This is lecture notes of Marius Dadarlat’s talks during the Master class on classification of $C^*$-algebras at the University of Copenhagen. The material he covered appears to be from the papers Continuous fields of $C^*$-algebras over finite dimensional spaces (Advances in Mathematics 222 (2009) 1850-1881) and Fiberwise KK-equivalence of continuous fields of $C^*$-algebras (J. K-Theory 3 (2009), 205-219).

1 First Lecture

In general $\mathcal{A}$ will denote a separable $C^*$-algebra and $X$ will denote a locally compact Hausdorff space.

**Definition 1.1** (Kasparov). $\mathcal{A}$ is a $C_0(X)$ algebra if a *-homomorphism from $C_0(X)$ to $Z(\mathcal{M}(\mathcal{A}))$ (the center of the multiplier algebra) is given (this means we can multiply elements from $C_0(X)$ with elements from $\mathcal{A}$) such that

$$C_0(X)\mathcal{A} = \mathcal{A}.$$  

Morphisms of $C_0(X)$-algebras $\gamma: \mathcal{A} \to \mathcal{B}$ commutes with the multiplication, that is $\gamma(fa) = f\gamma(a)$.

An equivalent definition would be that a surjective *-homomorphism going from $C_0(X) \otimes \mathcal{A}$ to $\mathcal{A}$, which is $\mathcal{A}$ linear, is given.

Another equivalent definition is that a continuous map from Prim$(\mathcal{A})$ to $X$ is given.

**Remark 1.2.** We can extend the map from $C_0(X)$ to $Z(\mathcal{M}(\mathcal{A}))$ to a map from $C_b(X)$ to $Z(\mathcal{M}(\mathcal{A}))$.

If $U \subseteq X$ is open, then by Cohens lemma $C_0(U)\mathcal{A} = C_0(U)\mathcal{A}$. This is an ideal in $\mathcal{A}$ and we denote it by $\mathcal{A}(U)$.

If $Y \subseteq X$ is closed, then we let $\mathcal{A}(Y)$ be the quotient $\mathcal{A}/\mathcal{A}(X \setminus Y)$. If $x \in X$ then the set $\{x\}$ is closed and $\mathcal{A}(x)$ denotes $\mathcal{A}(\{x\})$. This quotient is called the fiber at $x$ of $\mathcal{A}$.

We let $\pi_x$ denote the quotient map from $\mathcal{A}$ to $\mathcal{A}(x)$. If $a \in \mathcal{A}$ then we write $a(x)$ for $\pi_x(a)$. We have a *-homomorphism $\mathcal{A} \to \Pi_{x \in X}\mathcal{A}(x)$ given by $a \mapsto (\pi_x(a))_{x \in X}$.

**Lemma 1.3.** For all $a \in \mathcal{A}$ the map $x \mapsto \|\pi_x(a)\| = \|a(x)\|$ is upper semi-continuous.

**Proof.** We must show that for all $\alpha > 0$ the set

$$U = \{x \in X \mid \|\pi_x(a)\| < \alpha\}$$

is open. We have

$$\|\pi_x(a)\| = \inf \{\|a + z\| \mid z \in \mathcal{A}(X \setminus \{x\})\}$$

$$= \inf \{\|a + fb\| \mid f \in C_0(X \setminus \{x\}), b \in \mathcal{A}\}$$

$$= \inf \{\|a + (g - g(x))hb\| \mid g \in C_0(X), b \in \mathcal{A}, h \in C_0(X)\}.$$  

If $x \in U$ then $\|\pi_x(a)\| < \alpha$ so then there must exist $g \in C_0(X), b \in C_0(X), b \in \mathcal{A}$ such that

$$\|a + (g - g(x))hb\| < \alpha.$$
1.1 Examples

Example 1.6. \( A = C_0(X,D) = C_0(X) \otimes D \). This is called the trivial field. Note that \( A(x) \cong D \) for all \( x \in X \).

