The Kadison-Singer Problem in Mathematics and Engineering
Lecture 3: The Kadison-Singer Problem in Engineering
The Casazza/Tremain Conjecture and the Feichtinger Conjecture

Master Course on the Kadison-Singer Problem
University of Copenhagen

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The Defense Threat Reduction Agency

NSF-DMS

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The Air Force Office of Scientific Research

Signal Processing

The best way to understand signal processing is to attend a piano concert.

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Pictures by: Music With Ease http://www.musicwithease.com/

To the Pianist

We are hearing a continuous acoustical signal (If we ignore the banging of the keys).

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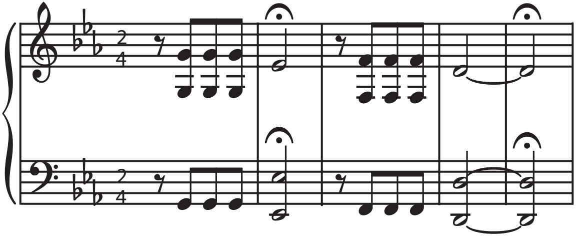
We are hearing a continuous acoustical signal (If we ignore the banging of the keys).

To the pianist, the concert is a collection of black dots on a piece of paper.

Sheet Music:



In the Audience

If we were fast enough, we could write the sheet music as the concert is being played.

In the Audience

If we were fast enough, we could write the sheet music as the concert is being played.

Then, when we get home, we could use our sheet music to replay (i.e. reconstruct) the concert.

Correcting Mistakes?

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Well, we could just erase the incorrect notes and play back a perfect concert.

If she left out some notes, we could just add them.

Dennis Gabor 1946

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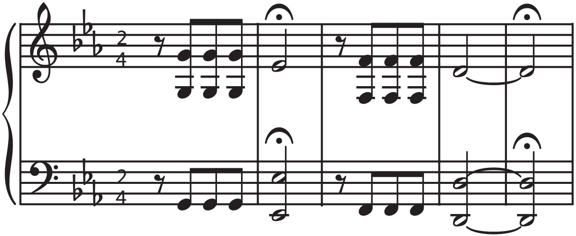
Sheet Music:



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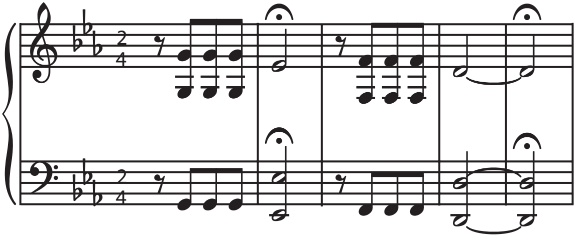
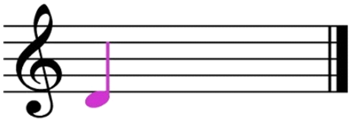


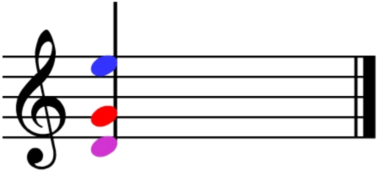
We take that note and change its modulation, then shift it in time and change its modulation and continue.

Our Basic Note

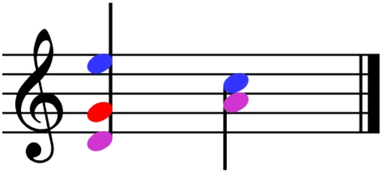


We Change the Modulation of Our Note

Change the Modulation again



Shift our note in time and change modulation



Gabor's Idea

We will write sheet music for a signal.

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Fix $g \in \left(L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} | \int_{\mathbb{R}} |f(t)|^2 \, dt < \infty\}\right) \cap L^\infty(\mathbb{R})$—our NOTE

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Let $f \in L^2(\mathbb{R})$–our SIGNAL

Fix $0 < a, \; b \leq 1$.

Take modulations of our "note" and compute the intensity of our signal for each of these:

$$\left(\left\langle f, e^{2\pi iant} g(t) \right\rangle\right)_{n \in \mathbb{Z}}$$

Continue

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Continue

Now, take a translation in time of our note and for all modulations of our "translated note", compute the intensity of our signal for each of these:

$$\left(\left\langle f, e^{2\pi iant} g(t-b)\right\rangle\right)_{n\in\mathbb{Z}}$$

Continuing, we digitalize our signal:

$$\left(\left\langle f, e^{2\pi iant} g(t-mb)\right\rangle\right)_{m,n\in\mathbb{Z}}$$

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This requires that

$$(e^{2\pi iant}g(t-mb))_{m,n\in\mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ called a **Gabor Frame** and denoted (g,a,b).

What Did Gabor Do?

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Unfortunately for Gabor, this is a case which doesn't work.

Balian-Low

Balian-Low Theorem

If $(g, 1, 1)$ is a Gabor frame for $L^2(\mathbb{R})$ then either $tg(t) \notin L^2(\mathbb{R})$ or $g' \notin L^2(\mathbb{R})$.

Time Frequency Analysis

Time frequency analysis is the mathematics of signal processing.

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Major Problem

Classify all (g, a, b) which give Gabor frames.

Classifying Gabor Frames is a Very Difficult Problem

Theorem (C/Kalton)

Classifying the Gabor Frames of the form $(\chi_E, 1, 1)$

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Classifying the Gabor Frames of the form $(\chi_E, 1, 1)$

$$\Longleftrightarrow$$

Littlewood's Problem

(1977) Classify the integer sets $\{n_1 < n_2 < \ldots < n_k\}$ so that

$$f(z) = \sum_{j=1}^{k} z^{n_j}$$

does not have any zeroes on the unit circle.

