Ref. (Elliot-Kucerovsky) We say that a $C^*$-algebra $A$ has the Corona Factorization Property (CFP) if any full projection in $M(A\otimes K)$ is properly infinite.

A projection is full if it generates all of $M(A\otimes K)$.

"p is properly infinite if $(p^*p)\leq (2,0)$" in the Murray-von Neumann sense.

Theorem (Kucerovsky-K Ng) Let $A$ be a separable $C^*$-algebra. Then $A$ has the CFP iff every norm-full extension is nuclearly absorbing.

Theorem (Kucerovsky-K, Kucerovsky-K Ng) Let $A$ be a separable $C^*$-algebra. TFAE:

(i) $A$ has the CFP

(ii) Every full projection in $M(A\otimes K)$ is properly infinite.

(iii) Every strictly full element in $M(A\otimes K)$ is properly infinite.

Strictly full element is every non-zero element in $C^*(A)$ is full.

Properly infinite: $(a,a)\leq (2,0)$ in the Glimm sense.

Examples (i) Simple $C^*$-algebras with Real Rank 0, stable rank 1 and weakly unperforated $K_0$ groups.

(ii) Purely infinite simple $C^*$-algebras.

(iii) $X$ compact Hausdorff metric space with finite dimension $\omega$, $C(X)$ satisfies the CFP.

(iv) $C(\mathbb{R}^2)$ does not have the CFP.
(vi) Rørdam's example of a simple $C^*$-algebra with a finite and an infinite projection does not have the CFP.

Rørdam constructed a $C^*$-algebra $A$ such that fits into

$$0 \to B \to C \to A \to 0$$

with $B, A$ stable, $C$ not stable.

Rørdam constructed a $C^*$-algebra $A$ with $M_n(A)$ stable and $A$ not stable.

**Theorem (Kucerovsky - Ng)**

(i) If $B$ has the CFP, then:

- $0 \to B \to C \to A \to 0$, $C$ stable when $B, A$ are.

(ii) $A$ has the CFP $\Rightarrow$ $D = A \otimes K$ hereditary and full implies $M_n(D)$ stable for some $n$.

Outline proof of part (ii)

(\Rightarrow) Assume $A$ has the CFP. Let $D = A \otimes K$ be a hereditary subalgebra of $A \otimes K$ and full and such that $M_n(D)$ is stable. We want to show that $D$ is stable.

We have

$$D \text{ full } \Rightarrow D \otimes K = A \otimes K \Rightarrow D \text{ has the CFP.}$$

In $M(D \otimes K)$, take $P = 1 \otimes e_n$, then $D = P(D \otimes K)P$ is full and hereditary in $D \otimes K$.

$M(D \otimes K)$ is properly infinite.

We can embed $Q_n \hookrightarrow M(D \otimes K)$ unitally. If $e_i$'s are the isometries coming from $Q_n$, take $Q := \sum_i e_i P e_i$.

Let $Q = M_n(P(D \otimes K)P) = M_n(D)$, full, stable.

This implies by Brown's theorem that

$$nP \sim 1 \Rightarrow \text{CFP}$$

It follows $1 \leq nP \leq P \Rightarrow P \sim 1 \Rightarrow P(D \otimes K)P \cong D$.
\((\leq)\) Take \(P\) a full projection \(P\) in \(M(A \otimes K)\)

Then
\[
P \text{ full } \Rightarrow \exists n \ni P^* P = T \Rightarrow M_n(P(A \otimes K)P) \cong A \otimes K
\]

for some \(n\)

\[P(A \otimes K)P\] is stable \(\Rightarrow M(P(A \otimes K)P)\) is properly infinite with unit \(P\)

\(\Rightarrow P\) is properly infinite

The CFP in the real rank zero situation

Give a \(C^*\)-alg. \(A\)

\(\mathcal{C}(A) = \overline{P M_{\infty}(A)}\) where \(P M_{\infty}(A) = \) projections in \(M_{\infty}(A)\)

\(\mathcal{C}(A)\) is an abelian semigroup with zero

\(CP_{\infty}\) equivalence class of \(P\)

