Some properties of QWEP C^*-algebras

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Sections

1. The original Connes-Conjecture
2. Relatively weakly injective morphisms.
3. Weakly injective C*-algebras
4. QWEP algebras
5. Special cases of the QWEP-conjecture

The original **Conjecture of A. Connes**:

A. Connes wrote in proof Theorem 5.1 (= Separable injective II_1 factors N are isomorphic to the hyperfinite II_1 factor $\mathcal{R}$):

... We now construct an approximate imbedding of N in $\mathcal{R}$.

Apparently such an imbedding ought to exist for all II_1 factors because it does for the regular representation of free groups. ...

(Perhaps, he deduced such an embedding of F_2 from the residual finiteness of $\mathrm{SL}(2,\mathbb{Z})$? – Using that the tensor products of a faithful group representation of a countable discrete group G into non-scalar unitaries in $\mathcal{R}$ define an embedding of $vN(G)$ into $\mathcal{R}^\omega$?)

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Let $F := F_n$ denote any free group on $n \in \{2, 3, \dots; \infty\}$ generators. (Local properties can be also checked with uncountably many generators.)

Definition (1)

We say that a C*-*mono*morphism $\varphi\colon A \hookrightarrow B$ is **relatively weakly injective** (r.w.i.), or that A **is r.w.i. in** B, if φ satisfies the following (equivalent !) conditions:

(rwi,1) $\varphi \otimes \mathrm{id}\colon A \otimes^{\max} C^*(F) \to B \otimes^{\max} C^*(F)$ is injective.

(rwi,2) $\varphi \otimes \mathrm{id}\colon A \otimes^{\max} C \to B \otimes^{\max} C$ is injective for every C*-algebra C.

(rwi,3) There exists a cp contraction $V\colon B \to A^{**}$ s.t. $V \circ \varphi = \mathrm{id}$ on $A \subset A^{**}$.

(rwi,4) There is a normal conditional expectation E from B^{**} onto the weak closure of $\varphi(A)$ in B^{**}, that is extremal among those conditional expectations.

Properties (rwi,1)-(rwi,4) depend from the chosen C*-monomorphism φ:

Example (2)

There are unital r.w.i. and (nuclear) non-r.w.i. embeddings of $C^*_{\mathrm{red}}(F)$ into the (norm-) ultra-power $(\mathcal{O}_2)_\omega$.

Some (permanence) properties of r.w.i. maps $A \hookrightarrow B$:

(1) For each factorial representation $\rho\colon A \to N \subset \mathcal{L}(H)$ there exists a projection $p \in M_\infty(N) \cong \mathcal{L}(\ell_2)\overline{\otimes}N$, a factorial representation $\rho'\colon B \to pM_\infty p$ such that $p \geq 1_N \otimes e_{11}$, $\rho = (1_N \otimes e_{11})\rho' \circ \varphi$ (under natural identifications) and that the u.c.p. map $X \in \rho'(B)'' \mapsto (1 \otimes p_{11})X(1 \otimes p_{11})$ is an extreme point in the u.c.p. maps.

(2) If $J \subset A \subset B$, $J \lhd B$ and $A/J \hookrightarrow B/J$ is r.w.i. then $A \hookrightarrow B$ is r.w.i.

(3) For each C^*-algebra B and every separable subspace $X \subseteq B$ there exists a separable sub-C^*-algebra $A \subseteq B$ such that $X \subseteq A$ and $A \hookrightarrow B$ is r.w.i.

(4) If $A_n \hookrightarrow B_n$ $(n = 1, 2, \ldots)$ are r.w.i., then $c_0(A_1, A_2, \ldots) \hookrightarrow c_0(B_1, B_2, \ldots)$, $\ell_\infty(A_1, A_2, \ldots) \subset \ell_\infty(B_1, B_2, \ldots)$, $\prod_\omega A_n \subset \prod_\omega B_n$ and $A_1 \otimes^{max} A_2 \otimes \cdots \to B_1 \otimes^{max} B_2 \otimes \cdots$ are r.w.i. *-monomorphisms.

