The Cuntz semigroup and its relation to classification

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We consider C*-algebras $A$ which are:

- separable
- unital
- nuclear (which is equivalent to being amenable)
- usually simple

$A$ is nuclear if for any other C*-algebra $B$ there is only one way to complete the algebraic tensor product $A \odot B$ to get a C*-algebra.

1.1. Example (cross products): Any cross product $A = C(X) \rtimes_{\alpha} \mathbb{Z}$ is nuclear, where $X$ is a compact Hausdorff space, $\alpha : X \to X$ is a homeomorphism. Recall that $C(X) \rtimes_{\alpha} \mathbb{Z} = C^*(C(X), u)$ where $u$ is a unitary which implements $\alpha$, i.e. $ufu^* = f \circ \alpha^{-1}$ for any $f \in C(X) \subset C(X) \rtimes_{\alpha} \mathbb{Z}$.

1.2. Example (recursive subhomogeneous algebras): Any recursive subhomogeneous algebras (RSH-algebra) $A$ is nuclear. Recall that these are defined as iterated pullbacks using the following data:

- compact metric spaces $X_1, \ldots, X_l$
- closed subspaces $X_i^{(0)} \subset X_i$
- numbers $n_1, \ldots, n_l \in \mathbb{N}$
- unital $*$-homomorphisms $\phi_k : A_{k-1} \to M_{n_k}(C(X_k^{(0)}))$ (attaching maps)

such that $A_1 = M_{n_1}(C(X_1))$, and the following is a pullback (for $k = 2, \ldots, l$):
Here $\partial_k$ is induced by the inclusion $X_k^{(0)} \to X_k$. Such a pullback is often written as $A_k = A_{k-1} \oplus M_{nk}(C(X_k))$, and the standard way to define that pullback algebra is as follows:

$$A_k = \{ (a, b) : a \in A_{k-1}, b \in M_{nk}(C(X_k)), \varphi_k(a) = b_{|X_k^{(0)}} \}$$

These algebras are interesting because one can try to extend results from homogeneous to RSH-algebras. Possibly all stably finite C*-algebras are directed limits of RSH-algebras. Note also that all RSH-algebras are of type $I$.

What kind of theorem do we want?

1.3. **Theorem:** Let $A, B$ be simple, unital, separable, nuclear C*-algebras in some class $\mathcal{C}$. There exists a functor $F : \mathcal{C} \to \mathcal{C}'$ such that if $\varphi : F(A) \to F(B)$ is an isomorphism, then there exists an $*$-isomorphism $\Phi : A \to B$ s.t. $F(\Phi) = \varphi$.

What is $F$ typically? It is K-theory and traces. (we do not need quasitraces, since we only consider nuclear C*-algebras, where every quasitrace is automatically a trace)

1.4 ($K_0$-group): For simplicity let us only consider the unital case. For projections $p, q \in A \otimes \mathbb{K}$ say

$$p \sim q :\iff \text{there exists some } v \in A \otimes \mathbb{K} \text{ s.t. } p = v^*v, vv^* = q$$

Set $V(A) := \{ \text{the projections in } A \otimes \mathbb{K} \}/_\sim$. For a projection $p \in A \otimes \mathbb{K}$ we denote its equivalence class in $V(A)$ by $[p]$. Define an addition on $V(A)$ by $[p] + [q] = \begin{bmatrix} 0 & p \\ s & q \end{bmatrix}$. In this way $V(A)$ becomes an abelian semigroup.

Use the Grothendieck completion process $\Gamma$ to define an abelian group $K_0(A) := \text{Gr}(V(A))$. This comes with a natural map $\Gamma : V(A) \to K_0(A)$ and we denote its image as $K_0(A)^+ := \Gamma(V(A))$. This is also called the positive part (or positive cone) in $K_0(A)$. Then $(K_0(A), K_0(A)^+, [1_A])$ is a pre-ordered, pointed abelian group.

A projection $p$ is called infinite if it is equivalent to a proper subprojection, otherwise it is called finite. We call $A$ stably finite, if all projections in $M_n(A)$ are finite (for all $n$). In that case $K_0$ is ordered.