\(CP_{\infty} + [q] = \overline{[0, \infty)} = [P + q]\)

\(P \leq q \Leftrightarrow p - p' \leq q\) defines an order on \(\mathcal{C}(A)\) which is algebraic \(CP_{\infty} + CP_{\infty} = [P] + [0] = [P]\)

Ref \(\mathcal{C}(A)\) an abelian semigroup \(V\) is said to be a \(C^*\)-algebra if:

\(a + b = c + d\) in \(V\) then \(\exists (x_{ij})\) such that

\[
\begin{array}{cccc}
  c & d \\
  a & x_{11} & x_{12} \\
  b & x_{21} & x_{22}
\end{array}
\]

\(\begin{array}{c}
  a \\
  b \\
  c \\
  d
\end{array} = \begin{array}{cc}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{array}\)
Theorem (Zhang, Ara–Pardo)
If \( A \) has real rank zero, then \( V(A) \) is a refinement semigroup.

- \( V = \{0, u, x, y, u^2\} \) where \( 2u = 0 \) is an example of a Riesz semigroup that is not a refinement semigroup for all \( a, c, d \in V \).

Def. A semigroup \( V \) is Riesz if for all \( a, c, d \in V \), there are \( x_1, x_2 \in V \) such that \( a = x_1 + x_2 \) and \( x_1 \leq c \), \( x_2 \leq d \).

Weak divisibility

Def. \( V \) is locally semiprime (with order given by the addition operation) if \( V \) is a subsemigroup of \( V \).

Example A \( C^* \)-alg., \( p \) is a projection if \( [p] \in V(A) \) is an order-unit if \( V \times V \leq V \).

Denote by \( V^* = \{ u \in V \mid u \text{ is an order-unit} \} \)

- \( V^* \) is a subsemigroup of \( V \)

Def. \( V \) is simple if \( V \) is simple and \( \{0\} \).

Example \( V(A) \) is simple if \( A \) is a simple \( C^* \)-alg.

Def. An element \( x \in V \) is weakly divisible if there are \( u, z \in V \) such that \( x = 2u + 3z \).

Say that \( V \) has weak divisibility of order-units when every order-unit is weakly divisible.

Remark: If \( V \) has weak divisibility (WD) of order-units, then given \( u, z \in V^* \), \( u = 2x + 3y \) and take \( u = x + y, w = x + 2y \).

Then \( u = v + w, u \leq w \), and \( v, w \in V^* \).

We will show that if \( A \) has real rank zero and if \( 1 \) is properly infinite, then \( V(A) \) has WD of order-units (this applies to \( M(A \otimes K) \) when \( A \) has real rank zero because \( V(M(A \otimes K)) \) has refinement and \( 1 \) is properly infinite).
Lemma: \( V \) is refinement with \( W \) d. of order-units. Then,

given \( a, b, c \in V \) such that \( W \leq a, b \)
\( (a, b \leq c \) if \( a, b \) are in an order-unit \)

Proof:
Split \( W = u_1 + u_2 \), \( V = v_1 + v_2 \) with \( u_1, v_1 \in V^* \)

\( u_1 \in V^* \Rightarrow v_2 \leq u_1 \) for some \( v_2 \)

Then, \( v_1 + v_2 = u_1 \), for some \( v_1 \)

Refinement such that:

\[
\begin{array}{c|ccccc}
  & u_1 & u_2 & \ldots & u_n & v_1 & v_2 & \ldots & v_m \\
  w & = & w_1 & + \ldots & + w_n & = & v_1 & \leq v_2 & \leq \ldots & \leq v_m \\
  t & = & t_1 & + \ldots & + t_n & & t_1 & > t_2 & > \ldots & > t_m \\
\end{array}
\]

Let \( v_n = W \). Then \( W \leq v_i, u_i \). \( W \) is an order unit because \( v_i \) is

\[ W + (v_i) = u_i. \]

\[ \Box \]

Def. An ideal \( I \) in \( V \) is a subring such which is order-
hereditary. \( \forall x, y \in I \Rightarrow x \land y \in I \)

