

The Kadison-Singer Problem in Mathematics and
Engineering
Lecture 3: The Kadison-Singer Problem in Engineering
The Casazza/Tremain Conjecture and the Feichtinger
Conjecture

Master Course on the Kadison-Singer Problem
University of Copenhagen

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Signal Processing

The best way to understand **signal processing** is to attend a piano concert.

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Pictures by: Music With Ease <http://www.musicwithease.com/>

To the Pianist

We are hearing a **continuous acoustical signal** (If we ignore the banging of the keys).

To the Pianist

We are hearing a **continuous acoustical signal** (If we ignore the banging of the keys).

To the pianist, the **concert** is a collection of black dots on a piece of paper.

Sheet Music:

The image shows a snippet of sheet music for a piano piece. It consists of two staves, a treble clef on top and a bass clef on the bottom. The key signature has two flats (B-flat and E-flat), and the time signature is 2/4. The music is written in a style that emphasizes the physical act of playing, with many notes beamed together in chords and some notes marked with a fermata. The first measure of each staff shows a series of chords. The second measure has a fermata over a single note. The third measure has another series of chords. The fourth measure has a fermata over a single note. The fifth measure has a series of chords. The sixth measure has a fermata over a single note. The seventh measure has a series of chords. The eighth measure has a fermata over a single note. The ninth measure has a series of chords. The tenth measure has a fermata over a single note. The eleventh measure has a series of chords. The twelfth measure has a fermata over a single note. 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In the Audience

If we were fast enough, we could write the sheet music as the concert is being played.



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If we were fast enough, we could write the sheet music as the concert is being played.



Then, when we get home, we could use our sheet music to replay (i.e. **reconstruct**) the concert.

Correcting Mistakes?

But what if the pianist made a **mistake**?

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If she left out some notes, we could just add them.

Dennis Gabor 1946

To Gabor, sheet music was made up of just **one note**.

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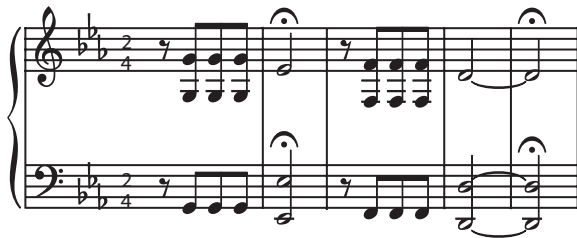
Sheet Music:

The image shows a sheet music score for a piece by Dennis Gabor, 1946. The music is written for piano in 2/4 time, featuring a single note (G4) repeated throughout. The score is presented in two staves, treble and bass clef, with a grand staff bracket on the left. The key signature is one flat (B-flat). The time signature is 2/4. The music consists of a single note (G4) repeated throughout, with various rhythmic patterns and articulations. The first measure is a quarter rest followed by a quarter note G4. The second measure is a quarter rest followed by a quarter note G4. The third measure is a quarter rest followed by a quarter note G4. The fourth measure is a quarter rest followed by a quarter note G4. The fifth measure is a quarter rest followed by a quarter note G4. The sixth measure is a quarter rest followed by a quarter note G4. The seventh measure is a quarter rest followed by a quarter note G4. The eighth measure is a quarter rest followed by a quarter note G4. 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Dennis Gabor 1946

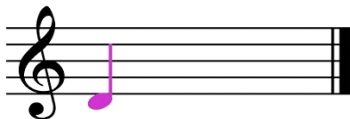
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Sheet Music:

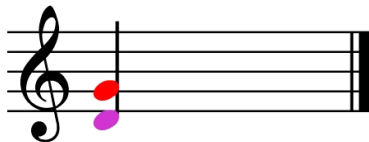


We take that note and change its **modulation**, then **shift it in time** and change its modulation and continue.

