Regularity properties and classification of nuclear
C*-algebras

1.1 Nuclearity

Theorem (Elliott) AF algebras are classified by
their scaled ordered K0-groups.

1.2 Conjecture (Elliott) Separable nuclear C*-alg
are classified by K-theoretic data

Why nuclear C*-algebras?

1.3 Theorem (Choi- Effros; Kirchberg)
A is nuclear iff A has the completely positive
approximation property (CPAP)

\[ \psi_1, \psi_2: A \rightarrow I \]

Finite-dimensional \( \psi_1, \psi_2 \) cpc maps

1.4 Remarks
- Finite-dimensional approximations seem promising
  but c* approximations are not a natural
  framework to study K-theoretic data
- Nuclearity is a flexible concept; it can
  be characterized in many different ways
  which make contact with many areas
  of operator algebras
1.5 Definition (Kadison - Kastler, Christensen)
A, B ∈ B(H) be C*-algebras acting on the same Hilbert space. Write d(A, B) < δ if, for each a ∈ A, ∃ b ∈ B with \|a - b\| < δ and vice versa.
Write A ≤ B if ∃ 0 < δ ≤ 1 such that for each a ∈ A, ∃ b ∈ B with \|a - b\| < δ.

1.6 Conjecture (Kadison - Kastler) If A, B ∈ B(H) are separable C*-algebras and d(A, B) < δ for some small enough δ, then A, B are unitarily isomorphic.

ie. ∃ u ∈ B(H) such that A = uBu*.

1.7 Theorem (Christensen - Sinclair - Smith - White - Winter)
Let A, B ∈ B(H) be C*-algebras with A separable and nuclear and \( d(A, B) < 10^{-11} \).
Then there is a unitary u ∈ B(H) such that A = uBu*.

1.8 Theorem (Christensen - Sinclair - Smith - White - Winter)
For each δ > 0 such that the following holds:
Let A, B ∈ B(H) be C*-algebras with A separable and dim(A) ≤ n and with A ≤ B.
Then there exists an embedding A ≤ B.

Remark: K-theoretic invariants tend to be preserved under closeness.
1.3 The purely infinite case

1.3. Definition (Cuntz): A simple C*-algebra is called purely infinite if for any a, b ∈ A
exists x ∈ A such that a = x*b*x

Being purely infinite means that positive elements can be compared.

1.10 Theorem (Kirchberg): Let A be separable, nuclear and simple C*-algebra. Then A is purely infinite if and only if $A \cong A \otimes \mathbb{O}_2$

$\mathcal{O}_2 = C^*(s_1, s_2 | \frac{1}{2} s_1 s_1^* s_2 s_2^* \leq 1 \forall n, s_1^* s_2 = 0)$

1.11 Theorem (Kirchberg): Let A be separable and exact. Then A embeds into $\mathcal{O}_2$, $A \hookrightarrow \mathcal{O}_2$

$\mathcal{O}_2 = C^*(s_1, s_2 | s_1^* s_2 = 1, s_1 s_2^* + s_2 s_1^* = 1)$

If moreover A is nuclear, then there are a-embedding.

1.12 Theorem (Kirchberg, Phillips): Kirchberg algebras (sep., simple, nuclear, purely infinite) with UCT are classified by their K-theory (K-theory isomorphic with isomorphism of algs. inducing the K-theoretic isomorphism).
1.4 The Stably finite case

There are many partial results, for example:

1.13 Theorem (Elliott–Gong–Li): Simple AH algebras of bounded topological dimension are classified by their Elliott invariants.

- $B$ homogeneous: $B = p(M_2 \otimes C(X))p$
- $B$ semihomogeneous: $B = \bigoplus p_i (M_2 \otimes C(X))p_i$
- $B$ AH (approximately homogeneous): direct limit of semihomogeneous $C^*$-algebras.

In fact, E-G-L show that very slow dimension growth is enough.

(bounded topological dimension means that the covering dimension of the $X_i$'s are bounded)

Task: Find stably finite versions of $O_2$ (maybe also $O_2$) to shed new light on existing classification results in the stably finite case and also to (hopefully) unify the purely infinite and stably finite case.