Example: A close ideal of \( A \) \( \Rightarrow V(A) \) in an ideal of \( V(A) \)

Def. \( \forall i = N \) where \( x \lor y \Rightarrow x \land y = y + t \) for some \( t \in I \)

Example: \( A \) has real range zero. \( V(A) \lor (I) \Rightarrow V(A) \ni I \)

Def. An element \( x \) is an atom if \( x \neq 0 \) and

\[ x = a + b \Rightarrow a = 0 \text{ or } b = 0 \]

If \( V \) has refinement,

\( x \) is an atom \( \Rightarrow \langle x \rangle = \{ 0, x, 2x, 3x, \ldots \} \) is an ideal.
If \( V \) is refinement, then \( u \) is weakly divisible \( \Rightarrow \)

\([u]\) in \( V_I \) for any \( I \) is not an atom

Proposition: If \( V \) contains a properly infinite order unit \( u \)

\((2u \leq u)\) then \( V \) has w.d. of order units

Proof: Take any order-unit \( v \).

\[ v \text{ order-unit } \Rightarrow v \leq su \]

\[ u \leq sv \] for some \( m \). Then \( 2(mv) = mv + mv \leq u + u < u < mv \)

\[ \Rightarrow mv \text{ is properly infinite} \]

If \( I \) is any ideal, look at \( V_I \). \([mv]\) is properly infinite

so \([v]\) cannot be an atom. Otherwise \( <[v]\> \cong \mathbb{Z}^+ \)

and contains a properly infinite element. \( \Rightarrow \subseteq \) \( \square \)
Lecture 2

V is a refinement semigroup \( \forall a, b, c, d \in V \) s.t. \( a + b = c + d \)

Then \( \exists (x_i)_{1 \leq i \leq 2} \) s.t.

\[
\begin{align*}
    a &= x_{11} + x_{12} \\
    b &= x_{21} + x_{22} \\
    c &= x_{11} + x_{21} \\
    d &= x_{12} + x_{22}
\end{align*}
\]

\( V^* = \{ u \in V \mid u \text{ is an order unit} \} \)

\( V \) has weak divisibility of order-units \( (\forall u \in V^*, u = u \cdot v \text{ with } v \in V^* \) and \( v \geq u \)

**Theorem**: If \( V \) has refinement and weak divisibility of order-units, then \( V^* \) has refinement.

**Outline**

**Lemma 1 (Goodzik)**

If \( \exists \) refinement \( a \mid a_1 \quad a_2 \), \( b \mid b_1 \quad b_2 \in V^+ \)

then \( \exists \) refinement \( c_1 \quad c_2 \)

\[
\begin{align*}
    a &= \frac{a_1}{a_2} \geq \frac{a_1}{a_2} \\
    b &= \frac{b_1}{b_2} \geq \frac{b_1}{b_2}
\end{align*}
\]

**Lemma 2**: If \( \exists \) refinement \( c \mid d \)

\( a \quad x_{11} \quad x_{12} \quad x_{11}, x_{22} \in V^+ \)

then \( \exists \) refinement \( e \mid f \)

\[
\begin{align*}
    a &= x_{11} \quad x_{12} \quad x_{11}^j \in V^+ \text{ for } 1 \leq j \leq 2
\end{align*}
\]

**Proof of the theorem**

Suppose \( a + b = c + d \) with \( a, b, c, d \in V^+ \)

Split \( a = x + y \), \( x \geq y \), \( c = u + v \), \( u \geq v \) with \( x, y, u, v \in V^+ \)

by **Lemma 1**

\[
\begin{align*}
    & x \\
    & a + b = c + d \quad x_1 \quad x_2 \quad x_{21} \in V^+
\end{align*}
\]
\[ v \leq u + d, \text{ then by Lemma 1 } \]
\[
\begin{array}{c|c|c}
0 & 1 & 0 \\
\hline
1 & 0 & 1 \\
\end{array}
\] with all \( x_{ij} \in V^* \)

By Lemma 2
\[
\begin{array}{c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

I have that \( b + x = x_{21} + x_{22} \). Find any refinement

\[
\begin{array}{c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

We have
\[
\begin{array}{c|c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

Applying Lemma 1
\[
\begin{array}{c|c|c|c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

After applying Lemma 2, may assume that all \( t_{ij} \)'s are ordered.