1.5 ($K_1$-group): Let $\mathcal{U}(A)$ denote the set of unitaries in $A$, and $\mathcal{U}_0(A) \subset \mathcal{U}(A)$ its connected component containing $1_A$. The map $u \mapsto \begin{bmatrix} 0 & u \\ 0 & 1_A \end{bmatrix}$ induces
a homomorphism \( \varphi_n : \mathcal{U}(M_n A)/\mathcal{U}_0(M_n A) \to \mathcal{U}(M_{n+1} A)/\mathcal{U}_0(M_{n+1} A) \). We set \( K_1(A) := \lim_n \mathcal{U}(M_n A)/\mathcal{U}_0(M_n A) \). This is an abelian group with addition defined via \([u] + [v] = [uv] \).

1.10. Example: \[ \text{Homological equivalence for stably finite algebras. Note that for any } \theta > 0, \text{ we have } M_n \otimes \mathbb{K}_+ \text{ is } \theta \text{-positive for all } n. \]

1.11. Example: \[ \text{The reason is that dim(} X \text{) } \geq 3 \text{ ensures that we can find } S^2 \text{ in } X. \text{ We can find projections } p,q \text{ in } M_2(\mathcal{C}(S^2)) \text{ that both have constant rank one, yet } p \sim q \text{ (e.g. the trivial line bundle, and the Bott line bundle). Extend this to a small neighborhood of } S^2 \to X, \text{ and then to positive elements } a,b \in M_2(\mathcal{C}(X)) \subset M_n(\mathcal{C}(X)). \]
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1.11. Example: \( C(X) \)

Let \( A = C(X) \) and \( f, g \in A_+ \). Then \( f \preceq G \) iff \( \text{supp}(f) \subseteq \text{supp}(g) \).

1.12 (The Cuntz semigroup):

Define \( \text{Cu}(A) := \{ \text{positive elements in } A \otimes \mathbb{K} \}/\sim \). We denote the equivalence class of \( a \in (A \otimes \mathbb{K})_+ \) in \( \text{Cu}(A) \) by \( \langle a \rangle \).

As before we define an addition \( \langle a \rangle + \langle b \rangle := \langle (a/b) \rangle \). If we define \( \langle a \rangle \preceq \langle b \rangle \) iff \( a \preceq b \), then we get an ordered abelian semigroup.

1.13. Example: \( M_n \)

Let \( A = M_n \). Then \( \text{Cu}(A) = \mathbb{N} \cup \{ \infty \} \) with \( x + \infty = \infty \), \( \infty + \infty = \infty \) and \( \langle 1_A \rangle = n \in \mathbb{N} \).

1.14. Example: \( M_n(C[0, 1]) \)

Let \( A = M_n(C[0, 1]) \). Then \( \text{Cu}(A) \) consists of all functions \( f : [0, 1] \to \mathbb{N} \cup \{ \infty \} \) that are the supremum of an increasing sequence of functions \( f^{(n)} : [0, 1] \to \{0, \ldots, n\} \).

We denote by \( \text{Aff}(T(A)) \) the continuous affine \( \mathbb{R} \)-valued functions on \( T(A) \), and by \( \text{L}(T(A)) \) the functions \( T(A) \to \mathbb{R} \cup \{ \infty \} \) that are the supremum of an increasing sequence of functions \( f^{(n)} \in \text{Aff}(T(A)) \).

Why are we interested in \( \text{Cu}(A) \)?

- if \( \text{Cu}(A) \) is nice, you can prove classification theorems for such \( A \)
- \( \text{Cu}(A) \) is more sensitive than K-theory and traces

Assume \( A \) is unital, exact and \( T(A) \neq \emptyset \). Then every \( \tau \in T(A) \) extends to an unbounded trace on \( A \otimes \mathbb{K} \) as follows: if \( a \in (A \otimes \mathbb{K})_+ \), then define \( d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n}) \).

This is an example of a dimension function on \( A \), i.e. an additive order-preserving map \( \iota : \text{Cu}(A) \to [0, \infty] \) s.t. \( \iota(1_A) = 1 \). (This gives exactly the lower semicontinuous dimension functions).