Example 1.7. Let \( D \) be a \( C^* \)-algebra and let \( \psi \in \text{End}(D) \). Let
\[
A = \{ (\alpha, d) \in C([0,1],D) \oplus D \mid \alpha(1) = \psi(d) \},
\]
\( A \) is \( C([0,1]) \)-algebra with multiplication of an \( f \in C([0,1]) \) given by
\[
f(\alpha,d) = (f\alpha,f(1)d).
\]
We will show that \( A(x) \cong D \) for all \( x \in X \). Observe that
\[
C_0([0,1] \setminus \{x\})A = \begin{cases} 
(\alpha,d) \in A \text{ with } \alpha(x) = 0, & \text{if } 0 \leq x < 1 \\
C_0([0,1],D) \oplus 0, & \text{if } x = 1
\end{cases}
\]
The extensions
\[
0 \rightarrow C_0([0,1],D) \rightarrow A \xrightarrow{(\alpha,d) \mapsto d} D \rightarrow 0
\]
and
\[
0 \rightarrow \{ (\alpha,d) \in A \mid \alpha(x) = 0 \} \rightarrow A \xrightarrow{\text{ev}} D \rightarrow 0
\]
show that indeed all \( A(x) \) are isomorphic to \( D \).

In this example the norm function is
\[
N(\alpha,d) = \begin{cases} 
\|\alpha(x)\|, & \text{if } 0 \leq x < 1 \\
\|d\|, & \text{if } x = 1
\end{cases}
\]
\( N \) is continuous if and only if \( \|d\| = \|\alpha(1)\| = \|\psi(d)\| \) for all \( d \in D \), that is \( N \) is continuous if and only if \( \psi \) is injective. So we have a continuous field \( C^* \)-algebra if and only if \( \psi \) is injective.

If \( \psi \) is injective, then
\[
A \cong \{ \alpha \in C([0,1],D) \mid \alpha(1) \in \psi(D) \},
\]
by an isomorphism that sends \( (\alpha,d) \) to \( \alpha \).
We will now try to find out when the field in the second example is trivial, i.e. when \( \mathcal{A} \cong C([0,1], D) \).

**Lemma 1.8.** Suppose that \( \psi \) is injective. Then \( \mathcal{A} \cong C([0,1], D) \) if and only if there exists a continuous map \( \theta : [0,1] \to \text{End}(\mathcal{A}) \) (where \( \text{End}(\mathcal{A}) \) has the point norm topology) such that \( \theta(s) \in \text{Aut}(\mathcal{A}) \) for all \( 0 \leq s < 1 \) and \( \theta(1) = \psi \).

**Proof.** Suppose \( \theta \) exists. By identifying \( \mathcal{A} \) with \( \{ \alpha \in C([0,1], D) \ | \ \alpha(1) \in \psi(D) \} \) we can define a map \( \eta : C([0,1], D) \to \mathcal{A} \) by

\[
\eta(\alpha)(s) = \theta(s)(\alpha(s)).
\]

This maps into \( \mathcal{A} \) since \( \eta(\alpha)(1) = \psi(\alpha(1)) \in \psi(D) \). One can check that \( \eta \) is an isomorphism of \( C([0,1]) \)-algebras.

For the other implication, assume that \( \eta : C([0,1], D) \to \mathcal{A} \) is an isomorphism of \( C([0,1]) \)-algebras. This gives us a family of injective homomorphisms \( (\eta_s)_s \in [0,1] \) from \( D \) to \( D \), such that \( s \mapsto \eta_s \) is a continuous map from \( [0,1] \) to \( \text{End}(D) \), \( \eta_s \) is an automorphism of \( D \) if \( 0 \leq s < 1 \) and \( \eta_1(D) = \psi(D) \). By the latter we can define \( \gamma \in \text{Aut}(D) \) by \( \gamma = \eta_1^{-1}\psi \). We now define \( \theta : [0,1] \to \text{End}(D) \) by \( \theta(s) = \eta^{-1}_s \gamma \). We note that if \( 0 \leq s < 1 \) then \( \theta(s) \in \text{Aut}(D) \) and that \( \theta(1) = \eta_1 \eta^{-1}_1 \psi = \psi \).

We can say more if we know more about \( D \).

**Corollary 1.9.** Suppose \( D \) is a stable Kirchberg algebra. Then \( \mathcal{A} \) is trivial if and only if \( [\psi] \in KK(D, D)^{-1} \).

