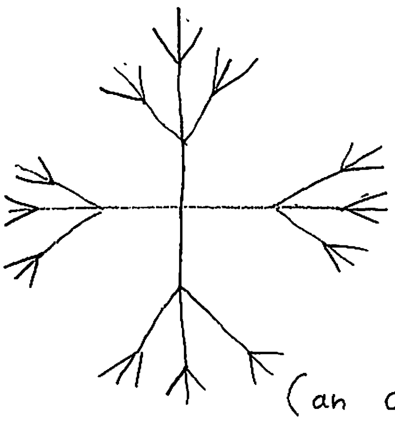
Kaimanovich 2 $\quad$ 1

Stone - Čech

βX : too big.



$F = F_2$

∂F = space of infinite words.

space of ends

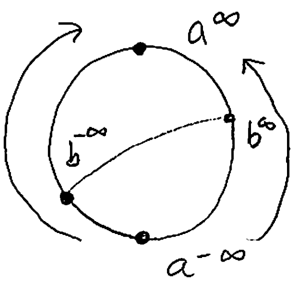
Busemann boundary

action of F on ∂F

a : generator of F

(an arbitrary element)

Action of $\langle a \rangle \cong \mathbb{Z}$ on ∂F is of North - South type



two fixed points. $a^{\pm\infty}$.

a^{∞} - attracting fixed pt $\qquad a^n g \xrightarrow{n\to\infty} a^{\infty}$

$a^{-\infty}$ - repelling fixed pt

<u>Corollary</u>

(There are no invariant measures on ∂F.)

<u>Corollary 2</u> $\quad$ prob. measure.

(For any $\lambda \in P(\partial F)$

$\overline{\{ g\cdot\lambda \}_{g\in F}}^{w^*}$ contains δ - measures.)

$a^n \cdot \lambda \xrightarrow{n\to\infty} \delta_{a^{\infty}}$ unless λ contains $\delta_{a^{-\infty}}$

<u>Def:</u> (Furstenberg '73)

B : a compact G - space

B is called a <u>boundary</u> if

1) The action on B is minimal.

(no nontrivial closed invariant subsets)

2) $\forall \lambda \in P(B)$ $\overline{\{ g\cdot\lambda \}_{g\in G}}$ contains δ - measures

Ex F : free group.

∂F is (a) Furstenberg boundary
the?

Prop If G is amenable then the only Furstenberg boundary (is a single point.)

<proof>

There is an invariant measure λ on B. ($G \curvearrowright B$ amenable.)

$$g \cdot \lambda = \lambda \quad \forall g$$

$\overline{\{g \cdot \lambda\}} = \{\lambda\}$ is a delta measure. $\lambda = \delta_x$.

$\Rightarrow B = \{x\}$ by invariance.

Lem

M : minimal G-space, B : G-boundary

$M \longrightarrow B$.

there exists at most one equivariant map.

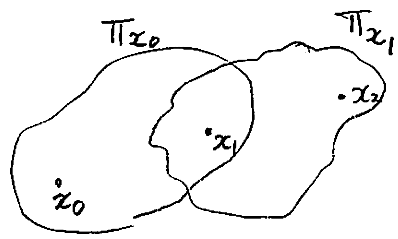
$B \subseteq \prod_\alpha B_\alpha$ is the universal boundary.

Markov Chains

$T : X \circlearrowleft \quad x \longmapsto Tx$ deterministic $\quad \bullet\, x \to Tx \to T^2x \to \cdots$

$x \longmapsto \Pi_x \in P(X)$

transition probability



$$x_0 \xrightarrow{\Pi_{x_0}} x_1 \xrightarrow{\Pi_{x_1}} x_2$$

(x_n) is the Markov chain determined by $\{\Pi_x\}$

$X^{\mathbb{Z}_+} = \{(x_0, x_1, \dots) ; x_i \in X\}$ = sample space
path space
space of trajectories

Kolmogorov extension theorem

$$X^{\{0\}} \leftarrow X^{\{0,1\}} \leftarrow X^{\{0,1,2\}} \longleftarrow \cdots \quad \leftarrow X^{\mathbb{Z}_+}$$

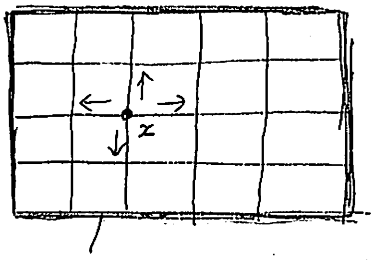
The family of prob. measures

$\pi^0_1 = \theta \quad \pi^{0,1} \quad \pi^{0,1,2} \cdots$ are consistent so $\mathbb{P}_\theta$ has an extension to $X^{\mathbb{Z}_+}$

$$\pi^{0,1}(x_0, x_1) = \pi^0(x_0)\pi_{x_0}(x_1) \qquad \pi^{0,1,2}(x_0,x_1,x_2) = \pi^{0,1}(x_0,x_1)\pi_{x_1}(x_2)$$

Write

$\mathbb{P}_x = \mathbb{P}_\theta$ if $\theta = \delta_x$.



boundary

Simple random walk on a graph.