1.5 Towards a structural conjecture

1.14 Question: In what way, and under what conditions, are finite topological dimension, Murray–von Neumann comparison
of positive elements, D-stability (for $D=O_2, O_2^*\ldots$)
and classifiability related?

Topological Dimension

2.1 Order zero maps

2.1 Definition. A cpc map $\varphi: A \to B$ has order zero if it respects orthogonality, i.e.

$$e f \in A_+ \Rightarrow \varphi(e) f \in B_+$$

$$\text{CPC}_0(A,B) := \{ \text{c.p.c. order zero maps } A \to B \}$$

For general $e,f \in A$,

$$e f = e f^* = e f^* = e f = e f^* = 0$$

2.2 Theorem (Winter–Jucharski; using results of Wolf)

Let $\varphi: A \to B$ be a CPC order zero map.

Then there is a $*$-homomorphism

$$\varphi_0: A \to M(C^*(\varphi(A))) \subset B^{**}$$

and an element

$$0 < h_\varphi \in M(C^*(\varphi(A))) \cap \varphi_0(A)$$

such that

$$\varphi(a) = h_\varphi \Pi_{\varphi_0}(a) \text{ for } a \in A$$

Note that $\|\varphi\| = \|h_\varphi\|

The universal $C^*$-algebra generated by a contraction $h$:

$$C^*(h \mid 0 \leq h \leq 1) = C_0((0,1])$$
2.3 Corollary. Let \( \varphi : A \to B \) be a c.p.c. order zero map. Then:

- \( \varphi^{(n)} : M_n(A) \to M_n(B) \) has order zero for all \( n \in \mathbb{N} \).
- There is an induced map \( W(\varphi) : W(A) \to W(B) \).
- For \( 0 \leq f \in C_0(0,1] \) we may define a c.p.c. order zero map \( f(\varphi)(x) = f(\varphi(x)) I_{M_n(x)} : A \to B \) (functional calculus for order zero maps).

Moreover, there is a 1-1 correspondence:

\[
CPC_1(A, B) \leftrightarrow \text{Hom}(C_0(0,1]) \otimes A, B)
\]

2.4 Theorem (Loring). C.p.c. order zero maps with finite-dimensional domains are given by weakly stable relations.

More precisely: Let \( F \) be a finite-dimensional \( C^* \)-algebra and let \( \xi_{\sigma_0} \). Then there is \( \delta > 0 \) such that the following holds:

If \( \varphi : F \to A \) is c.p.c. \( \delta \)-order zero, then there is a c.p.c. order zero map \( \tilde{\varphi} : F \to A \) such that \( \| \tilde{\varphi} - \varphi \| < \delta \).
2.2 Decomposition rank

2.5 Definition (Kirchberg, Winter): Let $A$ be a C*-algebra, $n \in \mathbb{N}$. We say $A$ has decomposition rank at most $n$, denoted $\text{dr}(A) \leq n$, if the following holds: For any FCA finite and any $\varepsilon > 0$ there is a finite-dimensional C.P.C. approximation

$$\varphi : A \rightarrow F \rightarrow A$$

with

$$\varphi \circ \varphi = F \circ \text{id}_A$$

and such that $F$ can be written as

$$F = F^{(0)} \oplus \ldots \oplus F^{(n)}$$

with

$$\varphi^{(i)} := \varphi \big|_{F^{(i)}}$$

having order zero.

2.6 Proposition: Let $X$ be a locally compact metrizable space, $\text{dr}(C_0(X)) = \text{dim} X$

Proof: Use partitions of unity and barycentric subdivision.

$$X = [0, 1] \quad \overset{\text{partition of unity}}{\longrightarrow} \quad \Delta^n$$

2.7 Proposition: ($A$ separable?) $\text{dr} A = 0$ if and only if $A$ is AF
Proof: If $A$ is unital, consider $A \to F \to A$, and note that almost unital order zero maps are almost $\times$-homomorphisms (by $\sim_1$). In the nonunital case, use an idempotent approximate unit for $A$ (e.g., $e_n$).

