VON NEUMANN ALGEBRAS WITH UNIQUE CARTAN DECOMPOSITION

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Abstract. These are the lectures notes from a minicourse given at the Masterclass on "Ergodic theory and von Neumann algebras" at the University of Copenhagen in October 2013.

In the Lectures 2, 3 and 4, we will follow S. Vaes's exposition for his mini-course at the " Intensive Month on Operator Algebra and Harmonic Analysis" at ICMAT in Madrid in May 2013. The exposition in itself is very close to the one in the articles by S. Popa and S. Vaes [PV11, PV12].

In the Lectures 5 and 6, we will follow the exposition of our joint article with S. Vaes [HV12].

Lecture 1

In the first lecture, we review some fundamental tools which arise in Popa's *deformation/rigidity theory*. These include for instance Popa's intertwining techniques and Connes's characterization of amenability. We state Popa-Vaes's dichotomy result for tracial actions of the free groups.

Popa's intertwining techniques in tracial von Neumann algebras. Let (M, τ) be a tracial von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be any von Neumann subalgebras. We denote by 1_A (resp. 1_B) the unit of A (resp. B). Popa discovered a very useful method to unitarily conjugate A into B inside M.

Theorem 1 (Popa)**.** *The following are equivalent:*

(1) *There exist projections $p \in A$, $q \in B$, a nonzero partial isometry $v \in pMq$ and a unital normal $*$-homomorphism $\theta : pAp \to qBq$ such that $xv = \theta(x)v$ for every $x \in pAp$.*
(2) *There is no net $w_k \in \mathcal{U}(A)$ such that $\lim_k \|E_B(x^* w_k y)\|_2 = 0$ for every $x, y \in 1_A M 1_B$.*

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If one of the equivalent conditions of Theorem 1 is satisfied, we say that A *embeds into* B *inside* M and denote $A \preceq_M B$. Otherwise, we denote $A \npreceq_M B$.

When dealing with tracial crossed product von Neumann algebras, we can give another useful characterization of the intertwining criterion. Let $\Gamma \curvearrowright (B, \tau)$ be a trace preserving action of a countable discrete group on a tracial von Neumann algebra. Put $M = B \rtimes \Gamma$ and let $E_B : M \to B$ be the unique trace preserving conditional expectation. Write $x = \sum_{g \in \Gamma} (x)_g u_g$ for the *Fourier expansion* of $x \in M$ where $(x)_g = E_B(x u_g^*)$.

Corollary 1. *The following are equivalent:*

1. $A \preceq_M B$
2. *There is no net* $w_k \in \mathcal{U}(A)$ *such that* $\lim_k \|(w_k)_g\|_2 = 0$ *for every* $g \in \Gamma$.

The following is a central concept in these lectures.

Definition 1. Let (M, τ) be a tracial von Neumann algebra. We say that $A \subset M$ is a *Cartan subalgebra* if A satisfies the following:

- A is maximal abelian in M, that is, $A' \cap M = A$.
- The group $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M.

When $\Gamma \curvearrowright (X, \mu)$ is a free pmp action, $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is a Cartan subalgebra. In the case when $A, B \subset M$ are Cartan subalgebras of a II_1 factor, one can upgrade the previous intertwining result in order to obtain a genuine conjugation by a unitary.

Theorem 2 (Popa). *Let* $A, B \subset M$ *be Cartan subalgebras in a* II_1 *factor. The following conditions are equivalent:*

1. $A \preceq_M B$.
2. *There exists* $u \in \mathcal{U}(M)$ *such that* $uAu^* = B$.

Connes's characterization of amenability.

Definition 2. Let (M, τ) be a tracial von Neumann algebra. We say that M is *amenable* if there exists an M-central state $\varphi \in \mathbf{B}(\mathrm{L}^2(M))$ such that $\varphi|_M = \tau$. We say that M is *hyperfinite* if there exists an increasing sequence of unital finite dimensional $*$-subalgebras $Q_n \subset M$ such that $M = \bigvee_n Q_n$.

Theorem 3 (Connes). *Let* (M, τ) *be a tracial von Neumann algebra with separable predual. The following are equivalent:*

1. M *is amenable.*
2. *There exists a norm one projection* $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to M$.
3. *There exists a net of unit vectors* $\xi_n \in \mathrm{L}^2(M) \otimes \mathrm{L}^2(M)$ *such that*
    - $\lim_n \|(x \otimes 1)\xi_n - \xi_n(1 \otimes x)\|_2 = 0$ *for every* $x \in M$ *and*

- $\lim_n \langle (x \otimes 1)\xi_n, \xi_n \rangle = \tau(x)$ *for every* $x \in M$.

(4) *For all* $a_1, \dots, a_k, b_1, \dots, b_k \in M$, *we have*

$$|\tau(\sum_{i=1}^{k} a_i b_i)| \leq \| \sum_{i=1}^{k} a_i \otimes b_i^{\mathrm{op}} \|_{\min}.$$

(5) M *is hyperfinite.*

Whenever $M = \mathrm{L}(\Gamma)$ *is the von Neumann algebra of a countable discrete group, the previous conditions are equivalent to:*

(6) Γ *is amenable.*

We will also need the notion of *relative amenability*. Recall that if $Q \subset (M, \tau)$ is an inclusion of tracial von Neumann algebras, the *basic construction* $\langle M, e_Q \rangle$ is the von Neumann algebra defined on $\mathrm{L}^2(M)$ and generated by M and the orthogonal projection $e_Q : \mathrm{L}^2(M) \to \mathrm{L}^2(Q)$. The basic construction $\langle M, e_Q \rangle$ is endowed with a canonical semifinite faithful normal trace Tr defined by

$$\mathrm{Tr}(x e_Q y) = \tau(xy), \ \forall x, y \in M.$$

Definition 3 (Ozawa-Popa)**.** Let (M, τ) be a tracial von Neumann algebra and $P, Q \subset M$ von Neumann subalgebras. We say that P is *amenable relative to* Q *inside* M if there exists a P-central state φ on $\langle M, e_Q \rangle$ such that $\varphi | M = \tau$.

We have a similar characterization of relative amenability for tracial von Neumann algebras.

Theorem 4 (Ozawa-Popa)**.** *Let* (M, τ) *be a tracial von Neumann algebra and* $P, Q \subset M$ *von Neumann subalgebras. The following are equivalent:*

(1) P *is amenable relative to* Q *inside* M.
(2) *There exists a norm one projection* $\Phi : \langle M, e_Q \rangle \to P$ *such that* $\Phi|_M = E_P$.
(3) *There exists a net of vectors* $\xi_n \in \mathrm{L}^2(\langle M, e_Q \rangle, \mathrm{Tr})$ *such that*
    - $\lim_n \langle x\xi_n, \xi_n \rangle_{\mathrm{Tr}} = \tau(x)$ *for every* $x \in M$ *and*
    - $\lim_n \|y\xi_n - \xi_n y\|_{2,\mathrm{Tr}} = 0$ *for every* $y \in P$.

We simply say that Q is *co-amenable inside* M if M is amenable relative to Q inside M. Observe that when $\Gamma \curvearrowright (Q, \tau)$ is a trace preserving action, Q is co-amenable inside $Q \rtimes \Gamma$ if and only if Γ is amenable.

We will be using the fact that when Q is amenable, then P is amenable relative to Q inside M if and only if P is amenable.

Popa-Vaes's dichotomy result. The first part of these lecture series will be devoted to proving the following theorem due to Popa and Vaes (2011). Denote by $\mathbf{F}_n$ the free group on n generators.

Theorem 5 (Popa-Vaes)**.** *Let $n \geq 2$ and $\mathbf{F}_n \curvearrowright (X, \mu)$ be any free ergodic pmp action on a standard probability space. Then $\mathrm{L}^\infty(X)$ is the unique Cartan subalgebra of the II_1 factor $\mathrm{L}^\infty(X) \rtimes \mathbf{F}_n$, up to unitary conjugacy.*

Combining this theorem with Gaboriau's results on cost (2000), we obtain the following.

Corollary 2. *Let $\mathbf{F}_n \curvearrowright (X, \mu)$ and $\mathbf{F}_p \curvearrowright (Y, \eta)$ be any free ergodic pmp actions. If $n \neq p$, then*

$$\mathrm{L}^\infty(X) \rtimes \mathbf{F}_n \ncong \mathrm{L}^\infty(Y) \rtimes \mathbf{F}_p.$$

Proof. Let $\pi : \mathrm{L}^\infty(X) \rtimes \mathbf{F}_n \to \mathrm{L}^\infty(Y) \rtimes \mathbf{F}_p$ be a surjective $*$-isomorphism. By Theorem 5, up to composing by an inner automorphism, we may assume that $\pi(\mathrm{L}^\infty(X)) = \mathrm{L}^\infty(Y)$. By a classical result of Singer, this implies that the pmp equivalence relations $\mathcal{R}(\mathbf{F}_n \curvearrowright X)$ and $\mathcal{R}(\mathbf{F}_p \curvearrowright Y)$ are orbit equivalent. Hence, by Gaboriau's result, we get

$$n = \mathrm{cost}(\mathcal{R}(\mathbf{F}_n \curvearrowright X)) = \mathrm{cost}(\mathcal{R}(\mathbf{F}_p \curvearrowright Y)) = p. \qquad \square$$

In fact, Theorem 5 will be a consequence of the following more general result.

Theorem 6 (Popa-Vaes)**.** *Let (B, τ) be any tracial von Neumann algebra and $\mathbf{F}_n \curvearrowright (B, \tau)$ any trace preserving action. Put $M = B \rtimes \mathbf{F}_n$. Let $A \subset M$ be any amenable von Neumann subalgebra. At least one of the following holds:*

- *$A \preceq_M B$.*
- *$\mathcal{N}_M(A)''$ is amenable relative to B.*

The proof of Theorem 6 will entirely occupy Lectures 2, 3 and 4. Let us explain why Theorem 6 implies Theorem 5.

Proof of Theorem 5 using Theorem 6. Put $B = \mathrm{L}^\infty(X)$ and $M = B \rtimes \mathbf{F}_n$. Let $A \subset M$ be another Cartan subalgebra. Since $\mathcal{N}_M(A)'' = M$ is not amenable and B is amenable, we have that $\mathcal{N}_M(A)''$ is not amenable relative to B inside M. Therefore $A \preceq_M B$ by Theorem 6. Since A and B are both Cartan subalgebras in the II_1 factor M, there exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$ by Corollary 2. $\square$

We mention that Theorem 5 holds for a more general class of countable discrete groups Γ. Such groups are called *Cartan-rigid*.

