\[ \text{Last time} \quad \text{Tucker-Dobr(V+VI)} \]

- \( \Delta(G) = \left\{ N \triangleleft G \mid G/C_N(G) \text{ is amenable} \right\} \)

- \( \Gamma(G) = \left\{ N \triangleleft G \mid N \times G \cap N \text{ amenable} \right\} \)

- Dani's Lemma

**Def (173 Kegel-Wehrfritz)**

A group \( G \) is said to satisfy **m.c.c.** (minimal condition on centralizers) if \( \{ C_G(B) \mid B \leq G \} \) satisfies D.C.C. (descending chain condition) (i.e., for any \( B \leq G \), \( \exists B_0 \leq B \) finite with \( C_G(B_0) = C_G(B) \))

**Prop 13**

(*Linear groups satisfy m.c.c.*)

**Pf** If \( H \leq G \) then \( C_H(B) = C_G(B) \cap H \)

so m.c.c. passes to subgroups.

So suffices to show m.c.c. for \( \text{GL}_n(F) \)

If \( B \leq \text{GL}_n(F) \),

\[ C_{\text{GL}_n(F)}(B) = \left\{ x \in \text{M}_n(F) \mid xb = bx \quad \forall x \in B \right\} \]

is exactly the set of solutions to the system of linear equations

\[ (xb - bx) = 0, \ b \in B \]

By linear alg / Hilbert basis theorem, \( \exists B_0 \subseteq B \) finite

s.t.

\[ C_{\text{GL}_n(F)}(B) = C_{\text{GL}_n(F)}(B_0) \]

Thus \( C_{\text{GL}_n(F)}(B) = C_{\text{GL}_n(F)}(B_0) \) \( \square \)
Remark
BS(m,n) not m.c.c. when |m1|, |m1| > 1 & |m1| + |n1| (so they are not linear)

Lemma
Suppose G is m.c.c. Then

(i) \( AC(G) = I(G) \).
(ii) \( G/C_G(Ac(G)) \) is amenable.
(iii) Every conjugation invariant mean on \( G \) lives on \( AC(G) \).

<pf> We'll show

(iii') Every conj-inv mean \( m \) on \( G \)
\[ \exists N \triangleleft G \quad G/C_G(N) \text{ amenable, s.t. } m(N) = 1 \]
(this implies (iii')).

Consider conjugation action \( G \acts G \)
\[ \{ GB \mid B \in G \} = \{ C_G(B) \mid B \in G \} \]
satisfies D.C.C.

So by Dani's Lemma, \( G/G_0 \) is amenable,
where \( G_0 = \{ g \in G \mid m(C_G(g)) = 1 \} \)

Let \( N = C_G(G_0) \) be m.c.c. \( \exists F \) finite
with \( N = C_G(F) = \bigcap_{g \in F} C_G(g) \)
by finite additivity \( m(N) = 1 \).
Also \( C_G(N) = G_0 \) so \( G/C_G(N) \) is amenable.
(i) & (ii) By yesterday
\[ \exists m \in M(I(G)) \text{ which is } I(G) \times G \text{- inv.} \]

By (iii) \[ \exists N \triangleleft G \text{ } G/C_G(N) \text{ amenable } & \text{ } m(N) = 1 \]

Then \[ N \leq AC(G) \leq I(G) \]

\[ m: \text{ left-invariant under } I(G), \text{ } m(N) = 1 \]

\[ \Rightarrow N = AC(G) = I(G) \square \]

Theorem Let \( G \) be m.c.c. Then TFAE

(1) \( G \) is inner-amenable.
(2) \( AC(G) = I(G) \) is infinite.
(3) \( \exists \) short exact sequence

\[ 1 \to N \to G \to K \to 1 \]

\[ K \text{ is amenable, and either } \mathbb{Z}(N) \text{ is infinite or} \]

\[ N = LM, \text{ where } L, M \triangleleft G \text{ commuting, } L \cap M \]

\[ \text{is finite, } M: \text{ infinite amenable.} \]

\[ \langle \text{Pf} \rangle (3) \Rightarrow (2) \Rightarrow (1) \checkmark \]

\[ (1) \Rightarrow (2) \text{ if } m \text{ is atomless, conj-inv on } G \]