2.8 Proposition: Decomposition rank behaves well with respect to quotients, limits, tensor products, hereditary subalgebras, and Morita equivalence.

\[
\operatorname{dr}(\varprojlim A_i) \leq \liminf_{i \to \infty} (\operatorname{dr} A_i)
\]

\[
\operatorname{dr} (A \otimes B) \leq ([\operatorname{dr} (A)+1][\operatorname{dr} (B)+1]) - 1
\]

\[
B \subseteq A, \quad \operatorname{dr} (B) \leq \operatorname{dr} (A)
\]

2.9 Corollary: If $A$ has continuous trace then $\operatorname{dr} A = \dim A$.

2.10 Theorem: A separable $C^*$-algebra $A$ is $\times$-subhomogeneous if $\operatorname{dr} A = \max_{k \in \mathbb{N}} \dim (\text{Prim}_k A)$

2.11 Corollary: $\operatorname{dr} A \leq \dim_{\text{ASH}} A \leq \dim_{\text{AH}} A$

(if $A$ is not ASH, $\dim_{\text{ASH}} A = 2$)
2.8 Proposition: $\sigma$ behaves well with quotients, limits, tensor products, hereditary subalgebras, Morita equivalence.

2.9 Corollary: If $A$ has continuous trace, then $\sigma(A) = \dim A$.

2.12 Definition: $A$ has locally finite decomposition rank if for any finite $F \subseteq A$, $\exists F \subseteq \beta \subseteq A$ s.t. $\dim F \leq \infty$ and $F \subseteq \beta$.

2.3 Revisiting

2.13 Lemma: $\phi: A \to B$ cp between $C^*$-algebras $A, B$. Then for any $x, y \in B$ we have

$$\|\phi(xy) - \phi(x)\phi(y)\| \leq \|\phi(x^*x) - \phi(x)\phi(x^*)\|^{1/2} \|y\|.$$

Proof: Use Stinespring's Theorem.
2.14 Lemma. \( A, B \subset C^* \)-algebras, \( a \in A \), \( \| a \| \leq 1 \) and \( \eta > 0 \), if

\[ A \to B \to A \]

are c.p.c. maps satisfying

\[ \| \varphi(a) - a \|, \| \varphi(a^2) - a^2 \| < \eta \]

then, for all \( b \in B \),

\[ \| \varphi(\varphi(a)b) - \varphi(\varphi(a))\varphi(b) \| \leq 3^{1/2} \eta^{1/2} \| b \| \]

Proof: We have

\[ \| \varphi(\varphi(a)^2) - (\varphi(\varphi(a))^2 \| \leq \| \varphi(\varphi(a^2)) - (\varphi(\varphi(a)^2) \| \leq 3 \eta \]

so

\[ \| \varphi(\varphi(a)b) - \varphi(\varphi(a))\varphi(b) \| \leq (3 \eta)^{1/2} \| b \| \]

for all \( b \in B \) by Lemma 2.13.

If \( b \) is a central projection in \( B = \mathbb{D}^\omega \oplus \mathbb{D} \), \( \varphi(b) = b \)

\[ B = \mathbb{D}^\omega \oplus \mathbb{D} \]

\[ \varphi(\varphi(a) \mathbb{D}^{\omega_1}) \cong \varphi(\varphi(a)) \mathbb{D}^{\omega_1} \]

\[ \varphi(i) \varphi(i)(a) \cong \varphi(i), \varphi(i) \]

c.p.c. order zero maps

2.15 Prop. (Kirchberg - Winter): If \( \text{dim} A < n < \infty \) then

\[ \exists \text{ a system } \left( A \to F_2 \to A \right) \]

of c.p.c. approximations for \( A \) with finite-dim \( F_2 \)

n-decomposable c.p.c. maps \( \varphi \) and

approximately multiplicative c.p.c. maps \( \varphi \)

In particular, \( A \) embeds into \( \mathbb{T}_n F_2 / \mathbb{D}_n F_2 \)
2.16 Corollary: if \( \text{dim } A < \infty \) then \( A \) is quasidiagonal (and hence totally finite).