Example 1. Here are examples of Cartan-rigid groups.

(1) Weakly amenable groups with a positive first ℓ^2-Betti number (Popa-Vaes 2011).

(2) Non-elementary hyperbolic groups and non-amenable discrete subgroups of connected simple Lie groups with finite center and real rank one (Popa-Vaes 2012).
(3) Free product groups $\Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ (Ioana 2012).
(4) Central quotients of braid groups $B_n/\mathcal{Z}(B_n)$ with $n \geq 3$ (Chifan-Ioana-Kida 2013).

Lecture 2

In the second lecture, we show that the complete metric approximation property (CMAP) implies the existence of almost invariant states.

Definition 4 (Haagerup)**.** Let Γ be a countable discrete group. We say that Γ has the *complete metric approximation property* (CMAP) if there exists a sequence $f_n : \Gamma \to \mathbf{C}$ of finitely supported functions such that $f_n \to 1$ pointwise and such that the corresponding multipliers $\mathrm{m}_{f_n} : \mathrm{L}(\Gamma) \to \mathrm{L}(\Gamma)$ defined by $\mathrm{m}_{f_n}(u_g) = f_n(g)u_g$ satisfy $\lim_n \|\mathrm{m}_{f_n}\|_{\mathrm{cb}} = 1$.

For every $f : \Gamma \to \mathbf{C}$, we will write $\|f\|_{\mathrm{cb}} = \|\mathrm{m}_f\|_{\mathrm{cb}}$ whenever it is well-defined. We will need the following well-known result due to Haagerup.

Theorem 7 (Haagerup)**.** *The free groups $\mathbf{F}_n$ have the CMAP.*

We will be using the following notation throughout this lecture. Let Γ be a countable discrete group with the CMAP. Let $\sigma : \Gamma \curvearrowright (B, \tau)$ be a trace preserving action of Γ on a tracial von Neumann algebra. We still denote by $\sigma : \Gamma \to \mathcal{U}(\mathrm{L}^2(B))$ the corresponding Koopman unitary representation. Put $M = B \rtimes \Gamma \subset \mathbf{B}(\mathrm{L}^2(B) \otimes \ell^2(\Gamma))$ and regard M as generated by $b \otimes 1$ for $b \in B$ and $u_g = \sigma_g \otimes \lambda_g$ for $g \in \Gamma$. We will identify b with $b \otimes 1$ for every $b \in B$. Recall that we have the following covariance relation:

$$u_g b u_g^* = \sigma_g(b), \forall g \in \Gamma, \forall b \in B.$$

Let $A \subset M$ be an amenable von Neumann subalgebra. Write $J : \mathrm{L}^2(M) \to \mathrm{L}^2(M)$ for the canonical conjugation.

Definition 5 (Popa-Vaes)**.** Define the *link* algebra $\mathcal{L} = B \vee JAJ \subset \mathbf{B}(\mathrm{L}^2(M))$.

Define the action $\alpha : \Gamma \curvearrowright \mathcal{L}$ by $\alpha_g = \mathrm{Ad}(u_g)$ for every $g \in \Gamma$. In particular, we have

$$\alpha_g(b) = \sigma_g(b) \text{ and } \alpha_g(JaJ) = JaJ, \ \forall a \in A, \forall b \in B.$$

Observe that $\Gamma \curvearrowright \mathcal{L}$ need not be trace preserving. Put $\mathcal{M} = \mathcal{L} \rtimes \Gamma \subset \mathbf{B}(\mathrm{L}^2(M) \otimes \ell^2(\Gamma))$ and regard $\mathcal{M}$ as generated by $Ja^*J \otimes 1$ for every $a \in A$, $b \otimes 1$ for every $b \in B$ and $u_g \otimes \lambda_g$ for every $g \in \Gamma$. Define the $*$-embedding $\pi : M \to \mathcal{M}$ by $\pi(b) = b \otimes 1$ for every $b \in B$ and $\pi(u_g) = u_g \otimes \lambda_g$ for every $g \in \Gamma$. Define the $*$-embedding $\rho : A^{\mathrm{op}} \to \mathcal{M}$ by $\rho(a^{\mathrm{op}}) = Ja^*J \otimes 1$. Observe that $\pi(M)$ and $\rho(A^{\mathrm{op}})$ commute and together they generate $\mathcal{M}$.

Put $\mathcal{G} = \mathcal{N}_M(A)$. We will regard $\mathcal{G}$ as a discrete group. Define the action $\beta : \mathcal{G} \curvearrowright \mathcal{M}$ by $\beta_u = \mathrm{Ad}(JuJ \otimes 1)$ for every $u \in \mathcal{G}$. In particular, we have

$$\beta_u(\pi(x)) = \pi(x) \text{ and } \beta_u(\rho(a^{\mathrm{op}})) = \rho((uau^*)^{\mathrm{op}}), \ \forall a \in A, \forall x \in M.$$

We will moreover use the notation $\overline{a} = (a^{\mathrm{op}})^*$ for every $a \in A$. The main technical result of this lecture is the following.

Theorem 8 (Popa-Vaes)**.** *There exists a sequence $\omega_n \in \mathcal{M}_*$ of normal states with the following properties:*

- $\lim_n \omega_n(\pi(x)) = \tau(x)$ *for every* $x \in M$.
- $\lim_n \omega_n(\pi(a)\rho(\overline{a})) = 1$ *for every* $a \in \mathcal{U}(A)$.
- $\lim_n \|\omega_n \circ (\mathrm{Ad}(\pi(u)) \circ \beta_u) - \omega_n\| = 0$ *for every* $u \in \mathcal{G}$.

We will need the following two lemmas. For every C*-algebra $\mathcal{A}$, every $a \in \mathcal{A}$ and every $\omega \in \mathcal{A}^*$, define $a\omega$ and $\omega a \in \mathcal{A}^*$ by

$$(a\omega)(x) = \omega(xa) \text{ and } (\omega a)(x) = \omega(ax), \forall x \in \mathcal{A}.$$

Lemma 1. *Let $\mathcal{A}$ be a unital C*-algebra, $u \in \mathcal{A}$ with $\|u\| \leq 1$ and $\omega \in \mathcal{A}^*$ a state. Then we have*

$$\max\{\|\omega - u\omega\|, \|\omega - \omega u\|\} \leq \sqrt{2|1 - \omega(u)|}.$$

Proof of Lemma 1. Let $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ be the GNS representation associated with the state ω on $\mathcal{A}$. Then $\omega(a) = \langle \pi_\omega(a)\xi_\omega, \xi_\omega\rangle$ for all $a \in \mathcal{A}$. We have

$$\|\omega - u\omega\| \leq \|\xi_\omega - \pi_\omega(u)\xi_\omega\| \leq \sqrt{2(1 - \Re\omega(u))} \leq \sqrt{2|1 - \omega(u)|}.$$

Likewise, we get $\|\omega - \omega u\| \leq \sqrt{2|1 - \omega(u)|}$. □

Lemma 1 implies in particular that when $\omega(u) = 1$, then

$$\omega = u\omega = \omega u.$$

Lemma 2. *Let $Q \subset (M, \tau)$ be an amenable von Neumann subalgebra and $u \in \mathcal{N}_M(Q)$. Then $\langle Q, u\rangle$ is amenable.*

Proof of Lemma 2. We may choose a Q-central state Ω on $\mathbf{B}(\mathrm{L}^2(M))$ such that $\Omega|_M = \tau$. Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbf{N}) \setminus \mathbf{N}$ and define the state $\Phi \in \mathbf{B}(\mathrm{L}^2(M))$ by the formula

$$\Phi(T) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=0}^{n-1} \Omega(u^k T u^{-k}), \ \forall T \in \mathbf{B}(\mathrm{L}^2(M)).$$

It is clear that Φ is $(Q \cup \{u^{\mathbf{Z}}\})$-central and $\Phi|_M = \tau$. By the Cauchy-Schwarz Inequality, we have

$$|\Phi(Tx)| \leq \|T\| \, \|x\|_2 \text{ and } |\Phi(xT)| \leq \|T\| \, \|x\|_2, \ \forall T \in \mathbf{B}(\mathrm{L}^2(M)), \forall x \in M.$$

This implies that Φ is a $\langle Q, u\rangle$-central state and so $\langle Q, u\rangle$ is amenable. □

Proof of Theorem 8. Since Γ has CMAP, let $f_n : \Gamma \to \mathbf{C}$ be a sequence of finitely supported functions such that $f_n \to 1$ pointwise and such that $\lim_n \|f_n\|_{\mathrm{cb}} = 1$. For every $n \in \mathbf{N}$, define the normal completely bounded map $\varphi_n : M \to M$ by $\varphi_n(bu_g) = f_n(g)bu_g$. We have $\lim_n \|\varphi_n(x) - x\|_2 = 0$ for every $x \in M$ and $\lim_n \|\varphi_n\|_{\mathrm{cb}} = 1$.

Claim. For every $n \in \mathbf{N}$, there exists a unique normal linear functional $\mu_n \in \mathcal{M}_*$ such that

$$\mu_n(\pi(x)\rho(a^{\mathrm{op}})) = \tau(\varphi_n(x)a),\ \forall a \in A, \forall x \in M.$$

Proof of the Claim. The uniqueness of μ_n follows from the facts that $\pi(M)$ and $\rho(A^{\mathrm{op}})$ commute and together generate $\mathcal{M}$.

Define $\mu_n \in \mathcal{M}_*$ by

$$\mu_n(S) = \sum_{h \in \mathrm{supp}(f_n)} f_n(h)\langle S(1 \otimes \delta_e), 1 \otimes \delta_h\rangle,\ \forall S \in \mathcal{M}.$$

Hence for every $a \in A$, every $b \in B$ and every $g \in \Gamma$, we have

$$\begin{aligned}\mu_n(\pi(bu_g)\rho(a^{\mathrm{op}})) &= \sum_{h \in \mathrm{supp}(f_n)} f_n(h)\langle bu_g\, Ja^*J1 \otimes \delta_g, 1 \otimes \delta_h\rangle \\ &= \delta_{g \in \mathrm{supp}(f_n)}\, f_n(g)\tau(bu_g\, a) \\ &= \tau(\delta_{g \in \mathrm{supp}(f_n)} f_n(g) bu_g\, a) \\ &= \tau(\varphi_n(bu_g)\, a).\end{aligned}$$

Since φ_n and μ_n are normal, we get $\mu_n(\pi(x)\rho(a^{\mathrm{op}})) = \tau(\varphi_n(x)a)$ for every $a \in A$ and every $x \in M$. This finishes the proof of the Claim. □

For the moment, we have little control on the norm $\|\mu_n\|$. Using the amenability of A, we will show now that $\lim_n \|\mu_n\| = 1$ and that $\omega_n = \frac{1}{\|\mu_n\|}|\mu_n| \in \mathcal{M}_*$ is a sequence of normal states which satisfies the conclusion of Theorem 8.