π_x is equidistributed on the set of neighbors of x.

$x \longmapsto \partial_x$ the Harmonic hitting distribution on ∂X.

$\{x_n\}_n \longrightarrow x_\infty$.

$\mathbb{P}_x \longmapsto \partial_x$

$$\partial_x = \sum_y \partial_y \pi_y(x)$$

$\hat{f}$ is a function on ∂X

$$f(x) = \langle \hat{f}, \partial_x \rangle$$

$$f(x) = \sum f(y)\pi_x(y)$$
$$= \langle f, \pi_x \rangle$$

X: State space. ∂X = boundary of X

$\{\partial_x\}$ hitting measures.

$\hat{f}$: function on ∂X

$\Rightarrow f(x) = \langle \hat{f}, \partial_x \rangle$ harmonic function on X.

X ← countable $\quad x \mapsto \pi_x \in P(X) \quad \{\pi_x(y)\}_{y \in X}$. transition probability

Markov operator

$$Pf(x) = \langle f, \pi_x \rangle$$

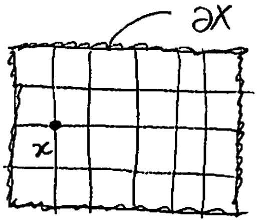
$$f \mapsto Pf$$

$$\lambda \mapsto \lambda P \qquad \langle \lambda P, f \rangle = \langle \lambda, Pf \rangle$$

$\lambda \in P(X)$.

$$\lambda = \sum_{x \in X} \lambda(x) \delta_x \rightsquigarrow P\lambda = \sum_{x \in X} \lambda(x) \pi_x$$

f is harmonic $\Leftrightarrow f = Pf$.



If a point hits ∂X it is absorbed.

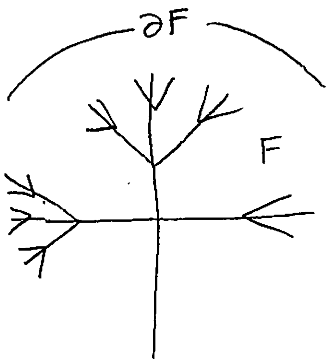
$x \mapsto \nu_x$ hitting distribution

$$\nu_x = \sum_y \nu_y \pi_x(y)$$

$\hat{f}$ on ∂X. $\quad f(x) := \langle \nu_x, \hat{f} \rangle$

$\Rightarrow Pf = f$.

$\{\nu_x\}$ solves the Dirichlet problem with boundary data on ∂X.

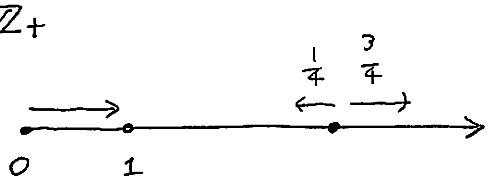


Simple random walk almost surely

$$x_n \longrightarrow x_\infty \in \partial F$$

radial chain

$\mathbb{Z}_+$



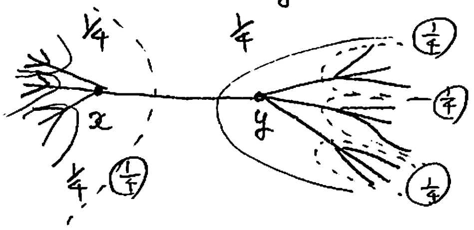
$e \longrightarrow \nu$: uniform measure on the boundary as seen from e.

$\nu_x = \sum \pi_x(y)\,\nu_y$ $\qquad \hat{f}$ on ∂F $\qquad \hat{f} \in L^\infty(\partial F, \nu)$

$f(x) = \langle \hat{f}, \nu_x \rangle$ bounded harmonic function $\qquad$ "$d(y\cdot\gamma) - d(x\cdot\gamma)$"

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\shortparallel$ Busemann cocycle

$\nu_x \sim \nu_y \quad \forall x, \forall y \in F$



$$\frac{d\nu_y}{d\nu_x}(\gamma) = \begin{cases} 3 \\ \frac{1}{3} \end{cases} = 3^{-\beta_\gamma(x,y)}$$

F : free group.

Consider the distribution s.t.

$g \mapsto \pi_g$ is equivariant

$e \mapsto \pi_e = \mu. \qquad g \mapsto g\cdot\mu$

$RW(G,\mu)$ $\quad x_0 = e \qquad x_1 = h_1 \qquad h_1 \sim \mu$

$\qquad\qquad\qquad x_2 = x_1 h_2 \qquad h_2 \sim \mu$

$x_n = h_1 \cdots h_n \qquad h_i \sim \mu$ i.i.d.

($h_1 + \cdots + h_n$ in the abelian case)

Do sample paths converge to ∂F for arbitrary μ ?

Previous case $\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$