Then \[ m(AC(G)) = 1 \]

\[ \Rightarrow AC(G) \text{ is infinite.} \]

(2) \( \Rightarrow \) (3) Let \[ N = C_G(AC(G))AC(G) \]

Then \[ K = G/N \text{ is amenable.} \]

Case 1: \[ C_G(AC(G)) \cap AC(G) \text{ is infinite.} \]

\[ = \mathbb{Z}(N) \]
Case 2 \[ C_2 \ (A C_2 G_2) \cap \ M \] is finite. M.

\[ \square \]

Cost of actions (Levitt)

Let \( G \times (x, \mu) \) be a probability measure preserving (pmp) action. A measurable graph \( G \) on \( X \) is a graphing of undirected no self-loop the action \( G \times (x, \mu) \) if the connected components of \( G \) are precisely the orbits of the action.

The cost of \( G \) is

\[ \text{Cost} (G) = \frac{1}{2} \int_x \text{deg}_G (x) \, d\mu(x) \]

\[ \text{Cost} (G \times (x, \mu)) = \inf \{ \text{Cost} (G) \mid G \text{ is a graphing of } G \times (x, \mu) \} \]

\( G \) is said to have fixed price if

\[ \text{Cost} (G \times (x, \mu)) = \text{Cost} (G \times (y, \nu)) \]

for any two free pmp actions of \( G \).

Fixed price conjecture

Every (countable) group has fixed price.

This is known to hold for many groups.

- Infinite amenable groups \( (f.p. = 1) \) (Levitt)
- Finite groups \( (f.p. = 1 - \frac{1}{|G|}) \)
- Free groups \( F_n \) \( (f.p. = n) \) (Gaboriau '00)
\[ \exists (G) \text{ infinite (s.p. = 1)} \]
\[ H \times K \text{, infinite amenable group, (s.p. = 1)} \]

**Thm (T-D)**
(Inner-amenable groups have fixed price = 1)

\[ \beta_1^{(s)}(G) \leq \text{Cost}(G) = \inf \left( \text{Cost}(G \cup X) \right) - 1 \]  
(Gaboriau)

Open: Is this an equality?

\[ \beta_1^{(s)}(\text{inner-amen}) = 0 \]  
(T-D)

(Chifan-Sinclair-Udrea)

(Ozawa?)

**Def (Popa)**
A subgroup \( H \leq G \) is called \( s \)-normal if the set
\[ \{ g \in G \mid gHg^{-1} \cap H \text{ is infinite} \} \]
generates \( G \).

**\( s \)-normality Lemma (Gaboriau-Furman)**
(If \( H \) is \( s \)-normal in \( G \) then

\[ \text{Cost}(G \cup (x, \mu)) \leq \text{Cost}(H \cup (x, \mu)) \]

\[ \text{finite} \]

**Lemma** If \( M \) is a normal subgroup of \( G \), then

\[ \sup_{G \cup (x, \mu)} \left\{ \text{Cost}(G \cup X) \right\} \leq \sup_{G/H \cup (x, \mu)} \left\{ 1 + \frac{\text{Cost}(G/H \cup X)}{1M} \right\} \]

\[ 1M (\text{Cost}(G) - 1) \leq \text{Cost}(G/M) - 1 \]
Let $G$ be inner-amenable. Let $H$ be a non-amenable subgroup of $G$. Then $\exists K \leq G$ with $H \triangleleft G$.

**Important Lemma**

$G \bowtie X$ amenable $G_x$ amenable $\forall x \in X$

$\Rightarrow G$ amenable.

**Improvement**

$G$ non-amenable, $G \bowtie X$ amenable

with invariant mean $m \in M(X)$.

$\Rightarrow m(\{x \in X \mid G_x \neq \text{amenable}\}) = 1$

$x_0$

Assume $m(x_0) < 1$ then $Y := X \setminus x_0$

$G \bowtie Y$ amenable $G_y$ amenable $\forall y \in Y$

$\Rightarrow G$ amenable, contradiction.