Put $P = \mathcal{G}''$. Let $Q \subset P$ be any amenable von Neumann algebra. Using Theorem 3(4), the functional $\Omega_Q : M \otimes_{\min} Q^{\mathrm{op}} \to \mathbf{C} : x \otimes y^{\mathrm{op}} \mapsto \tau(E_Q(x)y)$ is a well-defined state on the C*-algebra $M \otimes_{\min} Q^{\mathrm{op}}$. Moreover, the linear map $\varphi_n \otimes E_{Q^{\mathrm{op}}} : M \otimes_{\min} P^{\mathrm{op}} \to M \otimes_{\min} Q^{\mathrm{op}}$ is completely bounded. Put $\mu_n^Q = \Omega_Q \circ (\varphi_n \otimes E_{Q^{\mathrm{op}}}) : M \otimes_{\min} P^{\mathrm{op}} \to \mathbf{C}$ and observe that $\lim_n \|\mu_n^Q\| = 1$ since $\lim_n \|\varphi_n\|_{\mathrm{cb}} = 1$.

Define the $*$-homomorphisms

$$\pi \otimes \rho : M \otimes_{\max} A^{\mathrm{op}} \to \mathcal{M} : x \otimes a^{\mathrm{op}} \mapsto \pi(x)\rho(a^{\mathrm{op}})$$

and

$$\lambda : M \otimes_{\max} A^{\mathrm{op}} \to M \otimes_{\min} A^{\mathrm{op}} : x \otimes a^{\mathrm{op}} \mapsto x \otimes a^{\mathrm{op}}.$$

Since $\mu_n^A \circ \lambda = \mu_n \circ (\pi \otimes \rho)$, we have $\|\mu_n^A\| = \|\mu_n\|$ and so $\lim_n \|\mu_n\| = 1$. Indeed, on the one hand, since λ is surjective, we have $\|\mu_n^A\| \leq \|\mu_n\|$.

On the other hand, we have $\|\mu_n\|_{(\pi\otimes\rho)(M\otimes_{\max}A^{\mathrm{op}})} \leq \|\mu_n^A\|$. Since μ_n is normal and since $(\pi\otimes\rho)(M\otimes_{\max}A^{\mathrm{op}})$ is ultraweakly dense in $\mathcal{M}$, we have $\|\mu_n\| = \|\mu_n\|_{(\pi\otimes\rho)(M\otimes_{\max}A^{\mathrm{op}})}$ by Kaplansky's Density Theorem. Altogether, we obtain $\|\mu_n^A\| = \|\mu_n\|$.

By construction, we also have $\mu_n(1) = \tau(\varphi_n(1)) \to 1$ since $\varphi_n(1) \to 1$. Write $\mu_n = u_n|\mu_n|$ for the polar decomposition of μ_n. We have $\||\mu_n|\| = \|\mu_n\|$ and so $\omega_n = \frac{1}{\|\mu_n\|}|\mu_n| \in \mathcal{M}_*$ is a normal state. Moreover, thanks to Lemma 1, we have

$$\|\omega_n - \mu_n\| = \|\omega_n - \|\mu_n\|\, u_n\omega_n\| \to 0$$

since $\|\mu_n\| \to 1$ and $\omega_n(u_n) = \frac{1}{\|\mu_n\|}|\mu_n|(u_n) = \frac{1}{\|\mu_n\|}\mu_n(1) \to 1$.

By construction, we have $\mu_n(\pi(x)) = \tau(\varphi_n(x)) \to \tau(x)$ for every $x \in M$ and $\mu_n(\pi(a)\rho(\overline{a})) = \tau(\varphi_n(a)a^*) \to \tau(aa^*) = 1$ for every $a \in \mathcal{U}(A)$. Since $\|\omega_n - \mu_n\| \to 0$, the same applies to ω_n and so the sequence ω_n satisfies the first two items of Theorem 8.

Fix $u \in \mathcal{G}$. By Lemma 2, we know that $Q = \langle A, u\rangle$ is amenable and so we can define $\mu_n^Q \in (M \otimes_{\min} P^{\mathrm{op}})^*$. We then have $\limsup_n \|\mu_n^Q\| = 1$ and $\lim_n \mu_n^Q(1) = 1$ since $\lim_n \varphi_n(1) = 1$. Moreover,

$$\mu_n^Q(u\otimes\overline{u}) = \tau(\varphi_n(u)u^*) \to 1.$$

Another application of Lemma 1 yields

$$\left\|\mu_n^Q \circ \mathrm{Ad}(u\otimes\overline{u}) - \mu_n^Q\right\| \to 0.$$

But then we also have

$$\left\|\mu_n^Q \circ \mathrm{Ad}(u\otimes\overline{u}) \circ (\mathrm{id}\otimes E_{A^{\mathrm{op}}}) - \mu_n^Q \circ (\mathrm{id}\otimes E_{A^{\mathrm{op}}})\right\| \to 0.$$

Since $u \in \mathcal{N}_M(A)$, we have $\mathrm{Ad}(u^*) \circ E_A = E_A \circ \mathrm{Ad}(u^*)$. Since moreover $\mu_n^A = \mu_n^Q \circ (\mathrm{id}\otimes E_{A^{\mathrm{op}}})$, it follows that

$$\left\|\mu_n^A \circ \mathrm{Ad}(u\otimes\overline{u}) - \mu_n^A\right\| \to 0.$$

Observing that

$$\mu_n^A \circ \mathrm{Ad}(u\otimes\overline{u}) \circ \lambda = \mu_n \circ (\mathrm{Ad}(\pi(u)) \circ \beta_u) \circ (\pi\otimes\rho)$$

we obtain that

$$\left(\mu_n^A \circ \mathrm{Ad}(u\otimes\overline{u}) - \mu_n^A\right) \circ \lambda = (\mu_n \circ (\mathrm{Ad}(\pi(u)) \circ \beta_u) - \mu_n) \circ (\pi\otimes\rho)$$

and so $\lim_n \|\mu_n \circ (\mathrm{Ad}(\pi(u)) \circ \beta_u) - \mu_n\| = 0$ by the same reasoning as before. Since $\lim_n \|\omega_n - \mu_n\| = 0$, the same applies to ω_n and we get $\lim_n \|\omega_n \circ (\mathrm{Ad}(\pi(u)) \circ \beta_u) - \omega_n\| = 0$. This finishes the proof of Theorem 8. □

In order to prepare for the third lecture, we prove the following useful property of the free groups. Put $\Gamma = \mathbf{F}_n$. Denote by $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ (resp. $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$) the *left* (resp. *right*) regular representation. Denote by $\mathcal{S}_{\mathrm{fin}}$ the directed set of finite subsets of Γ. For every $\mathcal{F} \subset \Gamma$, write $P_{\mathcal{F}} : \ell^2(\Gamma) \to \ell^2(\mathcal{F})$ for the canonical orthogonal projection.

Proposition 1 (Akemann-Ostrand)**.** *There exists an isometry* $W : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ *such that*

$$\lim_{\mathcal{F} \in \mathcal{S}_{\mathrm{fin}}} \left\| \left((\lambda_g \otimes \rho_h) W - W \lambda_g \rho_h \right) P_{\Gamma \setminus \mathcal{F}} \right\| = 0, \ \forall g, h \in \Gamma.$$

Proof. For every $g \in \Gamma$, denote by $(g_0, \dots, g_n)$ the unique geodesic path from $g_0 = e$ to $g_n = g$ in the free group Γ. Put $c(g) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \delta_{g_i} \in \ell^2(\Gamma)$ and observe that $\|c(g)\|_2 = 1$. By construction, we have

$$\lim_{x \to \infty} \|c(gxh) - \lambda_g c(x)\|_2 = 0, \ \forall g, h \in \Gamma.$$

Define the isometry $V : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ by $V\delta_x = c(x) \otimes \delta_x$. By construction, we have

$$\lim_{\mathcal{F} \in \mathcal{S}_{\mathrm{fin}}} \left\| \left((\lambda_g \otimes \lambda_g \rho_h) V - V \lambda_g \rho_h \right) P_{\Gamma \setminus \mathcal{F}} \right\| = 0, \ \forall g, h \in \Gamma.$$

Define $U \in \mathcal{U}(\ell^2(\Gamma) \otimes \ell^2(\Gamma))$ by $U(\delta_x \otimes \delta_y) = \delta_x \otimes \delta_{x^{-1}y}$. We have that $U(\lambda_g \otimes \lambda_g \rho_h) U^* = \lambda_g \otimes \rho_h$ for every $g, h \in \Gamma$. Letting $W = UV$, we are done. $\square$

Lecture 3

In Lectures 3 and 4, we prove Theorem 6 in the special case when $B = \mathbf{C}$. This is only a technical simplification and conceptually the proof remains the same. In that particular case, the result is due to Ozawa-Popa.

Theorem 9 (Ozawa-Popa)**.** *Let* $A \subset \mathrm{L}(\mathbf{F}_n)$ *be a diffuse amenable von Neumann subalgebra. Then* $\mathcal{N}_{\mathrm{L}(\mathbf{F}_n)}(A)''$ *is amenable.*

A tracial von Neumann algebra satisfying the conclusion of Theorem 9 is called *strongly solid*. Put $\Gamma = \mathbf{F}_n$ and $M = \mathrm{L}(\mathbf{F}_n)$. Since we assumed that $B = \mathbf{C}$, the link algebra is simply $\mathcal{L} = JAJ \cong A^{\mathrm{op}} \subset \mathbf{B}(\ell^2(\Gamma))$. Recall that we have a sequence $\omega_n \in \mathrm{L}^1(A^{\mathrm{op}} \mathbin{\overline{\otimes}} M)_+$ which satisfies the conclusion of Theorem 8. If we take $\xi_n = \omega_n^{1/2} \in \mathrm{L}^2(A^{\mathrm{op}} \mathbin{\overline{\otimes}} M)_+$, we have $\|\xi_n\|_2 = 1$ for every $n \in \mathbf{N}$ and

- $\lim_n \langle (1 \otimes x)\xi_n, \xi_n \rangle = \tau(x)$ for every $x \in M$.
- $\lim_n \|(\overline{a} \otimes a)\xi_n - \xi_n\|_2 = 0$ for every $a \in \mathcal{U}(A)$.
- $\lim_n \| \operatorname{Ad}(\overline{u} \otimes u)\xi_n - \xi_n\|_2 = 0$ for every $u \in \mathcal{G}$.