1.15. Example: For \( a \in (M_n)_+ \) we get \( d_\tau(a) = \text{rank}(a)/n \).

For \( \langle a \rangle \in \text{Cu}(A) \) we define \( \iota(\langle a \rangle) : T(A) \to [0, \infty] \) by \( \iota(\langle a \rangle)(\tau) := d_\tau(a) \). Then:

- \( \iota(\langle a \rangle) \) is in \( \text{L}(T(A)) \) since \( \tau \mapsto \tau(a^{1/n}) \) is continuous and \( \tau(a^{1/n}) \leq \tau(a^{1/n+1}) \) (if \( \|a\| \leq 1 \), so rescale \( a \))
- if \( a \geq 0, f \in C^*(a) \), \( f \geq 0 \), then \( d_\tau(f(a)) = \mu_\tau(\text{supp}(f) \cap \sigma(a)) \) where \( \mu_\tau \) is the spectral measure induced by \( \tau \)
- \( a \preceq b \) iff \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( (a-\varepsilon)_+ \preceq (b-\delta)_+ \).
Question: When is \( \langle a \rangle = \langle p \rangle \) for some projection \( p \)?

1.16. Lemma: If \( A \) is unital, simple and \( T(A) \neq \emptyset \), then \( \langle a \rangle = \langle p \rangle \) for a projection \( p \) iff \( 0 \) is not a limit point of \( \sigma(a) \).

Proof:
\( \Leftarrow \): then \( a \sim \chi_X(a) \) where \( \chi_X \) is the characteristic function on the set \((0, \infty) \cap \sigma(a)\), and \( \chi_X(a) \) is a projection

\( \Rightarrow \): then \( p \sim (p-\varepsilon)_+ \not\lesssim (a-\delta)_+ \not\lesssim a \sim p \), whence \( d_\tau((a-\delta)_+) = d_\tau(p) \) for all \( \delta \) small enough. But \( (a-\delta)_+ \leq g(a)+(a-\delta)_+ \leq a \) for some small function \( g \) with \( \text{supp}(g) \subset [0, \delta] \). Then \( d_\tau((a-\delta)_+) = d_\tau(g(a)) + d_\tau((a-\delta)_+) \), and therefore \( d_\tau(g(a)) = 0 \) for all \( \tau \) while \( g(a) \neq 0 \). This is a contradiction. \( \Box \)

Now for \( A \) unital, simple with \( T(A) \neq \emptyset \) we have

\[ \text{Cu}(A) = V(A) \cup \text{Cu}(A)_+ \]

where \( \text{Cu}(A)_+ = \{ \langle a \rangle : 0 \text{ is a limit point of } \sigma(a) \} \). \( \text{Cu}(A)_+ \) is absorbing in the sense that \( x + y \in \text{Cu}(A)_+ \) whenever \( y \in \text{Cu}(A)_+ \).

1.17. Definition: Let \( A \) be unital. We say \( A \) has strict comparison of positive elements (often abbreviated by just saying ”strict comparison”) if \( \not\lesssim b \) whenever \( d_\tau(a) < d_\tau(b) \) for all \( \tau \in T(A) \) such that \( d_\tau(b) < \infty \).
2. Part 2 - Lecture from 17. November 2009

Let $A$ be simple, unital with $T(A) \neq \emptyset$. Then $\text{Cu}(A) = V(A) \sqcup \text{Cu}(A)_+$. We define a map

$$\varphi : \text{Cu}(A) \to V(A) \sqcup L(TA)$$

as $\varphi([a]) := [p]$ whenever $a \sim p$ for a projection $p$, and for $\langle a \rangle \in \text{Cu}(A)_+$ we set $\varphi(\langle a \rangle) := \varphi_\tau(\langle a \rangle) := d_\tau(a)$. When is this map injective, when is it surjective?

Suppose $A$ has strict comparison, $\langle a \rangle \in \text{Cu}(A)_+, \langle b \rangle \in \text{Cu}(A)$, and $d_\tau(a) \leq d_\tau(b)$ for all $\tau \in T(A)$ with $d_\tau(b) < \infty$. Since 0 is a limit point of $\sigma(a)$, we have $d_\tau((a-\varepsilon)_+) < d_\tau(b)$ for all $\varepsilon > 0$ small enough. From strict comparison of $A$ we get $(a-\varepsilon)_+ \preceq b$ for all $\varepsilon > 0$ small enough, and therefore also $a \preceq b$.

Thus, if $\langle a \rangle, \langle b \rangle \in \text{Cu}(A)_+$, then $\langle a \rangle = \langle b \rangle$ iff $d_\tau(a) = d_\tau(b)$ for all $\tau$. Now $\varphi$ is at least injective if $A$ has strict comparison.

