THE OTHER KADISON–SINGER PROBLEM

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Let H be ℓ_2 (over $\mathbb{C}$) and let $\mathcal{B}(H)$ denote the C*-algebra of all bounded operators on H. Fix an orthonormal basis (e_n) for H. The *atomic masa* corresponding to this basis is ℓ_∞; equivalently, the algebra of all operators that are diagonalized by the basis (e_n). The projections in ℓ_∞ are exactly the projections onto subspaces spanned by a subset of $\{e_n\}$. That is, $\mathcal{P}(\ell_\infty) \cong \mathcal{P}(\mathbb{N})$ (here $\mathcal{P}(A)$ denotes the set of all projections in a C*-algebra A). In particular, if we fix a basis, then the Boolean algebra $\mathcal{P}(\mathbb{N})$ is naturally a sublattice of $\mathcal{P}(\mathcal{B}(H))$. Given $X \subseteq \mathbb{N}$, we write $P_X^{(\vec{e})}$ for the projection onto $\overline{\operatorname{span}}\{e_n : n \in X\}$.

The following problem and conjecture appear to be very similar at the first sight.

Problem 0.1 (Kadison–Singer, 1959 [15])**.** *Assume ϕ is a pure state on the atomic masa in $\mathcal{B}(H)$. Does ϕ have a unique extension to a pure state of $\mathcal{B}(H)$?*

Conjecture 0.2 (Kadison–Singer, 1959 [15])**.** *For every pure state φ of $\mathcal{B}(H)$ there is a masa A such that $\varphi\restriction A$ is multiplicative.*

In case when ϕ is a vector state $\omega_\xi(a) = (a\xi|\xi)$ then Problem 0.1 has a positive answer since by Lemma 2.3 any extension ψ must satisfy $\psi(a) = (a\xi|\xi)$. The conjecture is trivially true in case when ϕ is a vector state.

We may therefore assume that the states are singular, mod out the compact operators, and work in the Calkin algebra $\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$.

At the first sight, these problems are about ultrafilters on $\mathbb{N}$ and even more complicated analogous objects on $\mathcal{B}(H)$. Indeed, some of the early research was trying to attack Kadison–Singer problem as a problem about ultrafilters. For example, in 1971 Reid ([20]) proved that for pure states diagonalized by certain specific type of ultrafilters the answer to Problem 0.1.

However, for quarter of a century it was known that a positive answer to the Kadison–Singer problem is equivalent to an arithmetic statement, and

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This note is based on a talk given in Masterclass on Ergodic Theory, von Neumann algebras (and Kadison–Singer problem) held at the University of Copenhagen in October 2013. Results of this note are taken from [13]. Every time I come back to this problem I have an impression that I am missing something simple; hopefully one of the readers will find out what it is and finally resolve this problem in ZFC..

therefore absolute between transitive models of ZFC and therefore highly unlikely to be independent from ZFC (see [21] and [10], respectively).

By the work of Weaver ([22]) and Marcus–Spielman–Srivastava [17] this problem has a positive solution (see Pete Casazza's lecture notes).

1. Extending pure states on masas

Assume B is a unital C*-algebra and A is its unital subalgebra. A little trick involving positivity and Hahn–Banach theorem shows that every state ϕ on A has an extension to a state on B. More precisely, $\{\psi \in \mathbb{S}(B) : \psi \restriction A = \phi\}$ is a compact weak*-convex set. If ϕ is pure, then the extreme points of this set are pure states of B.

Let A be a unital abelian C*-algebra and let ϕ be a state on A. By the Gelfand–Naimark theorem we have that $A \cong C(X)$ for a compact Hausdroff space X and by by Riesz Representation Theorem there is a Borel probability measure μ on X such that $\phi(f) = \int f\, d\mu$. From here one easily sees that ϕ is pure if and only if μ is a Dirac measure if and only if ϕ is multiplicative—i.e., a *-homomorphism into $\mathbb{C}$.

If A is moreover generated by projections then these assertions are also equivalent to the statement that $\phi(p) \in \{0,1\}$ for every projection p.

In order to give an another statement equivalent to the purity of ϕ, let us write

$$\mathcal{P}(A) = \{p \in A : \ p \text{ is a projection}\}$$

and note that in case when A is abelian $\mathcal{P}(A)$ forms a Boolean algebra under the operations $p \vee q = p + q - pq$, $p \wedge q = pq$ and $p^{\complement} = 1_A - p$.