(5) If $A_{n-1} \subset A_n \subset B_n \subset B_{n+1}$, and $A_n \hookrightarrow B_n$ is r.w.i. then $\mathrm{indlim}_n A_n \hookrightarrow \mathrm{indlim}_n B_n$ is r.w.i.

(6) $A \hookrightarrow B \hookrightarrow C$ r.w.i. implies $A \hookrightarrow C$ r.w.i.

(7) $A \hookrightarrow B$ r.w.i. if and only if $A^{**} \hookrightarrow B^{**}$ is r.w.i.

(8) If $M \subset N$ is sub-W*-algebra of a W*-algebra N, then $M \hookrightarrow N$ is r.w.i. if and only if there is a (not necessarily normal) conditional expectation $E \colon N \to M$.

(9) If $A \subset B$, $\varphi \colon A \hookrightarrow C$ is r.w.i. and φ extends to a contraction from B into C^{**} then $A \hookrightarrow B$ is r.w.i.

(10) (N.Ozawa) If $A \subset B := \ell_\infty(M_{k_1}, M_{k_2}, \ldots)/c_0(M_{k_1}, M_{k_2}, \ldots)$ is a unital simple sub-C^*-algebra with *unique* tracial state τ, such that $D_\tau \colon A \hookrightarrow N_\tau$ is r.w.i., then $A \hookrightarrow B$ is r.w.i.

Question (3)

Is the inclusion map $C^*_{\mathrm{red}}(G) \hookrightarrow vN(G)$ r.w.i. for every finitely presented (discrete) group G?

Question (4)

Let A is a simple unital MF-algebra in the sense of B. Blackadar and suppose that A has the Dixmier property.
Let τ the unital tracial state on A. When $A \hookrightarrow N_\tau$ is r.w.i.?

Question (5)

Let A is a unital separable exact C^*-algebra such that $A \otimes^{\min} B$ and $A \otimes^{\max} B$ are finite for each exact unital (separable) MF-algebra B. Is A an MF-algebra?

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Let $A \subset \mathcal{L}(H)$ a C^*-algebra.

Definition (6)

A is called **weakly injective** if it has the following equivalent (!) properties:

(wi,1) $A \hookrightarrow \mathcal{L}(H)$ is r.w.i.

(wi,2) For every *-monomorphism $\varphi\colon A \to B$, φ is r.w.i.

(wi,3) $A \otimes^{\max} C^*(F) = A \otimes^{\min} C^*(F)$, in the sense that there is a unique C*-norm on $A \odot C^*(F)$.

(wi,4) For every faithful *-representation $\rho\colon A \to \mathcal{L}(H)$ there exists a c.p. contraction $P\colon \mathcal{L}(H) \to d(A)''$ with $P \circ \rho = \rho$. I.e. A has the *weak expectation property* (**WEP** of Ch. Lance).

Notice that WEP is a typical C^*-algebra property: A^{**} is weakly injective if and only if A is nuclear !

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Definition (7)

A C^*-algebra B is has **QWEP** (B is a QWEP algebra) if B is a quotient of weakly injective C^*-algebra.

Some **elementary properties of QWEP** algebras, that follows step by step from the above given properties of r.w.i. maps and other above listed properties (using the short-exactness of $\otimes^{\mathrm{max}}$ etc.).

(q0) Unital A has QWEP, if and only if, for some surjective/any unital C*-morphism $\psi\colon C^*(F) \to A$ the *-morphism $\psi \otimes^{max} \mathrm{id}\colon C^*(F) \otimes^{max} C^*(F) \to A \otimes^{max} C^*(F)$ naturally factorizes over $C^*(F) \otimes^{min} C^*(F)$.

(q1) A has QWEP, if and only if, A^{**} has QWEP.

(q2) $A \oplus B$ has QWEP, if and only if, A and B have QWEP.