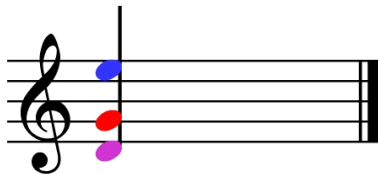
Our Basic Note



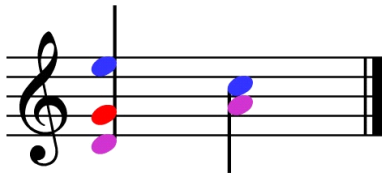
We Change the Modulation of Our Note



Change the Modulation again



Shift our note in time and change modulation



Gabor's Idea

We will write **sheet music** for a signal.

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Fix $g \in (L^2(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{\mathbb{R}} |f(t)|^2 dt < \infty\}) \cap L^\infty(\mathbb{R})$ —our **NOTE**

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Let $f \in L^2(\mathbb{R})$ —our **SIGNAL**

Fix $0 < a, b \leq 1$.

Take modulations of our “note” and compute the intensity of our signal for each of these:

$$\left(\langle f, e^{2\pi i a n t} g(t) \rangle \right)_{n \in \mathbb{Z}}$$

Continue

Now, take a **translation in time** of our note and for all modulations of our “translated note”, compute the intensity of our signal for each of these:

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$$\left(\left\langle f, e^{2\pi i a n t} g(t - b) \right\rangle \right)_{n \in \mathbb{Z}}$$

Continuing, we **digitalize our signal**:

$$\left(\left\langle f, e^{2\pi i a n t} g(t - mb) \right\rangle \right)_{m, n \in \mathbb{Z}}$$

For This to Work We Need

- 1 Our “digits” are **unique** to the signal.

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- ① Our “digits” are **unique** to the signal.
- ② We have **“fast” reconstruction** of the signal from its digits.

For This to Work We Need

- 1 Our “digits” are **unique** to the signal.
- 2 We have **“fast” reconstruction** of the signal from its digits.

This requires that

$$(e^{2\pi i a n t} g(t - mb))_{m,n \in \mathbb{Z}}$$

is a frame for $L^2(\mathbb{R})$ called a **Gabor Frame** and denoted (g, a, b) .

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Unfortunately for Gabor,

What Did Gabor Do?

Gabor used Gaussians $e^{-\pi t^2}$ and $a = b = 1$.

Unfortunately for Gabor, this is a case which **doesn't work**.

Balian-Low Theorem

If $(g, 1, 1)$ is a Gabor frame for $L^2(\mathbb{R})$ then either $tg(t) \notin L^2(\mathbb{R})$ or $g' \notin L^2(\mathbb{R})$.

Time Frequency Analysis

Time frequency analysis is the **mathematics** of signal processing.

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Time frequency analysis is the **mathematics** of signal processing.

Major Problem

Classify all (g, a, b) which give Gabor frames.

Classifying Gabor Frames is a **Very Difficult** Problem

Theorem (C/Kalton)

Classifying the Gabor Frames of the form $(\chi_E, 1, 1)$

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Littlewood's Problem

(1977) Classify the integer sets $\{n_1 < n_2 < \dots < n_k\}$ so that

$$f(z) = \sum_{j=1}^k z^{n_j}$$

does not have any zeroes on the unit circle.

Gabor Frames

Theorem (Rieffel)

If (g, a, b) is a Gabor frame then $ab \leq 1$.

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If (g, a, b) is a Gabor frame then $ab \leq 1$.

Theorem

If $ab = 1$ and (g, a, b) is a Gabor frame then it is a Riesz basis.

How do we clean up a signal?

One Possibility: **Thresholding.**

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See: K. Gröchenig, *Foundations of Time Frequency Analysis*, Birkhäuser (2000).

(2004) e-mail: Hans to Pete

Every Gabor frame I know can be partitioned into a finite number of Riesz basic sequences. Do you think this is always true?

Feichtinger Conjecture

Feichtinger Conjecture (FC)

Every unit norm frame is a finite union of Riesz Basic Sequences.

Equivalent Formulations

Every unit norm Bessel sequence is a finite union of Riesz basic sequences.

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Then $(\phi_i) \cup (e_i)$ is a unit norm frame.

