The Kadison-Singer Problem in Mathematics and Engineering
Lecture 5: BT $\Rightarrow$ Weaver Conjecture KS_r $\Rightarrow$ Weaver Conjecture $\Rightarrow$ PC $\Rightarrow$ KS

Master Course on the Kadison-Singer Problem
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Akemann-Anderson Conjecture

AA-Conjecture

There exist $0 < \epsilon,\ \delta < 1$ with the following property:

for any orthogonal projection P on ℓ_2^n with $|Diag\ P| \leq \delta$,

there is a diagonal projection Q such that

$$\|QPQ\| \leq 1 - \epsilon \text{ and } \|(I-Q)P(I-Q)\| \leq 1 - \epsilon.$$

AA and KS

Theorem (Akemann and Anderson)

The Akemann-Anderson Conjecture implies the Paving Conjecture.

Proof: See Akemann and Anderson, Lyapunov theorems for operator algebras, memoirs of the AMS, Vol. 94 (1991).

Weaver Conjecture

This is a weakening of AA:

Weaver Conjecture

There exist universal constants $0 < \epsilon,\ \delta < 1$ and an $r \in \mathbb{N}$ so that

for all n and all orthogonal projections P on ℓ_2^n with Diag $P \leq \delta$,

there is a paving $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that

$$\|Q_{A_j} P Q_{A_j}\| \leq 1 - \epsilon, \text{ for all } j = 1, 2, \ldots, r.$$

Weaver and AA

Theorem

The Weaver Conjecture implies the Paving Conjecture.

Proof: We just need to iterate the proof of C/Edidin/Kalra/Paulsen that paving projections with constant diagonal $1/2$ implies PC.

Weaver Conjecture KS_r

Weaver Conjecture KS_r for $r \in \mathbb{N}$

There exist universal constants $N \geq 2$ and $\epsilon > 0$ such that the following holds.

Given $(\phi_i)_{i=1}^n$ vectors in ℓ_2^k with $\|\phi_i\| \leq 1$ for all i and suppose

$$\sum_{i=1}^{n} |\langle \phi, \phi_i \rangle|^2 = N, \text{ for all } \phi \in \ell_2^n.$$

Then there is a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A_j} |\langle \phi, \phi_i \rangle|^2 \leq N - \epsilon, \text{ for all } \phi \in \ell_2^n \text{ and all } j.$$

Weaver Conjecture and Weaver Conjecture KS_r

Theorem

The Weaver Conjecture KS_r implies the Weaver Conjecture and hence the Paving Conjecture.

Proof: Let P be a projection with $|Diag\ P| \leq \frac{1}{N}$.

Define $\phi_i = \sqrt{N} \cdot Pe_i$ for $1 \leq i \leq n$ so that

$$\|\phi_i\|^2 = N \cdot \|Pe_i\|^2 = N\langle Pe_i, e_i\rangle \leq N\text{Diag } P \leq 1.$$

Also, for $\phi \in \ell_2^n$ we have

$$\sum_{i=1}^{n} |\langle \phi, \phi_i\rangle|^2 = \sum_{i=1}^{n} |\langle \phi, \sqrt{N}Pe_i\rangle|^2 = N \cdot \sum_{i=1}^{n} |\langle \phi, \phi_i\rangle|^2 = N.$$

Proof Continued

KS_r asserts there is a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ such that

$$\sum_{i \in A_j} |\langle \phi, \phi_i \rangle|^2 \leq N - \epsilon.$$

Letting Q_j be the diagonal projection given by A_j we have that

$$\sum_{j=1}^{r} Q_j = I_n.$$

Proof Completed

Now, for any $\phi \in \ell_2^n$ we have

$$\begin{aligned} \|Q_j P\phi\|^2 &= \sum_{i\in A_j} |\langle Q_j P\phi, e_i\rangle|^2 \\ &= \sum_{i\in A_j} |\langle \phi, PQ_j e_i\rangle|^2 \\ &= N^{-1}\sum_{i\in A_j} |\langle \phi, \phi_i\rangle|^2 \\ &= 1 - \frac{\epsilon}{N}. \end{aligned}$$