Letting

$$\mathbb{F}_\phi = \{p \in \mathcal{P}(A) : \phi(p) = 1\}$$

we see that under our assumptions μ is a Dirac measure if only if $\mathbb{F}_\phi$ s an ultrafilter of the Boolean algebra $\mathcal{P}(A)$. (Later on we shall be using this notation in case when ϕ is a state—not necessarily pure—on a C*-algebra—not necessarily abelian—A.)

Theorem 1.1 below shows that Conjecture 0.2 is not true in all models of ZFC. It follows from a stronger result, Theorem 3.6, whose proof will be sketched below.

Theorem 1.1 (Akemann–Weaver, 2005 [2])**.** *Continuum Hypothesis implies there is a pure state φ on $\mathcal{B}(H)$ that is not multiplicative on any atomic masa.*

2. Key trivialities

The proof of Theorem 3.6 uses a combination of a few trivialities and some basic set theory. Here is the key triviality, pointed out by Nik Weaver in a crucial moment.

Lemma 2.1. *For projections $p_1, p_2, \dots, p_n$ in $\mathcal{B}(H)$ the following are equivalent.*

(1) $\|p_1 \cdot \dots \cdot p_n\| = 1$,
(2) *For every* $\epsilon > 0$ *there exists a unit vector* ξ *such that* $\|p_j \xi\| \geq 1 - \epsilon$ *for all* j.

Proof. The first statement is equivalent to the assertion that for every $\delta > 0$ there exists a unit vector ξ such that $\|p_1 \cdots \cdot p_n \xi\| \geq 1-\delta$. This implies both $\|p_n \xi\| \geq 1 - \delta$ and $\|\xi - p_n \xi\| \leq \delta$. By choosing δ small enough and using a straightforward induction, one shows that this is equivalent to (2). □

Lemma 2.2. *Let* (e_n) *and* (ξ_n) *be orthonormal bases. Then there is a partition of* $\mathbb{N}$ *into finite intervals* (J_n) *such that for all* k,

$$\xi_k \in \overline{\operatorname{span}}\{e_i : i \in J_n \cup J_{n+1}\}$$

(modulo a small perturbation of ξ_k*) for some* $n = n(k)$.

Proof. Choose J_1 to be any finite initial segment of $\mathbb{N}$. If $J_1, \dots, J_n$ are chosen, let m be large enough to have

$$\|P^{(\vec{e})}_{[m,\infty)} \xi_j\| < 2^{-n}$$

for all $j \in \bigcup_{k \leq n} J_k$. Then choose J_{n+1} so that $\bigcup_{k \leq n+1} J_k$ covers the interval $\{1, \dots, m\}$. □

The following well-known lemma gives a motivation for introducing quantum filters.

Lemma 2.3. *If* φ *is a state on* A *and* $a \in A$ *is a positive contraction such that* $\varphi(a) = 1$, *then* $\varphi(b) = \varphi(aba)$ *for all* $b \in A$.

Proof. By the Cauchy–Schwartz inequality for states

$$|\varphi((1-a)b)| \leq \sqrt{\varphi(1-a)\varphi(b^*b)} = 0.$$

Since $b = ab + (1-a)b$, we have $\varphi(b) = \varphi(ab) + \varphi((1-a)b) = \varphi(ab)$. By applying the same argument to ab and multiplying by $1-a$ on the right one proves that $\varphi(ab) = \varphi(aba)$. □

In the following we write π for the quotient map from $\mathcal{B}(H)$ onto the Calkin algebra.

Lemma 2.4. *Assume* p_n, *for* $n \in \mathbb{N}$, *is a decreasing sequence of noncompact projections in* $\mathcal{B}(H)$. *Then there is a noncompact projection* p *such that* $\pi(p) \leq \pi(p_n)$ *for all* n.

Proof. Choose orthogonal vectors ξ_n in the range of P_n. Then let p be the projection to the closed linear span of these vectors. □

3. A pure state that is not multiplicative on any masa in $\mathcal{B}(H)$

The basic idea of constructing such a pure state is to encode pure states as "quantum ultrafilters"; a pure state on the atomic masa $\ell_\infty \subset \mathcal{B}(H)$ is equivalent to an ultrafilter.