When is $\text{im}(\varphi) = \text{LT}(A)_{>0}$?

2.1. Proposition: Let $A$ be simple, unital with strict comparison and $TA \neq \emptyset$. Suppose that for any $f \in \text{Aff}(TA)$, $\varepsilon > 0$ there exists $a \in (A \otimes \mathbb{K})_+$, s.t.

$$|d_\tau(a) - f(\tau)| < \varepsilon$$

for all $\tau \in TA$. Then for any $g \in \text{LT}(A)_{>0}$ there exists $b \in (A \otimes \mathbb{K})_+$ s.t. $d_\tau(b) = g(\tau)$.

Proof:
Let $g$ be given. There exists a sequence $(f_n) \subset \text{Aff}(TA)$ s.t. $f_n > 0$, $f_n < f_{n+1}$ and $\sup_n f_n(\tau) = g(\tau)$. Find a sequence $\varepsilon_n > 0$ s.t. $f_n - \varepsilon_n < f_{n+1} - \varepsilon_{n+1}$. Then find $a_n \in (A \otimes \mathbb{K})_+$ s.t. $|d_\tau(a_n) - f_n(\tau)| < \varepsilon_n$. Then $d_\tau(a_n) < d_\tau(a_{n+1})$ and $\sup_n d_\tau(a_n) = g(\tau)$. By strict comparison $a_n \preceq a_{n+1}$. Suprema of increasing sequences in $\text{Cu}(A)$ exist, and $d_\tau$ is sup-preserving. Let $\langle a \rangle = \sup(a_n) \in \text{Cu}(A)$. Then $d_\tau(a) = g(\tau)$.

So when do we have density (in the sense of the proposition)?

2.2. Definition: We say $\text{Cu}(A)$ is almost divisible if for any $x \in \text{Cu}(A)$, $n \in \mathbb{N}$ there exists $y \in \text{Cu}(A)$ s.t. $ny \leq x \leq (n+1)y$.

2.3. Proposition: Let $A$ be simple, unital with $T(A) \neq \emptyset$ and $\text{Cu}(A)$ almost divisible. It follows that for any $f \in \text{Aff}(TA)_{>0}$, $\varepsilon > 0$ there exists $a \in (A \otimes \mathbb{K})_+$, s.t. $|d_\tau(a) - f(\tau)| < \varepsilon$ for all $\tau \in T(A)$.

Proof:
We can assume $\|f\| \leq 1$. By a theorem of Lin / Cuntz, Pedersen there exists $b \in A_+$ s.t. $\tau(b) = f(\tau)$ and $\|b\| \leq 1 + \varepsilon$. Then:

$$f(\tau) = \tau(b) \approx \sum_{i=1}^{n} \frac{1}{n} \tau(\chi_{(i/n, \|b\|]}(b))$$

$$= \sum_{i=1}^{n} \frac{1}{n} d_\tau(f_i(b))$$

for functions $f_i$ with $\text{supp}(f_i) = (i/n, \|b\|]$

$$= \sum_{i=1}^{n} d_\tau(c_i)$$

Set $c = \bigoplus_{i=1}^{n} c_i$, then $d_\tau(c) \approx f(\tau)$. □

2.4. **Theorem:** Let $A$ be simple, unital with strict comparison, $T(A) \neq \emptyset$ and $\text{Cu}(A)$ almost divisible. Then $\text{Cu}(A) \cong V(A) \cup L(TA)_{>0}$ is an order-isomorphism. Here addition on the right hand side is as usual in each of $V(A)$ and $L(TA)_{>0}$, and if $x \in V(A)$, $y \in L(TA)_{>0}$ then $x + y = \iota(x) + y$.

Also, the order on the right hand side is the usual in each of $V(A)$ and $L(TA)_{>0}$, and if $x \in V(A)$, $y \in L(TA)_{>0}$ then $x \leq y$ if $\tau(x) < y$ in $L(TA)_{>0}$, and $y \leq x$ if $y \leq \iota(x)$.

2.5. **Example:** If $A$ is UHF-algebra with $K_0(A) \cong \mathbb{Q}$, then $\text{Cu}(A) \cong \mathbb{Q}^+ \sqcup (\mathbb{R}^+ \setminus \{0\}) \cup \{\infty\}$. Also $\text{Cu}(M_n) = \mathbb{N} \cup \{\infty\}$.