**Proof.** Suppose \( [\psi] \in KK(D, D)^{-1} \). Then by the Kirchberg-Phillips theorem, there exists an automorphism \( \phi \) of \( D \) and a family of unitaries \( u_s \in \mathcal{U}(1C + D), 0 \leq s < 1 \) such that \( [\psi] = [\phi] \) and

\[
\lim_{s \to 1} \|u_s \phi(d) u_s^* - \psi(d)\| = 0,
\]

for all \( d \in D \). Now the map \( \theta : [0,1] \to \text{End}(D) \) given by

\[
\theta(s)(d) = \begin{cases} 
  u_s \phi(d) u_s^*, & \text{if } 0 \leq s < 1, \\
  \psi(d), & \text{if } s = 1
\end{cases}
\]

and the above lemma combines to give the desired conclusion.

The converse is also true, since, by lemma 1.8, we then have that \( \psi \) is homotopic to an automorphism.

**Remark 1.10.** By the corollary we get: If \( \psi : K_*(D) \to K_*(D) \) is not bijective then \( \mathcal{A} \) is not a trivial field.

As a variation on this example we can fix \( x \in (0,1) \) and define

\[
\mathcal{A} = \{ \alpha \in C([0,1], D) \ | \ \alpha(x) \in \psi(D) \} = \{ (\alpha, d) \in C([0,1], D) \otimes D \ | \ \alpha(x) = \psi(D) \}.
\]

The short exact sequence

\[
0 \to C_0([0,1] \setminus \{x\}, D) \to \mathcal{A} \xrightarrow{\pi_x} D \to 0
\]
where \( \pi_z \) maps \((\alpha, d)\) to \(d\), is split with the split \( s: \mathcal{D} \to \mathcal{A} \) given by \( s(d) \mapsto (\psi(d), d) \) (\( \psi(d) \) means a function constantly taking that value). Hence we get a short exact sequence of \( K_0 \)-groups

\[
0 \to K_0(C_0([0,1] \setminus \{x\}), \mathcal{D}) \to K_0(\mathcal{A}) \xrightarrow{(\pi_z)_*} K_0(\mathcal{D}) \to 0
\]

Since \( K_0(C_0([0,1] \setminus \{x\}), \mathcal{D}) = 0 \), we get that \((\pi_z)_*\) is an isomorphism. It must have inverse \( s_* \). Consider now some point \( y \neq x \). The quotient map \( \pi_y: \mathcal{A} \to \mathcal{A}(y) \) is given by \( \pi_y((\alpha, d)) = \alpha(x) \). Hence we have a map

\[
(\pi_y)_*: K_0(\mathcal{A}) \to K_0(\mathcal{A}(y)) \cong K_0(\mathcal{D}).
\]

We have \((\pi_y)_* s_* \equiv \psi_*: K_0(\mathcal{D}) \to K_0(\mathcal{D}). \) Thus \( \psi_* \) is not bijective. This implies that \( \mathcal{A} \) is not trivial since \( K_0(\mathcal{A}) \cong K_0(\mathcal{A}(y)) \).

**Example 1.11** (Dadarlat & Elliott). Let \( \mathcal{D} \) be a unital Kirchberg algebra such that \( K_0(\mathcal{D}) = \mathbb{Z} \oplus \mathbb{Z}, \ [1_\mathcal{D}] = (1,0) \) and \( K_1(\mathcal{D}) = 0 \). Set

\[
\mathcal{B} = \mathcal{D}^{\otimes \infty} = \lim \left( \mathcal{D} \xrightarrow{d \otimes 1_\mathcal{D}} \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \to \cdots \right)
\]

We will construct a continuous field \( \mathcal{A} \) over \([0,1]\) such that \( \mathcal{A}(x) \cong \mathcal{B} \) for all \( x \in [0,1] \) and such that for all closed intervals \( I = [a, b] \subseteq [0,1], \ a < b, \)

\[
\mathcal{A}(I) \not\cong C(I, \mathcal{B}).
\]

Thus \( \mathcal{A} \) has all fibers isomorphic but is not locally trivial at any point.

Let \( \psi \) be an endomorphism of \( \mathcal{D} \) such that \( K_0(\psi) = \psi_*: K_0(\mathcal{D}) \to K_0(\mathcal{D}) \) is given by

\[
\psi_* = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).
\]

Let \( (x_n) \) be a dense sequence in \([0,1]\) with \( x_i \neq x_j \) if \( i \neq j \). Define

\[
\mathcal{D}_n = \{ \alpha \in C([0,1], \mathcal{D}) | \alpha(x_n) \in \psi(\mathcal{D}) \}.
\]