Lemma 3.1 (Weaver, 2007)**.** *For $\mathbb{F} \subseteq \mathcal{P}(\mathcal{B}(H))$ the following are equivalent:*

(A) $\|p_1 p_2 \cdots p_n\| = 1$ *for any* $p_1, \cdots, p_n \in \mathbb{F}$ *and* $\mathbb{F}$ *is maximal with respect to this property.*

(B) *For all* $\epsilon > 0$ *and for all finite* $F \subseteq \mathbb{F}$ *there is a unit vector* ξ *such that* $\|p\xi\| > 1 - \epsilon$ *for all* $p \in F$.

Proof. Since $\|p_1 p_2 \cdots p_n\| \leq \|p_1\| \cdot \|p_2\| \cdot \ldots \cdot \|p_n\| = 1$, clause (A) is equivalent to stating that for every $\epsilon > 0$ there is a unit vector ξ such that $\|p_1 p_2 \cdots p_n \xi\| > 1 - \epsilon$. The remaining calculations are left as an exercise to the reader. Keep in mind that, for a projection p, the value of $\|p\xi\|$ is close to $\|\xi\|$ if and only if $\|\xi - p\xi\|$ is close to 0. □

We call an $\mathbb{F}$ satisfying the conditions of Lemma 3.1 a *quantum filter*. Such an $\mathbb{F}$ is a *maximal quantum filter* if it is not properly included in another quantum filter. Quantum filters were studied (under a different name) in [5].

Theorem 3.2 (Farah–Weaver, 2007)**.** *Let $\mathbb{F} \subseteq \mathcal{P}(\mathcal{C}(H))$. Then the following are equivalent:*

(1) $\mathbb{F}$ *is a maximal quantum filter,*

(2) $\mathbb{F} = \mathbb{F}_\varphi = \{p : \varphi(p) = 1\}$ *for a unique pure state* φ.

Proof. To see that (1) implies (2), for a finite $F \subseteq \mathbb{F}$ and $\epsilon > 0$ let

$$X_{F,\epsilon} = \{\varphi \in \mathbb{S}(\mathcal{B}(H)) : \varphi(p) \geq 1 - \epsilon \text{ for all } p \in F\}.$$

If ξ is as in (B) then $\omega_\xi \in X_{F,\epsilon}$.

Since $X_{F,\epsilon}$ is weak*-compact, $\bigcap_{(F,\epsilon)} X_{F,\epsilon} \neq \emptyset$, and any extreme point of the intersection is a pure state with the desired property. Using a version of Kadison's Transitivity Theorem ([14]) one can prove that this intersection is actually a singleton (see also [5]).

Now we prove the reverse implication. If $\varphi(p_j) = 1$ for $j = 1, \ldots, k$, then $\varphi(p_1 p_2 \ldots p_k) = 1$ by Lemma 2.3, hence (A) holds. It is then not hard to show that $\mathbb{F}_\varphi$ also satisfies (B) and is maximal. □

Lemma 3.3. *Let $\mathbb{F}$ be a maximal quantum filter and let (ξ_n) be an orthonormal basis. If there exist a finite partition $\mathbb{N} = \bigcup_{j=1}^n A_j$ and $q \in \mathbb{F}$ such that $\|P^{(\vec{\xi})}_{A_j} q\| < 1$ for all j, then $\mathbb{F}$ is not diagonalized by (ξ_n). In particular, pure state $\phi_{\mathbb{F}}$ is not diagonalized by (ξ_n).*

Proof. Assume $\mathbb{F}$ is diagonalized by (ξ_n) and let $\mathcal{U}$ be such that $\mathbb{F} = \varphi^{(\vec{\xi})}_{\mathcal{U}}$. Then $A_j \in \mathcal{U}$ for some j, but $\|P^{(\vec{\xi})}_{A_j} q\| < 1$ for $q \in \mathbb{F}$, contradicting the assumption that $\mathbb{F}$ is a filter. □

For (J_n) as in Lemma 2.2 and $\epsilon > 0$ let

$$D_{\vec{J},\epsilon} = \{q : \|P^{(\vec{e})}_{J_n} q\| < 1 - \epsilon \text{ for all } n\}$$