2.6. **Theorem:** (Winter, Lin-Niu) Let $A, B$ be simple, unital with UCT and locally finite decomposition rank. Also suppose $\text{Cu}(A) = V(A) \cup L(TA)_{>0}$ (similarly for $B$) and projections separate traces. If there exists an isomorphism $\varphi : K_*(A) \rightarrow K_*(B)$, then there exists a $\ast$-isomorphism $\Phi : A \rightarrow B$ s.t. $K(\Phi) = \varphi$.

Note that these algebras will have real rank zero after tensoring with an UHF algebra.

2.7. **Example:** Let $A$ be simple, unital, exact, finite, $\mathcal{Z}$-stable. Then $A$ has strict comparison (the proof uses that strict comparison is equivalent to almost unperforation of $\text{Cu}(A)$, i.e. if $x, y \in \text{Cu}(A)$ with $(n + 1)x \leq y$ for some $n$, then $x \leq y$).

Also $\text{Cu}(A)$ is almost divisible. The proof uses:

1. Under the isomorphism $A \otimes \mathcal{Z} \cong A$ we have $\langle a \otimes 1_\mathcal{Z} \rangle = \langle a \rangle$

2. There exists an embedding $\gamma : C[0, 1] \rightarrow \mathcal{Z}$ s.t. the image of $\tau \in T(\mathcal{Z}) = \{\tau\}$ is the Lebesgue measure on $[0, 1]$. Thus, for any $0 < \lambda < 1$ there exists $a_\lambda \in C[0, 1]$ s.t. $d_\tau(a_\lambda) = \lambda$ for all $\tau \in T(A)$
(3) Compute \(d_r(a \otimes a_\lambda) = \lambda d_r(a)\) (so \(\text{Cu}(A)\) is a cone)

2.8. **Theorem:** If \(A\) is a simple, unital ASH-algebra with slow dimension growth, then \(\text{Cu}(A) \cong C(A) \sqcup L(TA)^{>0}\)

2.9. **Definition:** \(A\) has slow dimension growth (s.d.g.) if there exist RSH-algebras \(A_k\) and connecting maps \(\varphi_k : A_k \to A_{k+1}\) s.t. \(A \cong \lim_{\to} A_k\), and for the underlying spaces \(X_{k1}, X_{k2}, \ldots\) and matrix sizes \(n_{k1}, n_{k2}, \ldots\) of the RSH-algebras \(A_k\) we have:

\[
\limsup_k (\max_i \dim X_{ki}/n_{ki}) = 0
\]

How to prove strict comparison? Does s.d.g. imply \(\mathcal{Z}\)-stability for ASH-algebras?

For projections \(p, q \in M_n(C(X))\) with \(\text{rank}(p) + (\dim(X) - 1)/2 < \text{rank}(q)\), we have \(p \lesssim q\). We want to show that a similar result holds for positive elements.

Assume \(A = \lim A_k, A_k = M_{n_k}(C(X_k))\). Then s.d.g. means \(\dim(X_k)/n_k \to 0\). Assume \((n+1)a \leq nb\) for \(a, b \in A_k\). Does it follow that \(\text{rank}(a(x)) \leq \text{rank}(b(x))\) for all \(x \in X_k\)?

2.10. **Theorem:** If \(\text{rank}(a(x)) + \dim(X)/2 < \text{rank}(b(x))\) for all \(x \in X_k\), then \(a \lesssim b\).

The proceeding is a sketch why strict comparison holds for simple, unital ASH-algebras with s.d.g.

Why is \(\iota(\text{Cu}(A)^+)\) ”dense” in \(\text{Aff}(TA)^{>0}\) (in the above sense)? Consider \(M_n(C(X))\), and \(f \in \text{Aff}(T(M_n(C(X))))^{>0} \cong C_{\mathbb{R}}(X)\) (since \(T(\ldots)\) is a Bauer simplex, with compact boundary \(X\)). We want \(a \in M_n(C(X))^+\) s.t. \(|d_r(a) - f(\tau)| < 1/n\). Can assume \(\tau = \delta_x\) for some \(x \in X\), so \(d_r(a) = \text{rank}(a(x))/n\). Thus want \(|\text{rank}(a(x))/n - f(x)| < 1/n\). Take \(p = e_{11} \otimes \text{id}_x\), and fix \(f \in C(X)\) s.d. \(\text{supp}(f_i) = U_i\). Set \(a_i = f_i(p)\). Then \(a = a_1 \oplus \ldots \oplus a_n\) does the trick.
Are there simple, unital, separable, nuclear C*-algebras with the same K-theory and traces, but which are not isomorphic?