Then \( \mathcal{D}_n(x) \cong \mathcal{D} \) for all \( x \in [0,1] \). Now define \( \mathcal{A} \) by

\[
\mathcal{A} = \mathcal{D}^{\otimes \infty} \cong \lim (\mathcal{D}_1 \otimes \mathcal{D}_2 \otimes \cdots \otimes \mathcal{D}_n),
\]

where all tensor products are taken over \( C[0,1] \). That is

\[
\mathcal{D}_1 \otimes \cdots \otimes \mathcal{D}_n \cong \{ \alpha: [0,1] \to \mathcal{D}^{\otimes n} \mid \text{for } 1 \leq i \leq n \alpha(x_i) \in E_i \},
\]

where

\[
E_i = \mathcal{D} \otimes \mathcal{D} \otimes \cdots \otimes \psi(\mathcal{D}) \otimes \cdots \otimes \mathcal{D},
\]

with the \( \psi(\mathcal{D}) \) at the \( i \)th place.

For any \( I = [a, b] \subseteq [0,1] \) there exists an \( x \notin \{x_1, x_2, \ldots \} \) such that

\[
(\pi_x)_*: K_0(\mathcal{A}(I)) \to K_0(\mathcal{D}^{\otimes \infty})
\]

is not injective. This shows that there can be no \( I \) such that \( \mathcal{A}(I) \) is trivial, since for such an \( I \) all the maps \((\pi_x)_*\) would be isomorphisms.

**Theorem 1.12.** Let \( \mathcal{D} \) be a stable Kirchberg algebra. Let \( \mathcal{A} \) be a stable continuous field of stable Kirchberg algebras over a finite dimensional compact Hausdorff space. Suppose there exists \( \sigma \in KK(\mathcal{D}, \mathcal{A}) \) such that

\[
[\pi_x] \sigma \in KK(\mathcal{D}, \mathcal{A})^{-1},
\]

for all \( x \in X \). Then \( \mathcal{A} \cong C(X, \mathcal{D}) \).
2 Second Lecture

Example 2.1 (Due to Hirshberg, Rørdam & Winter). Let \( f \in M_2(C(S^2)) \) be the Bott projection and let \( e = 1_{C(S^2)} \). Denote by \( p \) the projection in \( M_3(C(S^2)) \) given by

\[
p = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}.
\]

For any \( x \in S^2 \) \( p(x) \) is a rank 2 projection. Define

\[
A = \otimes_{n=1}^{\infty} p M_2(C(S^2)) p.
\]

This is a continuous field \( C^* \)-algebra over \( \Pi_{n=1}^{\infty} S^2 \) with fibers

\[
\otimes_{n=1}^{\infty} M_2(C) = UHF(2^\infty).
\]

So all the fibers have \( \mathbb{Z}[[\frac{1}{2}]] \) as their \( K_0 \) group. We will now determine \( K_0(A) \). To ease the notation we put \( B = p M_3(C(S^2)) p \). Then \( K_0(B) = K_0(C(S^2)) \). Consider the map from \( C \oplus C \) to \( B \) that sends \( (0,1) \) to \( e \) and \( (0,1) \) to \( f \). It is a unital \( \ast \)-homomorphism and it induces a bijection on \( K_0 \) and \( K_1 \). Hence it is a \( KK \)-equivalence. So we get a \( KK \)-equivalence

\[
\otimes_{n=1}^{\infty} (C \oplus C) \to \otimes_{n=1}^{\infty} B = A,
\]

which sends \([1]\) to \([1]\). Letting \( K \) denote the set \( \Pi_{n=1}^{\infty} \{0,1\} \) (Cantor set) we then get a unital \( \ast \)-homomorphism from \( C(K) \) to \( A \) that induces a \( KK \)-equivalence mapping the class of the unit of \( A \) to the class of the function constantly taking the value 1. Hence

\[
K_0(A) \cong K_0(C(K)) = C(K,\mathbb{Z}).
\]

We now consider the \( C^\ast \)-algebra \( A \otimes O_3 \) (\( O_3 \) is the Cuntz-algebra with \( K_0(O_3) = \mathbb{Z}/2\mathbb{Z} \) and \( K_1(O_3) = 0 \)). We have that

\[
K_0(A \otimes O_3) = C(K,\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} = C(K,\mathbb{Z}/2\mathbb{Z}).
\]