Lemma 3.4. *Let $\mathbb{F}$ be a maximal quantum filter and let (J_n) be as in Lemma 2.2. If orthonormal basis (ξ_n) is blocked by (J_n) and $\mathbb{F}$ is a maximal quantum filter with nonempty intersection with $\bigcup_{\epsilon>0} D_{\vec{J},\epsilon}$ then the pure state $\phi_{\mathbb{F}}$ is not diagonalized by (ξ_n).*

Proof. This is an immediate consequence of Lemma 3.3. □

Lemma 3.5. *For every noncompact projection $p \in \mathcal{P}(\mathcal{B}(H))$, there is a noncompact projection $q \in D_{\vec{J},1/4}$ such that $\pi(q) \leq \pi(p)$.*

Proof. Take an orthogonal basic sequence η_n included in the range(p) which is 'thin' in the sense that for every k we have $\|p^{\vec{e}}_{\vec{J}_k}\eta_n\| \geq 2^{-k}$ for at most one n. Then let $\zeta_n = \frac{1}{\sqrt{2}}(\xi_{2n} + \xi_{2n+1})$ and let q be the projection to the closed linear span of $\{\zeta_n : n \in \mathbb{N}\}$. □

Let us consider the nonzero projections in the Calkin algebra $\mathcal{C}(H)$ as a partially ordered set. Let $\mathfrak{p}^*$ denote the minimal cardinal κ such that there is a family of nonzero projections in $\mathcal{C}(H)$ such that every finite subset has a lower bound, but the family has no lower bound. In particular, Lemma 2.4 implies $\mathfrak{p}^*$ is uncountable and therefore the Continuum Hypothesis implies that $\mathfrak{p}^* = 2^{\aleph_0}$ One should note that the same conclusion follows from Martin's Axiom ([23]). For more on cardinal invariants of the continuum see [7].

Theorem 3.6 (Farah–Weaver)**.** *Assume $\mathfrak{p}^* = 2^{\aleph_0}$. Then there exists a pure state on $\mathcal{B}(H)$ that is not diagonalized by any atomic masa.*

In particular, both Continuum Hypothesis and Martin's Axiom imply that both Conjecture 0.2 and Conjecture 4.3 are false.

Proof. We construct a corresponding maximal quantum filter. Enumerate all partitions $\vec{J}$ as $\vec{J}_\gamma$, for $\gamma < 2^{\aleph_0}$. We construct $\mathcal{F}_\gamma$, for $\gamma < 2^{\aleph_0}$, satisfying the following.

1. Each $\mathcal{F}_\gamma$ is a family of noncompact projections in $\mathcal{B}(\ell_2)$ such that $p_1 \cdot \dots \cdot p_n$ is a noncompact operator of norm one for all n and all $p_j \in \mathcal{F}_\gamma$.
2. If $\gamma < \lambda$ then $\mathcal{F}_\gamma \subseteq \mathcal{F}_\lambda$.
3. $|\mathcal{F}_\gamma| < 2^{\aleph_0}$ for all γ.
4. For every γ there exists $p \in \mathcal{F}_{\gamma+1} \cap D_{\vec{J}_\gamma,1/4}$.

Assume $\mathcal{F}_\gamma$ has been constructed. By $\mathfrak{p}^* = 2^{\aleph_0}$ we can find a noncompact projection q such that $\dot{q} \leq \dot{p}$ for all $p \in \mathcal{F}_\gamma$ (here $\dot{a}$ is the image of a under the quotient map in the Calkin algebra).

By Lemma 3.5 we can find a noncompact projection $r \leq q$ such that $r \in D_{\vec{J}_\gamma}$. Let $\mathcal{F}_{\gamma+1} = \mathcal{F}_\gamma \cup \{r\}$. If λ is a limit ordinal and all $\mathcal{F}_\gamma$ for $\gamma < \lambda$ were constructed let $\mathcal{F}_\lambda = \bigcup_{\gamma<\lambda} \mathcal{F}_\gamma$. This describes a recursive construction.

