Classification of C*-algebras

Consider C*-algebras A which are separable, unital, nuclear, amenable, usually simple.

A is nuclear if for any other C*-algebra B, there is only one way to complete A ⊗ B to get a C*-algebra.

Examples (1) C(x) x H is compact, Hausdorff
x: x → x a homeomorphism
C(x) x H = C*(C(x), u) if u = f * x

(2) Recursive subhomogeneous C*-algebra

X_1, ..., X_k compact metric spaces
X_i^{(n)} ⊂ X_i closed subsets
n_1, ..., n_k ∈ N

A_1 = M_{n_1}(C(X_1))
A_2 = A_1 ⊕ \frac{M_{n_2}(C(X_2))}{(M_{n_2}(C(X_2)), \phi_2)}

\phi_2: A_1 → M_{n_2}(C(X_2)) \text{ unital } x - \text{homomorphic}

A_2 = \left\{ (a, b) \mid a ∈ A_1, b ∈ M_{n_2}(C(X_2)) \left| \phi(a) = 5 \right. \right\}

A_3 = A_2 ⊕ \frac{M_{n_3}(C(X_3))}{(M_{n_3}(C(X_3)), \phi_3)}
\[
A = \begin{bmatrix}
(A_1 \otimes M_2(\mathbb{C}(x))) & \ldots \\
(C_1 \otimes \mathbb{C}) & \\
\end{bmatrix}
\] (Type I algebra)

What kind of theorem do we want?

"Theorem" Let \( A, B \) be simple unital separable amenable C*-algebras of same class \( \mathcal{C}_0 \).

An functor \( F : \mathcal{C} \to \mathcal{C}' \) such that if \( F(A) \cong F(B) \) is an isomorphism, then there exists a \( * \)-isomorphism \( \Phi : A \to B \) such that \( F(\Phi) = \Phi \).

What is \( F \) typically? \( K \)-theory and traces

\( K_0 \)-group A unital, \( K \)-compact operators on separable infinite dimensional Hilbert space \( p, q \) projections in \( A \otimes K \). Say \( pq \Leftrightarrow \exists q \in A \otimes K \) such that \( v^* v = p, vv^* = q \).

\[ V(A) = \{ \text{projections in } A \otimes K \}/_{\sim} \] \[ p \mapsto [p] \]

define an addition \([p] + [q] = [(0, 0)]\] \[ V(A) \] is a semigroup

Grothendieck

\[ V(A) \xrightarrow{\Gamma} K_0(A) \]

\[ K_0(A)^+ = \Gamma(V(A)) / [1_A] \]

\( (K_0(A), K_0(A)^+, [1_A]) \) is a preordered, pointed abelian group
A projection $p$ is infinite if $p 
eq 0$, finite otherwise.

A stably finite if all projections in $M_n(A)$ are finite.

In this case $K_0(A)$ is ordered.

The $K_1$-group $K_1(A)$ units in $A$, $U_0(A)$ contains component of $1_n$. The map $a 	o \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in A$ induces a homomorphism $\varphi_a : U(M_0(A)) \to U(M_n(A))$.

$$K_1(A) = \lim_{\to} C_{n+1}$$

$$[u]_1 + [v]_1 = [uv]_1$$

A tracial state on $A$ is a linear functional, $\tau : A \to \mathbb{C}$ such that $\tau(1_A) = 1$ and $\tau(xy) = \tau(yx)$ for $x, y \in A$. The set $T(A)$ of these is a metrizable Choquet simplex.

A trace defines a state on $(K_0(A), [1_A])$ via $\tau([p])$.

So we get a map $\varphi_A : T(A) \to S(K_0(A), [1_A])$.

A unital $\theta$-The Elliott invariant of $A$ is

$$\{(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), \varphi_A)\}$$

$$K_1(A) = K_0(A) \oplus K_1(A)$$

If projections separate traces, one would expect not to need $T(A)$, $\varphi_A$. 
On a good day, \((\mathcal{C}(A), \mathcal{C}(A)^+, (\mathcal{A}, \mathcal{F}), T(A), \mathcal{F})\) is equivalent to the Cuntz semigroup of \(A\).

