Spectra of C* algebras, classification.

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- Strategies for partial results
- Some Pimsner-Toeplitz algebras
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- Regular Abelian subalgebras
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Conventions and Notations

- Considered C*-algebras $A, B, \ldots$ are separable, ...
- ... except multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $Q(B) := \mathcal{M}(B)/B$, ...
- as e.g., $Q(\mathbb{R}_+, B) := C_b(\mathbb{R}_+, B)/C_0(\mathbb{R}_+, B) \subset Q(SB)$.
- $T_0$ spaces $X, Y, \ldots$ are second countable.
- $\mathcal{O}(X), \mathcal{F}(X)$ denote the (distributive) lattices of open and of closed subsets of $X$.
- Prim$(A)$ is the $T_0$ space of primitive ideals with kernel-hull topology (Jacobson topology).
- $\mathcal{I}(A)$ means the lattice of closed ideals of $A$ (It is naturally isomorphic to $\mathcal{O}(\text{Prim}(A))$).
- $\mathbb{Q}$ denotes the Hilbert cube (with its coordinate-wise order).
Spectra of C*-algebras

Let $A$ denote a separable $C^*$-algebra, $X := \text{Prim}(A)$.

- $X \cong \text{Prim}(A \otimes B)$ (naturally) for every simple exact $B$
  (e.g. $B \in \{O_2, O_\infty, U, \mathbb{Z}, \mathbb{K}, C^*_\text{reg}(F_2)\}$).
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- If $A$ is purely infinite then $(W(A), \leq, +)$ is naturally isomorphic to
  $(\mathcal{I}(A), \subset, +) \cong (\mathcal{O}(X), \subset, \cup)$.

- In general the Cuntz semi-group cannot detect whether p.i. $A$
tensorially absorbs $O_{\infty}$ or not (by “exact” counter-examples).
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- No p.i. amenable counter-example $A$ has been found until now.

- $X$ is $T_0$, sober (i.e., is point-complete), locally quasi-compact and is second countable (by separability of $A$).
  (The sobriety comes from the fact that $X$ has the Baire property, as an open and continuous image of a Polish space — the space of pure states on $A$ —.)
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The generalized Gelfand transforms \( a \in A \mapsto \hat{a} \in \ell_\infty(X) \) maps \( A \) \textit{onto} the set of all Dini functions on \( X \). (Here \( \hat{a}(J) := \|a + J\| \) for \( J \in \text{Prim}(A) \).)

All locally quasi-compact sober \( T_0 \) space \( X \) have also the above mentioned topol. properties. We get \textbf{three basic questions}:

1) Is every (second-countable) locally quasi-compact sober \( T_0 \) space \( X \) homeomorphic to the primitive ideal spaces \( \text{Prim}(A) \) of some (separable) \( A 

2) Is there a topological characterization of the Spectra \( \text{Prim}(A) \) of amenable \( A \) (up to homeomorphisms)?

3) Is there some uniqueness for the corresponding algebra \( A \) with \( \text{Prim}(A) \sim X \) (coming from 2), e.g. if we tensor \( A \) with \( O_2 \)?
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2) Is there a topological characterization of the Spectra \( \text{Prim}(A) \) of \textit{amenable} \( A \) (up to homeomorphisms)?

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Lemma

If $R \subset P \times P$ is a (partial) order relation ($y \leq x$ iff $(x, y) \in R$) on a locally compact Polish space $P$ such that the map $\pi_1 : (x, y) \in R \mapsto x \in P$ is open and $x^R := \{ y \in P ; (x, y) \in R \}$ is closed for all $x \in P$, then there is non-degenerate $*$-monomorphism $H_0 : C_0(P, \mathbb{K}) \to \mathcal{M}(C_0(P, \mathbb{K}))$, such that $\delta_\infty \circ H_0$ is unitarily equivalent to $H_0$, and $(x, y) \in R$ if and only if the irreducible representation $\nu_y \otimes \text{id}$ is weakly contained in $\mathcal{M}(\nu_x \otimes \text{id}) \circ H_0$.

