def: a type II_1 factor is a *-subalgebra $\mathcal{D}(H)$ s.t.

- $M$ is WOT-closed and unitary (von Neumann algebra) (factor)
- $\Delta(M) = C^1$ (factor) $M = M''$
- $\tau: M \to C$ always (type $\infty$)
  - $\tau(1) = 1$
  - $\tau(x^*x) > 0 \text{ if } x \neq 0$
  - WOT-continuous
  - $\tau(xy) = \tau(yx)$
  - $M \neq M''$

examples:

- $K^\Gamma$: let $\Gamma$ be a countable group
  $H = L^2\Gamma$
  $(\Delta(f))(h) = \int_\Gamma f(xg) g(h) d\mu(x)$
  - $K^\Gamma = \{ \chi \mid \chi \in C^1 \}$
  - $\tau(x) = \langle \delta_e, x \delta_e \rangle$
  - $K^\Gamma$ is a factor if $\Gamma$ is ICC

- $L^\infty(X, \mu, \Gamma)$: let $\Gamma/\Delta(X, \mu)$ be an equivalence
  $H = L^2(X, \mu) \otimes L^2\Gamma = L^2(X \times \Gamma)$

- $(x, f)(g) = (x \otimes g, f)$ $\forall x \in L^\infty(X)$
  $(\Delta f)(x, g) = \delta(x) (g^2, g)$ $\forall f \in L^2(X)$
  $(\psi \otimes f)(x, h) = \psi(x) f(x, g^2) g(h)$ $\forall g \in \Gamma$

then $\psi \otimes f \in L^2(X \times \Gamma)$

- $L^\infty(X, \mu, \Gamma) = \{ \psi \otimes f \mid \psi, f \in L^\infty(X, \mu), f \in \Gamma \}$

- $\tau(x) = \langle \delta_e, x \delta_e \rangle$

- $L^\infty(X, \mu, \Gamma)$ is a factor if $\Gamma/\Delta(X, \mu)$ is free and ergodic
  - free: $\forall x \in \Gamma, (\delta(x) \otimes f)(y) = 0$
  - ergodic: if $\mu(x) > 0$, then $\exists \psi \in L^2(X, \mu) \text{ s.t. } \tau(\psi) = 1$

$\Gamma$-invariant subset $\mathcal{K} \subseteq X$ has $\mu(\mathcal{K}) = 0$, $\mu(\mathcal{K}) = 1$
questions
- when \( \Gamma \cong \mathbb{Z} \)?
  \( L^{\infty} \mathcal{M} \cong L^{\infty} \mathcal{L} \mathcal{M} \)?
- if isomorphic, what are the isomorphisms?
  in particular: \( \text{Aut} (\mathcal{M}) = \{ \alpha : \mathcal{M} \to \mathcal{M} \} \)?
  obvious automorphisms: \( \alpha_\lambda : \mathcal{M} \to \mathcal{M}, x \mapsto \lambda x \lambda^{-1} \)
  \( \text{Im} (\mathcal{M}) = \{ \lambda \alpha \lambda^{-1} | \lambda \in \text{Aut} (\mathcal{M}) \} \) normal subgroup
  \( \text{Out} (\mathcal{M}) = \text{Aut} (\mathcal{M}) / \text{Im} (\mathcal{M}) \)
  \( \mathcal{F} (\mathcal{M}) = \{ (\mathcal{M}^+, \varphi) | \mathcal{M}^+ \cong \varphi (\mathcal{M}) \} \)
- what properties of \( \Gamma / \Gamma \) are invariants of the type \( \mathcal{F} \) factor?

