Quantum Minkowski Space

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Rieffel Deformation:
Rieffel Deformation:
- $C^*$-algebras
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- $C^*$-algebras
- $(A, \Delta)$ - quantum groups
Rieffel Deformation:
  ▶ $C^*$-algebras
  ▶ $(A, \Delta)$ - quantum groups
  ▶ $X \bowtie G$ - group actions
Deformation Data:
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- \((A, \rho, \Psi)\) - deformation data
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- $(A, \rho, \Psi)$ - deformation data
- $A$ - $C^*$-algebra

$\Psi(\gamma_1 + \gamma_2, \gamma_3, \gamma_2) = \Psi(\gamma_1, \gamma_2 + \gamma_3, \gamma_2 \gamma_3)$
Rieffel Deformation

Deformation Data:

- $(A, \rho, \Psi)$ - deformation data
- $A$ - $C^*$-algebra
- $\rho$ - $\Gamma$-action
- $\Psi$ - 2 cocycle on the dual group $\hat{\Gamma}$:
  
  $$\Psi(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3) \Psi(\hat{\gamma}_1, \hat{\gamma}_2) = \Psi(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}_3) \Psi(\hat{\gamma}_2, \hat{\gamma}_3)$$
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$$\Psi(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3)\Psi(\hat{\gamma}_1, \hat{\gamma}_2) = \Psi(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_2, \hat{\gamma}_3)$$
Deformation Procedure:

\[ A \rtimes \Gamma \text{- crossed product } \mathcal{C}^* \text{-algebra} \]

\[ \lambda : \Gamma \rightarrow M(A \rtimes \Gamma) \text{- unitary representation} \]

\[ \lambda \text{- extends to the embedding of } \mathcal{C}^*(\Gamma) \text{ into } M(A \rtimes \Gamma) \]

\[ \hat{\rho} \text{- dual action } \hat{\Gamma} \text{ on } A \rtimes \Gamma \]

\[ \hat{\rho} \hat{\gamma}(\lambda \gamma) = \langle \hat{\gamma}, \gamma \rangle \lambda \gamma \]
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- $A times \Gamma$ - crossed product $C^*$-algebra
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- $\hat{\rho}$ - dual action $\hat{\Gamma}$ on $A \rtimes \Gamma$
- $\hat{\rho}_{\gamma}(\lambda_{\gamma}) = \langle \hat{\gamma}, \gamma \rangle \lambda_{\gamma}$
Deformation Procedure:

\[ A \subset M(A \rtimes \Gamma) : \]

\[ \hat{\rho}(b) = b \]

\[ \Gamma \ni \gamma \mapsto \lambda \gamma b \lambda^{-1} \gamma \in M(A \rtimes \Gamma) \]

is $\| \cdot \|$-continuous.
Deformation Procedure:

\[ A \subset M(A \rtimes \Gamma) : \]

\[ A = \begin{cases} 
  b \in M(A \rtimes \Gamma) \\
  1. \hat{\rho}(b) = b \\
  2. \Gamma \ni \gamma \mapsto \lambda_\gamma b \lambda_\gamma^* \in M(A \rtimes \Gamma) \\
  \text{is } \| \cdot \| - \text{continuous} \\
  3. xb, bx \in A \rtimes \Gamma \text{ for any } x \in C^*(\Gamma) 
\end{cases} \]
Deformation procedure:

**Deformed $C^*$-algebra**
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Deformed $C^*$-algebra

$\hat{\rho} \rightsquigarrow \hat{\rho}^\Psi$ - by means of $\Psi$:
Deformation procedure:

Deformed $C^*$-algebra

- $\hat{\rho} \rightsquigarrow \hat{\rho}^\Psi$ - by means of $\Psi$:
- $\hat{\rho}^\Psi (b) = U\hat{\gamma} \hat{\rho}\hat{\gamma}(b) U^*$

$A^\Psi \subset M(A \rtimes \Gamma)$:

- $b \in M(A \rtimes \Gamma)$
- $\|\cdot\|\text{-continuous}$
- $xb, bx \in A \rtimes \Gamma$ for any $x \in C^*(\Gamma)$

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Quantum Minkowski Space
Deformation procedure:

Deformed $C^*$-algebra

- $\hat{\rho} \rightsquigarrow \hat{\rho}^\Psi$ - by means of $\Psi$:
- $\hat{\rho}_{\gamma}^\Psi(b) = U_{\gamma} \hat{\rho}_{\gamma}(b) U_{\gamma}^*$
- $A^\Psi \subset M(A \rtimes \Gamma)$:
Deformation procedure:

Deformed $C^*$-algebra

- $\hat{\rho} \rightsquigarrow \hat{\rho}^\Psi$ - by means of $\Psi$
- $\hat{\rho}^\Psi(\gamma)(b) = U_\gamma \hat{\rho}_\gamma(b) U_{\gamma}^*$
- $A^\Psi \subset M(A \rtimes \Gamma)$:

$$A^\Psi = \left\{ b \in M(A \rtimes \Gamma) \left| \begin{array}{l} 1. \hat{\rho}^\Psi(\gamma)(b) = b \\ 2. \Gamma \ni \gamma \mapsto \lambda_\gamma b \lambda_\gamma^* \in M(A \rtimes \Gamma) \\ \text{is } \| \cdot \| - \text{continuous} \\ 3. xb, bx \in A \rtimes \Gamma \text{ for any } x \in C^*(\Gamma) \end{array} \right. \right\}$$
Rieffel Deformation of locally compact groups

\[ C^*-algebra: \]
C*-algebra:

- $G$ - locally compact group
Rieffel Deformation of locally compact groups

**C*-algebra:**
- $G$ - locally compact group
- $\Gamma \subset G$ - abelian closed subgroup
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Rieffel Deformation of locally compact groups

\( C^*-\text{algebra:} \)

- \( G \) - locally compact group
- \( \Gamma \subset G \) - abelian closed subgroup
- \( \Psi \) - 2-cocycle on the dual group \( \hat{\Gamma} \)
- \( C_0(G) \) - is acted by \( \Gamma^2 \):
C*-algebra:

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- $C_0(G)$ - is acted by $\Gamma^2$:

$$\rho_{\gamma_1,\gamma_2}(f)(g) = f(\gamma_1^{-1}g\gamma_2)$$
Rieffel Deformation of locally compact groups

\[ C^*\text{-algebra:} \]

- \( G \) - locally compact group
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- \( \Psi \) - 2-cocycle on the dual group \( \hat{\Gamma} \)
- \( C_0(G) \) - is acted by \( \Gamma^2 \):
  \[
  \rho_{\gamma_1, \gamma_2}(f)(g) = f(\gamma_1^{-1} g \gamma_2)
  \]
- \( (C_0(G), \rho, \bar{\Psi} \otimes \Psi) \rightarrow C_0(G)\bar{\Psi} \otimes \Psi \)
Quantum Group Structure

Comultiplication:

\[ \Delta : \mathbb{C}_0(G) \rightarrow M(\mathbb{C}_0(G) \otimes \mathbb{C}_0(G)) \] - comultiplication

\[ \rho \text{- covariance properties of } \Delta : \Delta \circ \rho \gamma_1, \gamma_2 = (\rho \gamma_1, e \otimes \rho e, \gamma_2) \circ \Delta \]

\[ \Delta \Rightarrow \Delta \Gamma \text{- crossed product extension} \]

\[ \Delta \Gamma \in \mathbb{C}_0(G) \rtimes \Gamma \rightarrow M(\mathbb{C}_0(G) \rtimes \Gamma \otimes \mathbb{C}_0(G) \rtimes \Gamma) \]

\[ \Delta \Gamma \Rightarrow \Delta \Psi : \Delta \Psi(b) = V \Delta \Gamma(b) V^* \]

\[ \Delta \Psi : \mathbb{C}_0(G) \bar{\Psi} \otimes \Psi \rightarrow M(\mathbb{C}_0(G) \bar{\Psi} \otimes \Psi \otimes \mathbb{C}_0(G) \bar{\Psi} \otimes \Psi) \]

\[ (\mathbb{C}_0(G) \bar{\Psi} \otimes \Psi, \Delta \Psi) \text{- locally compact quantum group.} \]
Quantum Group Structure