Proof of 2.15 (Sketch): Assume \( A \) is finite. \( \mathcal{F} \subset A \), \( \exists \eta > 0 \)

Take \( A \xrightarrow{\gamma} F \xrightarrow{\phi} A \)

Since \( F \) is decomposable, can write \( F = F^{(1)} \oplus \ldots \oplus F^{(n)} \)

Find a projection \( p = p^{(1)} \oplus \ldots \oplus p^{(n)} \in F \) such that

\[ ||\phi - p|| < \eta, \quad ||p^{(i)}(e)|| < \eta \quad \forall e \in p^{(i)}F^{(i)}p^{(i)} \]

Set \( A \xrightarrow{\gamma} F \xrightarrow{\phi} A \)

Check that \( \gamma \) is \( F - \eta^{1/2} \)-multiplicative.

2.17 Examples: The Toeplitz algebra \( T \) and Cuntz algebra \( O_n \) have infinite \( \text{dim } A \).

2.4 Nuclear dimension

What to do when \( A \) is not necessarily finite?

2.18
2.18 Definition (Winter–Zacharias) 

A nuclear dimension, \( \dim_n A \leq n \), defined as decomposition rank, but in

\[
(f = f_1 \oplus \ldots \oplus f_n, \gamma, \varphi = \varphi_1 + \ldots + \varphi(n))
\]

only asking the \( \varphi^{(i)} \) be contractions (instead of \( \varphi \)).

2.19 Proposition

Nuclear dimension agrees with decomposition rank in the commutative and in zero-dim.

2.20 Proposition (Winter–Zacharias) If \( \dim_n A \leq n < \infty \), then there is a system

\[
(A \xrightarrow{\vartheta} F_2 \xrightarrow{\varphi_2} A) \in \Lambda
\]

of c.p. approximations for \( A \) with finite-dimensional \( F_2 \), \( n \)-decomposable c.p. maps \( \varphi_2 \) and approximately order zero embedding c.p. maps \( \vartheta \).

In particular, there is a c.p. order zero embedding of \( A \) into \( \vartheta(F_2) / \varphi_2(F_2) \).

2.12 Corollary: If \( A \) is unital
2.5 Kirchberg algebras

2.22 Theorem (Winter–Zacharias) For $n=2,3$, we have $\dim_{\text{nuc}} C_0 \leq 2$.

2.23 Corollary: Let $A$ be a UCT Kirchberg algebra. Then $\dim_{\text{nuc}} A \leq 5$.

3.1 Being strongly self-absorbing

3.1 Definition (Tom–Winter) A unital separable $C^*$-algebra $D$ is strongly self-absorbing if $D \cong D$ and there is a $\star$-isomorphism

$$\phi : D \cong D \otimes D$$

such that

$$\phi \circ \text{id} \otimes \phi$$

A $C^*$-algebra is $D$-stable if $A \cong A \otimes D$

3.2 Theorem (Effros–Rosenberg, Kirchberg) If $D$ is strongly self-absorbing then $D$ is simple and nuclear and $D$ either has a unique tracial state or is purely infinite.
3.3 Examples

(i) UHF algebras of infinite type \( M_{\infty}, M_{\infty} \).

(ii) \( \mathcal{G}_2 = \mathbb{C}^k \left( s_1, s_2 \mid s_i^* s_i = 1, \forall i \in \{1, 2\} \right) \)

(iii) \( \mathcal{O}_\infty \)

(iv) \( \mathcal{O}_\infty \otimes M_{\infty} \)