If we let \( x \in \Pi_{n=1}^{\infty} S^2 \) be given, then we can calculate the fiber at \( x \) as

\[
(A \otimes O_3)(x) \cong A(x) \otimes O_3 \cong UHF(2^\infty) \otimes O_3.
\]

So all the fibers are Kirchberg algebras, and we can compute their \( K \)-theory as

\[
K_0(UHF(2^\infty) \otimes O_3) = \mathbb{Z} \left[ \frac{1}{2} \right] \otimes \mathbb{Z}/2\mathbb{Z} = 0,
\]

and

\[
K_1(UHF(2^\infty) \otimes O_3) = 0.
\]

Hence all the fibers are \( O_2 \). However \( A \otimes O_3 \) is not a trivial continuous field \( C^\ast \)-algebra as it has \( K_0(A \otimes O_3) \cong C(K,\mathbb{Z}/2\mathbb{Z}) \neq 0 \).

The space used in the example to get at non-trivial field with all fibers isomorphic to \( O_2 \) were quite large. The following theorems tells us that small spaces can not exhibit that form of behavior.
Theorem 2.2. Let \( A \) be a separable unital continuous field over a compact Hausdorff space \( X \) of finite covering dimension. If \( A(x) \cong \mathcal{O}_2 \) for all \( x \in X \) then \( A \cong C(X, \mathcal{O}_2) \).

Theorem 2.3 (Dadarlat-Mayer). Suppose \( A \) is a separable continuous field of nuclear C*-algebras over a compact Hausdorff space \( X \). Suppose that for all ideals \( \mathcal{J} \) in \( A \) we have \( KK(\mathcal{J}, \mathcal{J}) = 0 \). Then

\[
A \cong_{KK} C(X, \mathcal{O}_2).
\]

If \( A(x) \) is a Kirchberg algebra for all \( x \in X \) then \( A \otimes \mathcal{O}_2 \otimes \mathcal{K} \cong A \otimes \mathcal{O}_2 \otimes \mathcal{K} \).

The rest of the lecture was devoted to giving an explanation of why the first theorem is true.

The key point is that \( \mathcal{O}_2 \) is semiprojective, which means that it has good perturbation properties.

Definition 2.4. A separable C*-algebra \( A \) is semiprojective, if for any C*-algebra \( B \) and any increasing chain of ideals \( \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \) in \( B \) and any *-homomorphism \( \phi: A \to B/\mathcal{J} \), where \( \mathcal{J} = \cup_n \mathcal{J}_n \), there exists an \( n \in \mathbb{N} \) and a *-homomorphism \( \psi: A \to B/\mathcal{J}_n \) such that the following diagram commutes

\[
\begin{array}{cccccc}
B/\mathcal{J}_1 & \longrightarrow & B/\mathcal{J}_2 & \longrightarrow & \cdots & \longrightarrow & B/\mathcal{J}_n & \longrightarrow & \cdots & \longrightarrow & B/\mathcal{J} \\
& & & & & & & & & & \\
& & & & & & & & & & \downarrow\phi \\
& & & & & & & & & & \downarrow\psi \\
A & \longrightarrow & B/\mathcal{J}_n \\
& & & & & & & & & & \uparrow\phi \\
& & & & & & & & & & \phi
\end{array}
\]

The definition is equivalent to requiring that for all \( B \) and \( \mathcal{J}_n \) as above, the canonical map from \( \lim_{\longrightarrow} \text{hom}(A, B/\mathcal{J}_n) \) to \( \text{hom}(A, B/\mathcal{J}) \) is surjective. We say that an algebra is weakly semiprojective if the the map has dense image in the point norm topology.

An algebra is said to be KK-semiprojective if the canonical map from the inductive limit \( \lim_{\longrightarrow} KK(A, B/\mathcal{J}_n) \) to \( KK(A, B/\mathcal{J}) \) is surjective. It turns out that this is equivalent to saying that the map is a bijection.

Example 2.5 (Examples of semiprojective C*-algebras). If \( A \) is a Kirchberg algebra satisfying the UCT, then \( A \) is weakly semiprojective if and only if \( K_1(A) \) is finitely generated.

If \( K_1(A) \) further is torsion free, then \( A \) is semiprojective. It is an open question whether we need \( K_1(A) \) to be torsion free.

