MIKAEL RØRDAM’S TALK AT MASTER CLASS ON CLASSIFICATION OF C*-ALGEBRAS

1. Simple C*-algebras

Theorem 1. For a simple C*-algebra $A \neq \mathbb{C}$ TFAE

1. $\forall B \subset A$ hereditary subalgebra, $B \neq \{0\}$, $B$ contains an infinite projection.
2. $\forall a, b \in A \{0\} \exists x, y \in A : b = xay$
3. $RR(A) = 0$ and all projections in $A$ are properly infinite ($p \oplus p \lesssim p$)
4. $W(A) = C(A) \cong [0, \infty]$

Definition 2. If one (and hence all) of the statements in the above theorem is true, we say that $A$ is a purely infinite simple C*-algebra.

Example 3. For $n \in \mathbb{N}$ set

$$O_n = C^*(s_1, s_2, \ldots, s_n | \forall j : s_j^*s_j = 1, \sum_{j=1}^{n} s_j s_j^* = 1)$$

and set

$$O_\infty = C^*(s-1, s_2, \ldots | \forall j : s_j^*s_j = 1, i \neq j \Rightarrow s_is_i^* \perp s_js_j^*).$$

Then $O_n$ is purely infinite and simple and $K_1(O_n) = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$ while $K_0(O_n) = \mathbb{Z}_{n-1}$ for $n \in \mathbb{N}$ and $K_0(O_\infty) = \mathbb{Z}$.

Theorem 4. For a simple C*-algebra $A$ we have that $A$ is purely infinite iff $T(A) = \emptyset$.

Theorem 5. It is possible to exhaust $(K_0, K_1)$ by purely infinite simple C*-algebras of the following two types:

- $A = \lim_{\to} \oplus_{n=1}^{\infty} M_{n\alpha}(O_n) \otimes C(\Pi)$ where simple implies purely infinite.
- $A \rtimes \mathbb{Z}$ for some simple stable II-algebra $A$.

Theorem 6. If $A$ is a simple, separable, exact, stable C*-algebra where $T(A) \neq \emptyset$ and $A \otimes \mathbb{Z} \cong A$, and $\alpha \in \text{Aut}(A)$ then $A \rtimes \mathbb{Z}$ purely infinite iff $A$ has no $\alpha$-invariant traces.

Definition 7. We say that a C*-algebra $A$ is a Kirchberg algebra if it is purely infinite, simple, separable and nuclear.

Definition 8. We say that a C*-algebra has the SP if $\forall B \subset A$ hereditary subalgebra, $B \neq \{0\}$, $B$ contains a nontrivial projection.

Theorem 9. There exists a stably infinite simple C*-algebra $A$ with $RR(A) \neq 0$ and not being purely infinite having the SP.

Question 10. If $A$ is a simple C*-algebra, do $RR(A) = 0$ and $A$ stably infinite imply $A$ purely infinite?

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Question 11. If $A$ is simple and all projections in $A$ are infinite, does $P(A) \neq 0$ imply $A$ purely infinite?

Question 12. If $A$ is stably infinite does that imply that $A$ has the SP?

Theorem 13 (Kirchberg). If $A, B$ simple not type I $C^*$-algebras then if both are stably infinite or one is stably finite and the other stably infinite $A \otimes_{\min} B$ is purely infinite. If both are stably finite and exact $A \otimes_{\min} B$ is stably finite, if they not both are exact the result is not known.

Theorem 14 (Kirchberg). If $A$ is a simple, separable, nuclear $C^*$-algebra $A$ is purely infinite iff $A \cong A \otimes O_\infty$.

Remark 15. $K_x(O_\infty) \cong K_x(C)$

Theorem 16 (Kirchberg). A $C^*$-algebra $A$ is simple, separable, unitary and nuclear iff $A \otimes O_2 \cong O_2$.

Remark 17. $K_x(O_2) = 0$

Theorem 18 (Kirchberg). A $C^*$-algebra $A$ is separable and exact iff $A \hookrightarrow O_2$.

Theorem 19 (Kirchberg, Phillips). If $A, B$ are Kirchberg algebras:
- $A \otimes K \cong B \otimes K$ iff $A \sim_K B$
- If $A, B$ have the UCT then $A \otimes K \cong B \otimes K$ iff $(K_0(A), K_1(A)) \cong (K_0(B), K_1(B))$.

2. Non-simple $C^*$-algebras

Theorem 20. If $A$ is a $C^*$-algebra with no nonzero abelian quotient $TFAE$

1. $\forall a, b \in A_+: a \in \overline{abA} \iff a \lesssim b$
2. $\forall a \in A_+$ $a$ is properly infinite $(a \oplus a \lesssim a)$

Definition 21. If $A$ fulfills one (and hence all) of the statements in the theorem we say that $A$ is purely infinite.

Definition 22. If $x \in W(A)$ the we say that $x$ is properly infinite if $2x \leq x$.

Definition 23. We say that $x \in W(A)$ is infinite if $\exists y \neq 0 \in W(A). x + y \leq x \Rightarrow x + y = x$

Remark 24. $A$ is purely infinite iff $W(A)$ is properly infinite, where $W(A)$ is properly infinite iff $\forall x \in W(A) x$ is properly infinite.

Remark 25. The function $W(A) \rightarrow \text{Ideal}(A)$ defined by $< a > \mapsto \overline{AaA}$ is well-defined and surjective. And this map is injective (equivalent: this map is an order isomorphism) iff $A$ is purely infinite.

Example 26. If $A \in M_n(\{0,1\})$ then is the Cuntz-Kneyr-algebra

\[ O_A = C^*(s_1, s_2, \ldots, s_n | \sum_{j=1}^{n} s_j s_j^* = 1, s_j^* s_j = \sum_{i=1}^{n} A_{i,j} s_i s_i^*) \]

properly infinite.

