Kesten's Thm
Tits Alternative
Powers (C*-simplicity)

Kesten's Thm (1950's)

\[ \Gamma : \text{finitely generated group} \]
\[ \mu : \text{probability measure on } \Gamma \]
\[ \text{symmetric } (\mu(x^{-1}) = \mu(x)) \]

Def: \( \mu \) is adapted to \( \Gamma \) if \( \langle \text{supp}(\mu) \rangle = \Gamma \).

\( \mu \rightarrow \) operator on \( L^2(\Gamma) \) by convolution \( f \rightarrow \mu * f \)

\[ \lambda(\gamma) f (x) = f (\gamma^{-1} x) \]

\[ \lambda(\mu) = \sum_{\gamma \in \Gamma} \lambda(\gamma) \mu(\gamma) \]

\[ \mu^\infty := \mu * \mu * \ldots * \mu \in \text{Prob}(\Gamma) \]

\( \mu^\infty \) is the distribution at time \( n \) of the random product

\[ S_n = X_1 \cdot \ldots \cdot X_n \quad (X_i)_{i \geq 1} \text{ are i.i.d. random variables with values in } \Gamma \]

\[ P (X_n = x) = \mu(x) \]

Fact: \( \mu(\delta_x) = \text{prob}(X \sim \delta_x) = \langle \lambda(\mu) \delta_x, \delta_y \rangle_{L^2(\Gamma)} \]

\[ \delta_x \in L^2(\Gamma) \quad \delta_x (\gamma) = \begin{cases} 1 & (y=x) \\ 0 & (y \neq x) \end{cases} \]

\[ \mu^n (x) = P(S_n = x) \]

\[ \mu^{2n} (e) \leq \mu^{2n} (e) \quad \forall x \in \Gamma \]

\[ \lambda(\mu^n) = \lambda(\mu)^n \quad \text{for bounded self-adjoint } \lambda(\mu) \]

\( \lambda(\mu) \) is symmetric.
\[ \langle pf \rangle. \quad \mu^{2n+2}(e) = \langle \lambda(\mu)^{2n+2} \delta_e, \delta_e \rangle \]
\[ = \langle \lambda(\mu)^{n+1} \delta_e, \lambda(\mu)^{n+1} \delta_e \rangle \]
\[ = \| \lambda(\mu)^{n+1} \delta_e \|^2 \]
\[ \leq \| \lambda(\mu)^n \delta_e \|^2 = \mu^{2n}(e) \]

since \[ \| \lambda(\mu) \| \leq 1. \]

\[ \mu^{2n}(x) = \langle \lambda(\mu)^{2n} \delta_e, \delta_x \rangle = \langle \lambda(\mu)^n \delta_e, \lambda(\mu)^n \delta_x \rangle \]
\[ \leq \| \lambda(\mu)^n \delta_e \| \| \lambda(\mu)^n \delta_x \| \]

\[ = \| \lambda(\mu)^n \delta_x \|^2 = \langle \lambda(\mu)^{2n} \delta_x, \delta_x \rangle \]
\[ = \mu^{2n}(x) = \mu^{2n}(e) \]
\[ = \| \lambda(\mu)^n \delta_x \|^2 \]

so \[ \mu^{2n}(x) \leq \| \lambda(\mu)^n \delta_x \|^2 = \mu^{2n}(e). \square \]

**Def.** The spectral radius of the random walk is defined as the spectral radius of \( \lambda(\mu) \) viewed as a bounded operator on \( l^2 \Gamma \):

\[ \rho(\mu) = \| \lambda(\mu) \| \]

= norm of \( \mu \) where \( \mu \) is viewed as an element of the reduced \( C^* \)-alg \( \Gamma^*(\Gamma) \).

= norm closure of span \( \{ \lambda(\mu)^n | n \in \mathbb{N} \} \) in \( l^2(\Gamma) \).

**Thm (Kesten)**

\[ \| \lambda(\mu) \| \leq \liminf_{n \to \infty} \mu^{2n}(e)^{1/n} = \mu^{2n}(e) = \rho(S_{2n} = e) \]

**Remarks**

(1) \( n \mapsto \frac{1}{\mu^{2n}(e)} \) is submultiplicative

i.e. \[ \mu^{2n+2n}(e) \geq \mu^{2n}(e) \mu^{2n}(e) \quad \forall n, m \]

\[ \rho(S_{2n+2n} = e) \geq \rho(S_{2m} = e) \times \rho(S_{2n} = e) \]

\[ = \rho(S_{2m} = e) \times \rho(S_{2n} = e) \]
The subadditivity lemma shows that
\[ \lim_{n \to \infty} \mu^{2n}(e) \frac{1}{2n} \text{ exists}. \]

(2) \[ \| \chi_\nu \|_2 \leq 1 \]
\[ \lim_{n \to \infty} \chi_\nu = \chi \]
\[ \| \chi \|_2 \leq 1 \]

Proof of Kesten's Thm

\[ \forall \epsilon > 0, \exists f \in C \text{ (finitely supported function)} \]
\[ \text{ s.t. } \| f \|_2 = 1 \]
\[ \langle \chi(\nu), f \rangle \geq \| \chi(\nu) \|_2 - \epsilon \]