\[ \text{Facts: } \] 
- if \( \Gamma \) is amenable, then \( \Gamma / \Gamma \) is amenable, i.e., \( \Gamma / \Gamma \cong \mathbb{Z} \cong L^{\infty} \mathcal{M} = R \) (Corry, 1978)
- for \( \Gamma \) amenable, \( \Gamma / \Gamma \cong \mathbb{Z} \cong L^{\infty} \mathcal{M} = R \) if \( \Gamma \) is amenable
- \( \mathcal{L} \mathcal{T} \neq R \)
- \( \mathcal{L} \mathcal{F} \neq \mathcal{L} \mathcal{F} \mathcal{F} \)
- \( \text{Out} (\mathcal{F} \mathcal{R}) \) is huge (contains \( \mathcal{F} \mathcal{R} \))
- \( \mathcal{F} (\mathcal{R}) = R^\infty \)

\[ \text{Corry 1980: } \]
- property (T) for type \( \mathcal{F} \) factor
  \( \Gamma \cong \mathbb{Z} \) if \( \Gamma \) has (T)
- \( \text{Out} (\Gamma) \) is countable if \( \Gamma \) has (T)
  thus at most countably many \( \Gamma / \Gamma \) a.e.
  \( \Gamma / \Gamma \cong \mathbb{Z} \) if \( \Gamma \) has (T)

\[ \text{(Oda 2001+): } \]
- \( \text{Malcev with (T)} \iff \text{deficiency of the algebra } M \)
- \( \mathcal{F} (\mathbb{Z}^2 \times \mathbb{Z}) = \{ -1 \} \)
- \( \mathbb{R} \mathbb{Z} \cong \mathbb{R} (\mathbb{Z}^2 \times \mathbb{Z}) \) has (T) \( \iff \mathbb{Z} \mathbb{Z} \) has (T)
Cone \( C(\mathcal{M}) = \{ x \} \) (FM)

**Core with expository theory:**

1. Let \( \Delta X \), \( \Delta Y \) be paths, two actions
2. \( \Delta : X \to Y, \delta : \Gamma \to \Lambda \) in a category of

\[ \Delta(x) = \delta(y) \Delta(x) \quad \text{a.e.} \]

3. \( \Delta : X \to Y \) is an \( \alpha \) of \( \Delta \Delta \)

\[ \Delta(\Gamma x) = \Lambda(\Delta x) \quad \text{a.e.} \]

4. Denote \( \alpha \) and \( \Delta \) as equivalent

\[ \Delta(x) = \omega(\delta(x)) \Delta(x) \quad \text{a.e.} \]

5. Denote \( \alpha \) as an \( \omega \) of \( \Delta \)

\[ \Delta(\Gamma x) = \omega(\delta(x)) \Gamma x \quad \text{a.e.} \]

6. \( L^2(X,\mu) \leq L^2(Y,\nu) \) if \( \Delta \) is \( \Lambda \) and \( \nabla \) is

\[ \Delta(x) = \omega(\delta(x)) \Delta(x) \quad \text{a.e.} \]

7. Define an action on \( \alpha \) of \( \Delta \)

\[ \Delta(\Gamma x) = \omega(\delta(x)) \Gamma x \quad \text{a.e.} \]

8. every \( \alpha \) is of this form

\[ \Delta(x) = \omega(\delta(x)) \Delta(x) \quad \text{a.e.} \]

9. every \( \alpha \) is of this form

\[ \Delta(x) = \omega(\delta(x)) \Delta(x) \quad \text{a.e.} \]

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39. every \( \alpha \) is of this form

\[ \Delta(x) = \omega(\delta(x)) \Delta(x) \quad \text{a.e.} \]

40. every \( \alpha \) is of this form

\[ \Delta(x) = \omega(\delta(x)) \Delta(x) \quad \text{a.e.} \]
If $R = \mathbb{R}^n \times \mathbb{R}^m$, then and $\Gamma_{\mathbb{R}^n}$ is true, then

$L^0(\mathbb{R}) = L^0$

**Proof:**

1. a subspace $A \subseteq M$ of the $\mathbb{R}$, factor is a compact subspaces of $L^0$.
   - note $a^t$, i.e., $N(a^t) = A$
   - relative, i.e., $N_b(a) = \{ x \in M \mid x \cdot a = 0 \}$ generally $M$