(q3) The class of QWEP-algebras is closed under extensions and inductive limites.

(q4) Every QWEP-algebra is the inductive limit of its separable sub-C^*-algebras with QWEP.

(q5) If $A \hookrightarrow B$ is r.w.i. and B has QWEP then A has QWEP. In particular all *hereditary* sub-C^*-algebras of B have QWEP.

(q6) If N is W*-algebra with separable predual N_*, then N has QWEP (as a C^*-algebra),
if and only if,
the central integral decomposition $N = \int N_x \, \mathrm{d}\mu(x)$ of N into Factors N_x – where x is a character of the center of N – has the property that N_x has QWEP μ-almost everywhere.

(q7) The class of QWEP-algebras is invariant under C*- (respectively W*-) crossed products by actions of amenable groups (or amenable quantum groups).

(q8) $\mathcal{R}^\omega$ has QWEP, – because it is a quotient of the weakly injective C^*-algebra $\ell_\infty(\mathcal{R})$.

One gets (as a sort of corollary):

Theorem (8)

Let N a II_1 factor with separable predual N_*. TFAE:

1. *N has QWEP.*
2. *N is a sub-C^*-algebra of $\mathcal{R}^\omega$.*
3. *For each *-morphism $\gamma\colon C^*(F_\infty) \to N$ with weakly dense image in N the (pure and positive) functional ρ on $C^*(F_\infty) \odot C^*(F_\infty)$ given by $\rho(a \otimes b) := \tau_N(\gamma(a)\gamma(b))$ (where we use the natural isomorphism $C^*(F) \cong C^*(F)^{op}$) is continuous with respect to $\|\cdot\|_{\min}$ on $C^*(F_\infty) \odot C^*(F_\infty)$.*

Corollary (9)

TFAE:

(a) *Connes Embedding Problem has positive answer.*

(b) *Every C^*-algebra has QWEP (=: QWEP-conjecture)*

(c) *There is only one C^*-algebra-norm on $C^*(F) \odot C^*(F)$.*

(d) *For each II_1 factor (N, τ), $n \in \mathbb{N}$, $u_1, \dots, u_n \in N$ unitary, $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and $v_1, \dots v_n \in M_m$ with $\max_{j,k} |\tau(u_j^* u_k) - m^{-1}\mathrm{Tr}(v_j^* v_k))| < \varepsilon$.*

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Theorem (N. Ozawa)

*The C^*-algebra $\mathcal{L}(\ell_2) \otimes^{\min} \mathcal{L}(\ell_2)$ is **not** weakly injective.*

(It implies that $\mathcal{L}(H) \otimes^{\min} \mathcal{L}(H)$ is not weakly injective for each Hilbert space H of infinite dimension.)

Question

Is $\mathcal{L}(\ell_2) \otimes^{\max} \mathcal{L}(\ell_2)$ a QWEP algebra?

Proposition (K.1992)

If A is a unital separable C^*-algebra with QWEP, then there exists a separable unital block diagonal C^*-algebra $B \subset \ell_\infty(\mathcal{U})$ (where $\mathcal{U} := M_2 \otimes M_3 \otimes \cdots$) such that $B^{op} \otimes^{\max} B = B^{op} \otimes^{\min} B$ and A is a quotient of B.
If $A^{op} \cong A$, then one can manage that $B^{op} \cong B$ (in addition).

Let $\mathcal{U} := M_2 \otimes M_3 \otimes \cdots$ denote the universal UHF algebra.

Remark (13)

By the above mentioned claim of Connes holds $vN(F) \subset \mathcal{R}^\omega$.