Equivalent Formulations

Every unit norm Bessel sequence is a finite union of Riesz basic sequences.

Proof: Given a unit norm Bessel sequence (ϕ_i) let (e_i) be an orthonormal basis for \mathbb{H} .

Then $(\phi_i) \cup (e_i)$ is a unit norm frame.

Partition this into a finite union of Riesz basic sequences.

Partial Answer

Theorem (C/Christensen)

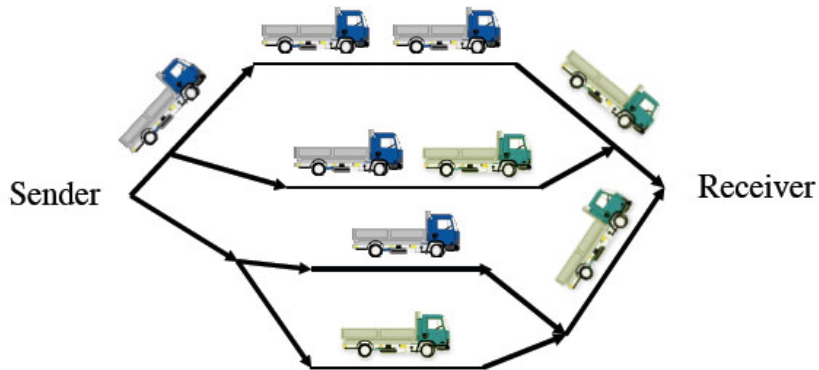
If ab is rational and (g, a, b) is a Gabor frame, then this is a finite union of Riesz basic sequences.

Kadison-Singer in Engineering: Internet Coding

“And again, the internet is not something you just dump something on. It’s not a truck. It’s a series of tubes. And if you don’t understand those tubes can be filled and if they are filled, when you put your message in, it gets in line and its going to be delayed by anyone that puts into that tube enormous amounts of material, enormous amounts of material.”

Ted Stevens, Senator, US Congress

Internet Coding



error control addressing
information

bits
timing

Goyal/Kovačević/Vetterly

Speeding up the Internet

Problem: Can we speed up the internet by encoding with a frame instead of an orthonormal basis?

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What we need are frames which provide efficient reconstruction after erasures.

Erasures

Goyal-Kovačević

An equal norm frame minimizes mean squared error after erasures **if and only if** it is tight.

Erasures

Goyal-Kovačević

An equal norm frame minimizes mean squared error after erasures **if and only if** it is tight.

Definition

A frame $(\phi_m)_{m \in I}$ is **robust to k -erasures** if for every $J \subset I$, $|J| = k$, the family $(\phi_m)_{m \in I \setminus J}$ is still a frame.

Major Problems

Problem

Find the equal-norm tight frames which are robust to k -erasures.

See [C/Kovačević/Bodmanm/Paulsen/Heath/Kutyniok/...](#)

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We also need good estimates on the behavior of reconstruction operators after erasures as well as accounting for quantization errors.

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There is a universal constant $0 < A$ and an integer $K > 2$ so that every unit norm K -tight frame $\{\phi_i\}_{i=1}^{KN}$ for \mathbb{H}_N can be partitioned into two subsets each of which have lower frame bound A .

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Important: A, K must be independent of N .

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We thought that CT might actually be **formally stronger** than KS.

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there is a partition $(A_j)_{j=1}^r$ of $\{1, 2, \dots, Kn\}$ so that

$$\sum_{i \in A_j} \phi_i \phi_i^* \leq (1 - \delta) \cdot I.$$

Recent Announcement

A consortium led by Muriel Medard, a Professor at MIT's Research Laboratory of Electronics and a leader in the effort, includes researchers at MIT, the University of Porto in Portugal, Harvard University, Caltech, and the Technical University of Munich is licensing a new technology designed to deal with lost packets (erasures) during wireless transmission and expects this to be a quantum leap forward in the area. This work rests on doing reconstruction in packet based wireless networks after erasures.