_1 is regular in $B \otimes \mathcal{O}_2$. Thus $C_1 \otimes \mathcal{O}_2 \subseteq B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2$ is regular by Lemma 6.8 (iii). There exists an Abelian C^*-algebra C such that $C_1 \otimes 1 \subseteq C \subseteq C_1 \otimes \mathcal{O}_2$ and such that $\dim(\widehat{C}) \leq 1$ by Proposition 3.7 (applied to $C_1 + \mathbb{C}1 \cong C(X)$). Observe that $C_1 \otimes 1$ is full in C because

$(C_1 \otimes 1)C(C_1 \otimes 1)$ is dense in C. It follows from Lemma 6.8 (iv) (with $A = \mathbb{C}1_{\mathcal{O}_2}$) that C is regular in $C_1 \otimes \mathcal{O}_2$. Therefore, C is also regular in $B \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \cong B \otimes \mathcal{O}_2$ by Lemma 6.8 (i). $\square$

It follows from Theorem 6.11 that if B is a separable nuclear C^*-algebra, then there is a locally compact second countable Hausdorff space X with $\dim(X) \leq 1$ such that the ideal lattice of B is order isomorphic to some sub-lattice $\mathcal{L}$ of the lattice of open subsets of X with $X, \emptyset \in \mathcal{L}$; and this lattice morphism also preserves least upper bounds and greatest lower bounds of infinite families. Indeed, let C be as in Theorem 6.11 and put $X = \widehat{C}$. The map $I \mapsto (I \otimes \mathcal{O}_2) \cap C$ is an injective lattice morphism from the lattice of ideals of B into the lattice of ideals of C, cf. the remark below Definition 6.7, and the latter is isomorphic to the lattice of open subsets of X.

Remark 6.12 Theorem 6.11 will in [3] be used by the first-named author and H. Harnisch to give several topological characterizations of primitive ideal spaces of separable nuclear C^*-algebras B, and to derive an almost functorial construction of nuclear C^*-algebras B from given T_0-spaces X in the described class. More precisely, a T_0-space X is the primitive ideal space of a separable nuclear C^*-algebra B if and only if X is a second countable point-wise complete T_0-space and there is a continuous, pseudo-open and pseudo-epic map ψ from a (one-dimensional) locally compact Polish space Y into X.

The map ψ defines a certain partial order relation R_ψ on Y (the pseudo-graph of ψ), which in turn defines a Hilbert $C_0(X)$–bimodule H such that X is naturally isomorphic to the primitive ideal space of the Cuntz-Krieger-Pimsner algebra $\mathcal{O}_H$.

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