(E.g. because $SL(2,\mathbb{Z}) \supset F$ *has a faithful group-representation into non-scalar unitaries of* $\ell_\infty(\mathcal{U}) \subset \mathcal{R}$*, and – then using that* $\mathcal{R}\overline{\otimes}\mathcal{R} = \mathcal{R}$ *–, it defines a unitary representation* $V\colon SL(2,\mathbb{Z}) \to \mathcal{R}^\omega$*, such that* $g \in SL(2,\mathbb{Z}) \mapsto V(g) \in \mathcal{R}^\omega$ *satisfies* $\mathrm{tr}_\omega(V(g)) = 0$ *for* $g \neq e$*.)*

A result of Haagerup implies that $C^*_{red}(F) \hookrightarrow vN(F)$ *is r.w.i.*

Together we get that $C^*_{red}(F)$ **has QWEP** *(and is* **exact***, because* F *is closed subgroup of the connected Lie group* $SL_2(\mathbb{R})$*).*

A combination of properties (2) and (10) of r.w.i. maps, and more involved observations of Haagerup and Thorbjørnsen show *directly*:

Theorem (14) (U. Haagerup, S. Thorbjørnsen, N. Ozawa)

*There are unital *-morphisms* $\rho_n \colon C^*(F) \to M_{k_n}$ *such that the *-morphism* $V \colon a \mapsto (\rho_1(a), \rho_2(a), \ldots)$ *of* $C^*(F)$ *into* $\ell_\infty(M_{k_1}, M_{k_2}, \ldots)$ *defines a *-monomorphism from* $C^*_{red}(F)$ *into* $\ell_\infty(M_{k_1}, M_{k_2}, \ldots)/c_0(M_{k_1}, M_{k_2}, \ldots)$.

The C^**-algebra* $B := V(C^*(F)) + c_0(M_{k_1}, M_{k_2}, \ldots)$ *is weakly injective.*

It is not known if $B^{op} \otimes^{\max} B = B^{op} \otimes^{\min} B$, or if one can modify the ρ_n such that $B \odot \mathcal{L}(\ell_2)$ has only one C*-norm.

Proposition (15)

*Let $\Phi \colon C^*(F) \to C^*(F) \otimes^{\min} C^*(F)$ the diagonal *-monomorphism and $\rho \colon C^*(F) \to C^*_{red}(F)$ the natural surjection.*

(1) $\rho \otimes^{\max} \rho \colon C^*(F) \otimes^{\max} C^*(F) \to C^*_{red}(F) \otimes^{max} C^*_{red}(F)$ *factorizes:*
$[\rho \otimes^{\max} \rho] \colon C^*(F) \otimes^{\min} C^*(F) \to C^*_{red}(F) \otimes^{max} C^*_{red}(F)$.

(2) $[\rho \otimes^{\max} \rho] \circ \Phi \colon C^*(F) \to C^*_{red}(F) \otimes^{max} C^*_{red}(F)$ *is induced by the diagonal embedding of F in $F \times F$ and is r.w.i. Moreover there is a conditional expectation E from $C^*_{red}(F) \otimes^{max} C^*_{red}(F)$ onto the image of $[\rho \otimes^{\max} \rho] \circ \Phi$.*

(3) *If $\psi_1, \psi_2 \colon C^*(F) \to \mathcal{L}(\ell_2)$ are faithful then*

$$(\psi_1 \otimes \psi_2) \circ \Phi \colon C^*(F) \to \mathcal{L}(\ell_2) \otimes^{\max} \mathcal{L}(\ell_2)$$

is r.w.i.

Proof:
(1) follows from the weak injectivity of B in Theorem 14.
(2) follows from the below given Theorem (16).
Ad(3):
Since a unital *-monomorphism $C^*(F) \hookrightarrow \mathcal{L}(\ell_2)$ can not contain non-zero compact operators in its image, all those are approximate unitarily equivalent (point-wise in operator-norm) by Voiculescu's theorem. Thus it suffices to consider one special injective unital *-morphism $\psi\colon C^*(F) \to \mathcal{L}(\ell_2)$.
We may suppose that $\prod_n M_{k_n}$ is a W*-subalgebra of $\mathcal{L}(\ell_2)$ (given by a partition of the integers $\mathbb{N}$, such that the complement X of union the disjoint subsets (with cardinalities k_n) is still infinite, – corresponding to a diagonal projection $p \in \mathcal{L}(\ell_2)$ –. Take any faithful unital *-monomorphism

$$\lambda\colon C^*(F) \hookrightarrow p\mathcal{L}(\ell_2)p\,.$$

Then let $\psi = V \oplus \lambda$.