From now on we will focus on a separable unital continuous field with fibers \( \mathcal{O}_2 \) over \([0, 1]\). Fix \( x \in [0, 1] \) and define \( U_n = [x - 1/n; x + 1/n] \cap [0, 1] \). Then \( \lim_{\longrightarrow} A(U_n) = A(x) \) (non-trivial fact). By the semiprojectivity of \( \mathcal{O}_2 \) we can get an \( n \) and a unital *-homomorphism \( \psi \) such that

\[
\begin{array}{cccccc}
A(U_1) & \longrightarrow & A(U_2) & \longrightarrow & \cdots & \longrightarrow & A(U_n) & \longrightarrow & \cdots & \longrightarrow & A(x) \\
& & & & & & & & & & \downarrow\psi \\
& & & & & & & & & & \psi \\
& & & & & & & & & & \mathcal{O}_2
\end{array}
\]
commutes. Moreover, given any finite set $\mathcal{F} \subseteq \mathcal{A}$ and any $\varepsilon > 0$ we can find a finite set $\mathcal{H} \subseteq \mathcal{O}_2$ such that the isomorphism from $\mathcal{O}_2$ to $\mathcal{A}(x)$ maps $\mathcal{H}$ to $\pi(x)(\mathcal{F})$ and such that $\psi(\mathcal{H}) \supseteq \varepsilon \pi(U^n(\mathcal{F}))$. We get the latter since $\lim_{\to A} A(U^n) = A(x)$.

We can extend $\psi$ to $\tilde{\psi}: C(U^n) \otimes O_2 \to A(U^n)$ by $C(U^n)$ linearity, and we will have $\pi(U^n(\mathcal{F})) \subseteq \varepsilon \tilde{\psi}(O_2)$.

Doing this for other $x$ we get closed sets $U_k$ covering all of $[0, 1]$ and maps from $C(U_k) \otimes O_2$ into $A(U_k)$ as above. The trick is the to paste them together. For that we use elementary fields.

Suppose we have 3 unital $C^*$-algebras $E_1, D, E_2$, and $*$-homomorphisms $\gamma_1: D \to E_1$ and $\gamma_2: D \to E_2$. Then the algebra

$$A = \{(\alpha, \beta, \gamma) \mid \alpha \in C([0, 1], E_1), \beta \in C([1, 2], D), \gamma \in C([2, 3], E_2) \text{ such that } \alpha(1) = \gamma_1(\beta(1)), \gamma_2(\beta(2)) = \gamma(2)\}$$

is built from elementary fields.

In our case we then have that for all finite sets $\mathcal{F} \subseteq \mathcal{A}$ and all $\varepsilon > 0$ there exists an elementary field $E \subseteq \mathcal{A}$ such that $E(x) \cong O_2$. The gluing morphisms $\gamma: O_2 \to O_2$ are $KK$-equivalent. We have seen that $E \cong C([0, 1], O_2)$. The idea is then to write $A$ as an inductive limit of elementary fields, and show that things extend nicely.
3 Third Lecture

The main theme of this lecture was the structure of continuous fields, restricted to the case where the fibers are Kirchberg algebras satisfying the UCT.

Definition 3.1. A sequence of sub-$C^*$-algebras $(D_n)$ of a $C^*$-algebra $D$ is called exhaustive if for all finite subsets $\mathcal{F} \subseteq D$ and all $\varepsilon > 0$ there exists $n$ such that $\mathcal{F} \subseteq_{\varepsilon} D_n$.

Note that we do not assume $D_1 \subseteq D_2 \subseteq \cdots$. If we did, then $(D_n)$ would be exhaustive if and only if $\bigcup_n D_n = D$.

We will now define $n$-pullbacks. They are continuous fields obtained by gluing $n+1$ locally trivial fields together.

Definition 3.2. Suppose we have $X = Y_0 \cup Y_1 \cup \cdots \cup Y_n$, where each $Y_i$ is closed. Suppose also that we have locally trivial $C(Y_i)$ algebras $E_i$ and fiberwise injective $C(Y_i \cap Y_j)$ maps $\gamma_{ij}: E_i|_{Y_i \cap Y_j} \to E_j|_{Y_i \cap Y_j}$ such that

$$(\gamma_{jk})_x \circ (\gamma_{ij})_x = (\gamma_{ik})_x,$$

for all $x \in Y_i \cap Y_j \cap Y_k$, $i \leq j \leq k$.