(v) \( \mathcal{Z} \) the Jiang-Su algebra, a finite analogue of \( \mathcal{O}_\infty \); \( \mathcal{Z} \) can be written as a stationary inductive limit

\[
\lim_{\rightarrow} (Z_{2\infty}, 3\omega, \alpha)
\]

where

\[
Z_{2\infty, 3\omega} = \{ f \in \mathcal{C}(\mathbb{C}, M_{\infty} \otimes M_{\infty}) \mid f(1) \in M_{\infty} \otimes 1 \} ;
\]

\[
f(1) \in 1 \otimes M_{\infty}
\]

and \( \alpha \) is trace-collapsing endomorphism of \( Z_{2\infty, 3\omega} \) (Rørdam-Winter)

\[
\alpha : Z_{2\infty, 3\omega} \rightarrow Z_{2\infty, 3\omega}
\]

\[
T(\ ) \leftrightarrow T(\ ) \cdot \alpha_f
\]

all traces are sent by \( \alpha_f \) to a single trace
3.4 Theorem (Rørdam; Winter) If $D$ is strongly self-absorbing then $D \cong D \otimes 2$

Remarks
- Any strongly self-absorbing $C^*$-alg is $K_1$-injective (using Gong-Jiang-Su); respective hypotheses in earlier papers are obsolete
- The known strongly self-absorbing examples form a hierarchy with $D_2$ at the top and $\mathbb{Z}$ at the bottom (everything embeds into $D_2$, $\mathbb{Z}$ embeds into everything)

3.2 D-stability

3.5 Theorem (Rørdam; Toms-Winter) Let $A$ be separable and $D$ strongly self-absorbing. Then $A$ is $D$-stable if and only if there exists a $*$-homomorphism $\varphi : A \otimes D \to \mathbb{M}_n(A) \bigoplus_n A$ such that $\varphi \otimes D = \iota_A$

If $A$ is unital and $D = \lim D_i$ then $A$ is $D$-stable if and only if for each $i$ there is a unital $*$-homomorphism $\varphi_i : D_i \to (\mathbb{M}_n(A) \bigoplus_n A) \otimes A'$

3.6 Theorem (Toms-Winter) For any strongly self-absorbing $D$, $D$-stability passes to limits, quotients, hereditary subalgebras, and extension
Extensions:

\[ 0 \to J \to A \to B \to 0 \]

D-stable \quad \text{D-stable}

Assume that $A$ is unital

\[ \sigma : B \to A \quad \text{and} \quad \sigma : B \otimes D \to A \]

\[ \sigma / D : D \to A \]

Let $(\mathcal{D})$ a quasicentral approximate unit

\[ 0 \leq h \leq 1, \quad h \in A^{\prime} \cap A \]

\[ C((0,1)) \subseteq A \]

Remark: There are results on the structure of $C(X)$-algebras with strongly self-absorbing fibres on D-stable fibres

3.2 Z-stability

3.7 Proposition (Tom\-s- Winter)

Approximate divisibility $\Rightarrow$ Z-stability

3.8 Theorem (Tom\-s- Winter)

All $C^*$-algebras classified to date are Z-stable
Pure finiteness and 2-stability

4.1 Strict comparison

4.1 Definition: A has strict comparison of positive elements if
\[ d_T(a) < d_T(b) \quad \forall z \in \mathcal{T}(A) \]
\[ \Rightarrow <a> \leq <b> \]
for all \( 0 \neq a, b \in A^+ \).

4.2 Theorem (Rørdam) If A is exact, unital and 2-stable then A has strict comparison (A also simple).

4.3 Lemma (Toms-Winter): Suppose A is a unital C*-algebra with \( \text{dr} \ A > 0 \). Given \( a, d^{(0)}, \ldots, d^{(n)} \in A^+ \) such that
\[ d_T(a) < d_T(d^{(i)}) \quad \text{for } i = 0, \ldots, n \quad \text{and } A \in \mathcal{T}(A) \]
Then
\[ <a> \leq <d^{(0)}> + \ldots + <d^{(n)}> \quad \text{in } W(A). \]

4.4 Remark: One can show directly that if \( \text{dr} \ A > 0 \), then quasi-traces are traces.

4.2 The conjecture:

4.5 Conjecture (Toms-Winter): For a nuclear, separable simple finite unital and non-elementary C*-algebra A

(\text{FA})