**The Cuntz Semigroup**

A unital, \(a, b \in A \otimes K\). Say \(a\) is Cuntz dominated by \(b\) if \(\exists (v_n) \in A \otimes K\) such that \(v_n b v_n^* \xrightarrow{n \to \infty} a\).

Write \(a \preceq b\).

We say \(a \lesssim b\), \(a\) is Cuntz equivalent to \(b\), if \(a \preceq b\) and \(b \preceq a\).

**Example:** \(a \preceq 2a\), \(a \succ 0\).

**Ex.:** \(A = M_n(C)\); \(a \preceq b\) iff \(\text{rank}(a) \leq \text{rank}(b)\).

**Ex.:** \(A = M_n(C([0, 1])\); \(a \preceq b\) iff \(\text{rank}(a(t)) \leq \text{rank}(b(t))\) \(\forall t \in [0, 1]\).

Why? Because \(a, b\) can be approximately unitarily diagonalized.

**Ex.:** \(X\) a \(C^*\)-complex, \(\dim X = 3\), \(n = 2\). Then \(\exists a, b \in M_n(C(X))^+\) such that \(\text{rank}(a(x)) = \text{rank}(b(x))\).

\(\forall x \in X\) yet \(a \preceq b\).

**Ex.:** \(f, g \in C(X)\), \(f, g \geq 0\) then \(f \leq g \iff \text{supp}(f) \subseteq \text{supp}(g)\).

Define \(\mathcal{C}_u(A) = \{\text{positive elements in } A \otimes K\} / \mathcal{A}\).

\[a \sim \langle a \rangle\]

As before define \(\langle a \rangle + \langle b \rangle = \langle (a, b) \rangle\) and \(\langle a \rangle \preceq \langle b \rangle \iff a \preceq b\).

We get an ordered abelian semigroup called the Cuntz semigroup.
\[ A = M_n(C^*(\mathbb{C})) : \text{Cu}(A) = NU[0,0] \text{ for } \mathcal{A} = \mathbb{C}^n \]

\[ \text{Ex: } A = M_n(C[0,1]) \]

\[ \text{Cu}(A) = f : [0,1] \to NU[0,0] \text{ if } f \text{ is supremum of an increasing sequence } (f_n) \text{ of functions } f_n : [0,1] \to [0,1], n \]

\text{Definition: } T(A) \text{ trace simplex, } \text{aff}(T(A)) = \text{cts. affine } \mathbb{R} \text{-valued functions on } T(A) \}

\[ \text{L}(T(A)) = \{ \text{sups of increasing sequences } (f_n) \text{ in } \text{aff}(T(A)) \}

\text{Why Cu(A)?}

(i) if Cu(A) is "nice" you can prove classification theorems for such A
(ii) Cu(A) is more sensitive as an invariant than K-theory and traces.

A unital, exact T(A) ≠ ∅.

\[ T(A) \text{ extends to an unbounded trace on } A \otimes K, \text{ if } A \in A \otimes K, \text{ define } \]

\[ d_f(a) = \lim_{n \to \infty} T(a^{1/n}) \]

This is an example of a dimension function on A, i.e. an additive order-preserving map \( \phi : \text{Cu}(A) \to [0, \infty] \) such that \( \phi([1_n]) = 1 \).
Ex. \( a \in M_n(\mathbb{C})^+ \), \( d_\tau(a) = \text{rank}(a) \)

For \( \langle a \rangle \in \text{Cu}(A) \) we define \( L(\langle a \rangle) : T(A) \to [0, \infty] \)

by \( L(\langle a \rangle)(T) = d_\tau(T) \)

**Facts:**

1. \( L(\langle a \rangle) \) is in \( L(T(A)) \) since \( T \mapsto T(\langle a \rangle) \)
   is continuous and \( T(\langle a \rangle) \leq T(\langle a \rangle^+) \)
   (since we can choose to represent \( a \) by a contraction)

2. If \( f \in C^*(A) \), \( a \geq 0 \), \( f \geq 0 \) then
   \[ d_\tau(f(\langle a \rangle)) = \mu_f(\text{supp}(f) \cap \Gamma(a)) \]
   where \( \mu_f \) is the spectral measure induced by \( f \).