Then we define the $n$-pullback $E$ as

$$E = \{(e_0, \ldots, e_n) \in E_0 \oplus \cdots \oplus E_n \mid e_j(x) = (\gamma_{ij})_x(e_i(x)) \text{ for all } x \in Y_i \cap Y_j\}.$$

Theorem 3.3. Let $A$ be a separable nuclear continuous $C(X)$-algebra over a compact metrizable space $X$ of finite covering dimension, $\dim(X) = n$.

Suppose each fiber $A(x)$ is a Kirchberg algebra which is $KK$-equivalent to a commutative $C^*$-algebra (i.e. satisfies the UCT). Then $A$ admits an exhaustive sequence $(A_n)$, where each $A_n$ is an $n$-pullback. Moreover, if $K_1(A(x))$ is torsion free for all $x$, then one can get $A_1 \subseteq A_2 \subseteq \cdots$. Hence, $A = \bigcup_n A_n$.

Outline. Fix a fiber $A(x)$. Write $A(x) = \lim_k D_k$, where the $D_k$ are Kirchberg algebras with finitely generated $K$-theory. By choice of the $D_k$ they are weakly semiprojective. So for a given $k$ we can find a closed neighborhood $V$ of $A$ and an approximate lifting $\rho: D_k \to A(V)$ such that the diagram

$$\begin{array}{ccc}
A(V) & \to & A(x) \\
\downarrow & & \downarrow \\
D_k & \to & A(x)
\end{array}$$

commutes. Using these liftings in a clever way, we can get $n$-pullbacks.

If $K_1$ is torsion free then we can choose the $D_k$ such that they also have torsion free $K_1$. Then they will be semiprojective, and the liftings will be exact.

You do not need Kirchberg algebras. One only needs that every fiber is a limit of direct sums of simple semiprojective algebras, e.g. $AF$-algebras.
What is \( KK_X \)? \( \mathcal{A} \) and \( \mathcal{B} \) two \( C(X) \)-algebras, \( X \) a compact Hausdorff space, then if \( \phi \) is a \( C(X) \)-linear \( * \)-homomorphism it will induce a class \([\phi] \in KK_X(\mathcal{A}, \mathcal{B})\). \( KK_X \) is a sort of fiberwise \( KK \)-theory. It consists of Fredholm-Kasparov bimodules \( _{\mathcal{A}}E_{\mathcal{B}} \) subject to the condition \((fa)\xi b = (a)\xi(fb)\) for all \( a \in \mathcal{A}, b \in \mathcal{B}, \xi \in E, f \in C(X) \).

Observe that while
\[
KK(C_0((0; 1]), C_0((0; 1])) = 0,
\]
we have
\[
KK_{[0, 1]}(C_0((0; 1]), C_0((0; 1])) = \mathbb{Z}[id],
\]
since one cannot contract fiberwise.

Let \( A \) and \( B \) be separable nuclear continuous field over \( X \) compact metrizible space.

\( \Rightarrow \)

Recall that if \( A \) be separable nuclear continuous \( C(X) \)-algebras with injections
\[
\mathcal{B}_1 \xrightarrow{\sigma_1} \mathcal{B}_2 \xrightarrow{\sigma_2} \cdots
\]
and \( B = \lim_{\to \mathcal{B}_n}. \) Then we have the following short exact sequence
\[
0 \to \lim_{\to \mathcal{B}_1} KK_X^1(\mathcal{B}_i) \to KK_X(\mathcal{B}, A) \to \lim_{\to \mathcal{B}_1} KK_X(\mathcal{B}_i, A) \to 0
\]
Recall that if
\[
G_1 \xrightarrow{\lambda_1} G_2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{i+1}} G_{i+1} \xrightarrow{\lambda_{i+2}} \cdots
\]
and we define a map \( \text{id} - S : \Pi_{i=1}^\infty G_i \to \Pi_{i=1}^\infty G_i \) by
\[
(g_1, g_2, \ldots) \mapsto (g_1 - \lambda_1(g_2), g_2 - \lambda_2(g_3), \ldots),
\]
then
\[
\ker(\text{id} - S) = \lim_{\to \mathcal{B}_1}(G_i, \lambda_i)
\]
and
\[
\coker(\text{id} - S) = \lim_{\to \mathcal{B}_1}(G_i, \lambda_i).
\]