(i) A has finite decomposable rank
(ii) A is 2-stable
(iii) A has strict comparison of positive elements.
Remarks.
(a) (i) $\Rightarrow$ (iii) Rørdam
(b) (i) $\Rightarrow$ (i) known in many cases, using classification results
(c) Conjecture can also be formulated in the commutative and non-simple case
(d) in the simple but not necessarily finite case

4.3 Finite decomposition rank and $\mathbb{Z}$-stability

4.6 Theorem (Winter): A separable simple unital with $\ell_1(A) < \infty$, then $A$ is $\mathbb{Z}$-stable

Ingredients of Proof

1. An approx. central sequence of unital $\mathbb{Z}$-homomorphisms $\varphi: C_0([0,1], M_p \otimes M_{p+1}) \to A$
   for any $p \in \mathbb{N}$ will do
2. By Rørdam-Winter, need to find $x \in C(A)$ such that
   $p x \leq \|a\| \leq (p+1)x$
   (in an approximately central way)
3. Finite decomposition rank implies "enough" comparison (by Lemma 4.3) to find $x$ as above
4. In-decomposable approxs allow to do things in an approximately central way.
4.4 Pure finiteness

4.7 Definition A separable simple nuclear C*-algebra is purely finite if it has strict comparison and if $W(A)$ is almost divisible.

$W(A)$ almost divisible.

$\forall p \in \mathbb{N}, x \in W(A) \exists y \in W(A): py \leq x \leq (p+1)y$

Question: Is almost divisibility implied by strict comparison?

4.8 Theorem (Winter, in progress)

Let $A$ be simple, separable unital with locally finite decomposition rank.

If $A$ is purely infinite then $A$ is 2-stable.

Partial confirmation of the question

strict comparison

$\Rightarrow$ 2-stability

Ingredient of proof: As in pf of Thm 4.6, an approx. central sequence of unital $\ast$-homomorphisms

$\mathbb{Z}_{p,p+1} \rightarrow A$

for any $p \in \mathbb{N}$ will do ($p \cdot x \leq 1_A \leq (p+1)x$)

By Jordan-Winter, need to find a c.p.c. order zero map

$\mathbb{F} \rightarrow A$
and $v \in A$ such that
\[ vv^* = 1_A - \Phi(1_{mp}) \quad \text{and} \quad v^* v \leq \Phi(e_u) \]
and such that $\Phi(mp)$ and $v$ are approximately central.

Key result for constructing $\Phi$ and $v$:

Lemma: For every $k \in \mathbb{Z}$, there exists $n$ such that

Let $A$ be a separable, simple, unital, purely infinite
Let $\{e_u\}_{u \in U}$ be a C*-subalgebra with $\operatorname{tr} B S m$
and let $k, l \in \mathbb{Z}$.

If
\[ \Phi : M_k \to A_{\infty} \Lambda B' \]
is a c.p.c. order zero map then there is a c.p.c. order zero map
\[ \psi : M_k \to A_{\infty} \Lambda B' \wedge \Phi(M_l) \]
such that
\[ \tau (\psi(1_k) \varphi(1_l)b) \geq \alpha_m \cdot \tau (\psi(1_k)b) \]
$\forall b \in B$ and $\varphi \in T(A)$ ($\varphi$ if $\lim \tau_n (\varphi) = 1$ for

some $(1_n)_{x = 1}^\infty \in T(A)$ so $T(A) \subseteq T(A)_{\text{max}}$ is a free ultrafilter on

use a geometric series argument after

repeated application of lemma $\alpha_m \cdot \sum_{k=1}^{\infty} (1 - \alpha_m)^k = 1$

TAS Algebras

5.1 Definition Let $S$ be a class of separable unital C*-algebras. Let $A$ be simple, separable and unital. We say $A$ is TAS if the following holds: Given $0 \neq e \in A^+$, $F \subseteq A$ finite, $\varepsilon > 0$ and $B \in S$ and

(i) $1_{(1, 1)} \Rightarrow e$ \quad (ii) $(1_k - 1_l) \leq \varepsilon$

(ii) $1_k F 1_l \subseteq B$
If $S$ is the class of finite-dimensional C*-algebras (or tensor products of such with closed intervals) we write TAF (or TAI, respectively).