Yes, first examples have been given by Rørdam, and there are even examples in the stably finite case.

Strategy: Construct $A$ as inductive limit $A = \lim_{\rightarrow} M_{n_k}(C(X_k))$ with each $X_k$ contractible. Then $K_0(A_k) = \mathbb{Z}$ and $K_1(A_k) = 0$, so also $K_1(A) = 0$. Assume we can achieve that the elements of $K_0(A_k)$ get divisible in the limit, i.e. for each $n$ and $k$ there is some $N > k$ such that $1 \in K_0(A_k)$ is divisible by $n$ in $A_N$. Then $K_0(A) = \mathbb{Q}$, and hence $\text{St}(K_0(A)) = \{\tau\}$, so the pairing between traces and $K_0$ is uninteresting.

Let $Q$ be the universal UHF-algebra (i.e. $K_0(Q) = \mathbb{Q}$), then $(K_0(A \otimes Q), K_1(A \otimes Q), T(A \otimes Q), \rho_A \otimes \rho_Q) \cong (K_0(A), K_1(A), T(A), \rho_A)$

For a counterexample we just need $A \not\cong A \otimes Q$. We will show that AUP (almost unperforation property) fails in $\text{Cu}(A)$, but $\text{Cu}(A \otimes Q)$ has AUP.

Let us first see how AUP can fail in $M_n(C(X))$ using the fact that AUP is equivalent to:

$$(n+1)x \leq ny \Rightarrow x \leq y$$

How do we show that $p \not\leq q$ for projections $p, q \in M_n(C(X))$? View $p, q$ as VB (vector bundles) over $X$: the fibre of $p$ at $x \in X$ is $p(x)\mathbb{C}^n$. Villadsen used Chern classes to get comparability obstructions.

3.1 (Chern classes): The (full) Chern class is a map $c(\cdot) : \text{Vect}(X) \to H^{ev}(X : \mathbb{Z})$ with the following properties:

(i) $c(\xi \oplus \xi') = c(\xi) \cup c(\xi')$
(ii) $c(e_r) = 1 \in H^0(X)$ where $e_r = X \times \mathbb{C}^r$ is the trivial VB
(iii) if $f : X \to Y$ is continuous, then $c(f^*(\xi)) = f^*(c(\xi))$
(iv) $c(\xi) = 1 + c_1(\xi) + \ldots + c_{\dim(\xi)}(\xi)$ with $c_i(\xi) \in H^{2i}(X)$

3.2. Lemma: (Villadsen) Let $\gamma, e_r$ be VB over $X$. Assume $c_j(\gamma) \neq 0$ for some $k > \dim(\gamma) - r$. Then $e_r \not\leq \gamma$.

Proof:
If $e_r \preceq \gamma$, then there exists $\omega$ s.t. $e_r \oplus \omega \cong \gamma$. Then $c(e_r \oplus \omega) = c(e_r) \cup c(\omega) = c(\omega) = c(\gamma)$, but $\dim(\omega) < \dim(\gamma) - i$. \qed
On the other hand, if \( \text{rank}(\omega) + (\dim(X) - 1)/2 < \text{rank}(\gamma) \), then \( \omega \preceq \gamma \). Thus, if \( \text{rank}(\omega) < \text{rank}(\gamma) \), then \( (n+1)\langle \omega \rangle \leq n\langle \gamma \rangle \) for large enough \( n \).

3.3. Example: Let \( \rho \) be the Bott bundle over \( S^2 \). Then \( c(\rho) = 1 + 1 \in H^0(S^2) \oplus H^2(S^2) \). \( \rho \times \rho \) is a bundle over \( S^2 \times S^2 \) defined by \( \pi_1^*(\rho) \oplus \pi_2^*(\rho) \) where \( \pi_i : S^2 \times S^2 \to S^2 \) are the coordinate projections. Then

\[
c(\pi_1^*(\rho) \oplus \pi_2^*(\rho)) = \pi_1^*(c(\rho)) \pi_2^*(c(\rho))
\]

in particular \( c_2(\rho \times \rho) \neq 0 \): Thus \( e_1 \nleq \rho \times \rho \).