**(pf) (of Prop)** Fix $m^i : G\text{-conj} \text{ inv, atom-less}$

Then $H \bowtie G$ is amenable w/ inv-mean $M$.

By improvement $\{g \mid C_H(g) \text{ is non-amenable} \}$ has measure 1.

Let $K = \langle H, \{g \in G \mid C_H(g) \text{ is non-amenable} \} \rangle$

Then $H \triangleleft K$ since $gHg^{-1} \cap H \geq C_H(g)$ is infinite for all $g \in G$ s.t. $C_H(g)$ is amenable.

$m(K) = 1 \Rightarrow m(gKg^{-1}) = 1$

$\Rightarrow m(K \cap gKg^{-1}) = 1$

$m$: atom-less

so $K \cap gKg^{-1}$ is infinite.
Goal now is to find $(C_{H_n})_{n=1}^{\infty}$ of non-amenable subgroups of $G$ with $\text{Sup Cost } (C_{H_n}) \xrightarrow{n \to \infty} 1$. 

Since then by $g$-normality Lemma we get $\text{Sup Cost } (G) \leq 1$.

**Prop**

Let $G$ be a non-amenable, inner-amenable group.

Then either

1) $\exists$ infinite amenable subgroup $K \leq G$ with $C_g(K)$ non-amenable, 

or 2) $\exists$ sequence of finite subgroups $M_n (n \geq 1)$ with $|M_n| \to +\infty$ and $C_g(M_n)$ is non-amenable for all $n$.

**Proof**

Atomless\footnote{conj. invariant mean on $G$}.

Fix atomless mean on $G$.

By Improvement, (since $G$ is non-amenable)

\[ G \overset{\text{coj}}{\underset{G}{\amenable}} \]

\[ M : \text{atomless} \]

\[ H_1 \]

\[ \exists g \in G \setminus \{1\} \text{ with } C_g(g_1) \text{ is non-amenable}. \]

If $\langle g_1 \rangle$ is infinite then (2) holds and we're done.

Otherwise, $\text{m}(\langle g_1 \rangle) = 0$. \footnote{finite set}

So non-amenable of $C_g(g_1) = H_1$ and the improvement, we can find $g_2 \in H_1 \setminus \langle g_1 \rangle$ with $C_{H_1}(g_2)$ non-amenable.

Keep going until we set $|M_n| = \infty$ for $\text{some } n \text{ of } M_1 \supset M_2 \supset M_3 \supset \ldots \quad 1_{M_n} \to \infty$ $C_g(M_1)$ non-amenable.
If \( \mu_n \) is finite, this procedure stops, and we get \((\mu_n)_{n=1}^\infty\) as in (2).

If \( |\mu_n| = \infty \) at some \( n \), \( \mu_n \): amenable subgroup.

Then (1).

Finally if \( \mu_n \) is non-amenable then \( \mu_n \& \mu_n \)

are commuting non-amenable subgroups.

\[
H^{(2)} := H_n, \quad H^{(3)} := \mu_n.
\]

By Improved Lemma

\[
\exists g \mid C_{\mu_n}(g) \text{ is non-amenable} \Rightarrow C_{\mu_n}(g) \text{ is } \delta
\]

has measure 1

Fix some \( g_1 \neq 1 \) in this set, and let \( \mu_1 = \langle g_1 \rangle \)

either something good happens, or \( \forall n \exists \) pairwise commuting

non-amenable subgroups \( H_1^{(n)}, \ldots, H_n^{(n)} \)

Define \( \mu_n \) by taking \( g_1 \in H_1^{(n)} - \{1\} \)

\( g_{n+1} \in H_c^{(n)} - \langle g_1, \ldots, g_c \rangle \)

\( \mu_n = \langle g_1, \ldots, g_n \rangle \)

Then \(|\mu_n| \geq 2^{n-1}\)

and \( H^{(n)} \subseteq C_{\mu_n}(\mu_n) \Rightarrow C_{\mu_n}(\mu_n) \) non-amenable.
Known: \( G \times H \) has some action with cost 1
- And \( G \times H \) has fixed price = 1 whenever \( G \) or \( H \) has an infinite amenable subgroup.

Open Problem
Does \( G \times H \) have fixed price = 1?