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**Fact:** \( a \leq b \iff \forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } (a-\epsilon)_+ \leq (b-\delta)_+ \)

where \( (a-\epsilon)_+ = f(a) \) where \( f = \begin{cases} 0 & [0, \epsilon] \\ t-\epsilon & [\epsilon, \infty) \end{cases} \)

**Question:** When is \( \langle a \rangle = \langle p \rangle \) for some projection \( p \)?

**Lemma:** If \( A \) is unital, simple, \( T(A) \) is then

\( \langle a \rangle \leq \langle p \rangle \iff p \text{ is a projection } \iff 0 \text{ is not a limit point of } \sigma(a) \)

**Proof:** \( (\Leftarrow) \) \( \forall \epsilon \geq 0 \text{ a limit point of } \sigma(a) \)

\[ \langle a \rangle \leq \langle 1 \times \epsilon(a) \rangle \]
Assume $0$ a limit pt.

(⇒) Suppose $<p> =<a>$. Pick $0<\varepsilon<1$. Find $S>0$ such that $(p-\varepsilon)_+ \subseteq (a-S)_+ \land \lambda \cap p$.

But also $p \sim (p-\varepsilon)_+$

⇒ $(a-S)_+ \sim p$ ∀s sufficiently small

⇒ $d_\tau ((a-S)_+) = d_\tau (p)$ ∀s small

\[ (a-S)_+ \subseteq g(a) \land (a-S)_+ \subseteq \alpha. \]

⇒ $d_\tau ((a-S)_+) \leq d_\tau (g(a)) + d_\tau ((a-S)_+) \subseteq \text{g}(a)$

⇒ $d_\tau (g(a)) = 0 \implies I[a]$

Note: A stably finite ⇒ $p \sim q \iff p \sim q$.

Now for a united simple $T(A) \neq \emptyset$, we have

$\text{Cu}(A) = V(A) \cup \text{Cu}(A)_+$ where

$\text{Cu}(A)_+ = \{ <a> | 0 \text{ is a limit point of } o(a) \}$

$\text{Cu}(A)_+$ is absorbing in that $x+y \in \text{Cu}(A)_+$ if $y \in \text{Cu}(A)_+$.
Definition: A unit $l$ of a partially ordered vector lattice $(A, \leq, \mathbb{K})$ has strict comparison of positive elements if
\[ a \leq b, \quad a, b \in (A \cap \mathbb{K})^+ \quad \text{whenever} \]
\[ d_{\mathbb{K}}(a) < d_{\mathbb{K}}(b) \quad \forall \, l \quad \text{s.t.} \quad d_{\mathbb{K}}(b) < \varepsilon \]
A simple unital, T(A) ≠ ϕ

\[ \text{From} \quad Cu(A) = V(A)L_iW(A) \]
\[ Cu(A) = V(A)L_iW(A) \]

\[ (a) = \lim_{n \to \infty} a^{1/n} \]

So we get a map \( \varphi: Cu(A) \to V(A)L_iW(A) \)
\[ \varphi(a) = \begin{cases} [p] & \text{if } a \text{ a proj.} \\ 1 & \text{otherwise} \end{cases} \]

**Definition.** A has strict comparison if \( a, b \in (A_{\geq 0})^+ \) satisfy \( a \preceq b \) whenever \( d_T(a) < d_T(b) \) for \( \forall T \).