We have \( u + d = t_{12} + t_{22} \) \( u, d, t_{12}, t_{22} \in V^* \)

Refine this equality using first part of the proof up to
\[
\begin{array}{c|c|c|c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

Now:
\[
\begin{array}{c|c|c|c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

and Lemma 2 \( \implies \) we can get them all
\[
\begin{array}{c|c|c|c|c|c}
0 & 1 & 2 \\
\hline
1 & 0 & 3 \\
\end{array}
\]

Theorem: if \( V \) is a refinement, and if \( u \in V^* \) such that \( uu \) is properly infinite, then \( u = s + t \) with \( s \in V^* \) and \( u, s \), \( t \) properly infinite.
Proof (Sketch) (1) Prove it when \( V \) is simple (\( V = V^* V^0 \)).

(2) \( V \) is properly infinite if \( V \) has weak divisibility for order units.

So by the theorem \( V^* \) is a refinement and simple. Now apply step 1.

Corollary. If \( V \) is a refinement and \( n \geq 1 \) if \( n V \) is such that \( \phi(nV) \) is properly infinite. Then \( \exists t, t_2, \ldots \in V^+ \) such that \( t + t_2 + \cdots + t_k \leq n \).

Proof. By induction.

The CFP for (algebraically ordered) semigroups.

Ref. (i) \( V \) has the strong CFP if \( \forall x, y \in V, m \geq 1 \) and \( x \leq m y \), then \( x \leq y + \cdots + y \) for some \( k \).

(ii) A sequence \( x, x, \ldots \) in \( V \) is called full if \( \forall y \in V, \exists m_n \leq m \) s.t. \( y \leq y_n \).

(c) \( V \) has the CFP if \( \forall x, y, z \in V, m \geq 1 \) and \( x \leq m y, y \leq m z \).

Remark. Strong CFP \( \Rightarrow \) CFP.

If \( V \) is simple, then they are equivalent.

Ref. Give \( x, y \in V \), \( x \leq y \) if \( (k+1) x \leq k y \) for \( k \geq 1 \).

We say that \( V \) has \( n \)-comparation if given \( x, y, \ldots \in V \) with \( x \leq y, \forall j \), then \( x \leq y_0 + \cdots + y_k \).

\( n \)-comparation is the same as almost unperforation.
\( n \)-comparison \( \Rightarrow \) \( m \)-comparison \( \forall m \leq n \)

If \( V \) is simple and refinement, then they are equivalent.

\textbf{Example}

Take \( W_n = \langle 0, n+1, n+2 \rangle \leq \mathbb{Z}^+ \). Then \( W_n \) has \( n \)-comparison

\( \text{but not} \ (n-1) \)-comparison.

If \( V \) has \( n \)-comparison \( \Rightarrow \) \( V \) has the strong CFP

\( \Rightarrow \) \( V \) has CFP

\textbf{Proof}

Suppose \( x, x_1, x_2, \ldots, x_\ell, \ldots \) \( m \geq 1 \) and \( x \leq m x_i \ \forall n \)

Take \( z_j = y_j (m+1)^j + \cdots + y_j (m+1)^j ) = 0, \ldots, n \)

Then \( m+1 \leq m z \) \( \forall j \)

\( \Rightarrow \) \( x \leq z (m+1)^j + \cdots + \frac{x}{m} \)

\textbf{Theorem}

If \( A \) is separable and has B R Q, then \( A \) has the CFP \( \iff \) \( V(A) \) has the CFP

\textbf{Proof (Outline)}

(\( \Rightarrow \)) Assume \( V(A) \) does not have the CFP

\( V(A \otimes k) \)

Can find \( \mathbb{R} \) \( x, x_\ell \) \( m \geq 1 \), \( \{ x \} \) full, \( x \leq m x_i \ \forall n \)

but \( x \neq x_1, \ldots, x_\ell \ \forall k \)

We have \( x = [x_\ell], \ x = [x] \)