Idea of proof: Bounded $*$-weakly cont. maps $x \in P \rightarrow \gamma(x) \in B^*_+$ and c.p. maps $V : B \rightarrow C_b(P)$, are 1-1-related by $\nu_x \circ V = \gamma(x)$. If $x \in P \rightarrow F(x) \in \mathcal{F}(\text{Prim}(B))$ is lower semi-cont. (e.g. $F(x) := x^R$, $B := C_0(P)$), then supports of the $\gamma(x)$ can be chosen in $F(x)$ and $\gamma(x_0) = f$ for $f \in B^*_+$ supported in $F(x_0)$ (by Michael selection).
If such a partial order relation \((x, y) \in R \iff y \leq x\) on \(P\) is given, then one can introduce a (not necessarily separated) topology \(\mathcal{O}_R(P)\) — (a “continuous” variant of the so-called D. Scott topology, or “way below” topology) — that is given by the family of those open subsets of \(P\) that are upward hereditary. I.e., \(U \in \mathcal{O}_R(P)\), iff, \(U = \uparrow U\), iff, \(U \in \mathcal{O}(P)\) and, \(\forall (y \in U), ((x, y) \in R)\) implies \(x \in U\).

The interior \(\bigcap_n U_n\) of the intersection \(\bigcap_n U_n\) of a sequence \(U_1, U_2, ... \in \mathcal{O}_R(P)\) is again in \(\mathcal{O}_R(P)\), i.e., \(\mathcal{O}_R(P)\) is a sup-inf–closed sub-lattice of \(\mathcal{O}(P)\).

If we introduce on \(P\) the equivalence relation \(x \sim y\) if \(x \leq y\) and \(y \leq x\), then one finds that \(x \sim y\), if and only if, for all \(U \in \mathcal{O}_R(P)\), \(x \in U\) iff \(y \in U\).

It follows that \(X := P / \sim\) with the quotient-topology (defined by the images of the \(U \in \mathcal{O}_R(P)\) in \(X\)) is a (not necessarily sober) \(T_0\) space such that \(\pi: x \in P \to [x] \sim \in P / \sim\) is continuous (with respect to the l.c. topology of \(P\)) and satisfies \(\{\pi^{-1}(W) : W \in \mathcal{O}(X)\} = \mathcal{O}_R(P)\).
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Now we can calculate the primitive ideal space of the Toeplitz algebra $\mathcal{T}_H (\cong \mathcal{O}_H$, the Cuntz-Pimsner algebra), where $H := C_0(P, \mathbb{K})$ is the Hilbert $C_0(P, \mathbb{K})$ bi-module that is defined by $H_0$ of Lemma 1.

Proposition (H.Harnisch,K.)

With above assumptions, $\mathcal{T}_H \cong \mathcal{O}_H$, and $\mathcal{T}_H$ is a stable separable nuclear strongly purely infinite algebra. Its ideal lattice is isomorphic to $\mathcal{O}_R(P) \cong \mathcal{O}(P/\sim)$ and the natural embedding $C_0(P, \mathbb{K}) \hookrightarrow \mathcal{T}_H$ defines $KK$-equivalence in $KK(O_R(P); C_0(P, \mathbb{K}), \mathcal{T}_H)$.

It leads to the problem to find — for given $A$ — a l.c. Polish space $P$ and a continuous map $\pi: P \to X := \text{Prim}(A)$, such that the relation $(x, y) \in R \iff \pi(y) \in \{\pi(x)\}$ satisfies the conditions of Lemma 1, and that $\pi(P)$ is “sufficiently dense” in $X$ in the sense that $\pi^{-1}: O(X) \to O(P)$ is injective: $X \sim = \pi(P)$ c-in notation below.
Now we can calculate the primitive ideal space of the Toeplitz algebra $T_{\mathcal{H}} (\cong \mathcal{O}_{\mathcal{H}}$, the Cuntz-Pimsner algebra), where $\mathcal{H} := C_0(P, \mathbb{K})$ is the Hilbert $C_0(P, \mathbb{K})$ bi-module that is defined by $H_0$ of Lemma 1.

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Now we can calculate the primitive ideal space of the Toeplitz algebra $\mathcal{T}_\mathcal{H} \cong \mathcal{O}_\mathcal{H}$, the Cuntz-Pimsner algebra), where $\mathcal{H} := \mathcal{C}_0(P, \mathbb{K})$ is the Hilbert $\mathcal{C}_0(P, \mathbb{K})$ bi-module that is defined by $H_0$ of Lemma 1.

**Proposition (H.Harnisch,K.)**

With above assumptions, $\mathcal{T}_\mathcal{H} \cong \mathcal{O}_\mathcal{H}$, and $\mathcal{T}_\mathcal{H}$ is a stable separable nuclear strongly purely infinite algebra. Its ideal lattice is isomorphic to $\mathcal{O}_R(P) \cong \mathcal{O}(P/\sim)$ and the natural embedding $\mathcal{C}_0(P, \mathbb{K}) \hookrightarrow \mathcal{T}_\mathcal{H}$ defines KK-equivalence in $\text{KK}(\mathcal{O}_R(P); \mathcal{C}_0(P, \mathbb{K}), \mathcal{T}_\mathcal{H})$.