5.2 Theorem (Lin): The class of all TAF algebras satisfies the Elliott conjecture.

5.2 The 1-fdr, r-r, 2-stable case

5.3 Theorem: A separable, simple, unital and 2-stable with locally finite decomposition rank and real rank zero. Then $A$ is TAF.

Assume $A$ has finite decomposition rank, and that $\tau(A) = 1$; i.e., it is just a point. Take c.p. approximation $A \rightarrow \varphi = \varphi^{(0)} + \ldots + \varphi^{(n)}$ order zero map, $F^{(0)} \Theta \otimes F^{(n)}$.

Can assume $\varphi^{(i)} = \bigoplus \lambda_j \pi_j$. $\pi_j$ a *-homomorphism.

There generated finite-dimensional subalgebras. At least one has to be at least $\frac{1}{n}$ in trace. Repeat.
end up with geometric series argument

5.3 Localizing at \( 2 \)

5.4 Theorem (Winter): Let \( \Lambda \) be a class of separable, simple, nuclear unital C*-algebras such that, for any \( A, B \in \Lambda \) and any isomorphism of invariants

\[
\Lambda : \text{Inv}(A) \to \text{Inv}(B)
\]

there are prime integers \( p, q \) such that \( \Lambda \) can be lifted along \( 2p, 2q \)

\[
\begin{align*}
\Phi: A \otimes \mathbb{Z}_{2p} \otimes \mathbb{Z}_{2q} & \to B \otimes \mathbb{Z}_{2p} \otimes \mathbb{Z}_{2q} \\
\Phi & \text{ as } C^*(\mathbb{Z})\text{-algebras}
\end{align*}
\]

such that \( \Phi \) induces \( \Lambda \otimes \text{id}_{\mathbb{Z}_{2p}} \)

\( \Phi \), \( \Phi \) unitarily intertwined

Then

\[
A^2 := \{ A \otimes 2 \mid A \in \Lambda \}
\]

satisfies the Elliott conjecture.

5.5 Theorem (Lin, Niu): Let \( \mathcal{B} \) denote the class of separable, simple nuclear unital C*-algebras with UCT and such that tensor products with UHF algebras are TAF. Then \( B^2 = \{ B \otimes 2 \mid B \in \mathcal{B} \} \) satisfies the Elliott conjecture.
5.6 Corollary (Using Dinh and Phillips)
$C^*$-algebras associated to minimal uniquely ergodic, smooth dynamical systems are classified.

5.7 Corollary (Using Gong and Theorems 4.8 and 5.5)
Simple, unital AH algebras with slow dimension growth are classified.

Remark: The elements of $B$ have rationally Riordan $K_0$-groups.

Minimal dynamical systems

6.1 The Set-up
$X$ a compact metrizable space (infinite)
$\alpha: X \to X$ a homeomorphism
The crossed product is given by
$C^*(X) \rtimes \mathbb{Z} = C^*(C(X), \alpha, \nu, \nu^f) \rtimes \mathbb{Z}$
$u^x = f(x^\alpha(\cdot))$

6.1 Proposition: If $\alpha$ is minimal then
$C^*(X) \rtimes \mathbb{Z}$ is simple unital nuclear with tracial state. If $\alpha$ is uniquely ergodic then the tracial state is unique.

6.2 Problem
(i) Determine the structure of such crossed products
(ii) Classify them
(iii) Draw conclusions about the underlying dynamical systems
To do: If $\mathcal{A}_1 \otimes U$ is TAF for one $x \in X$ then $\mathcal{A}_1 \otimes U$ TAF?