**Comultiplication:**

\[ \Delta : \mathcal{C}_0(G) \rightarrow M(\mathcal{C}_0(G) \otimes \mathcal{C}_0(G)) \] - comultiplication

\[ \Delta \circ \rho_{\gamma_1,\gamma_2} = (\rho_{\gamma_1}, e \otimes \rho_{e}, \gamma_2) \circ \Delta \]

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\[ \Delta \Gamma \Rightarrow \Delta \Psi : \Delta \Psi(\mathcal{b}) = V_{\Delta \Gamma}(\mathcal{b}) V^{\ast} \]

\[ \Delta \Psi : \mathcal{C}_0(G) \bar{\Psi} \otimes \Psi \rightarrow M(\mathcal{C}_0(G) \bar{\Psi} \otimes \Psi \otimes \mathcal{C}_0(G) \bar{\Psi} \otimes \Psi) \]

\[ (\mathcal{C}_0(G) \bar{\Psi} \otimes \Psi, \Delta \Psi) \] - locally compact quantum group.
Comultiplication:

- $\Delta : C_0(G) \rightarrow M(C_0(G) \otimes C_0(G))$ - comultiplication
- $\rho$ - covariance properties of $\Delta$:
Quantum Group Structure

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Comultiplication:

- \( \Delta : C_0(G) \to M(C_0(G) \otimes C_0(G)) \) - comultiplication
- \( \rho \) - covariance properties of \( \Delta \):
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  \]
- \( \Delta \leadsto \Delta^\Gamma \) - crossed product extension
Comultiplication:

- $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G))$ - comultiplication
- $\rho$ - covariance properties of $\Delta$:
  \[ \Delta \circ \rho_{\gamma_1,\gamma_2} = (\rho_{\gamma_1,e} \otimes \rho_{e,\gamma_2}) \circ \Delta \]
- $\Delta \rightsquigarrow \Delta^\Gamma$ - crossed product extension
- $\Delta^\Gamma \in C_0(G) \rtimes \Gamma^2 \to M(C_0(G) \rtimes \Gamma^2 \otimes C_0(G) \rtimes \Gamma^2)$
Quantum Group Structure

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- \( \Delta : C_0(G) \rightarrow M(C_0(G) \otimes C_0(G)) \) - comultiplication
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- \( \Delta^\Gamma \leadsto \Delta^\Psi : \Delta^\Psi(b) = V \Delta^\Gamma(b) V^* \)
Comultiplication:

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- \( \Delta^\Gamma \rhd \Delta^\Psi : \Delta^\Psi (b) = V \Delta^\Gamma (b) V^* \)
- \( \Delta^\Psi : C_0(G) \bar{\Psi} \otimes \Psi \to M(C_0(G) \bar{\Psi} \otimes \Psi \otimes C_0(G) \bar{\Psi} \otimes \Psi) \)
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- \( \Delta : C_0(G) \rightarrow M(C_0(G) \otimes C_0(G)) \) - comultiplication
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  \]
- \( \Delta^\Gamma \rightsquigarrow \Delta^\psi : \Delta^\psi(b) = V\Delta^\Gamma(b)V^* \)
- \( \Delta^\psi : C_0(G)\bar{\Psi} \otimes \Psi \rightarrow M(C_0(G)\bar{\Psi} \otimes \Psi \otimes C_0(G)\bar{\Psi} \otimes \Psi) \)
- \( (C_0(G)\bar{\Psi} \otimes \Psi, \Delta^\psi) \) - locally compact quantum group.
Group Actions:
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- $X \rhd G : X \times G \ni (x, g) \mapsto xg \in X$
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- $X \triangleleft G : X \times G \ni (x, g) \mapsto xg \in X$
- $\Delta_X : C_0(X) \to M(C_0(X) \otimes C_0(G)) : \Delta_X(f)(x, g) = f(xg)$
Group Actions:

- $X \trianglerighteq G : X \times G \ni (x, g) \mapsto xg \in X$
- $\Delta_X : C_0(X) \to M(C_0(X) \otimes C_0(G)) : \Delta_X(f)(x, g) = f(xg)$
- $\Delta_X$ - continuous coaction:
Group Actions:

- $X \lhd G : \ X \times G \ni (x, g) \mapsto xg \in X$
- $\Delta_X : C_0(X) \to M(C_0(X) \otimes C_0(G)) : \Delta_X(f)(x, g) = f(xg)$
- $\Delta_X$ - continuous coaction:
  - coassociativity $\sim (\Delta_X \otimes \iota) \circ \Delta_X = (\iota \otimes \Delta) \circ \Delta_X$
Rieffel Deformation of group acting on spaces

Group Actions:

- \( X \curvearrowright G : \ X \times G \ni (x, g) \mapsto xg \in X \)
- \( \Delta_X : C_0(X) \rightarrow M(C_0(X) \otimes C_0(G)) : \Delta_X(f)(x, g) = f(xg) \)
- \( \Delta_X - \) continuous coaction:
  - coassociativity \( \leadsto (\Delta_X \otimes \iota) \circ \Delta_X = (\iota \otimes \Delta) \circ \Delta_X \)
  - cancellation \( \leadsto [\Delta_X(C_0(X))(1 \otimes C_0(G))] = C_0(X) \otimes C_0(G) \)
Deformation Procedure:
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▶ $C_0(X)$ is acted by $\Gamma$:
Deformation Procedure:

- $C_0(X)$ is acted by $\Gamma$:

\[ \rho_\gamma(f)(x) = f(x\gamma) \]
Rieffel Deformation of group acting on spaces

**Deformation Procedure:**

- \( C_0(X) \) is acted by \( \Gamma \):
  \[
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  \]

- \((C_0(X), \rho, \Psi) \rightsquigarrow C_0(X)^\Psi\)
Rieffel Deformation of group acting on spaces

**Deformation Procedure:**

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- $(C_0(X), \rho, \Psi) \sim C_0(X)^\Psi$

- $\Delta_X \sim \Delta_X^\Psi$
Rieffel Deformation of group acting on spaces

**Deformation Procedure:**

- $C_0(X)$ is acted by $\Gamma$:
  \[ \rho_\gamma(f)(x) = f(x\gamma) \]

- $(C_0(X), \rho, \Psi) \simto C_0(X)\Psi$

- $\Delta_X \simto \Delta^\Psi_X$

- $\Delta^\Psi_X : C_0(X)\Psi \to M(C_0(X)\Psi \otimes C_0(G)\tilde{\Psi} \otimes \Psi)$
Deformation Procedure:

1. $C_0(X)$ is acted by $\Gamma$:
   \[ \rho_\gamma(f)(x) = f(x\gamma) \]

2. $(C_0(X), \rho, \Psi) \rightsquigarrow C_0(X)\Psi$

3. $\Delta_X \rightsquigarrow \Delta_X^\Psi$

4. $\Delta_X^\Psi : \mathcal{C}_0(X)^\Psi \to \mathcal{M}(\mathcal{C}_0(X)^\Psi \otimes \mathcal{C}_0(G)\bar{\Psi} \otimes \Psi)$

5. $\Delta_X^\Psi$ - continuous coaction:
Deformation Procedure:

- $C_0(X)$ is acted by $\Gamma$:
  $$\rho_\gamma(f)(x) = f(x\gamma)$$

- $(C_0(X), \rho, \Psi) \leadsto C_0(X)^\Psi$

- $\Delta_X \leadsto \Delta_X^\Psi$

- $\Delta_X^\Psi : C_0(X)^\Psi \to M(C_0(X)^\Psi \otimes C_0(G)\bar{\Psi} \otimes \Psi)$