Let $\mathbb{F}$ be a maximal quantum filter extending $\bigcup_\gamma \mathcal{F}_\gamma$. We shall prove that it is as required. Given a basis (ξ_k) by Lemma 2.2 we can find (J_n) such that $\xi_k \in J_{n(k)} \cup J_{n(k)+1}$ (modulo a small perturbation) for all k. Let

$A_i = \{k \mid n(k) = i \bmod 4)\}$ for $0 \leq i < 4$. Then $q \in \mathbb{F} \cap D_{\vec{J},1/4}$ and by Lemma 3.4, $\mathbb{F}$ is not diagonalized by (ξ_n). □

4. Motivation

By the Akemann–Weaver result, Anderson's and Kadison–Singer conjectures are 1/2-resolved. We know that a negative answer is consistent with ZFC, but this leaves open two possibilities:

(1) There is a ZFC refutation of the conjectures.
(2) One can construct a model of ZFC in which conjectures are true.

(Actually, (2) is slightly misstated and it is a subtle consequence of Gödel's Incompleteness Theorem that there is a third possibility. However, this possibility is a bit less likely and we can safely ignore it for the time being.)

Both of these possibilities happened for known open problems in the field of operator algebras. Notably, the assertion that the Calkin algebra has outer automorphisms is independent from ZFC ([19] and [9]). The situation with this problem is somewhat typical for consistency results in analysis (or more generally, for consistency results in 'not obviously set-theoretic' mathematics); see e.g., [8] One answer to the problem (the existence of an outer automorphism) is given by Continuum Hypothesis, while the other answer (all automorphisms are inner) is given by *Forcing Axioms*—far-reaching strengthenings of the Baire Category Theorem. The situation with the Kadison–Singer/Anderson conjecture is different. As proved above, both the Continuum Hypothesis and a rather mild forcing axiom (i.e., Martin's Axiom) imply the negation of the conjecture.

Again, because the statement of Problem 0.1 is equivalent to an arithmetic statement, finding its proof using the Continuum Hypothesis would be as good as proving it, by using a standard absoluteness argument (see e.g., [10]).[1]

Moreover, by a better bookkeeping argument one sees that the same conclusion follows from cardinal inequality $\mathfrak{d} \leq \mathfrak{p}^*$, an assertion a bit weaker than the Martin's Axiom. Let $\mathfrak{d}$ denote the minimal cardinality of a cofinal subset of $\mathbb{N}^{\mathbb{N}}$ under the pointwise order:

$$f \leq g \text{ if and only if } f(n) \leq g(n) \text{ for all } n.$$

An argument of [9, §2] shows that in order to perform the construction of Theorem 3.6 we only need to take care of $\mathfrak{d}$ partitions, hence the inequality $\mathfrak{p}^* \leq \mathfrak{d}$ suffices for the conclusion of this theorem. The following result appears in [13].

Theorem 4.1 (Farah–Weaver). *If $\mathfrak{p}^* \leq \mathfrak{d}$ then there exists a pure state on $\mathcal{B}(H)$ that is not diagonalized by any atomic masa.*

[1] Not that it matters now that the problem is solved, but there are other problems of the similar kind—Connes Embedding Problem, free group factor problem, and so on...

There are models in which this inequality fails. For example, Cohen's original model for the consistency of the negation of the Continuum Hypothesis with ZFC is of this form. However, the Kadison–Singer/Anderson conjecture is false in this model as well. This can be proved by a different argument (this is my unpublished result).

We conclude that a model in which this conjecture is true would be rather specific, if there is such a model at all. The most natural candidate is a model in which the Near Coherence of Filters holds (see [6] and [16]).

The situation with Naimark's problem is rather different. By an another result of Akemann and Weaver ([1]), a set-theoretic axiom (Jensen's diamond principle) implies the existence of a counterexample to Naimark's problem. This axiom is a strengthening of the Continuum Hypothesis, and it is possible that forcing axioms imply there is no counterexample to Naimark's problem. This could lead to an analogue of Glimm's theorem and an interesting representation theory for nonseparable C*-algebras (see the introduction to [1]),

4.1. **Ultrafilters vs. pure states.** Here is a motivation for this question of a rather different kind. Pure states on ℓ_∞ correspond to ultrafilters on $\mathbb{N}$, and the latter are undoubtedly one of the most important tools of infinitary mathematics.