Gabor Frames

Theorem (Rieffel)

If (g, a, b) is a Gabor frame then $ab \leq 1$.

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Theorem

If $ab = 1$ and (g, a, b) is a Gabor frame then it is a Riesz basis.

How do we clean up a signal?

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See: K. Gröchenig, *Foundations of Time FrequencyAnalysis*, Birkhäser (2000).

Hans Feichtinger

(2004) e-mail: Hans to Pete

Every Gabor frame I know can be partitioned into a finite number of Riesz basic sequences. Do you think this is always true?

Feichtinger Conjecture

Feichtinger Conjecture (FC)

Every unit norm frame is a finite union of Riesz Basic Sequences.

Equivalent Formulations

Every unit norm Bessel sequence is a finite union of Riesz basic sequences.

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Proof: Given a unit norm Bessel sequence (ϕ_i) let (e_i) be an orthonormal basis for $\mathbb{H}$.

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Then $(\phi_i) \cup (e_i)$ is a unit norm frame.

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Proof: Given a unit norm Bessel sequence (ϕ_i) let (e_i) be an orthonormal basis for $\mathbb{H}$.
Then $(\phi_i) \cup (e_i)$ is a unit norm frame.
Partition this into a finite union of Riesz basic sequences.

Partial Answer

Theorem (C/Christensen)

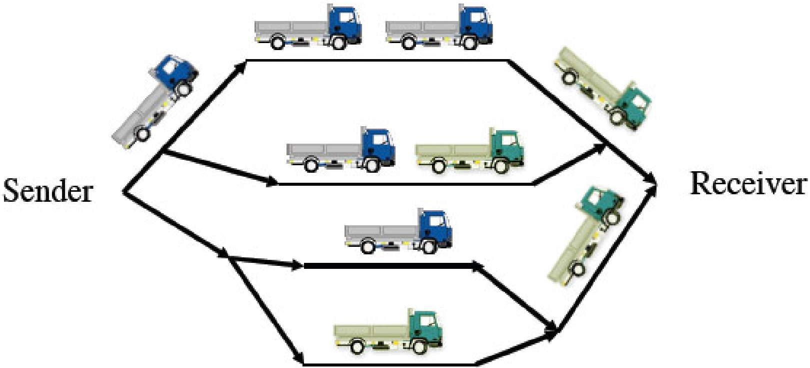
If ab is rational and (g, a, b) is a Gabor frame, then this is a finite union of Riesz basic sequences.

Kadison-Singer in Engineering: Internet Coding

"And again, the internet is not something you just dump something on. It's not a truck. It's a series of tubes. And if you don't understand those tubes can be filled and if they are filled, when you put your message in, it gets in line and its going to be delayed by anyone that puts into that tube enormous amounts of material, enormous amounts of material."

Ted Stevens, Senator, US Congress

Internet Coding



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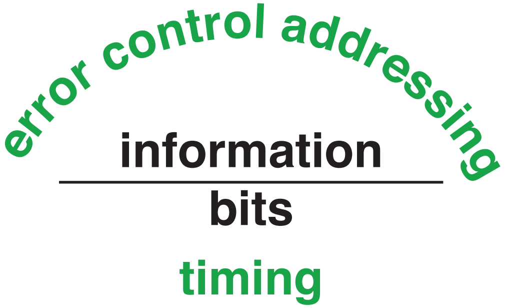

Goyal/Kovačević/Vetterly

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What we need are frames which provide efficient reconstruction after erasures.

Erasures

Goyal-Kovačević

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Definition

A frame $(\phi_m)_{m\in I}$ is robust to k-erasures if for every $J \subset I$, $|J| = k$, the family $(\phi_m)_{m\in I\setminus J}$ is still a frame.

Major Problems

Problem

Find the equal-norm tight frames which are robust to k-erasures.

See C/Kovačević/Bodmanm/Paulsen/Heath/Kutyniok/...

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We need low computational complexity.

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Find the equal-norm tight frames which are robust to k-erasures.

See C/Kovačević/Bodmanm/Paulsen/Heath/Kutyniok/...

Bigger Problem

We need low computational complexity.

Biggest Problem

We also need good estimates on the behavior of reconstruction operators after erasures as well as accounting for quantization errors.

A Strengthening of KS

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There is a universal constant $0 < A$ and an integer $K > 2$ so that every unit norm K-tight frame $\{\phi_i\}_{i=1}^{KN}$ for $\mathbb{H}_N$ can be partitioned into two subsets
each of which have lower frame bound A.

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Important: A, K must be independent of N.

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if and only if

$$\sum_{i\in J^c} |\langle \phi, \phi_i\rangle|^2 \le (K-A)\|\phi\|^2.$$

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But we did not know if they were equivalent.
We thought that CT might actually be formally stronger than KS.

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there is a partition $(A_j)_{j=1}^{r}$ of $\{1, 2, \ldots, Kn\}$ so that

$$\sum_{i \in A_j} \phi_i \phi_i^* \leq (1 - \delta) \cdot I.$$

Recent Announcement

A consortium led by Muriel Medard, a Professor at MIT's Research Laboratory of Electronics and a leader in the effort, includes researchers at MIT, the University of Porto in Portugal, Harvard University, Caltech, and the Technical University of Munich is licensing a new technology designed to deal with lost packets (erasures) during wireless transmission and expects this to be a quantum leap forward in the area. This work rests on doing reconstruction in packet based wireless networks after erasures.