Spectral Thm

\[ \| f \|_2 \leq \max \{ t | t \in \text{ supp}(f) \} \geq \int_{-1}^{1} t \, dV_f(t) \]
\[ \frac{1}{2n} \left( \int_{-1}^{1} t \, dV_f(t) \right) \]
\[ \downarrow n \to \infty \]
\[ \| f \|_2 \leq \mu^{2n}(e) \]
\[ \sum_{\Lambda} \mu^{2n}(x) f(x^{-1}y) \tilde{f}(y) \]
\[ \leq \mu^{2n}(e) \]
\[ \lim_{n \to \infty} \sup \langle \chi^{2n} \mu, f \rangle \frac{1}{2n} \leq \lim_{n \to \infty} \left( \frac{\mu^{2n}(e)}{2n} \right) \]
\[ \| \chi(\nu) \|_2 \leq \epsilon \]

\[ \epsilon > 0 \text{ arbitrary, done } \]
If $\Gamma$ is a free group on $k$-generators,

$$\Gamma = \langle S_1, \ldots, S_k \rangle$$

Then

$$\lambda(\mu_{S_k}) \ll = \sqrt{\frac{2k-1}{k}}$$

$$\mu_{S_k} = \frac{1}{2k} \sum_{c=1}^{k} (\delta_{Sc} + \delta_{Sc^{-1}})$$

Indeed, by Kesten 1,

$$\mu^{|2n}(e) = \frac{\text{loops of length } 2n \text{ around } e}{\text{paths of length } 2n \text{ from } e}$$

$2k+1$ regular tree.

To go right

$$\frac{8}{8+1}$$

To go left

$$\frac{8}{8+1}$$

 número de $2n$:

$$|\text{loops around } e| \sim \left(\frac{2n}{8n}\right)^{2n}$$

$$|\text{paths from } e| \sim (8+1)^{2n}$$

$$\mu^{|2n}(e) \sim 2^{2n} \left(\frac{8}{8+1}\right)^{2n}$$

$$\lambda(\mu) = \frac{2^{2n} \sqrt{8}}{8+1} = \sqrt{\frac{2k-1}{k}}$$

Then $k \geq 3$

$\Gamma$ is amenable $\iff \rho(\mu) = 1$ (\mu adapted and symmetric)

If $H \triangleleft \Gamma$ then $H$ is amenable

$$\|\lambda_H(\mu)\| = \|\lambda_{\Gamma/H}(\mu)\|$$
Remark
\[ \| \lambda \Gamma (\mu) \| = 1 \iff \exists \text{ almost } \Gamma \text{- invariant sequence in } L^2 \Gamma \]
i.e. if \( \Gamma \) is amenable,
\[
\begin{align*}
\exists f_n &\in L^2 \Gamma \quad \| f_n \|_2 = 1 \\
\| \lambda \Gamma (\mu) f_n \|_2 &\to 1 \\
\Rightarrow \| \lambda \Gamma (\nu) f_n - f_n \|_2 &\to 0 \quad \forall \nu \in \Gamma
\end{align*}
\]
\[ \Gamma \text{ amenable} \iff \lim_{n \to \infty} \| \chi_n \| = 1 \]

\[ \forall H \leq \Gamma \quad \| \lambda \Gamma/H (\mu) \| \geq \| \lambda H (\mu) \| \]
indeed by Kesten 1.

\[ \exists \text{ quasi-regular representation } \Gamma \mapsto \text{ weak containment } \text{ relative to } H \]

II. Tits Alternative

1972

Theorem (Tits)

If \( \Gamma \) is a finitely generated linear group.

Then either \( \Gamma \) has a free subgroup of some non-abelian \( \text{GL}_n(\mathbb{F}) \) \( n \in \mathbb{N} \), or \( \Gamma \) is virtually solvable.

Corollary: \( \Gamma \) is finitely generated linear group

Then \( \Gamma \) is amenable \( \iff \) \( \Gamma \) is virtually solvable.
A linear rep of $\Gamma$ is proximal if $\rho(\Gamma)$ contains a proximal element.

$$\rho : \Gamma \to \text{GL}_n(k)$$

is I-P if $\rho$ is proximal

$\rho$ is strongly irreducible

$$\Leftrightarrow \left( \overline{\rho(\Gamma)} \right)^{\text{Buniki}} \text{ is irreducible}$$

Remark

If $\gamma \in \text{GL}_d(k)$ is proximal, then $\gamma^n$ $(n \geq 1)$ behave like

"contractions" in $\text{IP}(k^d)$

Let $V_\gamma = \text{eigen direction corresponding to the max eigenvalue}$

$H_\gamma = \text{sum of remaining generalized eigenspaces}$

Then $\forall x \in H_\gamma$, $\gamma^n x \to V_\gamma$
Main Lemma of Tits

If $\Gamma$ is not virtually solvable, then $\exists$ a local field
$\exists V$ a $k$-vector space (finite-dim $k$)
and $\exists \rho: \Gamma \to \text{GL}(V)$ I-P.

Ideas.