2. $L^0(\mathbb{R}) = L^0(\mathbb{R}^t)$ if $\Gamma_{\mathbb{R}^n}$ is true

$L^0(\mathbb{R}) = L^0(\mathbb{R}^t)$

**Remark:** Every compact subspaces is often of the form $L^0(\mathbb{R}) = L^0(\mathbb{R}^t)$

**Strategy to study one pattern:**

[\( \psi \in L^0(\mathbb{R}) \rightarrow L^0(\mathbb{R}) \text{ is unique} \)]

shown that \( \psi^\prime \in L^0(\mathbb{R}) = L^0(\mathbb{R}^t) \)

Study orbit equivalence $\Delta : X \rightarrow Y$

[Domain]

**Technical Tool:** Popov's method (by bimodules)

The goal is to prove that:

$A, B \in C(M, \mathbb{R}) \rightarrow \text{goal is to prove } A = B$ from $\mathbb{R} \times M \rightarrow M$

**Remark:** Let $A, B \in C(M, \mathbb{R})$ TFAE

1. \( \exists \xi \in M, \eta \in M(b), \alpha - \epsilon \in M_\eta^*(M), \theta \in A, \pi \in B \in M(b), \alpha - \epsilon 
\)

\( \in \mathbb{R} \Rightarrow \varphi(x), \quad \forall x \in A \)

2. \( \exists \xi \in M, \eta \in M(b) 
\)

3. \( \exists \xi \in M, \eta \in M(b) 
\)

4. \( \exists \xi \in M, \eta \in M(b) 
\)

5. \( \exists \xi \in M, \eta \in M(b) 
\)

**Proof:**

(1) \( \Rightarrow \) (2):

suggest that $f, g, \alpha, \gamma$ are $A$ or $B$ or $C$

\( \Rightarrow \) (1) or $\rho \in C$

Thus, \( \| E_{\rho}(b_{\rho}(u)) \|_2 = \| E_{\rho}(b_{\rho}(w)) \|_2 \quad \Rightarrow \quad \| E_{\rho}(b_{\rho}(u)) \|_2 = \| E_{\rho}(b_{\rho}(w)) \|_2 \Rightarrow 0 \)
Basic construction: on $L^2(M)$, consider $\mathcal{D}_B(\mathcal{L}^2(M))$ etc.

$\mathcal{D}_B(\mathcal{L}^2(M)) \ni B(\mathcal{L}^2(M)) \rightarrow \text{trace } T_B = T_B : \mathcal{D}_B \rightarrow \mathcal{H}$

$B(\mathcal{L}^2(M)) = \langle M, \mathcal{D}_B \rangle$, $T_B(\mathcal{D}_B) = I$, $\mathcal{D}_B = \mathcal{E}_B(\mathcal{D}_B)$ $\forall \mathcal{D}_B, M$

$K \in \mathcal{L}^2(M)$ is an $A$-$B$ subminded $\iff P_K \in \langle M, \mathcal{D}_B \rangle \cap \mathcal{A}$

$K$ is finite dimensional $\iff T_B(P_K) = 0$

$\exists P_K \in K$ s.t. $K$ is finitely generated.

$\exists x \in \mathcal{H}(M) \ni A x, w : \mathcal{E}_B(x, u, x) \rightarrow 0$

Thus $\exists x, y, z \in \mathcal{H}(M) \ni \forall x \in \mathcal{H}(M) : \|E_B(x, u, y)\|_2 \geq \varepsilon$ for some $i = 1, \ldots, 2^n$.

Let $X = \sum_{i=1}^{2^n} a_i \delta_{a_i} \in \langle M, \mathcal{D}_B \rangle$, $T_B(X) \leq \varepsilon$, $X > 0$.

$\implies X \in L^2(\langle M, \mathcal{D}_B \rangle, T_B)$

$C = \inf \{ \|x \|_2^2 \mid x \in \mathcal{H}(M) \}

\text{all elements have finite trace.}$

Take $w \in C$ with minimum $\|w\|_{T_B}$.