Suppose \( a, b \in Cu(A) \), \( d_T(a) < d_T(b) \) \( \forall T \text{ such that } d_T(b) < \infty \)

Since 0 is a limit point of \( \sigma(a) \), \( d_T((a-\varepsilon)_+) < d_T(a) \)

If \( A \) has strict comparison
\[ \Rightarrow (a-\varepsilon)_+ \preceq b \quad \forall \varepsilon > 0 \Rightarrow a \preceq b \]

Thus if \( a < b \in Cu(A)^+ \) then \( \langle a \rangle \preceq \langle b \rangle \)
\[ \Leftrightarrow d_T(a) = d_T(b) \]

Now \( \varphi \) is at least injective (assuming strict comparison)
When is $\text{Im}(I) = \text{L}(T(A))$?

**Proposition**: Let $A$ be unital, simple, $T(A) \neq 0$, strict comparison. Suppose that for any $f \in \text{Aff}(T(A))$, \( f > 0 \) and any $\varepsilon > 0$, $\exists g \in (A \otimes K)^+$ s.t. $|d_I(a) - f(\tau)| < \varepsilon \forall \tau \in T(A)$. Then for any $g \in \text{L}(T(A))$, \( f \in (A \otimes K)^+ \) s.t. $d_I(b) = g(\tau)$.

**Proof**: Let $g$ be given $f \in \text{Aff}(T(A))$ such that $f > 0$, $f < f_{n+1}$, $\sup_n f_n(\tau) = g(\tau)$.

Find sequence $\varepsilon_n \downarrow 0$ such that $f_n + \varepsilon_n < f_{n+1} - \varepsilon_n$

Then find $a_n \in (A \otimes K)^+$ such that $|d_I(a_n) - f_n(\tau)| < \varepsilon_n$.

Then $d_I(a_n) < d_I(a_{n+1})$ and $\sup_n d_I(a_n) = g(\tau)$.

By strict comparison $a_n < a_{n+1}$.

**Fact (Coward, Elliott, Ivanescu)**: Suprema of increasing sequences exist in $\text{Cu}(A)$ and $d_I(\cdot)$ is sup-preserving.

Let $<a> = \sup_n <a_n>$, then $d_I(<a>) = \sup_n d_I(<a_n>) = g(\tau)$.

So when do we have density in the sense of the Proposition?

**Definition**: $\text{Cu}(A)$ is almost divisible if for any $x \in \text{Cu}(A)$, any $n \in \mathbb{N}$, $\exists y \in \text{Cu}(A)$ such that $ny < x \leq (n+1)y$. 


Proposition: Let $A$ be unital, simple, $T(A) 
eq \emptyset$. Assume $\text{Cu}(A)$ is almost divisible. It follows that for any $f \in \text{Aff}(T(A))$, $f \neq 0$ and any $\varepsilon > 0$, there exists $x \in (A \otimes \mathbb{K}^+)_+$ such that $|d_x(x) - f(x)| < \varepsilon$ for all $x$.

Proof: By Theorem of Lin/Cuntz-Pedersen, there exists $x \in A^+$ such that $f(x) = f(x)$ and $\|x\| < 1 + \varepsilon$

$f(x) = f(x) = \sum_{i=1}^{N} \frac{e_i}{N} \left( b_i, b_i \right) (b_i)$ (Remark: not $\text{tr}$ in the strict sense in general)

Set $a = \sum_{i=1}^{N} b_i$ and $d_x(a) = \sum_{i=1}^{N} d_x(b_i, b_i) (b_i) \approx f(x)$

Theorem: Let $A$ be unital, simple, $T(A) 
eq \emptyset$, strict comparison, $\text{Cu}(A)$ almost divisible. It follows that

$\text{Cu}(A) = V(A) L(I L(T(A))) > 0$

where addition in RHS is as usual in each of $V(A)$, $L(I L(T(A)))$, and if $x \in V(A)$, $y \in L(I L(T(A))) > 0$

then $x + y = (x + y)$

where order in $V(A)$, $L(I L(T(A))) > 0$ is as usual and if $x \in V(A)$, $y \in L(I L(T(A))) > 0$, then $x \leq y$ if $x(y) \leq y(x)$ in $L(I L(T(A))) > 0$ and $y \leq x$ if $y(x) \leq x$ in $L(I L(T(A))) > 0$.