Example 27. If $A$ is a $C^*$-algebra then $A \otimes O_\infty$ is purely infinite.

Example 28. $C_0(\mathbb{R}) \otimes O_\infty$ is purely infinite.
Theorem 29.  
(1) If we have $0 \to I \to A \to B \to 0$ then $I, B$ are purely infinite iff $A$ is purely infinite.

(2) If $A = \lim_{\to} A_i$ and all $A_i$ are purely infinite then $A$ is purely infinite.

(3) If $A, B$ are purely infinite and $A$ is exact then $A \otimes_{\min} B$ is purely infinite.

Question 30. Does $A$ or $B$ purely infinite imply that $A \otimes_{\min} B$ is purely infinite?

Question 31. Does $A$ purely infinite imply that $A \otimes \mathcal{O}(0,1)$ is purely infinite?

Definition 32. We say that $A$ is strongly purely infinite if
\[
\forall \begin{pmatrix} a & \lambda^* \\ \lambda & b \end{pmatrix} \in M_2(A) \forall \epsilon > 0 \exists d_1, d_2 \in A : \\
\| \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^* \begin{pmatrix} a & \lambda^* \\ \lambda & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \| < \epsilon
\]

Definition 33. We say that $A$ is weakly purely infinite if $\exists k \in \mathbb{N} \forall x \in W(A) : kx$ properly infinite.

Remark 34. $W(A)$ has no dimension function iff $\forall x \in W(A) \exists k \in \mathbb{N} : kx$ is properly infinite.

Theorem 35. Let $A$ be a separable, exact $C^*$-algebra and look at these properties:

(1) $A \cong A \otimes \mathcal{O}_\infty$

(2) $A$ strongly purely infinite

(3) $A$ purely infinite

(4) $A$ weakly purely infinite

(5) $l_\infty(A)/C_0(A)$ traceless

(6) $A$ traceless

We have that

- In general (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6).
- If $A$ is separable and nuclear (2) $\Rightarrow$ (1), but not true in general.
- If $A$ is simple, $RR(A) = 0$ or $A \cong A \otimes \mathbb{Z}$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2), unknown if true in general.
- If $A \cong A \otimes \mathbb{Z}$ (6) $\Rightarrow$ (5), but not in general.

Definition 36. Let $A$ be a $C^*$-algebra the we define $\text{Prim}(A) = \{ \ker(\pi) | \pi \text{irreducible representation of } A \}$, and equip it with the Jacobsen topology giving us a $T_0$-space.

Example 37. If $X$ is a locally compact Hausdorff space then $\text{Prim}(C(X)) = X$.

Theorem 38 (Kirchberg). If $A, B$ are separable nuclear $C^*$-algebras and $X := \text{Prim}(A) = \text{Prim}(B)$ we have that $A \otimes \mathcal{O}_\infty \otimes K \cong B \otimes \mathcal{O}_\infty \otimes K$ iff $A \sim_K B$. If $A, B$ are strongly purely infinite we can skip $\mathcal{O}_\infty$.

Corollary 39. If $A, B$ are separable nuclear $C^*$-algebras TFAE

(1) $A \otimes \mathcal{O}_2 \otimes K \cong B \otimes \mathcal{O}_2 \otimes K$

(2) $\text{Prim}(A) \cong \text{Prim}(B)$

(3) $\text{Ideal}(A) \cong \text{Ideal}(B)$

Question 40. Which $T_0$-spaces can arise as $\text{Prim}(A)$ for $A$ a separable $C^*$-algebra? Which if $A$ is nuclear?
Example 41. Let \((t_n)_{n \in \mathbb{N}} \subseteq [0, 1]\) and look at the sequence
\[(2.3) \quad C_0([0,1]) \rightarrow_{\phi_1} M_2(C_0([0,1])) \rightarrow_{\phi_2} M_4(C_0([0,1])) \rightarrow_{\phi_4} \ldots\]
where
\[(2.4) \quad \phi_n(f)(t) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t \land t_n) \end{pmatrix}.
\]
Let \(A\) be the inductive limit of this sequence, then \(A\) is an \(AH_0\)-algebra and \(\text{Ideal}(A) \approx [0,1]\) totally ordered.

We have \(A \cong A \otimes M_{2^n} \Rightarrow A \cong A \otimes Z\) and \(A \cong A \otimes O_{\infty} \iff A\) traceless, which is the case, so we have that \(A\) is purely infinite.

\[(2.5) \quad A \sim_{h, \text{ideal} 0} 0 \iff \exists \phi_t : A \rightarrow A, \phi_t \leq h, t \in [0,1], \phi_1 = \text{id}, \phi_0 = 0, \forall J \triangleleft A, \phi_t(j) \subseteq J\]

A strongly purely infinite, separable, nuclear then \(A \sim_{h, \text{ideal} 0} 0 \Rightarrow A \cong A \otimes O_2\).

Fact 42. If \(A\) has no projections then \(\forall I \triangleleft A : A/I\) has no projections.

Theorem 43. If \(A\) is nuclear, separable, stable, strongly purely infinite \(C^*\)-algebra then \(A \sim_{h, \text{ideal} 0} 0\) implies that \(A \cong A \otimes O_2\) and \(A\) is an \(AH_0\)-algebra.

Fact 44. If \(A\) is an \(AH_0\)-algebra then \(A\) can be embedded in an \(AF\)-algebra \(D\) and \(A\) is quasidiagonal. If \(A \cong A \otimes O_2\) and \(B\) separable and exact then \(CB \hookrightarrow CO_2 \hookrightarrow A \hookrightarrow D\).