Hence:

$$\|Q_j P Q_j\| = \|Q_j P\|^2 \leq 1 - \frac{\epsilon}{N}.$$

Recall the Bourgain-Tzafriri Conjecture

(strong) Bourgain-Tzafriri Conjecture

There exists a universal constant $A > 0$ so that

for every $0 < B$ there is a natural number $r = r(B)$

so that for every natural number "n" and every operator $T : \ell_2^n \to \ell_2^n$ with $\|Te_i\| = 1$ and $\|T\| \le B$,

there exists a partition $(A_j)_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all j and all scalars $(a_i)_{i \in A_j}$

$$\left\| \sum_{i \in A_j} a_i Te_i \right\|^2 \ge A \sum_{i \in A_j} |a_i|^2$$

BT and Weaver

Theorem

The Bourgain-Tzafriri Conjecture implies the Weaver Conjecture.

Proof: Choose A and $r = r(2)$ in BT. Given a projection P with Diag $P \leq \delta = \frac{3}{4}$.

Note that

$$\|(I-P)e_i\|^2 \geq 1 - \delta \geq 1/4.$$

Let

$$Te_i = \frac{(I-P)e_i}{\|(I-P)e_i\|}.$$

Proof Continued

Given scalars $(a_i)_{i=1}^n$ we have

$$\begin{aligned}\left\|\sum_{i=1}^n a_i Te_i\right\|^2 &= \left\|\sum_{i=1}^n \frac{a_i}{\|(I-P)e_i\|}(I-P)e_i\right\|^2 \\ &\le \sum_{i=1}^n \left|\frac{a_i}{\|(I-P)e_i\|}\right|^2 \\ &\le 4\sum_{i=1}^n |a_i|^2\end{aligned}$$

By BT, we get a partition $(A_j)_{j=1}^r$ of $\{1,2,\ldots,n\}$ so that for all r and scalars $(a_i)_{i\in A_j}$ we have

$$\|\sum_{i\in A_j} a_i Te_i\|^2 \ge A\|\sum_{i\in A_j} |a_i|^2.$$

Proof Continued

Hence,

$$\begin{aligned}\|\sum_{i\in A_j} a_i(I-P)e_i\|^2 &= \|\sum_{i\in A_j} a_i\|(I-P)e_i\|Te_i\|^2\\ &\geq A\sum_{i\in A_j}|a_i|^2\|(I-P)e_i\|^2\\ &\geq \frac{A}{4}\sum_{i\in A_j}|a_i|^2.\end{aligned}$$

So if $\phi=\sum_{i\in A_j}a_ie_i$, we have

$$\|PQ_{A_j}\phi\|^2=\|\sum_{i\in A_j}a_iPe_i\|^2\leq\left(1-\frac{A}{4}\right)\sum_{i\in A_j}|a_i|^2.$$

Hence,

$$\|Q_{A_j}PQ_{A_j}\|=\|PQ_{A_j}\|^2\leq 1-\frac{A}{4}.$$

Our Tour of the Kadison-Singer Problem

Marcus/Spielman/Srivastava	$\Rightarrow$	Casazza/Tremain Conjecture
	and	Weaver Conjecture KS_r
	$\Rightarrow$	Weaver Conjecture
	$\Rightarrow$	Paving Conjecture
	$\Rightarrow$	R_ϵ-Conjecture
	$\Rightarrow$	Bourgain-Tzafriri Conjecture
	$\Rightarrow$	Feichtinger Conjecture
	$\Rightarrow$	Sundberg Problem

Finally:

Bourgain-Tzafriri Conjecture	$\Rightarrow$	Weaver Conjecture
	$\Rightarrow$	Paving Conjecture
	$\Leftrightarrow$	The Kadison-Singer Problem