Example: A UHF, $K_0(A) = \mathbb{N}$, $\text{Cu}(A) = \mathbb{Q}^+ L(I R^+ \{0 \} \cup \{0 \})^*$

But $A = \lim_{n \to \infty} M_n(C)$, $\text{Cu}(M_n(C)) = IR^+ \{0 \}$
Theorem (Winter, Lin-Niu) Let $A, B$ unital UCT
simple, separable with locally finite decomposition
rank. Also suppose $\text{Cu}(A) = \text{V}(A) \cup L(T(A))_0$
similarity for $B$, and that projections
separate traces. If $\varphi : \text{K}_* (A) \to \text{K}_* (B)$
then $\text{Ext}$-isomorphism \( \varphi : A \to B \) such that
\( \text{K}_* (\varphi) = \varphi \).

Ex.: $\mathbb{A}$ simple unital exact finite, $A \otimes \mathbb{Z} = A$
($\mathbb{Z}$ is the Jiang-Su algebra). Then $A$
has strict comparison (Rordam) (Proof uses
that strict comparison is equivalent to
almost unperforation of the Cantor semigroup
ie if $a, y \in \text{Cu}(A)$, $(n+1)x < y$ new then $x \leq y$)

\( \text{Cu}(A \otimes \mathbb{Z}) \) is almost divisible:
1. $A \otimes \mathbb{Z} \cong A$ and \( \langle a \otimes 1_\mathbb{Z} \rangle = \langle a \rangle 
2. \exists$ embedding $\gamma : \mathbb{C}[0,1] \to \mathbb{Z}$ such that the
image of $\text{Ext}(\mathbb{Z}) = \{ pt \}$ under $\gamma^\text{tr}$ is Lebesgue
Thus, for any $0 < \theta < 1$, \( \exists \lambda \in \mathbb{C}[0,1] 
\) such that $d_\lambda (\delta (a_\lambda)) = \lambda$ \( \forall \text{Ext}(A) 
3. \text{One computes } d_\lambda (a \otimes \delta (a_\lambda)) = \lambda d_\lambda (a)

Theorem: $\text{Cu}(A) = \text{V}(A) \cup L(T(A))_0$ if $A$
is a unital simple ASH algebra with slow dimension
growth.

Ex. $C(M) \times \mathbb{R}$, $M$ compact manifold a minimal
diffeomorphism (Q. Lin, Phillips)
Definition: A has slow dimension growth if there exist subhomogeneous algebras $A_i$ and maps $\phi_i : A_i \to A_{i+1}$ such that $A = \lim (A_i, \phi_i)$ and

$$\limsup_{i \to \infty} \max_{j \in E_i} \left( \frac{\dim x_{ij}}{n_{ij}} \right) = 0$$

How to prove strict comparison?

Fact: If $p, q$ are projections in $M_n(C(X))$ and $\text{rank}(p) + \frac{\dim(X) - 1}{2} < \text{rank}(q)$, then $p \preceq q$.

Assume $A = \lim (A_i, \phi_i)$, $A_i = M_{n_i}(C(X_i))$, $\dim X_i \to 0$.

Assume $(a+b)(x) \leq n < (b-a) x \in A$;

$$\Rightarrow \text{rank}(a(x)) + 1 \leq \text{rank}(b(x)) \quad \forall x \in X$$

Theorem: If $\text{rank}(a(x)) + \frac{\dim(X) - 1}{2} < \text{rank}(b(x)) \forall x \in X$;

$$\Rightarrow a \preceq b$$

$\phi_{ij} : A_i \to A_j$, $\phi_{ij}(a)(y) = \bigoplus_{k=1}^{n_{ij}} a(x_k) \quad x_k \in X_k$

$$\Rightarrow \text{rank}(\phi_{ij}(a)(y)) + \frac{n_{ij}}{n_i} \leq \text{rank}(\phi_{ij}(b)(y)) \quad \forall y \in X_j$$

$\langle \phi_{ij} \phi_{ij}(a) \rangle \leq \langle \phi_{ij} \phi_{ij}(b) \rangle$. 
The preceding is a sketch of why strict comparability holds for a unital, simple, $AH$ algebra with strict slow dimension growth.