The c.p. map $V\colon C^*(F) \to B \subset (1-p)\mathcal{L}(\ell_2)(1-p)$ extends to a c.p. map $W\colon \mathcal{L}(\ell_2) \to \prod_n M_{k_n}$ with $W \circ \psi = V$. The multiplicative domain of W is

$$\mathrm{Mult}(W) := p\mathcal{L}(\ell_2)p \oplus \prod_n M_{k_n} .$$

It contains $\psi(C^*(F))$, and $C := \mathrm{Mult}(W) \cap W^{-1}(B) =$ is r.w.i. in $\mathrm{Mult}(W)$.
Then $C \otimes^{\max} C$ is r.w.i. in $\mathcal{L}(\ell_2) \otimes^{max} \mathcal{L}(\ell_2)$, and maps onto $C^*_{red}(F) \otimes^{max} C^*_{red}(F)$. Thus, the diagonal image of $C^*(F)$ in $C \otimes^{\max} C$ is r.w.i. Then this image is also r.w.i. in $\mathcal{L}(\ell_2) \otimes^{max} \mathcal{L}(\ell_2)$.

Q.E.D.

Theorem (16)

Let G is a discrete group that acts on a C^*-algebra A. Then the map $A \rtimes_{\max} G \to (A \rtimes_{red} G) \otimes^{\max} C^*_{red}(G)$, – that given by the diagonal embedding of G into $G \times G$ –, is injective and there is a conditional expectation E onto the image of this map.

Remark: The observation $C^*(F) \subset C^*_{red}(F) \otimes^{\max} C^*_{red}(F)$ is due to G.Pisier (communicated to me several years ago).

Proof: Let $\lambda, \rho \colon C^*_{red}(G) \to \mathcal{L}(\ell_2(G))$ the (commuting) left and right regular representations, and let γ the vector state on $\mathcal{L}(\ell_2(G))$ given by $\delta_e \in \ell_2(G)$. Then $\gamma(\lambda(g)\lambda(g)) = 1$ for all $g \in G$.
Choose a faithful non-degenerate *-representation $D_1 \colon A \rtimes_{\max} G \to \mathcal{L}(H)$ on some Hilbert space H.

D_1 comes from a non-degenerate *-representation $d\colon A \to \mathcal{L}(H)$ and a unitary representation $U\colon G \to \mathcal{L}(H)$ such that $D_1(a) = \sum_{g \in G} d(a(g))U(g)$, if we consider the dense subset of $A \rtimes_{\max} G$ given by maps $a\colon G \to A$ of finite support $F := \{g \in G\,;\ a(g) \neq 0\}$.

It is easy to see (with help of conjugation by the U-defining unitary "multiplier" $\mathcal{U}$ on $H \otimes_2 \ell_2(G)$ of the pentagon unitary on $\ell_2(G \times G)$), that

$$D_2(a) := \sum_{g \in G} (d(a(g)) \otimes 1)(U(g) \otimes \lambda(g))$$

defines a *-representation $D_2\colon A \rtimes_{red} G \to \mathcal{L}(H \otimes_2 \ell_2(G))$ of the *reduced* crossed product $A \rtimes_{red} G$ on $H \otimes_2 \ell_2(G)$.