Take $w \in C$ with minimum $\|w\|_{T_B}$.

$\implies w = 0$.

Let $w = 0$ and $u = 0$.

$\langle x, u \rangle_{T_B} = T_B(x, u) = \sum_{i=1}^{2^n} T_B(a_i \delta_{a_i} u, \delta_{a_i} u) = \sum_{i=1}^{2^n} \mathcal{E}_B(a_i \delta_{a_i} u, \delta_{a_i} u) = \sum_{i=1}^{2^n} \mathcal{E}_B(a_i \delta_{a_i} u, \delta_{a_i} u) \geq \varepsilon$

Take a special projection $P \in \mathcal{H}(M) \ni \mathcal{D}_B(\mathcal{L}^2(M)) = K$. 


\[ K_B = \{ g \in L^2(B) \mid g \text{ is a unitary} \} \]

Let \( f \in M_n(L^2(M)) \) be
\[
\begin{bmatrix}
  u_1 \otimes \phi_{x_1}, & \ldots, & u_n \otimes \phi_{x_n}
\end{bmatrix}
\]
with \( \phi(x) = u(x) \otimes \phi_{x} \in M_n(C_0(M)) \), \( \forall x \in M \).

Define \( \mu f = f \Theta(x) \)

\( \Theta(x) \) is a linear function such that
\[
\begin{align*}
\Theta: & \ L^2(M) \to \mathbb{C}(X) \times L^2(M) \\
\quad & \ x \mapsto (x, x)
\end{align*}
\]

when \( \Theta(x) \) is a doubly indexed operator on \( L^2(M) \) defined by
\[
\Theta(x) = \begin{bmatrix}
  T_x & 0 \\
  0 & T_x
\end{bmatrix}
\]

and \( \mu f = T_x f \).

For \( \alpha, \beta \in \mathbb{R} \), \( \alpha f = \beta f \).

Define \( \Phi \) on \( L^2(M) \):
\[
\Phi(x) = \left\{ \begin{array}{ll}
  1 & \text{if } x = 0, \\
  0 & \text{otherwise}.
\end{array} \right.
\]

Consider \( f \in M_n(M) \) such that \( \alpha f = \alpha f \).

Apply \( \Phi(x) \) to \( L^2(M) \):
\[
\Phi(x) = \begin{cases}
  1 & \text{if } x = 0, \\
  0 & \text{otherwise}.
\end{cases}
\]

Note \( \Phi(x) \) is a \( C_0 \) function and \( \phi_0 = \phi \).

\[ \forall x \in M \]: \( \| E_{[x]} \Phi(x) \|_2 \to 0 \).
in particular, \( \| E_{z_2}(u_0 v_k) \|_2 \to 0 \quad \forall z_2 \in C_2 \)

Moreover, we know that \( \| x(v_k) \|_2^2 = \frac{1}{2} \quad \forall v_k \).

Moreover, this is not the case.

Take \( n \to 1 \cdot \| x_0 (v_k) - x(v_k) \|_2 < \frac{1}{5} \quad \forall v_k \).

Let \( x_0 \in C_1 \). Take \( F \in S_L \quad \forall \frac{1}{8} \| v_k \|_2 < \frac{1}{8} \quad \forall v_k \in F \).

Which implies \( \| E_{z_2}(u_0 v_k) \|_2 < \frac{1}{8} \quad \forall v_k \in F \).

\[ \| x(v_k) \|_2^2 = \sum \| x_i (v_k) \|_2^2 \leq \sum \| x_i \|_2^2 + \frac{1}{5} \quad \forall v_k \in F \]

\[ \leq \frac{1}{8} + \frac{1}{8} \sum \| x_i \|_2^2 \]

\[ \Rightarrow \| x(v_k) \|_2 < \frac{1}{4} \quad \Rightarrow \| x \|_2 < \frac{1}{2} \]