Why is $c(Cu(A^+))$ "dense" in $\text{Aff}(T(A))_0$?

Consider $M_n(C(x))$, $f \in \text{Aff}(T(M_n(C(x))))_0 \implies C(x)_0$

restrict to extreme boundary.

Want $a \in M_n(C(x))_+$ such that $|d_\epsilon(a) - f(\epsilon)| \leq \frac{1}{n}$.

Can assume $\epsilon = \delta x$, $x \in x$ so that $d_\epsilon(a) = \frac{\text{range}(a)}{n}$.

Thus want $\left| \frac{\text{range}(a(x))}{n} - f(x) \right| \leq \frac{1}{n}$.

Take $e_i \otimes 1_{\text{Id}_x}$ (rank one projection) and fix $f_i \in C(x)$ such that $\text{supp}(f_i) = u_i$. Set $a_i = f_i(p)$.

Then $a = \bigoplus a_i$ does the trick.
Q: Are there separable simple unital nuclear C*-algebras with the same K-theory and traces but not isomorphic?

A: Yes (Rordam) and even in the stably finite case.

Strategy: Construct $A = \lim_{\rightarrow} M_n((C(X_i)) = \lim_{\rightarrow} A_i$,
- if contractible $\Rightarrow K_0(A) = \{0\}$, $K_0(A_i) = \mathbb{Z}$
- for any $k \in \mathbb{N}$, $k | n_i$; for all $i$ large $\Rightarrow K_0(A) = \mathbb{Q}$

$A = \text{universal UHF algebra } (K_0(A) = \mathbb{Q})$,
$(K_0(A \otimes \mathcal{Q}), K_1(A \otimes \mathcal{Q}), T(A \otimes \mathcal{Q}), \mathcal{P}_{A \otimes \mathcal{Q}})$

$\Rightarrow$ just need $A \not\cong A \otimes \mathcal{Q}$

We will show that almost unperforation fails in $C_u(A)$ but $C_u(A \otimes \mathcal{Q})$ has almost unperforation property.

We need
First see how A.u.p can fail in $M_n((C(X))$
using projections
$(a \leq \text{up} \iff (a+1) x \leq n y \Rightarrow x \leq y)$

How to show $p \not\leq q$ for projections $p, q \in M_n((C(X))$
View $p, q$ as vector bundles over $X$: the
define of $p$ at $x$ is $p(x)C^n$. Villadson used
Chern class to get comparability obstructions
Chem. class \( C(\cdot) : \text{Vec}(X) \to H^{2\chi}(X;\mathbb{Z}) \)

(i) \( c(d \oplus w) = c(d)c(w) \)
(ii) \( c(\theta_1) = 1 \in H^\ast(X) \), or trivial of rank \( r \), i.e. \( \theta_1 \cong X \times \mathbb{C}^r \)
(iii) \( f : Y \to X \) continuous then \( c(f^\ast(d)) = f^\ast(c(d)) \)
(iv) \( c(d) = 1 + c_1(d) + c_2(d) + \ldots + c_{\dim(d)}(d) \) \( c_i(d) \in H^{2i}(X) \)

Lemma (Villadsen): Let \( \theta, \theta' \) be bundles over \( X \). Assume \( c_j(\theta) \neq 0 \) for some \( j > \dim(\theta) + 1 \). Then \( \theta \nmid \theta' \).

Proof: If \( \theta \nmid \theta' \) then \( \exists \) bundle \( w \) such that \( \theta \oplus w = \theta' \) \( \Rightarrow c(\theta) = c(\theta \oplus w) = c(\theta)c(w) = c(\theta) \neq 0 \)

On the other hand, if \( \text{rank}(w) + \dim(X) - 1 < \text{rank}(\theta) \)

then \( \theta \nmid \theta' \). Thus if \( \text{rank}(w) < \text{rank}(\theta) \), then \( n + 1 < \dim(X) \) for large enough \( n \).