The third item is obtained by applying the Powers-Størmer Inequality.

Step 1. For every finite subset $\mathcal{F} \subset \Gamma$, we have $\lim_n \|(1 \otimes P_{\mathcal{F}})\xi_n\|_2 = 0$.

We prove Step 1 by contradiction and assume that there exists a finite subset $\mathcal{F}_0 \subset \Gamma$ such that $\limsup_n \|(1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 > 0$. Up to replacing ξ_n by a subsequence, we may assume that there exists $\delta > 0$ such that

$$\liminf_n \|(1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 > \delta.$$

Using the fact that A is diffuse, we will construct by induction an increasing sequence of finite subsets $\mathcal{F}_k \subset \Gamma$ such that

$$\liminf_n \|(1 \otimes P_{\mathcal{F}_k})\xi_n\|_2 > 2^{k/2}\delta.$$

This will lead to a contradiction. We only need to construct a finite subset $\mathcal{F}_1 \subset \Gamma$ such that $\mathcal{F}_0 \subset \mathcal{F}_1$ and $\liminf_n \|(1 \otimes P_{\mathcal{F}_1})\xi_n\|_2 > 2^{1/2}\delta$.

Claim. For every $x \in M$, we have

$$\limsup_n \|(1 \otimes x)(1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 \leq |\mathcal{F}_0| \, \|x\|_2.$$

Proof of the Claim. Observe that in order to prove the claim, it suffices to show that for every $g \in \Gamma$ and every $x \in M$ we have

$$\limsup_n \|(1 \otimes x)(1 \otimes P_g)\xi_n\|_2 \leq \|x\|_2.$$

We simply write $P_g = P_{\{g\}}$. Observe that $u_g P_e u_g^* = P_g$. With $y = u_g^* x^* x u_g$, we have

$$\begin{aligned}\|(1 \otimes x)(1 \otimes P_g)\xi_n\|_2^2 &= \langle (1 \otimes P_e)(1 \otimes y)(1 \otimes P_e)(1 \otimes u_g^*)\xi_n, (1 \otimes u_g^*)\xi_n\rangle \\ &= \tau(y)\langle (1 \otimes P_e)(1 \otimes u_g^*)\xi_n, (1 \otimes u_g^*)\xi_n\rangle \\ &= \tau(y)\|(1 \otimes P_e)(1 \otimes u_g^*)\xi_n\|_2^2.\end{aligned}$$

It follows that

$$\limsup_n \|(1 \otimes x)(1 \otimes P_g)\xi_n\|_2 \leq \tau(y) \limsup_n \|(1 \otimes u_g^*)\xi_n\|_2^2 = \|x\|_2^2.$$

This finishes the proof of the Claim. ∎

By assumption, we can take $\varepsilon > 0$ such that

$$\limsup_n \|\xi_n - (1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 < \sqrt{1 - \delta^2} - \varepsilon.$$

Write $x = \sum_{g \in \Gamma} (x)_g u_g$ for the Fourier expansion of $x \in M$ where $(x)_g = \tau(x u_g^*)$.

Claim. There exists $a \in \mathcal{U}(A)$ and $v \in \mathbf{C}[\Gamma]$ such that

$$\|a - v\|_2 \leq \frac{\varepsilon}{|\mathcal{F}_0|} \text{ and } (v)_g = 0, \forall g \in \mathcal{F}_0\mathcal{F}_0^{-1}.$$

Proof of the Claim. Since A is diffuse, we can find $a \in \mathcal{U}(A)$ such that

$$|(a)_{gh^{-1}}| \leq \frac{\varepsilon}{3|\mathcal{F}_0|^3}, \forall g, h \in \mathcal{F}_0.$$

This implies that $\|P_{\mathcal{F}_0\mathcal{F}_0^{-1}}(a)\|_2 \leq \varepsilon/(3|\mathcal{F}_0|)$. Take $a' \in \mathbf{C}[\Gamma]$ such that $\|a - a'\|_2 \leq \varepsilon/(3|\mathcal{F}_0|)$ and put $v = a' - P_{\mathcal{F}_0\mathcal{F}_0^{-1}}(a')$. Then we have $(v)_g = 0$ for every $g \in \mathcal{F}_0\mathcal{F}_0^{-1}$, $\|P_{\mathcal{F}_0\mathcal{F}_0^{-1}}(a')\|_2 \leq 2\varepsilon/(3|\mathcal{F}_0|)$ and

$$\|a - v\|_2 = \|a - a'\|_2 + \|a' - v\|_2 \leq \frac{\varepsilon}{|\mathcal{F}_0|}.$$ ∎

Since $\lim_n \|\xi_n - (\overline{a} \otimes a)\xi_n\|_2 = 0$, it follows that

$$\begin{aligned}\limsup_n \|\xi_n - (\overline{a} \otimes a)(1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 &= \limsup_n \|(a^{\mathrm{op}} \otimes a^*)\xi_n - (1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 \\ &= \limsup_n \|\xi_n - (1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 \\ &< \sqrt{1-\delta^2} - \varepsilon.\end{aligned}$$

Combining the two Claims, we get

$$\limsup_n \|\xi_n - (\overline{a} \otimes v)(1 \otimes P_{\mathcal{F}_0})\xi_n\|_2 < \sqrt{1-\delta^2}.$$

Define the finite subset $\mathcal{S} = \{g \in \Gamma : (v)_g \neq 0\}$. By assumption, we have $\mathcal{S} \cap \mathcal{F}_0\mathcal{F}_0^{-1} = \emptyset$, that is, $\mathcal{S}\mathcal{F}_0 \cap \mathcal{F}_0 = \emptyset$. We have that

$$(\overline{a} \otimes v)(1 \otimes P_{\mathcal{F}_0})\xi_n = (1 \otimes vP_{\mathcal{F}_0})(\overline{a} \otimes 1)\xi_n$$

lies in the range of $1 \otimes P_{\mathcal{S}\mathcal{F}_0}$. This implies that

$$\limsup_n \|\xi_n - (1 \otimes P_{\mathcal{S}\mathcal{F}_0})\xi_n\|_2 < \sqrt{1-\delta^2}$$

and hence

$$\liminf_n \|(1 \otimes P_{\mathcal{S}\mathcal{F}_0})\xi_n\|_2 > \delta.$$

If we put $\mathcal{F}_1 = \mathcal{S}\mathcal{F}_0 \cup \mathcal{F}_0$, since the vectors $(1 \otimes P_{\mathcal{F}_0})\xi_n$ and $(1 \otimes P_{\mathcal{S}\mathcal{F}_0})\xi_n$ are orthogonal, we get

$$\liminf_n \|(1 \otimes P_{\mathcal{F}_1})\xi_n\|_2 > \sqrt{2}\delta.$$

This proves the induction and leads to a contradiction. This finishes the proof of Step 1.

Lecture 4

We continue the proof of Theorem 9.

Step 2. We have that $\mathcal{N}_M(A)''$ is amenable.

Recall that $\xi_n \in \mathrm{L}^2(A^{\mathrm{op}} \mathbin{\overline{\otimes}} M)_+$ for every $n \in \mathbf{N}$. Put $P = \mathcal{N}_M(A)''$ and $\mathcal{M} = P^{\mathrm{op}} \mathbin{\overline{\otimes}} M$. Put $\mathcal{H} = \mathrm{L}^2(\mathcal{M}) = \mathrm{L}^2(P^{\mathrm{op}}) \otimes \mathrm{L}^2(M)$ endowed with the identity $\mathcal{M}$-$\mathcal{M}$-bimodule structure:

$$x\xi y = x\mathcal{J}y^*\mathcal{J}\xi,\ \forall x, y \in \mathcal{M}, \forall \xi \in \mathcal{H},$$

where $\mathcal{J} : \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ is the canonical conjugation. Denote by $\pi_{\mathcal{H}} : \mathcal{M} \otimes_{\mathrm{alg}} \mathcal{M}^{\mathrm{op}} \to \mathbf{B}(\mathcal{H})$ the corresponding unital $*$-representation. We will regard $\xi_n \in \mathcal{H}$ for every $n \in \mathbf{N}$.

Likewise, put $\mathcal{K} = \mathcal{H} \otimes \ell^2(\Gamma)$ endowed with the following $\mathcal{M}$-$\mathcal{M}$-bimodule structure. Put

$$(y^{\mathrm{op}} \otimes u_g) \cdot (\xi \otimes \delta_h \otimes \delta_k) = y^{\mathrm{op}}\xi \otimes \delta_{gh} \otimes \delta_k$$

and

$$(\xi \otimes \delta_h \otimes \delta_k) \cdot (y^{\mathrm{op}} \otimes u_g) = \xi y^{\mathrm{op}} \otimes \delta_h \otimes \delta_{kg}$$

for every $y \in P$, every $\xi \in \mathrm{L}^2(P^{\mathrm{op}})$ and every $g, h, k \in \Gamma$. Denote by $\pi_{\mathcal{K}} : \mathcal{M} \otimes_{\mathrm{alg}} \mathcal{M}^{\mathrm{op}} \to \mathbf{B}(\mathcal{K})$ the corresponding unital $*$-representation.

We moreover define the following unital $*$-subalgebras of $\mathcal{M} \otimes_{\mathrm{alg}} \mathcal{M}^{\mathrm{op}}$ by

$$\mathcal{D} = P^{\mathrm{op}} \otimes_{\mathrm{alg}} M \otimes_{\mathrm{alg}} P \otimes_{\mathrm{alg}} M^{\mathrm{op}}$$

and

$$\mathcal{D}_0 = P^{\mathrm{op}} \otimes_{\mathrm{alg}} M_0 \otimes_{\mathrm{alg}} P \otimes_{\mathrm{alg}} M_0^{\mathrm{op}}$$

with $M_0 = \mathbf{C}[\Gamma]$.

Choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbf{N}) \setminus \mathbf{N}$ and define the state $\Phi \in \mathbf{B}(\mathcal{H})^*$ by

$$\Phi(T) = \lim_{\mathcal{U}} \langle T\xi_n, \xi_n\rangle, \ \forall T \in \mathbf{B}(\mathcal{H}).$$

Proposition 2. *The state $\Phi \in \mathbf{B}(\mathcal{H})^*$ satisfies the following properties:*

1. $\Phi(1^{\mathrm{op}} \otimes x) = \Phi(\mathcal{J}(1^{\mathrm{op}} \otimes x^*)\mathcal{J}) = \tau(x)$ *for every* $x \in M$.
2. $|\Phi(T(1^{\mathrm{op}} \otimes x))| \leq \|T\| \, \|x\|_2$ *for every* $T \in \mathbf{B}(\mathcal{H})$ *and every* $x \in M$.
3. $\Phi(\pi_{\mathcal{H}}(\overline{u} \otimes u \otimes u \otimes \overline{u})) = 1$ *for every* $u \in \mathcal{N}_M(A)$.
4. $\Phi(T) = \Phi(T(1 \otimes P_{\Gamma \setminus \mathcal{F}}))$ *for every* $T \in \mathbf{B}(H)$ *and every finite subset* $\mathcal{F} \subset \Gamma$.
5. $|\Phi(\pi_{\mathcal{H}}(S))| \leq \|\pi_{\mathcal{K}}(S)\|$ *for every* $S \in \mathcal{D}_0$.

Proof. For (1), observe that for every $x \in M$, we have

$$\Phi(1 \otimes x) = \lim_{\mathcal{U}} \langle (1 \otimes x)\xi_n, \xi_n\rangle = \tau(x).$$

Since $\xi_n \in \mathrm{L}^2(A^{\mathrm{op}} \mathbin{\overline{\otimes}} M)_+$ is positive, we have $\mathcal{J}\xi_n = \xi_n$ and so

$$\Phi(\mathcal{J}(1 \otimes x^*)\mathcal{J}) = \lim_{\mathcal{U}} \langle \mathcal{J}(1 \otimes x^*)\mathcal{J}\xi_n, \xi_n\rangle = \lim_{\mathcal{U}} \langle \xi_n, (1 \otimes x^*)\xi_n\rangle = \tau(x).$$

For (2), apply the Cauchy-Schwarz Inequality. For (3), observe that for every $u \in \mathcal{N}_M(A)$, we have

$$\Phi(\pi_{\mathcal{H}}(\overline{u} \otimes u \otimes u \otimes \overline{u})) = \lim_{\mathcal{U}} \langle (\overline{u} \otimes u)\xi_n(\overline{u} \otimes u)^*, \xi_n\rangle = 1.$$

For (4), observe that thanks to Step 1, for every $T \in \mathbf{B}(H)$ and every finite subset $\mathcal{F} \subset \Gamma$, we have

$$\begin{aligned} \Phi(T(1 \otimes P_{\Gamma \setminus \mathcal{F}})) &= \lim_{\mathcal{U}} \langle T(1 \otimes P_{\Gamma \setminus \mathcal{F}})\xi_n, \xi_n\rangle \\ &= \lim_{\mathcal{U}} \langle T\xi_n, \xi_n\rangle \\ &= \Phi(T). \end{aligned}$$

For (5), observe that thanks to Proposition 1, for every $S \in \mathcal{D}_0$, we have

$$\lim_{\mathcal{F} \in \mathcal{S}_{\mathrm{fin}}} \|\pi_{\mathcal{H}}(S)(1 \otimes P_{\Gamma \setminus \mathcal{F}}) - (1 \otimes W^*)\pi_{\mathcal{K}}(S)(1 \otimes W)(1 \otimes P_{\Gamma \setminus \mathcal{F}})\| = 0.$$

In particular, using moreover item (3) above, we obtain

$$\begin{aligned}|\Phi(\pi_{\mathcal{H}}(S))| &= \limsup_{\mathcal{F}\in\mathcal{S}_{\mathrm{fin}}} |\Phi(\pi_{\mathcal{H}}(S)(1\otimes P_{\Gamma\setminus\mathcal{F}}))| \\ &\leq \limsup_{\mathcal{F}\in\mathcal{S}_{\mathrm{fin}}} \|\pi_{\mathcal{H}}(S)(1\otimes P_{\Gamma\setminus\mathcal{F}})\| \\ &\leq \|\pi_{\mathcal{K}}(S)\|.\end{aligned}$$

□

Observe that *a priori* the last item in Proposition 2 only holds for every $S \in \mathcal{D}_0$. The subtle part of the proof consists in showing that in fact it holds for every $S \in \mathcal{D}$.

Lemma 3. *We have $|\Phi(\pi_{\mathcal{H}}(S))| \leq \|\pi_{\mathcal{K}}(S)\|$ for every $S \in \mathcal{D}$.*

Proof. Since Γ has the CMAP, we may choose a sequence of finitely supported functions $f_n : \Gamma \to \mathbf{C}$ such that $f_n \to 1$ pointwise and $\limsup_n \|f_n\|_{\mathrm{cb}} = 1$. Denote by $\mathrm{m}_n : M \to M$ the corresponding Herz-Schur multiplier defined by $\mathrm{m}_n(u_g) = f_n(g)u_g$ for every $g \in \Gamma$. Likewise, define $\overline{\mathrm{m}}_n : M^{\mathrm{op}} \to M^{\mathrm{op}}$ by $\overline{\mathrm{m}}_n(\overline{u_g}) = f_n(g)\overline{u_g}$ for every $g \in \Gamma$.

Observe that for every $x \in M$, we have $\lim_n \|x - \mathrm{m}_n(x)\|_2 = 0$ and $\lim_n \|x^{\mathrm{op}} - \overline{\mathrm{m}}_n(x^{\mathrm{op}})\|_2 = 0$. Moreover, for every $S \in \mathcal{D}$, we have

$$(\mathrm{id}\otimes \mathrm{m}_n \otimes \mathrm{id} \otimes \overline{\mathrm{m}}_n)(S) \in \mathcal{D}_0.$$

Claim. For every $S \in \mathcal{D}$, we have

$$\Phi(\pi_{\mathcal{H}}(S)) = \lim_n \Phi\left(\pi_{\mathcal{H}}((\mathrm{id}\otimes \mathrm{m}_n \otimes \mathrm{id} \otimes \overline{\mathrm{m}}_n)(S))\right), \forall S \in \mathcal{D}.$$

Proof of the Claim. Let $S = y^{\mathrm{op}} \otimes x_1 \otimes z \otimes x_2^{\mathrm{op}} \in \mathcal{D}$ for some $x_1, x_2 \in M$ and $y, z \in P$. Using Proposition 2 (2), we have

$$\begin{aligned}&\limsup_n |\Phi\left(\pi_{\mathcal{H}}(y^{\mathrm{op}} \otimes (x_1 - \mathrm{m}_n(x_1)) \otimes z \otimes \mathrm{m}_n(x_2^{\mathrm{op}}))\right)| \\ &\leq \limsup_n \|\pi_{\mathcal{H}}\left(y^{\mathrm{op}} \otimes 1 \otimes z \otimes \mathrm{m}_n(x_2^{\mathrm{op}})\right)\| \, \|x_1 - \mathrm{m}_n(x_1)\|_2 \\ &\leq \limsup_n \|\mathrm{m}_n(x_2^{\mathrm{op}})\| \, \|y\| \, \|z\| \, \lim_n \|x_1 - \mathrm{m}_n(x_1)\|_2 = 0.\end{aligned}$$

Likewise, we show that

$$\lim_n \Phi\left(\pi_{\mathcal{H}}(y^{\mathrm{op}} \otimes x_1 \otimes z \otimes (x_2^{\mathrm{op}} - \mathrm{m}_n(x_2^{\mathrm{op}})))\right) = 0.$$

The claim now follows by linearity. □

Observe that for every $S \in \mathcal{D}$ and every $n \in \mathbf{N}$, we have

$$\pi_{\mathcal{K}}((\mathrm{id}\otimes \mathrm{m}_n \otimes \mathrm{id} \otimes \overline{\mathrm{m}}_n)(S)) = (\mathrm{id}\otimes \mathrm{m}_n \otimes \overline{\mathrm{m}}_n)(\pi_{\mathcal{K}}(S)).$$

It follows that for every $S \in \mathcal{D}$, we get

$$\begin{aligned}
|\Phi(\pi_{\mathcal{H}}(S))| &= \lim_n |\Phi\left(\pi_{\mathcal{H}}((\mathrm{id} \otimes \mathrm{m}_n \otimes \mathrm{id} \otimes \overline{\mathrm{m}}_n)(S))\right)| \\
&\leq \limsup_n \|\pi_{\mathcal{K}}((\mathrm{id} \otimes \mathrm{m}_n \otimes \mathrm{id} \otimes \overline{\mathrm{m}}_n)(S))\| \\
&\leq \limsup_n \|(\mathrm{id} \otimes \mathrm{m}_n \otimes \overline{\mathrm{m}}_n)(\pi_{\mathcal{K}}(S))\| \\
&\leq \limsup_n \|f_n\|_{\mathrm{cb}}^2 \, \|\pi_{\mathcal{K}}(S)\| \\
&= \|\pi_{\mathcal{K}}(S)\|.
\end{aligned}$$

This finishes the proof of the lemma. □

We can then uniquely define a bounded linear functional $\Psi : \pi_{\mathcal{K}}(\mathcal{D}) \to \mathbf{C}$ by the formula $\Psi(\pi_{\mathcal{K}}(S)) = \Phi(\pi_{\mathcal{H}}(S))$. Observe that $\|\Psi\| = 1 = \Psi(1)$. By the Hahn-Banach Theorem and since $\pi_{\mathcal{K}}(\mathcal{D}) \subset \mathbf{B}(\mathcal{K})$, we may extend Ψ to $\mathbf{B}(\mathcal{K})$ without increasing the norm. Since $\mathbf{B}(\mathcal{K})$ is a unital C*-algebra and $\|\Psi\| = 1 = \Psi(1)$, it follows that Ψ is a state on $\mathbf{B}(\mathcal{K})$. Observe that

$$\Psi(\pi_{\mathcal{K}}(\overline{u} \otimes u \otimes u \otimes \overline{u})) = \Phi(\pi_{\mathcal{H}}(\overline{u} \otimes u \otimes u \otimes \overline{u})) = 1, \forall u \in \mathcal{N}_M(A).$$

Put $\Omega(S) = \Psi(1 \otimes S \otimes 1)$ for every $S \in \mathbf{B}(\ell^2(\Gamma))$ and observe that Ω is a state on $\mathbf{B}(\ell^2(\Gamma))$. For every $x \in M$, we have

$$\begin{aligned}
\Omega(x) &= \Psi(1 \otimes x \otimes 1) \\
&= \Psi(\pi_{\mathcal{K}}(1^{\mathrm{op}} \otimes x \otimes 1 \otimes 1^{\mathrm{op}})) \\
&= \Phi(\pi_{\mathcal{H}}(1^{\mathrm{op}} \otimes x \otimes 1 \otimes 1^{\mathrm{op}})) \\
&= \Phi(1 \otimes x) = \tau(x).
\end{aligned}$$