\( \mathbb{R}, \mathbb{P} \) can be taken to be pairwise orthogonal, set

\( \mathbb{E} \mathbb{P}_n = \mathbb{P}, \ \mathbb{E} \mathbb{P}_n = \mathbb{Q} \ \text{in} \ M(A \otimes k) \)

\( \{ x \} \) full \( \Rightarrow \) \( 1_{M(A \otimes k)} \leq \mathbb{Q} \)

since \( \mathbb{Q} \) is full, then \( \mathbb{Q} \) is properly infinite and \( \mathbb{P} = \mathbb{Q} \)

\text{where} \( \mathbb{E} \mathbb{P}_n = \mathbb{P} \)

\( [x] = [x] \leq m x = m [x] \Rightarrow \mathbb{P} \text{ in full} \Rightarrow \mathbb{P} \text{ in full} \Rightarrow \mathbb{P} \text{ in proper CFP}

\text{full and properly infinite} \Rightarrow \mathbb{Q} \leq P \)

But it is not true because we should have:

\( \frac{x}{m} = \left[ \frac{x}{m} \right] + \cdots + \left[ \frac{x}{m} \right] = \frac{x}{m} x \)
(⇐) Suppose \( V(A) \) has the CFP and \( P \) is a full projection in \( M(A \otimes K) \). Then \( \text{mp} = 1 \). 

As before \( p = I_p = Z_p \), and \( m_p = Z_p \). \( [p_1] = m \). \( [p_n] \)

Take \( e_n \) such that \( Z e_n = 1 \). Let \( x_n = \sum_{k=1}^{n} [p_k] \). Then \( \sum_{k=1}^{n} x_n \) is a full sequence in \( V(A) \).

\[
\text{mp} = \sum_{i=1}^{n} x_i = [p_{k_1}] + \ldots + [p_{k_{m+1}}] \\
= m \left( [p_{k_1}] + \ldots + [p_{k_{m+1}}] \right) \\
\]

\( V(A) \) has CFP \( \Rightarrow x_i \leq \frac{1}{2^1} + \ldots + \frac{1}{2^i} \)

\( x_2 \leq \frac{1}{2^2} + \ldots + \frac{1}{2^2} \)

\( x_4 \leq \frac{1}{2^4} + \ldots + \frac{1}{2^4} \)

This implies \( 1 \leq p \).

\( \Rightarrow P \) is properly infinite.

Theorem If \( A \) has RRO, and the CFP, and if \( P \) is a full projection in \( A \) such that \( \text{mp} \) in properly infinite, then \( P \) is properly infinite.

\textbf{Sketch:} \( A \) RRO and CFP \( \Rightarrow V(A) \) is refinement and has the property CFP.

\( u = [p] \) is an order-unit such that \( m u \) is properly infinite.

\( \Rightarrow \) If \( \text{order-unit} \ t_1 \) such that \( m t_1 \) are properly infinite.

\( t_1 + \ldots + t_k \leq u \)

\( u = m t_1 \)

\( \text{CFP} \ u = t_{k+1} + \ldots + t_k \)

\( 2u \leq t_1 + \ldots + t_k \text{ SU } \Rightarrow \) \( u \) is properly infinite.
Corollary (Zhang) If $A$ has RRO, the CFP and is simple, then $A$ is either stably finite or purely infinite.

Proof: If all elements in $V(A)$ are finite, then $A$ is stably finite.

If not, $\exists u \in V(A), u \neq 0$ infinite, then $A$ is properly infinite.

If $v \in V(A) \setminus \{0\}$, then $u \leq u v \Rightarrow u v$ is properly infinite.

If $u$ properly infinite, then $A$ is purely infinite. \qed

Question: If $RR(A) = 0$, does $A$ have the CFP?

Theorem: If $A$ has RRO, then $V(A)$ has the strong CFP if and only if all ideals in $A$ have the CFP.

Idea: $V(A)$ is refinement, and $I \mapsto V(I)$ is a lattice isomorphism.

The result follows if:

$V$ refinement: $V$ has the strong CFP if and only if all order ideals have the CFP.