Ex. If bundle over \( S^2 \) then \( c(p) = 1 + 1 \)

\( p \times p \) is a bundle over \( S^2 \times S^2 \) isomorphic to

\( \pi_1^\ast(p) \oplus \pi_2^\ast(p) \)

\( \pi_i : S^2 \times S^2 \to S^2 \) \( i \)-order projections

\( c(\pi_1^\ast(p) \oplus \pi_2^\ast(p)) = \pi_1^\ast(c(p)) \pi_2^\ast(c(p)) = (1 + 1)(1 + 1) \)

\( \Rightarrow c_2(\cdot) \neq 0 \)

Thus \( \theta \nmid p \times p \).

Consider \( S_2 \times S_2 \subseteq [0,1]^3 \times [0,1]^3 = X \).
Extend \( p \times p \) to an open neighbourhood \( U \) of \( S_2 \times S_2 \).

Choose \( f : X \to [0,1] \)

\( f|_{S^2 \times S^2} = 1 \)

\( f|_{\partial X} = 0 \)

Set \( a = f(\theta) \), \( b = f(p \times p) \in M_n(C(V)) \)
and $(n+1) \leq a < n < b \quad \forall n \text{ large}

But $a < b$ since $a < b \Rightarrow \frac{a}{s^2 x s^2 z} \leq \frac{b}{s^2 x s^2 z}$

$X_2 = X_1^{x_m}$, what should $\phi$ of $M_n(\tau(1)) \to M_n(\tau(2))$ be?

\[
\phi(f) = \begin{bmatrix}
\phi_1
\phi_2
\phi_3
\phi_4
\end{bmatrix}
\]

\[
\phi(b) = (\rho \phi)^{x_1}_m
\]

$C_{2m}(\rho \phi)^{x_1}_m = 0$ (same argument)

Thus $\phi_1(a) \neq \phi_1(b)$ and similarly for all forward images. In fact $\exists S > 0$ such that

$\forall i, \forall x \in A_i, \|x \phi_i \cdot (b) x^x - \phi_i(a) x^x\| \leq S$

So $\phi_1(a) \neq \phi_1(b)$ and $A$ is proper.

\[
\text{Definition: A unital, exact. Define the radius of comparison for } A \text{ to be}
\]

\[
rc(A) = \inf \left\{ t > 0 \mid a \leq b \ (a, b \in \mathbb{R^+}) \text{ whenever } \right. \\
\left. d(c(a) + r < d(c(b)) \forall c \in A \right\}
\]

\[
rc(A) = \inf \left\{ \frac{m}{n} \mid n \times m \leq 1 \leq n y \Rightarrow x \leq y \quad x, y \in C(A) \right\}
\]

If $A$ is simple,

\[
rc(A) = \inf \left\{ \frac{m}{n} \mid (n+1) x + m \leq 1 \leq n y \Rightarrow x \leq y \quad x, y \in C(A) \right\}
\]
a CW-complex

Proposition. If $X$ has dimension $d < \infty$ then

$$\frac{d-2}{2} \leq \text{rc}(C(X)) \leq \frac{d-1}{2}$$

Sketch of Proof: Upper bound → already discussed

Lower bound → can immerse $S^{d-2}$ made even

Build positive elements from $n$-dimensional Bott bundle $S_n$ and $E_n$. These are not comparable but differ in rank by $\frac{d-2}{2}$

Properties

(i) $\text{rc}(\lim_{i \to \infty} (A_i, \phi_i)) \leq \liminf_{i \to \infty} \text{rc}(A_i)$