Moreover, using Lemma 1, for every $T \in \mathbf{B}(\ell^2(\Gamma))$ and every $u \in \mathcal{N}_M(A)$, we have

$$\begin{aligned}
\Omega(uT) &= \Psi(1 \otimes uT \otimes 1) \\
&= \Psi\left(\pi_{\mathcal{K}}(\overline{u} \otimes u \otimes u \otimes \overline{u})\,(u^{\mathrm{op}} J_{P^{\mathrm{op}}} u^{\mathrm{op}} J_{P^{\mathrm{op}}} \otimes T \otimes J_M u^* J_M)\right) \\
&= \Psi\left((u^{\mathrm{op}} J_{P^{\mathrm{op}}} u^{\mathrm{op}} J_{P^{\mathrm{op}}} \otimes T \otimes J_M u^* J_M)\,\pi_{\mathcal{K}}(\overline{u} \otimes u \otimes u \otimes \overline{u})\right) \\
&= \Psi(1 \otimes Tu \otimes 1) = \Omega(Tu).
\end{aligned}$$

Since $\Omega|_M = \tau$, it follows from the Cauchy-Schwarz Inequality that

$$|\Omega(Sx)| \leq \|S\| \, \|x\|_2 \text{ and } |\Omega(xS)| \leq \|S\| \, \|x\|_2, \forall S \in \mathbf{B}(\ell^2(\Gamma)), \forall x \in M.$$

Since $\Omega(xS) = \Omega(Sx)$ for every $S \in \mathbf{B}(\ell^2(\Gamma))$ and every x in the unital $*$-algebra generated by $\mathcal{N}_M(A)$, it follows that Ω is P-central. Therefore P is amenable. This concludes the proof of Theorem 9.

Lecture 5

In this lecture, we review some basic tools which arise in the study of type III von Neumann algebras. We then present a useful strategy to prove structural results for type III von Neumann algebras.

Connes-Takesaki's flow of weights. Let (M, φ) be a von Neumann algebra together with a faithful normal state. Denote by M^φ the centralizer of φ and by $\mathrm{c}_\varphi(M) = M \rtimes_\varphi \mathbf{R}$ the *continuous core* of M, that is, the crossed product of M with the modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbf{R}}$ associated with the faithful normal state φ. We have a canonical $*$-embedding $\pi_\varphi : M \to M \rtimes_\varphi \mathbf{R}$ and a canonical group of unitaries $(\lambda_\varphi(s))_{s\in\mathbf{R}}$ in $M \rtimes_\varphi \mathbf{R}$ such that

$$\pi_\varphi(\sigma_s^\varphi(x)) = \lambda_\varphi(s)\, \pi_\varphi(x)\, \lambda_\varphi(s)^*, \, \forall x \in M, s \in \mathbf{R}.$$

The unitaries $(\lambda_\varphi(s))_{s\in\mathbf{R}}$ generate a copy of $\mathrm{L}(\mathbf{R})$ inside $M \rtimes_\varphi \mathbf{R}$.

We denote by $\widehat{\varphi}$ the *dual weight* on $M \rtimes_\varphi \mathbf{R}$, which is a semifinite faithful normal weight on $M \rtimes_\varphi \mathbf{R}$ whose modular automorphism group $(\sigma_t^{\widehat{\varphi}})_{t\in\mathbf{R}}$ satisfies

$$\sigma_t^{\widehat{\varphi}}(\pi_\varphi(x)) = \pi_\varphi(\sigma_t^\varphi(x)), \, \forall x \in M \quad \text{and} \quad \sigma_t^{\widehat{\varphi}}(\lambda_\varphi(s)) = \lambda_\varphi(s), \, \forall s \in \mathbf{R}.$$

We denote by $(\theta_t^\varphi)_{t\in\mathbf{R}}$ the *dual action* on $M \rtimes_\varphi \mathbf{R}$, given by

$$\theta_t^\varphi(\pi_\varphi(x)) = \pi_\varphi(x) \text{ for all } x \in M \quad \text{and} \quad \theta_t^\varphi(\lambda_\varphi(s)) = \exp(\mathrm{i}ts)\lambda_\varphi(s), \, s \in \mathbf{R}.$$

Denote by h_φ the unique nonsingular positive selfadjoint operator affiliated with $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\varphi \mathbf{R}$ such that $h_\varphi^{\mathrm{i}s} = \lambda_\varphi(s)$ for all $s \in \mathbf{R}$. Then $\mathrm{Tr}_\varphi = \widehat{\varphi}(h_\varphi^{-1}\cdot)$ is a semifinite faithful normal trace on $M \rtimes_\varphi \mathbf{R}$ and the dual action θ^φ *scales* the trace Tr_φ:

$$\mathrm{Tr}_\varphi \circ \theta_t^\varphi = \exp(t)\mathrm{Tr}_\varphi, \forall t \in \mathbf{R}.$$

Note that Tr_φ is semifinite on $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\varphi \mathbf{R}$. Moreover, the canonical faithful normal conditional expectation $E_{\mathrm{L}(\mathbf{R})} : M \rtimes_\varphi \mathbf{R} \to \mathrm{L}(\mathbf{R})$ defined by $E_{\mathrm{L}(\mathbf{R})}(x\lambda_\varphi(s)) = \varphi(x)\lambda_\varphi(s)$ preserves the trace Tr_φ, that is,

$$\mathrm{Tr}_\varphi \circ E_{\mathrm{L}(\mathbf{R})} = \mathrm{Tr}_\varphi.$$

Thanks to Connes's Radon-Nikodym cocycle theorem, the semifinite von Neumann algebra $\mathrm{c}_\varphi(M)$, together with its trace Tr_φ and trace-scaling action θ^φ, "does not depend" on the choice of φ in the following precise sense. If ψ is another faithful normal state on M, there is a canonical surjective $*$-isomorphism $\Pi_{\psi,\varphi} : \mathrm{c}_\varphi(M) \to \mathrm{c}_\psi(M)$ such that $\Pi_{\psi,\varphi} \circ \pi_\varphi = \pi_\psi$, $\mathrm{Tr}_\psi \circ \Pi_{\psi,\varphi} = \mathrm{Tr}_\varphi$ and $\Pi_{\psi,\varphi} \circ \theta^\varphi = \theta^\psi \circ \Pi_{\psi,\varphi}$. Note however that $\Pi_{\psi,\varphi}$ does not map the subalgebra $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\varphi \mathbf{R}$ onto the subalgebra $\mathrm{L}(\mathbf{R}) \subset M \rtimes_\psi \mathbf{R}$. We will sometimes simply denote by $(\mathrm{c}(M), \mathrm{Tr}, \theta)$ the triple consisting in the continous core of M together with its canonical trace and canonical trace scaling dual action.

We summarize what we have explained above in the following proposition that we will use later.

Proposition 3. *Let $A, B \subset M$ be von Neumann subalgebras with expectation. Let φ_A (resp. φ_B) be a faithful normal state on M such that A (resp. B) is globally invariant under $(\sigma_t^{\varphi_A})$ (resp. $(\sigma_t^{\varphi_B})$). Put $\mathrm{c}(M) = M \rtimes_{\varphi_B} \mathbf{R}$, $\mathrm{c}(B) = B \rtimes_{\varphi_B} \mathbf{R}$ and $\mathrm{c}(A) = \Pi_{\varphi_B, \varphi_A}(A \rtimes_{\varphi_A} \mathbf{R})$.*

Then $\mathrm{c}(A) \subset \mathrm{c}(M)$ and $\mathrm{c}(B) \subset \mathrm{c}(M)$ are trace preserving inclusions of semifinite von Neumann algebras.

Definition 6. The restriction of the dual action $\theta : \mathbf{R} \curvearrowright \mathcal{Z}(\mathrm{c}(M))$ to the center of the continuous core $\mathrm{c}(M)$ is called the *flow of weights*. If M is a factor, then the flow of weights is ergodic.

A somewhat more concrete description of the flow of weights can be given when $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ arises from a free ergodic nonsingular action $\Gamma \curvearrowright (X, \mu)$ of a countable discrete group on a standard measure space. Indeed, define the *Radon-Nikodym cocycle* $\omega : \Gamma \times X \to \mathbf{R}$ by

$$\omega(g, x) = \log \left(\frac{\mathrm{d}g_* \mu}{\mathrm{d}\mu}(x) \right).$$

Then ω satisfies the 1-cocycle relation $\omega(gh, x) = \omega(g, hx)\omega(h, x)$ for every $g, h \in \Gamma$ and almost every $x \in X$. Endow $X \times \mathbf{R}$ with the σ-finite measure $m = \mu \otimes \exp(t)\mathrm{d}t$. Define the *Maharam extension* $\Gamma \curvearrowright (X \times \mathbf{R}, m)$ by

$$g \cdot (x, t) = (gx, t + \omega(g, x)).$$

The action of $\mathbf{R}$ by translation on the second variable induces an ergodic action on the standard measure space (Y, η) of the ergodic components of $\Gamma \curvearrowright (X \times \mathbf{R}, m)$. This flow $\mathbf{R} \curvearrowright (Y, \eta)$ is called the *Radon-Nikodym flow*. Observe that $\mathrm{c}(\mathrm{L}^\infty(X) \rtimes \Gamma) \cong \mathrm{L}^\infty(X \times \mathbf{R}) \rtimes \Gamma$ and the Radon-Nikodym flow $\mathbf{R} \curvearrowright (Y, \eta)$ coincides with the flow of weights $\mathbf{R} \curvearrowright \mathcal{Z}(\mathrm{c}(\mathrm{L}^\infty(X) \rtimes \Gamma))$.

By Takesaki's duality theorem, we have that $\mathrm{c}(M) \rtimes_\theta \mathbf{R} \cong M \mathbin{\overline{\otimes}} \mathbf{B}(\mathrm{L}^2(\mathbf{R}))$. In particular, M is amenable if and only if $\mathrm{c}(M)$ is amenable.

Definition 7. Let M be any von Neumann algebra. We say that $A \subset M$ is a *Cartan subalgebra* if A satisfies the following properties:

- There exists a faithful normal conditional expectation $E_A : M \to A$.
- A is maximal abelian in M, that is, $A' \cap M = A$.
- The group $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M.