- $\Delta_X^\Psi$ - continuous coaction:
  - coassociativity $(\Delta_X^\Psi \otimes \iota)\Delta_X^\Psi = (\iota \otimes \Delta^\Psi)\Delta_X^\Psi$
Deformation Procedure:

- $C_0(X)$ is acted by $\Gamma$:
  \[ \rho_\gamma(f)(x) = f(x\gamma) \]

- $(C_0(X), \rho, \Psi) \sim \to C_0(X)\Psi$

- $\Delta_X \sim \to \Delta_X^\Psi$

- $\Delta_X^\Psi : C_0(X)^\Psi \to M(C_0(X)^\Psi \otimes C_0(G)\bar{\Psi} \otimes \Psi)$

- $\Delta_X^\Psi$ - continuous coaction:
  - coassociativity $(\Delta_X^\Psi \otimes \iota)\Delta_X^\Psi = (\iota \otimes \Delta_X^\Psi)\Delta_X^\Psi$
  - $[\Delta_X^\Psi(C_0(X))(1 \otimes C_0(G)\bar{\Psi} \otimes \Psi)] = C_0(X)^\Psi \otimes C_0(G)\bar{\Psi} \otimes \Psi$
Quantum Minkowski Space

Classical Minkowski Space:

\[ G = \text{SL}(2, \mathbb{C}) = \{ (\alpha \beta \gamma \delta) : \alpha \delta - \beta \gamma = 1 \} \]

\[ X = H = \{ (x \bar{w} w y) : x, y \in \mathbb{R}, w \in \mathbb{C} \} \]

\[ H \times G \ni (h, g) \mapsto g^* h g \in H \]

\[ \Gamma = \{ (e^z 0 0 e^{-z}) : z \in \mathbb{C} \} \]

\[ \Psi(z_1, z_2) = e^{is \overline{z}_1 z_2} \]
Quantum Minkowski Space

Classical Minkowski Space:

\[ G = \text{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha\delta - \beta\gamma = 1 \right\} \]
Quantum Minkowski Space

Classical Minkowski Space:

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- \( X = \mathcal{H} = \left\{ \begin{pmatrix} x & w \\ \bar{w} & y \end{pmatrix} : x, y \in \mathbb{R}, w \in \mathbb{C} \right\} \)
Quantum Minkowski Space

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- \( \mathcal{H} \times G \ni (h, g) \mapsto g^* hg \in \mathcal{H} \)

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Classical Minkowski Space:

- $G = SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha \delta - \beta \gamma = 1 \right\}$
- $X = \mathcal{H} = \left\{ \begin{pmatrix} x & w \\ \bar{w} & y \end{pmatrix} : x, y \in \mathbb{R}, w \in \mathbb{C} \right\}$
- $\mathcal{H} \times G \ni (h, g) \mapsto g^* hg \in \mathcal{H}$
- $\Gamma = \left\{ \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} : z \in \mathbb{C} \right\}$
Quantum Minkowski Space

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- \( G = SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha \delta - \beta \gamma = 1 \right\} \)

- \( \mathcal{X} = \mathcal{H} = \left\{ \begin{pmatrix} x & w \\ \bar{w} & y \end{pmatrix} : x, y \in \mathbb{R}, w \in \mathbb{C} \right\} \)

- \( \mathcal{H} \times G \ni (h, g) \mapsto g^* hg \in \mathcal{H} \)

- \( \Gamma = \left\{ \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix} : z \in \mathbb{C} \right\} \)

- \( \Psi(z_1, z_2) = e^{is\Im(z_1 \bar{z}_2)} \)
Quantum Minkowski space

\[ (C_0(G)\bar{\Psi} \otimes \Psi, \Delta \Psi) \text{ - quantum Lorentz group:} \]
Quantum Minkowski space

\[(C_0(G)\bar{\Psi} \otimes \Psi, \Delta \Psi) - \text{quantum Lorentz group:}\]