(ii) $\text{rc}(A/I) \leq \text{rc}(A)$

(iii) $\text{rc}(M_n(A)) = \frac{1}{n} \text{rc}(A)$

Theorem: For a family $A_r$, $r \in [0, \infty]$ of simple AH algebras such that

(i) K-theory and traces same $A_r$

(ii) $\text{rc}(A_r) = r$

$K_0(A_r) = \mathbb{Q}$, $	ext{sr}(A) = 1$

Uncountably many Morita equivalence classes among $(A_r)_{r \in [0, \infty]}$

Mean dimension

$(X, d)$ compact metric, $\phi$ homeomorphism

$U$ an open cover of $X$. Define $\text{ord}(U) = \sup_{x \in X} \frac{1}{\text{dim}_d(x)}$

Write $V \preceq U$ if $V$ is an open cover which refines $U$

$D(U) = \min_{V \preceq U} \text{ord}(V)$
Fact (Lindenstrauss): \( D(W) \leq D(U) + D(V) \).
Since one can show that \( D(U) \leq d \) if \( f \) is a \( d \)-dimensional \( f \)-compatible map \( f : X \to U \), \( U \) is \( d \)-dimensional, \( f \) compatible with \( U \).

Set \( W_n = UV_n^{-1}(W) \cdot V_n^{-\alpha}(\cdot \cdot \cdot)(W) \).

If \( W_n \) is finite,
\[
\dim(X, \alpha) = \sup_{n \to \infty} \lim_{n \to \infty} D(W_n)
\]

Ex. \( Y \) a CW-complex, \( X = Y^{\infty} \), \( \alpha \) the bilateral shift on \( X \). Then \( \dim(X, \alpha) = \dim Y \).

Problem: If \( \dim(X) < \infty \) then the mean dimension
\( \dim(X, \alpha) = 0 \) \( \forall \alpha \).

Theorem (Giol-Kerr): For any \( k > 0 \), \( \exists \) a minimal system \((X_k, \alpha_k)\) such that
\[
\dim[X_k] = k \leq \inf C(X_k) \cdot Y_k \cdot Z
\]

If \( x : Y^{\infty} \to Y^{\infty} \) is the bilateral shift, then
\[
C(Y^{\infty}) \cdot Y_k \cdot Z
\]
\( Y_k \) \( 2 \)-periodic points

\[
\mathcal{C}(Y^{\infty}) \cdot X_k \cdot Z
\]

\[
\mathcal{C}(Y_k) \cdot X_k \cdot Z \to \mathcal{C}(Y^{\infty}) \cdot X_k \cdot Z
\]

\[
\text{rk} \left( \mathcal{C}(Y_k) \cdot X_k \cdot Z \right) = \dim Y = \frac{\dim(X, \alpha)}{2}
\]
Proposal: Define a dynamic dimension dimension
\( \overline{\text{dim}}(X, G) \) (\( G \) countable, discrete) by
\[ \overline{\text{dim}}(X, G) = \text{rc}(c(X) \times G) \]

Why? 1. Looks like we could recover \( \text{min} \) for unital time shift
2. If \( G = \mathbb{K}^2 \) then \( \overline{\text{dim}}(X, G) = \text{dim}^X \)
3. If \( G = \mathbb{K}^2 \) acting trivially, then
\[ \overline{\text{dim}}(X, G) = \frac{\text{dim}^X + 2}{2} \]
4. \( x = y^n \), \( \alpha \) a cyclic shift on co-ords, then
\[ \overline{\text{dim}}(X, \alpha) = \frac{\text{dim}^X}{2} \]

Outlook: Hopeful for minimal systems \( (X, \alpha) \)
that \( \overline{\text{dim}}(X, \alpha) < \text{min}(X, \alpha) \) and that this is sharp (Gies-Kerr)

Why hope?
\[ c^X(c(X), \alpha \cdot c(X \setminus \{y\}) = A_{1y^3} \text{ is ASH}, \text{ but RSH} \]
\[ \alpha \text{-subalgebras have infinite dimensional spectrum} \]

Idea: for \( a, b \in A_{y^3} \),
\[ a \equiv \frac{a}{x_i} + a' \]

Take \( U \) finite open cover, iterate under \( \alpha^{-1} \)
\[ \mapsto \text{cover } Y \text{, ord } - \text{n(dim)} \]