We next show that Cartan inclusions behave well when passing to the continuous core.

Proposition 4. *Let $A \subset M$ be a Cartan subalgebra. Then $\mathrm{c}(A) \subset \mathrm{c}(M)$ is still a Cartan subalgebra.*

Proof. Let $E_A : M \to A$ be the (unique) faithful normal conditional expectation. Fix any faithful normal state τ on A and put $\varphi = \tau \circ E_A$. We will simply write $\mathrm{c}(M) = \mathrm{c}_\varphi(M)$ and $\mathrm{c}(A) = \mathrm{c}_\varphi(A)$. There exists a trace preserving faithful normal conditional expectation $E : \mathrm{c}(M) \to \mathrm{c}(A)$

Regard $\mathrm{c}(M) \subset M \mathbin{\overline{\otimes}} \mathbf{B}(\mathrm{L}^2(\mathbf{R}))$ and observe that $\mathrm{c}(A) = A \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{R})$. Since both $A \subset M$ and $\mathrm{L}(\mathbf{R}) \subset \mathbf{B}(\mathrm{L}^2(\mathbf{R}))$ are masas, $A \mathbin{\overline{\otimes}} \mathrm{L}(\mathbf{R}) \subset M \mathbin{\overline{\otimes}} \mathbf{B}(\mathrm{L}^2(\mathbf{R}))$ is a masa and so is $\mathrm{c}(A) \subset \mathrm{c}(M)$.

It remains to show that $\mathrm{c}(A) \subset \mathrm{c}(M)$ is regular. It suffices to show that $u\mathrm{c}(A)u^* = \mathrm{c}(A)$ for every $u \in \mathcal{N}_M(A)$. Let $u \in \mathcal{N}_M(A)$ and $t \in \mathbf{R}$. For every $a \in A$, using the fact that $A \subset M^\varphi$, we have

$$\begin{aligned} u\lambda_t u^* \lambda_t^* \, a &= u\lambda_t u^* a \lambda_t^* \\ &= u\lambda_t (u^* a u) u^* \lambda_t^* \\ &= u (u^* a u) \lambda_t u^* \lambda_t^* \\ &= a \, u\lambda_t u^* \lambda_t^*. \end{aligned}$$

It follows that $u\lambda_t u^* \lambda_t^* \in A' \cap M = A$ and so $u\lambda_t u^* \in A\lambda_t \subset \mathrm{c}(A)$. □

Intertwining techniques in arbitrary von Neumann algebras. We will need the following generalization of Popa's intertwining techniques to arbitrary von Neumann algebras.

Theorem 10 (H-Vaes)**.** *Let M be any σ-finite von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation $E_B : 1_B M 1_B \to B$. The following are equivalent.*

(1) *There exist projections $p \in A$, $q \in B$, a nonzero partial isometry $v \in pMq$ and a unital normal $*$-homomorphism $\theta : pAp \to qBq$ such that $xv = \theta(x)v$ for every $x \in pAp$.*
(2) *There is no net of unitaries (w_i) in $\mathcal{U}(A)$ such that $E_B(x^* w_i y) \to 0$ $*$-strongly for all $x, y \in 1_A M 1_B$.*

If moreover $A, B \subset M$ are Cartan subalgebras and M is a factor, the above conditions are equivalent to:

(3) *There exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

Definition 8. Let M be any σ-finite von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be von Neumann subalgebras such that B is finite and with expectation. We say that A *embeds into* B *inside* M and denote $A \preceq_M B$ if one of the equivalent conditions of Theorem 10 is satisfied. Note that this forces A to have a finite direct summand.

Intertwining techniques, crossed products and the core. Let Γ be any countable discrete group, B any tracial von Neumann algebra and $\Gamma \curvearrowright B$ any action. We do not assume that the action $\Gamma \curvearrowright B$ is trace preserving. Put $M = B \rtimes \Gamma$.

Let $A \subset M$ be any other tracial subalgebra with expectation. We keep the notation of Proposition 3. We moreover regard $\mathrm{c}(M) = \mathrm{c}(B) \rtimes \Gamma$ where the action $\Gamma \curvearrowright \mathrm{c}(B)$ is trace preserving. We simply denote by Tr the faithful normal trace on $\mathrm{c}(M)$.

Proposition 5 (H-Vaes). *Assume that $A \not\preceq_M B$. There exists a net of unitaries $w_k \in \mathcal{U}(\mathrm{c}(A))$ such that*

$$\lim_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{2,\mathrm{Tr}} = 0, \ \forall x, y \in \mathrm{c}(M) \cap \mathrm{L}^2(\mathrm{c}(M), \mathrm{Tr}).$$

In particular, let $p \in \mathcal{Z}(\mathrm{c}(A))$ and $q \in \mathrm{c}(B)$ be any nonzero projections with finite trace. Then $\mathrm{c}(A)p \not\preceq_{(p\vee q)\mathrm{c}(M)(p\vee q)} q\mathrm{c}(B)q$ in the usual sense for finite von Neumann algebras.

Proof. Since $A \not\preceq_M B$, there exists a net of unitaries $w_k \in \mathcal{U}(A)$ such that $E_B(x^* v_k y) \to 0$ $*$-strongly for every $x, y \in M$. Regard $w_k \in \mathcal{U}(\mathrm{c}(A))$. We have $E_{\mathrm{c}(B)}(w_k u_g^*) = E_B(w_k u_g^*) \to 0$ $*$-strongly for every $g \in \Gamma$. We will denote by $(w_k)_g = E_{\mathrm{c}(B)}(w_k u_g^*)$ the gth Fourier coefficient of w_k.

Denote by $\mathcal{W} = \mathrm{span}\{bu_g : b \in \mathrm{c}(B) \cap \mathrm{L}^2(\mathrm{c}(B), \mathrm{Tr}), g \in \Gamma\}$. Then $\mathcal{W}$ is a $\|\cdot\|_{2,\mathrm{Tr}}$-dense subspace of $\mathrm{L}^2(\mathrm{c}(M), \mathrm{Tr})$.

Claim. We have $\lim_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{1,\mathrm{Tr}} = 0$ for every $x, y \in \mathcal{W}$.

Proof of the Claim. It suffices to take $x = au_g$ and $y = u_h b$ with $a, b \in \mathrm{c}(B) \cap \mathrm{L}^2(\mathrm{c}(B), \mathrm{Tr})$ and $g, h \in \Gamma$. In that case, we have

$$E_{\mathrm{c}(B)}(xw_k y) = aE_{\mathrm{c}(B)}(u_g w_k u_h)b = a\,\sigma_g((w_k)_{g^{-1}h^{-1}})\,b.$$

Using the Cauchy-Schwarz Inequality, we get

$$\begin{aligned}\|E_{\mathrm{c}(B)}(xw_k y)\|_{1,\mathrm{Tr}} &\leq \|a\|_{2,\mathrm{Tr}}\,\|\sigma_g((w_k)_{g^{-1}h^{-1}})b\|_{2,\mathrm{Tr}} \\ &= \|a\|_{2,\mathrm{Tr}}\,\|(w_k)_{g^{-1}h^{-1}}\sigma_{g^{-1}}(b)\|_{2,\mathrm{Tr}}.\end{aligned}$$

Since $(w_k)_{g^{-1}h^{-1}} \to 0$ strongly as a bounded net of operators on $\mathrm{L}^2(\mathrm{c}(B), \mathrm{Tr})$, we get $\lim_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{1,\mathrm{Tr}} = 0$. □

Next, let $x, y \in \mathrm{c}(M) \cap \mathrm{L}^2(\mathrm{c}(M), \mathrm{Tr})$ and $\varepsilon > 0$. We may choose $x_1, y_1 \in \mathcal{W}$ such that $\|x - x_1\|_{2,\mathrm{Tr}}\|y\|_{2,\mathrm{Tr}} < \varepsilon$ and $\|y - y_1\|_{2,\mathrm{Tr}}\|x_1\|_{2,\mathrm{Tr}} < \varepsilon$. We get

$$\limsup_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{1,\mathrm{Tr}} \leq \limsup_k \|E_{\mathrm{c}(B)}(x_1 w_k y_1)\|_{1,\mathrm{Tr}} + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\lim_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{1,\mathrm{Tr}} = 0$. Since the net $(E_{\mathrm{c}(B)}(xw_k y))_k$ is uniformly bounded in $\mathrm{c}(B)$, we have

$$\lim_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{2,\mathrm{Tr}} = 0.$$

This finishes the proof of the proposition. □

Lecture 6

In the last lecture, we will prove the result which generalizes Popa-Vaes's unique Cartan decomposition result to arbitrary nonsingular actions of the free groups. This is our joint work with S. Vaes (2012).

Definition 9. We will use the following terminology.

- We say that a von Neumann $M \subset \mathbf{B}(H)$ is *amenable* if there exists a norm one projection $\Phi : \mathbf{B}(H) \to M$.
- Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular action of a countable discrete group on a standard measure space. We say that the action $\Gamma \curvearrowright (X, \mu)$ is *amenable* if there exists a Γ-equivariant norm one projection $\Phi : \mathrm{L}^\infty(\Gamma \times X) \to \mathrm{L}^\infty(X)$.

Proposition 6. *Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular action of a countable discrete group on a standard measure space. The following conditions are equivalent.*

1. *The action $\Gamma \curvearrowright (X, \mu)$ is amenable.*
2. *The crossed product von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma$ is amenable.*

Proof. We will only use (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Let $\Phi : \mathrm{L}^\infty(X \times \Gamma) \to \mathrm{L}^\infty(X)$ be a Γ-equivariant norm one projection. Regard the crossed product von Neumann algebra $\mathrm{L}^\infty(X) \rtimes \Gamma \subset \mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma))$. Now we regard $\mathbf{B}(\ell^2(\Gamma)) \cong \ell^\infty(\Gamma) \rtimes \Gamma$ and so we have

$$\mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma)) \cong \mathrm{L}^\infty(X \times \Gamma) \rtimes \Gamma.$$

Since $\Phi : \mathrm{L}^\infty(X \times \Gamma) \to \mathrm{L}^\infty(X)$ is Γ-equivariant, we may extend Φ to a norm one projection $\Phi : \mathrm{L}^\infty(X \times \Gamma) \rtimes \Gamma \to \mathrm{L}^\infty(X) \rtimes \Gamma$ (this fact is not entirely trivial since Φ is *a priori* not normal). Since $\mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma))$ is amenable, there exists a norm one projection $E : \mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma)) \to \mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} \mathbf{B}(\ell^2(\Gamma))$. Then $\Phi \circ E : \mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma)) \to \mathrm{L}^\infty(X) \rtimes \Gamma$ is a norm one projection, hence $\mathrm{L}^\infty(X) \rtimes \Gamma$ is amenable.