It leads to the problem to find — for given $A$ — a l.c. Polish space $P$ and a continuous map $\pi: P \to X := \text{Prim}(A)$, such that the relation $(x, y) \in R \iff \pi(y) \in \overline{\{\pi(x)\}}$ satisfies the conditions of Lemma 1, and that $\pi(P)$ is “sufficiently dense” in $X$ in the sense that $\pi^{-1}: \mathcal{O}(X) \to \mathcal{O}(P)$ is injective: $X \cong \pi(P)^c$ in notation below.
Functorial passage to sober $T_0$ spaces:

If $X$ is any topological $T_0$ space then the lattice $\mathcal{F}(X)$ of closed subsets order anti-isomorphic to $\mathcal{O}(X)$ by $F \mapsto X \setminus F$. The set $\mathcal{F}(X)$ becomes a $T_0$ space with the topology generated by the complements $\mathcal{F}(X) \setminus [\emptyset, F]$ of the order intervals $[\emptyset, F]$ (for $F \in \mathcal{F}(X)$).
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The map $\eta: x \in X \mapsto \{x\} \in \mathcal{F}(X)$ is a topological homeomorphism from $X$ onto $\eta(X)$. The image $\eta(X)$ is contained in the set $X^c$ of \lor-prime elements of $\mathcal{F}(X)$, and $X^c$ is a sober subspace of $\mathcal{F}(X)$, such that $\eta^{-1}$ defines a lattice isomorphism from $\mathcal{O}(X^c)$ onto $\mathcal{O}(X)$. If $X$ is sober, iff, $\eta(X) = X^c$. 

Result: The lattice $\mathcal{O}(X)$ and the top. space $X$ define each other up to isomorphisms in a natural (functorial) way, if and only if, $X$ is sober.
Functorial passage to sober $T_0$ spaces:

If $X$ is any topological $T_0$ space then the lattice $\mathcal{F}(X)$ of closed subsets order anti-isomorphic to $\mathcal{O}(X)$ by $F \mapsto X \setminus F$. The set $\mathcal{F}(X)$ becomes a $T_0$ space with the topology generated by the complements $\mathcal{F}(X) \setminus [\emptyset, F]$ of the order intervals $[\emptyset, F]$ (for $F \in \mathcal{F}(X)$).

The map $\eta: x \in X \mapsto \overline{\{x\}} \in \mathcal{F}(X)$ is a topological homeomorphism from $X$ onto $\eta(X)$. The image $\eta(X)$ is contained in the set $X^c$ of $\lor$-prime elements of $\mathcal{F}(X)$, and $X^c$ is a sober subspace of $\mathcal{F}(X)$, such that $\eta^{-1}$ defines a lattice isomorphism from $\mathcal{O}(X^c)$ onto $\mathcal{O}(X)$. If $X$ is sober, iff, $\eta(X) = X^c$.

Result: The lattice $\mathcal{O}(X)$ and the top. space $X$ define each other up to isomorphisms in a natural (functorial) way, if and only if, $X$ is sober. The passage $X \to X^c$ is functorial.
Strategies (2): Regular Abelian subalgebras

An abelian $C^*$-subalgebra $C \subset A$ is regular, iff, for $J_1, J_2 \in \mathcal{I}(A)$,
- $C \cap (J_1 + J_2) = (C \cap J_1) + (C \cap J_2)$ and,
- $C$ separates the ideals of $A$ (i.e., $C \cap J_1 = C \cap J_2$ implies $J_1 = J_2$).
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If $P := \text{Prim}(C) = X(C)$ and $Y := \text{Prim}(A)$, then $J \mapsto C \cap J$ defines maps $\psi : \mathcal{O}(Y) \leftarrow \mathcal{O}(P)$ and $\pi : P \rightarrow Y$, with $\pi^{-1}|\mathcal{O}(Y) = \psi$. 
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If $P := \text{Prim}(C) = X(C)$ and $Y := \text{Prim}(A)$, then $J \mapsto C \cap J$ defines maps $\Psi: \emptyset(Y) \hookrightarrow \emptyset(P)$ and $\pi: P \to Y$, with $\pi^{-1}|\emptyset(Y) = \Psi$. The $\pi$ is **pseudo-open** (i.e., relation $(x, y) \in R \iff \pi(y) \in \{\pi(x)\}$ satisfies the assumptions on $R$ in Lem. 1) and is **pseudo-epimorphic** (i.e., $U \subset V \in \emptyset(Y)$ and $\pi(P) \cap (V \setminus U) = \emptyset$ imply $U = V$).
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There are regular $C \subset A$ in AH-algebras (AF if $A$ is AF). Regular comm. $C \subset A$ are in general not maximal, and $C \cap J$ does not necessarily contain an approximate unit of $J$. 
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- $C \cap (J_1 + J_2) = (C \cap J_1) + (C \cap J_2)$ and,
- $C$ separates the ideals of $A$ (i.e., $C \cap J_1 = C \cap J_2$ implies $J_1 = J_2$).