- will focus on (i) and (ii) with minimal actions
- will solve (ii) in the finite dimensional minimal uniquely ergodic case
  (there is little hope for a complete solution
  by $K$-theory in the infinite dimensional case)
- joint with Tows and A. Strong and Danish furniture

6.2 Classification up to $\mathcal{Z}$-stability

6.3 Theorem: (Strong - based on Lin-Phillips).

$x$ compact, metrizable, infinite. Let $\phi: x \rightarrow X$
be a minimal homeomorphism. Let $S$ be a
class of separable unital $C^*$-algebras which
is closed under taking hereditary unital $C^*$-subalgebras
Let $U$ be a UHF algebra and $y \in X$
Set

$$\mathcal{A}_y = C^*(C(x), C_0(x \setminus y \overline{y})) \otimes U$$

If $\mathcal{A}_y \otimes U$ is TAF then $(C(x) \otimes \mathcal{Z}) \otimes U$ TAF

Remarks
- In the above situation, $\mathcal{A}_y$ is simple, ASH
  and $d(\mathcal{A}_y) \leq \dim X$

  ($\mathcal{A}_y$ subhomogeneous if $y \neq \emptyset$ and is closed
  if $y_1, y_2 \in \overline{y}$, $y_1 \neq y_2$
  then $\lim_{n \to \infty} \mathcal{A}_y \subseteq \mathcal{A}_y$)

- $\mathcal{A}_y \otimes U$ is TAF if projections separate traces
  (by theorem 5.3)
6.4 Corollary  Let $X$ compact, metrizable infinite 
and let $\alpha : X \to X$ be minimal uniquely ergodic homeomorphism. Then 
\[(C(X) \times \mathbb{Z}) \otimes \mathcal{U} \text{ is TAF}\]
for any UHF algebra $\mathcal{U}$.

6.5 Corollary The result yields classification  
up to 2-stability without any dimension  
restriction on $X$.

6.6 Theorem (Toms - Winter) Let $X$ be compact  
metrizable, infinite with finite topological dimension. 
Let $\alpha : X \to X$ be a minimal homeomorphism. 
Then $C(X) \times \mathbb{Z}$ is 2-stable.

6.7 Corollary 
\[E = \mathbb{Z} C(X) \times \mathbb{Z} \]
\[\text{Let } X \text{ compact, metrizable, fin.-dim., } \alpha \text{ uniquely ergodic, minimal
homeomorphism.}\]
is classified by ordered K-theory.

\text{Proof of Theorem 6.6}

Remark: one also proves that 
\[\dim_{\text{nuc}} \left( (C(X) \times \mathbb{Z}) \right) \leq \dim X + 1\]
in a similar fashion.
Proof of \( b \): (Sketch) Need \( Z \to A \), \( Y \to A' \).

For \( Y \), \( X \), set \( A_Y = C_0(\mathcal{C}(X), \mathcal{C}(X \times Y)) \leq C(\mathcal{C}(X \times Y)) \).

Proposition (Putnam, Lin-Phillips, Toms-Winter):

If \( A_Y \) is finite, then \( A_Y \) is simple, ASH with finite der.

\[ \Rightarrow A_Y \text{ is } 2\text{-stable} \]

Choose \( x_0, x_1 \in X \) with disjoint orbits. Set \( A_t = A_{x_0} \), \( A_{t/2} = A_{x_0, x_1} \), \( A_1 = A_{x_1, x_1} \). Then \( A_{t/2} \subset A_1 \) and \( A_t \subset A_0 \).

Let \( F \in \mathbb{C} \), \( \varepsilon > 0 \), \( G \in \mathbb{C} \).

Choose \( 0 < \varepsilon \leq 1 \) in \( C(\mathcal{C}) < A \) such that

\( h(x_0) = 0 \), \( h(x_1) = 1 \), \( \| [h, a] \| < \varepsilon \) \forall a \in F \).

(this uses minimality of \( a \).

\( h \) is almost invariant under \( a \), hence almost commutes with \( V_{a+x} \).

For \( i = 0, \frac{1}{2}, 1 \) define

\[ d_i = d_i(h) \]

Then \( d_i : a \in \mathbb{C} \to A_i \).

\[ A_0 \oplus \mathbb{Z} \quad A_i \oplus \mathbb{Z} \quad A_1 \oplus \mathbb{Z} \]