**Proposition 3.4.** Let \( A \) be a separable and nuclear continuous field over a compact metrizible space \( X \). Then there exists \( A^\# \) a separable nuclear continuous field over \( X \) with \( A^\#(x) \) Kirchberg for all \( x \in X \) and \( C(X) \)-linear map \( \phi : \mathcal{A} \hookrightarrow A^\# \) such that
\[
[\phi] \in KK_X(A, A^\#)^{-1}.
\]

**Theorem 3.5.** Let \( \mathcal{A}, \mathcal{B} \) be separable nuclear continuous \( C(X) \)-algebras over a finite dimensional compact metrizible space \( X \). Let \( \sigma \in KK_X(\mathcal{A}, \mathcal{B}) \) (e.g. \( \sigma = [\phi] \) where \( \phi \) is \( C(X) \) linear map from \( \mathcal{A} \) to \( \mathcal{B} \)). Suppose that for all \( x \in X \) we have \( \sigma_x \in KK(\mathcal{A}(x), B(x))^{-1} \), then \( \sigma \in KK_X(\mathcal{A}, \mathcal{B}) \).

**Proof.** Consider the mapping cone
\[
C_\phi = \{(f, a) \mid f \in C_0((0; 1], B), a \in A, f(1) = \phi(a)\}.
\]
It is a continuous \( C(X) \)-algebra with fibers \( (C_\phi)_x = C_{\phi_x} \). We have a Puppe sequence
\[
KK_X(C, \phi) \to KK_X(C, A) \to KK_X(C, B) \to KK_X^1(C, C_\phi)
\]
Corollary 3.6. Let $\mathcal{B}$ be as in the previous theorem. Suppose $\mathcal{D}$ is a separable nuclear $C^*$-algebra with an element $\sigma \in KK(\mathcal{D}, \mathcal{B})$ such that $\sigma_x \in KK(\mathcal{D}, \mathcal{B}(x))^{-1}$ for all $x$, then $C(X) \otimes \mathcal{D} \sim_{KKX} \mathcal{B}$.

Proof.

$$KK_X(C(X) \otimes \mathcal{D}, \mathcal{B}) \cong KK(\mathcal{D}, \mathcal{B}).$$

Corollary 3.7. Let $\mathcal{A}$ be a unital separable continuous field over a finite dimensional compact metrizable space $X$. Suppose $\mathcal{A}(x) \cong \mathcal{O}_n$ for all $x$ (a fixed, $2 \leq n \leq \infty$). Then

1. If $n = 2$ or $n = \infty$ then $\mathcal{A} \cong C(X) \otimes \mathcal{O}_n$.

2. In all cases $\mathcal{A}$ is locally trivial. Moreover $\mathcal{A} \cong C(X) \otimes \mathcal{O}_n$ if and only if $(n - 1)[1, \mathcal{A}] = 0$ in $K_0(\mathcal{A})$.

"Proof". Locally trivial: Fix $x_0 \in X$. It suffices to find $V$ a closed neighborhood of $x_0$ such that $C(V) \otimes \mathcal{O}_n \sim_{KKX} \mathcal{A}(V)$ and $\sigma_x[1] = [1]$. For that it suffices to find a closed neighborhood $V$ and a unital $*$-homomorphism $\phi: \mathcal{O}_n \to \mathcal{A}(V)$. Indeed if that is the case, then $[\phi] \in KK(\mathcal{O}_n, \mathcal{A}(V))$, and if $x \in X$ then $\phi_x \in KK(\mathcal{O}_n, \mathcal{A}(x))^{-1}$ since the map $KK(\mathcal{O}_n) \to KK(\mathcal{O}_n, \mathcal{A}(x))$ is bijective (it is unital). As there is no $K_1$ equivalence.

To get such a $V$, we consider a decreasing set of neighborhoods $V_1 \supseteq V_2 \supseteq \cdots$ such that $\cap_m V_m = \{x\}$. Then, by the semiprojectivity of $\mathcal{O}_n$, we get an $n$ and a unital $*$-homomorphism $\psi$ such that the following diagram commutes

$$\begin{array}{ccc}
A(V_1) & \longrightarrow & A(V_2) \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow \psi & & \downarrow \\
\mathcal{O}_n & & A(x)
\end{array}$$

To get global triviality we need to find unital $\phi: \mathcal{O}_n \to \mathcal{A}$. For that it is enough to find a map $K_0(\mathcal{O}_n) \to K_0(\mathcal{A})$ mapping $[1]$ to $[1]$ and then lift it up to the level of algebras.