- Choose $k$ s.t. $\Gamma \subseteq \text{GLd}(k)$ unbounded.
- Pass to a wedge power $\Lambda^m \otimes k^d$ then it becomes proximal but may be not irr.
- Pass to irreducible quotient.

Ping-pong Lemma

Suppose $\gamma_1, \gamma_2 \in \text{GLd}(k)$, $\gamma_1^i, \gamma_2^{-1}$ proximal and assume $V_{\gamma_1^i} \cap \mathcal{H}_{\gamma_1} \cup \mathcal{H}_{\gamma_2}^{-1}$

Then $<\gamma_1^n, \gamma_2^n>$ is free for $n$ large.

$$w(\gamma_1^n, \gamma_2^n) = \gamma_1^n \gamma_2^{-n} (\gamma_1^n)^3$$

$w \neq x$

$\downarrow$

$w \neq 1$

$\downarrow$

$<\gamma_1^n, \gamma_2^n>$ is free.
Q? Can one get an estimate on the spectral radius $\| \lambda_\Gamma (\mu) \|$ ?

$\mu \in \text{Prob}(\Gamma)$

$\lambda(\mu_{sk}) = \frac{\sqrt{2k-1}}{k} \xrightarrow{k \to \infty} 0$

Kesten's bound

($\Gamma$ Uniform spectral gap for linear groups (AG))

Given $d \geq 1, \exists \beta = \beta(d) < 1$

s.t. $\forall \gamma_1, \ldots, \forall \gamma_k \in \text{GL}_d$ (some field)

- either $\langle \gamma_1, \ldots, \gamma_k \rangle$ is amenable
- or $\| \lambda_\Gamma \left( \frac{1}{2k} \sum_{k=1}^{n} (\delta_{\gamma_k} + \delta_{\gamma_k^{-1}}) \right) \| < \beta$

Deduced from

($\Gamma$ Uniform Tits alternative)

Given $d$, $\exists N = N(d) \in \mathbb{N}$

s.t. $\forall F \subseteq \text{GL}_d$ (some field) $F = F^{-l}, 1 \in F$

- either $\langle F \rangle$ is virtually solvable
- or $(F)^N$ contains generators of a non-abelian free subgroups.

(cf. connected to Eskin-Mozes-Oh)

$\Gamma \subset N^\Lambda$

$N = \text{N}(\Gamma)$
Random walks on $\text{GL}_d$. \text{Bougerol, Benoist} - \text{Quint}

$\mu \in \text{Prob}(\text{GL}_d(k))$

$S_n = X_1 \cdots X_n$ iid $\sim \mu$.

Guivarc'h Poié?

If $\rho : \Gamma \to \text{GL}(V)$ is $\text{I-P}$ then $\rho(S_n)$ is proximal with probability 1, as $n \to \infty$.

Tum (Aoun 2011)

If $(S_n^1, \ldots, S_n^k)$ are $\mathbb{R}$-independent adapted RW on a non-amenable linear group, then

$\langle S_n^1, \ldots, S_n^k \rangle$ is free of rank $k$.

III, Powers

$C^*$-simplicity of groups.

$\Gamma$ a discrete group is $C^*$-simple if $C^*_\alpha(\Gamma)$ is simple.

$\Leftrightarrow \left( \forall \pi \text{ unitary repr } \pi \not\sim \lambda_\Gamma \Rightarrow \pi \sim \lambda_\Gamma \right)$

If $N \triangleleft \Gamma$, $N$ is amenable then $\Gamma$ not $C^*$-simple

$\lambda_{\Gamma/N} \not\sim \lambda_\Gamma$

but $\lambda_{\Gamma/N} \not\sim \lambda_\Gamma$, $\langle \lambda_\Gamma(x), \delta_e \rangle$
Open Problem
Is this the only obstruction?

\[ \text{Def: } \text{Rad}(\Gamma) = \langle N \triangleleft G \mid N \text{ amenable} \rangle \]
\[ (\text{amenable radical}) \]

Powers 1975
Free groups are $C^*$-simple.

- Bekka - Cowling - de la Harpe '90s: center-free Zariski-dense subgroups of semi-simple algebraic groups.
- Poynzusky: any linear group with $\text{Rad}(\Gamma) = 1$ is $C^*$-simple.
- Gromov hyperbolic groups, Baumslag - Solitar groups, Free Burnside groups, Osin, Olshanski.

**Theorem (Kalantar - Kennedy)**
\[
\Gamma \text{ is } C^*\text{-simple} \iff \Gamma \text{ has a topologically free boundary action.}
\]

Powers Lemma \( \Gamma \) discrete group. Assume \( \forall \varepsilon > 0 \) \( \forall F \subseteq \Gamma \setminus \{e\} \)
\[ \exists g_1, \ldots, g_k \in \Gamma \text{ s.t.} \]
\[ \| \lambda g \left( \frac{1}{k} \sum_{i=1}^{k} g_i \circ x \circ g_i^{-1} \right) \| < \varepsilon \]
\[ \forall x \in F \]
Then \( \Gamma \) is $C^*$-simple (and has a unique trace).

Kesten's Thm \( \rightarrow \) This estimate is satisfied for free groups.