The image of D_2 commutes element-wise with $1_H \otimes \rho(C^*_{red}(G))$. Thus, the bilinear map

$$a \times b \mapsto \sum_{g \in G} (d(a(g)) \otimes 1)(U(g) \otimes \lambda(g))(1_H \otimes \rho(b))$$

extends uniquely to a *-representation

$$\psi \colon (A \rtimes_{red} G) \otimes^{\max} C^*_{red}(G) \to \mathcal{L}(H \otimes_2 \ell_2(G))\,.$$

Now let

$$\varphi \colon A \rtimes_{\max} G \to (A \rtimes_{red} G) \otimes^{\max} C^*_{red}(G)$$

the *-morphism defined by

$$\varphi \colon a \mapsto \sum_{g \in G} (a(g) \otimes 1)(\lambda(g) \otimes \rho(g))\,.$$

Then $\psi \circ \varphi(a) = \sum_{g \in G} (d(a(g))U(g)) \otimes (\lambda(g)\rho(g)))$.

The vector-state γ defines a normal conditional expectation E_γ from $\mathcal{L}(H \otimes_2 \ell_2(G))$ onto $\mathcal{L}(H)$ with $E_\gamma(S \otimes T) = \gamma(T) \cdot S$. It is easy to see (on generators) that E_γ maps the image of ψ into the image of D_1, and that $E_\omega(\psi \circ \varphi(a)) = D_1(a)$.
Thus, φ is faithful, and $P := \varphi \circ D_1^{-1} \circ E_\omega \circ \psi$ is a conditional expectation from $(A \rtimes_{red} G) \otimes^{\max} C^*_{red}(G)$ onto $\varphi(A \rtimes_{\max} G)$.
Q.E.D.

Definition (17)

A W*-algebra N is **weakly exact** if for every C^*-algebra A, ideal $J \lhd A$ and nuclear $T\colon A \to N$ with $T(J) = \{0\}$, the class cp map $[T]_J\colon A/J \mapsto N$ is still a nuclear map.

All bi-duals of exact C^*-algebras are weakly exact, but not conversely (see B in Theorem 14).
The class of weakly exact W*-algebras has nice permanence properties:
E.g., is invariant under normal conditional expectation, spatial W* tensor products with injective W*-algebras, integrals of families of W*-algebras, decomposition into factors (a.e.), crossed products by amenable groups, ...
N.Ozawa: If G is a discrete group with weakly exact $vN(G)$ then G is exact.
Thus: It is still a "small" class of W*-algebras.
(It is unknown if locally reflexive C^*-algebras have weakly exact bi-duals.)

Corollary

TFAE

(i) $\mathcal{L}(H) \otimes^{\max} \mathcal{L}(H)$ *has QWEP.*

(ii) $C^*(F)$ *has WEP.*

(iii) *Connes' embedding conjecture holds.*

(iv) *If* $N \subset M$ *are W*-factors with separable predual that are both weakly exact (in sense of Def. 17) QWEP-algebras, and if* $N' \cap M$ *is a* II_1 *factor, then* $N' \cap M$ *is a QWEP algebra.*

Remark

*Since for all *-morphisms $\varphi, \psi \colon C^*(F) \to \mathcal{L}(H)$ (with $F = F(X)$ any free group on generators X with any cardinality ≥ 2) the morphism*

$$\varphi \otimes^{\max} \psi \colon C^*(F) \otimes^{\max} C^*(F) \to \mathcal{L}(H) \otimes \mathcal{L}(H)$$

factorizes naturally over $C^*(F) \otimes^{\min} C^*(F)$ by property (q0), one can deduce with help of properties (2) of r.w.i. maps and (wi,1):

There is an epimorphism $\lambda \colon D \to \mathcal{L}(H)$ for some weakly injective C^*-algebra D such that epimorphism

$$\lambda \otimes^{\max} \lambda \colon D \otimes^{\max} D \to \mathcal{L}(H) \otimes^{\max} \mathcal{L}(H)$$

*contains the kernel of the epimorphism $D \otimes^{\max} D \to D \otimes^{\min} D$ in its kernel, i.e., there is a natural *-epimorphism*

$$[\lambda] \colon D \otimes^{\min} D \to \mathcal{L}(H) \otimes^{\max} \mathcal{L}(H) \,.$$