Consider \( S^2 \times S^2 \subset [0,1]^3 \times [0,1]^3 = \mathbb{R}^6 \). Extend \( \rho \times \rho \) to an open neighborhood \( U \) of \( S^2 \times S^2 \), choose \( f : X_1 \to [0,1] \) with \( f = 1 \) on \( S^2 \times S^2 \) and \( f = 0 \) on \( U^c \) (the complement of \( U \)). Set \( a = f : e_1, b = f : \rho \times \rho \). Then \( a, b \in M_{n_1}(C(X_1)) \) and \( (n+1)\langle a \rangle \leq n\langle b \rangle \) for large \( n \), but \( \langle a \rangle \nleq \langle b \rangle \) since otherwise \( \langle a, e_1 \rangle = \langle e_1 \rangle \leq \langle \rho \times \rho \rangle = \langle b, e_1 \rangle \).

Set \( X_2 := X_1^{m_2} \). Define \( \varphi_1 : M_{n_1}(C(X_1)) \to M_{n_2}(C(X_2)) \) as:

\[
\varphi_1(f) = \begin{pmatrix}
    f \circ \pi_1 \\
    \ldots \\
    f \circ \pi_{m_1} \\
    f(x_i) \\
    \ldots
\end{pmatrix}
\]

Note that we add the evaluations at points \( x_i \) to ensure simplicity of the limit. (so want these points to be eventually dense). Then:

\[
\varphi_1(b)_{(S^2 \times S^2)^{\times m_1}} = (\rho \times \rho)^{\times m_1}
\]

and \( c_{2m_1}((\rho \times \rho)^{\times m_1}) \neq 0 \). Thus \( \langle \varphi_1(a) \rangle \nleq \langle \varphi_1(b) \rangle \). If we proceed this way, a similar result will hold for all forward images. In fact there exists \( \delta > 0 \) such that for all \( i \) and \( x \in A_i \): \( \| x \varphi_{1,i}(b)x^* - \varphi_{1,i}(a) \| \geq \delta \), so \( \langle \varphi_{1,\infty}(a) \rangle \nleq \langle \varphi_{1,\infty}(b) \rangle \). Thus AUP fails in \( A \).

3.4. Definition: Let \( A \) be unital, exact. Define the radius of comparison for \( A \) to be:

\[
\text{rc}(A) := \inf \{ r > 0 : a \preceq \rho b \text{ whenever } d_\tau(a) + r < d_\tau(b) \forall \tau \}
\]

(\text{where } \tau \text{ runs over all normalized traces, and } a, b \in (A \otimes K)_+ \}).

One can show that

\[
\text{rc}(A) = \inf \{ m/n : a \preceq \rho b \text{ whenever } na + m\langle 1_A \rangle \leq ny \}
\]

3.5. Proposition: If \( X \) is a CW-complex with \( \dim(X) = d < \infty \), then:

\[
(d - 2)/2 \leq \text{rc}(C(X)) \leq (d - 1)/2
\]
**Proof:**
The upper bound was already discussed (and it works for all $X$, not just CW-complexes). To get the lower bound note that one can immerse $S^{2d'}$ into $X$ (for some large $d'$).

If $A$ is simple, then $rc(A) = 0$ if and only if $Cu(A)$ is almost unperforated. We also have the following properties:

(i) $rc(\lim_k A_k) \leq \liminf_k rc(A_k)$
(ii) $rc(A/I) \leq rc(A)$
(iii) $rc(M_n(A)) = 1/n \cdot rc(A)$

3.6. **Theorem:** There exists a family $A_r$ of simple AH-algebras indexed over $r \in [0, \infty]$ s.t.:

1) The Elliott invariant of $A_r$ (K-theory and traces) is the same for all $r$

2) $rc(A_r) = r$, so the algebras are pairwise not isomorphic

The algebras $A_r$ of the theorem are all shape equivalent, since they are constructed as AH-algebras over contractible spaces, so all homotopy invariant continuous functors agree on the $A_r$. Further $K_0(A_r) = \mathbb{Q}$ and $sr(A_r) = 1$. This means we have uncountably many different Morita equivalence classes among the $A_r$.