- \(C_0(G)\bar{\Psi} \otimes \Psi = \mathbb{C}^*-\text{algebra generated by } \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}\)
Quantum Minkowski space

\((C_0(G)\bar{\Psi} \otimes \psi, \Delta \psi)\) - quantum Lorentz group:

- \(C_0(G)\bar{\Psi} \otimes \psi = C^*\)-algebra generated by \(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}\)
- Commutation relations

\[
\begin{align*}
\hat{\alpha}\hat{\alpha}^* &= \hat{\alpha}^* \hat{\alpha} \\
\hat{\alpha}\hat{\beta}^* &= t\hat{\beta}^* \hat{\alpha} \\
\hat{\alpha}\hat{\gamma}^* &= t^{-1}\hat{\gamma}^* \hat{\alpha} \\
\hat{\alpha}\hat{\delta}^* &= \hat{\delta}^* \hat{\alpha} \\
\hat{\beta}\hat{\beta}^* &= \hat{\beta}^* \hat{\beta} \\
\hat{\beta}\hat{\gamma}^* &= \hat{\gamma}^* \hat{\beta} \\
\hat{\gamma}\hat{\gamma}^* &= \hat{\gamma}^* \hat{\gamma} \\
\hat{\beta}\hat{\delta}^* &= t^{-1}\hat{\delta}^* \hat{\beta} \\
\hat{\gamma}\hat{\delta}^* &= t\hat{\delta}^* \hat{\gamma} \\
\hat{\delta}\hat{\delta}^* &= \hat{\delta}^* \hat{\delta},
\end{align*}
\]

and \(\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} = 1\), \(\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha}\), \(\hat{\alpha}\hat{\delta} = \hat{\delta}\hat{\alpha}\)

- \(t = e^{-8s}\)
Quantum Minkowski space

**Comultiplication**

\[ \Delta \Psi \in \text{Mor}(C_0(G)\bar{\psi} \otimes \psi, C_0(G)\bar{\psi} \otimes \psi \otimes C_0(G)\bar{\psi} \otimes \psi) : \]

\[ \Delta \Psi (\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha} + \hat{\beta} \otimes \hat{\gamma}, \]

\[ \Delta \Psi (\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\delta}, \]

\[ \Delta \Psi (\hat{\gamma}) = \hat{\gamma} \otimes \hat{\alpha} + \hat{\delta} \otimes \hat{\gamma}, \]

\[ \Delta \Psi (\hat{\delta}) = \hat{\gamma} \otimes \hat{\beta} + \hat{\delta} \otimes \hat{\delta}. \]
Comultiplication

\[ \Delta^\psi \in \text{Mor}(C_0(G)^\Psi \otimes \psi, C_0(G)^\Psi \otimes \psi \otimes C_0(G)^\Psi \otimes \psi) : \]

\[ \Delta^\psi(\hat{\alpha}) = \hat{\alpha} \otimes \hat{\alpha} + \hat{\beta} \otimes \hat{\gamma}, \]
\[ \Delta^\psi(\hat{\beta}) = \hat{\alpha} \otimes \hat{\beta} + \hat{\beta} \otimes \hat{\delta}, \]
\[ \Delta^\psi(\hat{\gamma}) = \hat{\gamma} \otimes \hat{\alpha} + \hat{\delta} \otimes \hat{\gamma}, \]
\[ \Delta^\psi(\hat{\delta}) = \hat{\gamma} \otimes \hat{\beta} + \hat{\delta} \otimes \hat{\delta}. \]
Quantum Minkowski space (\(C_0(X)^\Psi, \Delta^\Psi\)):
Quantum Minkowski space

Quantum Minkowski space \((C_0(X)\Psi, \Delta^\Psi)\):

- \(C_0(X)\Psi\) - C\(^*\) - algebra generated by \(\hat{x}, \hat{y}, \hat{w}\)
Quantum Minkowski space $(C_0(X)\Psi, \Delta_X^\Psi)$:

- $C_0(X)\Psi$ - $C^*$ - algebra generated by $\hat{x}, \hat{y}, \hat{w}$
- Commutation relations:
  \[
  \hat{x}\hat{y} = \hat{y}\hat{x}, \quad \hat{w}\hat{x} = t\hat{w}\hat{x}, \quad \hat{w}\hat{y} = t^{-1}\hat{y}\hat{w}
  \]
Quantum Minkowski space \((C_0(X)^\Psi, \Delta_X^\Psi)\):

- \(C_0(X)^\Psi\) - \(C^*\) - algebra generated by \(\hat{x}, \hat{y}, \hat{w}\)
- Commutation relations:
  \[\hat{x}\hat{y} = \hat{y}\hat{x}, \quad \hat{w}\hat{x} = t\hat{w}\hat{x}, \quad \hat{w}\hat{y} = t^{-1}\hat{y}\hat{w}\]
- \(\Delta_X^\Psi \in \text{Mor}(C_0(X)^\Psi, C_0(X)^\Psi \otimes C_0(G)^\Psi \otimes \psi)\)
Quantum Minkowski space \((C_0(X)^\Psi, \Delta^\Psi)\):

- \(C_0(X)^\Psi\) - \(C^*\) algebra generated by \(\hat{x}, \hat{y}, \hat{w}\)
- Commutation relations:
  \[
  \hat{x}\hat{y} = \hat{y}\hat{x}, \quad \hat{w}\hat{x} = t\hat{w}\hat{x}, \quad \hat{w}\hat{y} = t^{-1}\hat{y}\hat{w}
  \]
- \(\Delta^\Psi\in Mor(C_0(X)^\Psi, C_0(X)^\Psi\otimes C_0(G)^\bar{\Psi}\otimes\Psi)\)
- \(\Delta^\Psi\) - continuous coaction of \((C_0(G)^{\bar{\Psi}\otimes\Psi}, \Delta^\Psi)\) on \(C_0(X)^\Psi\)
Quantum Minkowski space (\(C_0(X)\), \(\Delta_X\)):

- \(C_0(X)\) - \(C^*\) - algebra generated by \(\hat{x}, \hat{y}, \hat{w}\)
- Commutation relations:
  \[
  \hat{x}\hat{y} = \hat{y}\hat{x}, \quad \hat{w}\hat{x} = t\hat{w}\hat{x}, \quad \hat{w}\hat{y} = t^{-1}\hat{y}\hat{w}
  \]
- \(\Delta_X \in Mor(C_0(X), C_0(X) \otimes C_0(G)\bar{\Psi} \otimes \Psi)\)
- \(\Delta_X\) - continuous coaction of \((C_0(G)\bar{\Psi} \otimes \Psi, \Delta\Psi)\) on \(C_0(X)\)

\[
\Delta_X(\hat{x}) = \hat{x} \otimes \hat{\alpha}^*\hat{\alpha} + \hat{w} \otimes \hat{\alpha}^*\hat{\gamma} + \hat{w}^* \otimes \hat{\gamma}^*\hat{\alpha} + \hat{y} \otimes \hat{\gamma}^*\hat{\gamma},
\]
\[
\Delta_X(\hat{w}) = \hat{x} \otimes \hat{\alpha}^*\hat{\beta} + \hat{w} \otimes \hat{\alpha}^*\hat{\delta} + \hat{w}^* \otimes \hat{\gamma}^*\hat{\beta} + \hat{y} \otimes \hat{\gamma}^*\hat{\delta},
\]
\[
\Delta_X(\hat{y}) = \hat{x} \otimes \hat{\beta}^*\hat{\beta} + \hat{w} \otimes \hat{\beta}^*\hat{\delta} + \hat{w}^* \otimes \hat{\delta}^*\hat{\beta} + \hat{y} \otimes \hat{\delta}^*\hat{\delta},
\]