If $P := \text{Prim}(C) = X(C)$ and $Y := \text{Prim}(A)$, then $J \mapsto C \cap J$ defines maps $\Psi: \mathcal{O}(Y) \hookrightarrow \mathcal{O}(P)$ and $\pi: P \to Y$, with $\pi^{-1}|\mathcal{O}(Y) = \Psi$. The $\pi$ is pseudo-open (i.e., relation $(x, y) \in R \iff \pi(y) \in \overline{\{\pi(x)\}}$ satisfies the assumptions on $R$ in Lem. 1) and is pseudo-epimorphic (i.e., $U \subset V \in \mathcal{O}(Y)$ and $\pi(P) \cap (V \setminus U) = \emptyset$ imply $U = V$).

There are regular $C \subset A$ in AH-algebras (AF if $A$ is AF). Regular comm. $C \subset A$ are in general not maximal, and $C \cap J$ does not necessarily contain an approximate unit of $J$. For w.p.i. $B$ and separable $E \subset Q(\mathbb{R}_+, B)$, there exists separable $E \subset A \subset Q(\mathbb{R}_+, B)$ such that $EAE = A$ and $A$ contains a regular abelian subalgebra.
Theorem (On Prim(A), H.Harnisch, E.Kirchberg, M.Rørdam)

Let $X$ a point-complete $T_0$-space. TFAE:

(i) $X \cong \text{Prim}(E)$ for some exact $C^*$-algebra $E$.

(ii) The lattice of open sets $\mathcal{O}(X)$ is isomorphic to an sup–inf invariant sub-lattice of $\mathcal{O}(P)$ for some l.c. Polish space $P$.

(iii) There is a locally compact Polish space $P$ and a pseudo-open and pseudo-epimorphic continuous map $\pi: P \to X$.

If $X$ satisfies (i)–(iii), then there is a stable nuclear $C^*$-algebra $A$ with $A \cong A \otimes O_2$, and a homeomorphism $\psi: X \to \text{Prim}(A)$, such that, for every nuclear stable $B$ with $B \otimes O_2 \cong B$ and every homeomorphism $\phi: X \to \text{Prim}(B)$, there is an isomorphism $\alpha: A \to B$ with $\alpha(\psi(x)) = \phi(x)$ for $x \in X$.

This $\alpha$ is unique up to unitary homotopy.
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Above Theorem 3 answers Questions 2) partially and 3) (almost) completely.
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$$X \cong \text{Prim}(A) \iff X \text{ is a sober } T_0 \text{ space that has a base (!!) of its topology consisting of open quasi-compact sets.}$$
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The latter spaces are special cases of coherent spaces. A sober $T_0$ space $X$ is called “coherent” if the intersection $C_1 \cap C_2$ of two saturated quasi-compact subsets $C_1, C_2 \subset X$ is again quasi-compact. A subset $C$ of $X$ is “saturated” if $C = \text{Sat}(C)$, where $\text{Sat}(C)$ means the intersection of all $U \in \mathcal{O}(X)$ with $U \supset C$. 
Proposition

The image \( \eta(X) \cong X \) in \( \mathcal{F}(X) \setminus \{\emptyset\} \) of a l.q-c. second countable sober \( T_0 \) space \( X \) is closed in the Fell-Vietoris topology on \( \mathcal{F}(X) \), if and only if, \( X \) is coherent, if and only if, the set \( \mathcal{D}(X) \) of Dini functions on \( X \) is convex, if and only if, \( \mathcal{D}(X) \) is min-closed, if and only if, \( \mathcal{D}(X) \) is multiplicatively closed.
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Corollary

If there is a coherent sober l.c. space $X$ that is not homeomorphic to the primitive ideal space of an amenable C*-algebra, then there is $n \in \mathbb{N}$ and a finite union $Y$ of (Hausdorff-closed) cubes in $[0, 1]^n$ such that $Y$ with induced order-topology is not the primitive ideal space of any amenable C*-algebra.