3.7 (Mean dimension): Let $X$ be compact, metric, $\alpha : X \to X$ a homeomorphism, and $\mathcal{U}$ an open cover of $X$. Define

$$ord(\mathcal{U}) := \sup\{\left(\sum_{U \in \mathcal{U}} \chi_U(x)\right) - 1 : x \in X\}$$

and write $\mathcal{V} > \mathcal{U}$ if $\mathcal{V}$ refines $\mathcal{U}$. Set:

$$D(\mathcal{U}) := \min\{ord(\mathcal{V}) : \mathcal{V} > \mathcal{U}\}$$

We have $D(\mathcal{U} \cup \mathcal{V}) \leq D(\mathcal{U}) + D(\mathcal{V})$, since one can show that $D(\mathcal{U}) \leq d$ if and only if there exists a continuous map $f : X \to K$ with $\dim(K) \leq d$ such that $f$ is compatible with $\mathcal{U}$.

Set $\mathcal{U}^n := \mathcal{U} \vee \alpha^{-1}(\mathcal{U}) \vee \ldots \vee \alpha^{-(n-1)}(\mathcal{U})$ where $\mathcal{V} \vee \mathcal{W}$ means the union and also all intersections of set in $\mathcal{V}, \mathcal{W}$. Set

$$mdim(X, \alpha) := \sup \lim_{n \to \infty} D(\mathcal{U}^n)/n$$

3.8. **Example:** Let $Y$ be a CW-complex, $X = Y^{\mathbb{Z}}$, and $\alpha : X \to X$ the bilateral shift. Then $mdim(X, \alpha) = \dim(Y)$.

Problem: If $\dim(X) < \infty$, then $mdim(X, \alpha) = 0$ for all $\alpha$. 
3.9. **Theorem:** (Kerr, Giol) For any \( k > 0 \) there exists a minimal system \((X_k, \alpha_k)\) s.t. \( k \leq \text{rc}(C(X_k) \rtimes_{\alpha_k} \mathbb{Z}) \). Also \( \text{mdim}(X, \alpha) / 2 \approx k \).

If \( \alpha : Y^\infty \to Y^\infty \) is the bilateral shift, then let \( Y_{2^n} \) be the \( 2^n \)-periodic points. Then:

\[
\cdots \rightarrow C(Y_{2^n}) \rtimes_{\alpha} \mathbb{Z} \rightarrow C(Y_{2^{n+1}}) \rtimes_{\alpha} \mathbb{Z} \rightarrow \cdots
\]

**Proposal:** Define a dynamical dimension \( \text{ddim}(X, G) \) for a countable, discrete group \( G \) acting on \( X \) via:

\[
\text{ddim}(X, \alpha) := \text{rc}(C(X) \rtimes_{\alpha} G)
\]

The reasons are:

1. It looks like one could recover \( \text{mdim} \) for the bilateral shift
2. If \( G = \{1\} \), then \( \text{ddim}(X, G) \approx \dim(X) / 2 \)
3. If \( G = \mathbb{Z} \) acting trivially, then \( \text{ddim}(X, G) = (\dim(X) + 1) / 2 \)
4. If \( X = Y^m \) with \( \alpha \) the cyclic shift, then \( \text{ddim}(X, \alpha) \approx \dim(Y) / 2 \)

**Outlook:** Hopefully for minimal systems \((X, \alpha)\) we have \( \text{ddim}(X, \alpha) \leq \text{mdim}(X, \alpha) / 2 \) and that this is sharp (see results of Kerr and Giol). Why are we hopeful?

We have that \( C^*(C(X), uC(X \setminus \{y\})) = A_{\{y\}} \) is ASH, but the RSH-subalgebras have infinite dimension. Idea: fix \( a, b \in A_{\{y\}^+}, a = \sum_{i=1}^N f_i u \).

Take \( \mathcal{U} \) a finite open cover, iterate under \( \alpha^{-1} \), get covers \( \mathcal{V}_n \) s.t. \( \text{ord}(\mathcal{V}_n) = n \cdot \text{ddim} \), thus \( u \) corresponds to the size of the matrices.