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and (e_n) be an orthonormal basis for H. Then

$$\varphi_{\mathcal{U}}^{(\vec{e})}(a) = \lim_{n\to\mathcal{U}} (ae_n|e_n)$$

is a state on $\mathcal{B}(H)$. Assume for a moment $\mathcal{U}$ is a *principal ultrafiler*: for some $k \in \mathbb{N}$ we have that $\mathcal{U} = \{X \subseteq \mathbb{N} : k \in X\}$. Then $\phi_{\mathcal{U}}^{(\vec{e})}(a) = (ae_k|e_k)$. By Lemma 2.3, for any state extension ψ of ϕ we have that (writing p_k for the projection to the span of e_k) $\psi(a) = \phi(p_k a p_k) = (ae_k|e_k)$ for all a, and therefore ψ is unique. We may therefore assume that the ultrafilter $\mathcal{U}$ is nonprincipal.

A state of the form $\varphi_{\mathcal{U}}^{(\vec{e})}$ for some basis (e_n) and some ultrafilter $\mathcal{U}$ is *diagonalizable*. As noted above, the restriction of a diagonalizable state to the corresponding atomic masa is a pure state of the masa, and every pure state of an atomic masa is of this form.

It was proved by Anderson ([3]) that diagonalizable states are pure. The proof relies on a clever combinatorial lemma that is a 'quantized' version of the following simple fact.

Lemma 4.2. *If $f\colon \mathbb{N} \to \mathbb{N}$ is a function with no fixed points, then there is a partition $\mathbb{N} = A_1 \sqcup A_2 \sqcup A_3$ such that $f[A_i] \cap A_i = \emptyset$ for all i.* □

Conjecture 4.3 (Anderson, 1981 [4])**.** *Every pure state on $\mathcal{B}(H)$ is diagonalizable.*

Clearly Conjecture 4.3 is stronger than Conjecture 0.2. A moment of thought reveals that a positive answer to Problem 0.1 implies that the two conjectures are equivalent.

The Kadison–Singer/Anderson conjecture roughly asks whether pure states on $\mathcal{B}(H)$ correspond to ultrafilters in some way. Let me make this more precise, after introducing a definition.

If $\mathcal{U}$ is a filter on $\mathbb{N}$ and $h\colon \mathbb{N} \to \mathbb{N}$, then we write

$$h(\mathcal{U}) = \{A \subseteq \mathbb{N} : h^{-1}(A) \in \mathcal{U}\}.$$

If $\mathcal{U}$ is an ultrafilter then $h(\mathcal{U})$ is an ultrafilter as well. We write $\mathcal{V} \leq_{RK} \mathcal{U}$ if there is h such that $\mathcal{V} = h(\mathcal{U})$. This is the *Rudin–Keisler ordering* on the ultrafilters.

Two nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ are *Rudin–Keisler equivalent* if $\mathcal{U} \leq_{RK} \mathcal{V}$ and $\mathcal{V} \leq_{RK} \mathcal{U}$. This is equivalent to the existence of a bijection $h\colon \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{U}$ if and only if $h^{-1}(A) \in \mathcal{V}$ for all $A \subseteq \mathbb{N}$ (this is immediate from Lemma 4.2). Although all nonprincipal ultrafilters 'look the same' this ordering is surprisingly complicated (see e.g., [18]). Furthermore, the structure of ultrapowers and even relative commutants may depend on the choice of ultrafilters (see [11] and [12]).

Theorem 4.1 (B. Minaud, 2012). *Assume ψ is a pure state on $\mathcal{B}(H)$ which is diagonalized in two ways, so that $\psi = \varphi_{\mathcal{U}}^{(\vec{e})}$ and $\psi = \varphi_{\mathcal{V}}^{(\vec{f})}$, for some orthonormal bases (e_n), (f_n) and ultrafilters $\mathcal{U}, \mathcal{V}$. Then $\mathcal{U}$ and $\mathcal{V}$ are Rudin–Keisler isomorphic.* □

This suggests that the Rudin–Keisler equivalence naturally extends to the space of pure states of $\mathcal{B}(H)$. Two states ϕ and ψ are RK-equivalent if there is a unitary u such that $\phi = \psi \circ \operatorname{Ad} u$ (recall that all automorphisms of $\mathcal{B}(H)$ are inner).

Is there a natural extension of the Rudin–Keisler ordering to the space of pure states on $\mathcal{B}(H)$?

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