(2) $\Rightarrow$ (1) Let $\Phi : \mathbf{B}(\mathrm{L}^2(X) \otimes \ell^2(\Gamma)) \to \mathrm{L}^\infty(X) \rtimes \Gamma$ be a norm one projection. Put $\Psi = E_{\mathrm{L}^\infty(X)} \circ \Phi|_{\mathrm{L}^\infty(X \times \Gamma)}$. Then $\Psi : \mathrm{L}^\infty(X \times \Gamma) \to \mathrm{L}^\infty(X)$ is a Γ-equivariant norm one projection. □

Theorem 11 (H-Vaes)**.** *Let $\mathbf{F}_n \curvearrowright (X, \mu)$ be a nonamenable free ergodic nonsingular action on a standard measure space. Then $\mathrm{L}^\infty(X)$ is the unique Cartan subalgebra of $\mathrm{L}^\infty(X) \rtimes \mathbf{F}_n$, up to unitary conjugacy.*

Proof. Put $B = \mathrm{L}^\infty(X)$ and $M = B \rtimes \Gamma$. Denote by $E : M \to B$ the canonical faithful normal conditional expectation. Let $A \subset M$ be another Cartan subalgebra such that $A \npreceq_M B$. We will show that M is amenable, hence $\Gamma \curvearrowright (X, \mu)$ is amenable by Proposition 6. Together with Theorem 10, this will prove Theorem 11.

Using Propositions 3 and 4, we get that $\mathrm{c}(A)$ and $\mathrm{c}(B)$ are both Cartan subalgebras in $\mathrm{c}(M)$ with trace preserving expectation. Fix nonzero finite trace projections $p \in \mathrm{c}(A)$ and $q \in \mathrm{c}(B)$. Since $\mathrm{c}(B) \subset \mathrm{c}(M)$ is a masa, we may moreover assume that there exists a partial isometry $v \in \mathrm{c}(M)$ such that $p = v^*v$ and $q = vv^*$.

Define the trace preserving *dual coaction* $\Delta : q\mathrm{c}(M)q \to q\mathrm{c}(M)q \mathbin{\overline{\otimes}} \mathrm{L}(\Gamma)$ by

$$\Delta(qbu_g q) = qbu_g q \otimes u_g.$$

We regard $\mathcal{M} = q\mathrm{c}(M)q \mathbin{\overline{\otimes}} \mathrm{L}(\Gamma)$ as the tracial crossed product von Neumann algebra $q\mathrm{c}(M)q \rtimes \Gamma$ with respect to the trivial action. Put $\mathcal{A} = \Delta(v\mathrm{c}(A)v^*)$ and observe that $\mathcal{A}$ is regular inside $\Delta(q\mathrm{c}(M)q)$.

Since $A \npreceq_M B$, Proposition 5 yields a net of unitaries $w_k \in \mathcal{U}(A)$ such that $\lim_k \|E_{\mathrm{c}(B)}(xw_k y)\|_{2,\mathrm{Tr}} = 0$ for every $x, y \in \mathrm{c}(M) \cap \mathrm{L}^2(\mathrm{c}(M), \mathrm{Tr})$. Put $v_k = vw_k v^* \in \mathcal{U}(v\mathrm{c}(A)v^*)$. Proposition 5 yields $\lim_k \|(v_k)_g\|_{2,\mathrm{Tr}} = \lim_k \|E_{\mathrm{c}(B)}(vw_k v^* u_g^*)\|_{2,\mathrm{Tr}} = 0$ for every $g \in \Gamma$. We next prove the following.

Claim. For every $x, y \in \mathcal{M}$, we have

$$\lim_k \|E_{q\mathrm{c}(M)q\otimes 1}(x\Delta(w_k)y)\|_{2,\mathrm{Tr}\otimes\tau} = 0.$$

Proof of the Claim. It suffices to take $x = 1 \otimes u_g$ and $y = 1 \otimes u_h$ with $g, h \in \Gamma$. In that case, we have

$$\begin{aligned} E_{q\mathrm{c}(M)q\otimes 1}((1 \otimes u_g)\Delta(v_k)(1 \otimes u_h)) &= \sum_{s\in\Gamma} E_{q\mathrm{c}(M)q\otimes 1}((v_k)_s u_s \otimes u_{gsh}) \\ &= (v_k)_{g^{-1}h^{-1}} u_{g^{-1}h^{-1}}. \end{aligned}$$

Therefore, we obtain

$$\lim_k \|E_{q\mathrm{c}(M)q\otimes 1}((1\otimes u_g)\Delta(v_k)(1\otimes u_h))\|_{2,\mathrm{Tr}\otimes\tau} = \lim_k \|(v_k)_{g^{-1}h^{-1}}\|_{2,\mathrm{Tr}} = 0. \quad \square$$

We have that $\mathcal{A}$ is amenable and $\mathcal{A} \npreceq_{\mathcal{M}} q\mathrm{c}(M)q$. By Theorem 6, we get that $\Delta(q\mathrm{c}(M)q)$ is amenable relative to $q\mathrm{c}(M)q$ inside $\mathcal{M}$. There exists a norm one projection $\langle \mathcal{M}, e_{q\mathrm{c}(M)q}\rangle \to \Delta(q\mathrm{c}(M)q)$. Since $\Delta \circ E_{\mathrm{c}(B)q} = E_{q\mathrm{c}(M)q} \circ \Delta$, the $*$-homomorphism $\Delta : q\mathrm{c}(M)q \to \mathcal{M}$ extends to

$$\Psi : \langle q\mathrm{c}(M)q, e_{\mathrm{c}(B)q}\rangle \to \langle \mathcal{M}, e_{q\mathrm{c}(M)q}\rangle$$

by letting $\Psi(x) = \Delta(x)$ if $x \in q\mathrm{c}(M)q$ and $\Psi(e_{\mathrm{c}(B)q}) = e_{q\mathrm{c}(M)q}$. Then

$$\Delta^{-1} \circ \Psi : \langle q\mathrm{c}(M)q, e_{\mathrm{c}(B)q}\rangle \to q\mathrm{c}(M)q$$

is a norm one projection. Hence $q\mathrm{c}(M)q$ is amenable relative to $\mathrm{c}(B)q$ and so $q\mathrm{c}(M)q$ is amenable. If z denotes the central support of q in $\mathrm{c}(M)$, we get that $\mathrm{c}(M)z$ is amenable. Since the dual action is ergodic, it follows that $\mathrm{c}(M)$ is amenable as well. By Takesaki's Duality Theorem, M is amenable. $\square$

With the following explicit examples of nonsingular actions of the free group $\mathbf{F}_2$, we obtain factors with unique Cartan subalgebra, having any possible type and any possible flow of weights in the type III_0 case.

Corollary 3 (H-Vaes)**.** *Let $\mathbf{F}_2 = \langle a, b\rangle$. Denote by $\pi : \mathbf{F}_2 \to \mathbf{Z}$ the group homomorphism given by $\pi(a) = 1$ and $\pi(b) = 0$. Choose a free probability measure preserving action $\mathbf{F}_2 \curvearrowright (X, \mu)$ such that $\ker \pi$ acts ergodically.*

Choose an arbitrary properly ergodic nonsingular action $\mathbf{Z} \curvearrowright (Y, \eta)$ on a standard measure space. Define the action $\Gamma \curvearrowright X \times Y$ given by $g \cdot (x, y) = (g \cdot x, \pi(g) \cdot y)$ and put $M = \mathrm{L}^\infty(X \times Y) \rtimes \mathbf{F}_2$.

Then M is a nonamenable factor and $\mathrm{L}^\infty(X \times Y)$ is the unique Cartan subalgebra of M, up to unitary conjugacy. The factor M has the same type and the same flow of weights as the amenable factor $\mathrm{L}^\infty(Y) \rtimes \mathbf{Z}$.

Proof of Corollary 3. Put $\Gamma = \mathbf{F}_2$. We first claim that $M = \mathrm{L}^\infty(X\times Y) \rtimes \Gamma$ is nonamenable. Indeed, M contains $N = \mathrm{L}^\infty(X \times Y) \rtimes \ker \pi \cong (\mathrm{L}^\infty(X) \rtimes \ker \pi) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(Y)$ as a von Neumann subalgebra with expectation. Since $\ker \pi \curvearrowright (X, \mu)$ is probability measure preserving and since $\ker \pi$ is a nonamenable group, it follows that N is nonamenable. So, M follows nonamenable as well.

By Theorem 11, $\mathrm{L}^\infty(X \times Y)$ is the unique Cartan subalgebra of M, up to unitary conjugacy.

It remains to determine the type and the flow of weights of M. Put $P = \mathrm{L}^\infty(Y) \rtimes \mathbf{Z}$. First consider the trivial cases. If Y admits an equivalent $\mathbf{Z}$-invariant probability measure, both M and P are of type II_1. If Y admits an equivalent $\mathbf{Z}$-invariant infinite measure, both M and P are of type II_∞. So, assume that P is of type III. It remains to prove that M and P have an isomorphic flow of weights.

Let $\mathbf{Z} \curvearrowright (\widetilde{Y}, m)$ be the Maharam extension of $\mathbf{Z} \curvearrowright (Y, \eta)$. Consider the action $\Gamma \curvearrowright X \times \widetilde{Y}$ given by $g \cdot (x, y) = (g \cdot x, \pi(g) \cdot y)$. Since $\Gamma \curvearrowright X$ is measure preserving, the action $\Gamma \curvearrowright X \times \widetilde{Y}$ can be identified with the Maharam extension of $\Gamma \curvearrowright X \times Y$. So, the flow of weights of M can be identified with the natural action of $\mathbf{R}$ on the von Neumann algebra $\mathrm{L}^\infty(X \times \widetilde{Y})^\Gamma$ of Γ-invariant functions. Since $\ker \pi \curvearrowright X$ is ergodic, we get that

$$\mathrm{L}^\infty(X \times \widetilde{Y})^\Gamma = 1 \otimes \mathrm{L}^\infty(\widetilde{Y})^{\mathbf{Z}}.$$

Since the flow of weights of P is given by the natural action of $\mathbf{R}$ on $\mathrm{L}^\infty(\widetilde{Y})^{\mathbf{Z}}$, we have found the required isomorphism between the flow of weights of M and P. □

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